

The Master Field for Large N Matrix Models and Quantum Groups

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Abstract

In recent works by Singer, Douglas and Gopakumar and Gross an application of results of Voiculescu from non-commutative probability theory to constructions of the master field for large N matrix field theories have been suggested. In this note we consider interrelations between the master field and quantum groups. We define the master field algebra and observe that it is isomorphic to the algebra of functions on the quantum group $SU_q(2)$ for $q = 0$. The master field becomes a central element of the quantum group Hopf algebra. The quantum Haar measure on the $SU_q(2)$ for any q gives the Wigner semicircle distribution for the master field. Coherent states on $SU_q(2)$ become coherent states in the master field theory.

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1 Introduction

Recently it has been a reveal of interest in the theory of master field describing the large N limit of matrix models [1]-[7]. Much of the interest in large N expansions is motivated by the desire to find reliable methods for analyzing the dynamics of QCD [8]. QCD at large N provides phenomenologically an appealing picture of strong interactions. In fact $\frac{1}{N}$ provides the only known expansion parameter which can be used in calculations of hadronic properties [9, 10]. Different methods has been proposed for finding the large N limit of various theories [11]-[18]. However all of them are effective only for low-dimensional models. Mathematical structures appearing in the large N limit are related with free independent algebras and so called free or Boltzmannian Fock space [13, 19] deserve a thorough study. An investigation of these questions have been performed in non-commutative probability theory [20, 6]. A model of quantum field theory with interaction in the free Fock space has been considered in [7].

The master field Φ for the Gaussian matrix model is defined by the relation

$$\lim_{N \rightarrow \infty} \frac{1}{Z_N} \int \frac{1}{N^{1+k/2}} \text{tr} M^k e^{-S(M)} dM = \langle 0 | \Phi^k | 0 \rangle \quad (1.1)$$

for $k = 1, 2, \dots$, where the action $S(M) = \frac{1}{2} \text{tr} M^2$ and M is an Hermitian $N \times N$ matrix.

The operator $\Phi = a + a^*$ acts in the so called free or Boltzmannian Fock space with creation and annihilation operators satisfying the relation

$$aa^* = 1 \quad (1.2)$$

The relations (1.1), (1.2) have been obtained in physical [13] and mathematical [19] works. It can be interpreted as a central limit theorem in non-commutative probability theory, for a review see [6]. The basic notion of non-commutative probability theory is an algebraic probability space, i.e a pair (A, h) where A is an algebra and h is a positive linear functional on A . An example of the algebraic probability space is given by the algebra of random matrix with (1.1) being non-commutative central limit theorem. As another example one can consider quantum groups. Theory of quantum groups have received in the last years a lot of attention [22]-[25]. In this case A is the Hopf algebra of functions on the quantum group and h is the quantum Haar measure.

In this note we discuss relations of theory of the master field in the Boltzmannian Fock space with quantum groups. If one has q -deformed canonical commutation relations (see for example [26])

$$aa^* - qa^*a = 1 \quad (1.3)$$

then for $q = 0$ one gets the relation (1.2) in the Boltzmannian Fock space.

We will show that in fact one has more. In Sect.2 we define an algebra describing the master field (the master field algebra) and show that this algebra is isomorphic to the algebra of functions on the quantum group $SU_q(2)$ for $q = 0$. In fact the master field algebra coincides with the algebra of so called central elements of quantum group Hopf algebra. It is interesting, that the transfer matrix in quantum inverse transform method [23] is the central element of Yangian Hopf algebra. In this sense the transfer matrix is an analog of the master field. In Sect.3 we show how the canonical master field algebra is related with quantum groups. In Sect.4 we demonstrate how the quantum group methods can be used to perform calculations in the large N limit of matrix models. Namely, the

quantum Haar measure on the $SU_q(2)$ for any q gives the Wigner semicircle distribution for the master field. Coherent states on $SU_q(2)$ become coherent states in the master field theory.

2 The Master Field Algebra and $Fun(SU_q(2))$

The free (or Boltzmannian) Fock space F over the Hilbert space H is just the tensor algebra

$$F = \bigoplus_{n=0}^{\infty} H^{\otimes n}.$$

Creation and annihilation operators are defined as

$$a^*(f)f_1 \otimes \dots \otimes f_n = f \otimes f_1 \otimes \dots \otimes f_n$$

$$a(f)f_1 \otimes \dots \otimes f_n = \langle f, f_1 \rangle \otimes f_2 \otimes \dots \otimes f_n$$

where $\langle f, g \rangle$ is the inner product in H . We shall consider the simplest case $H = C$. One has the vacuum vector $|0\rangle$,

$$a|0\rangle = 0 \tag{2.1}$$

and the relations

$$aa^* = 1, \tag{2.2}$$

$$a^*a = 1 - |0\rangle\langle 0|. \tag{2.3}$$

We shall reformulate equations (2.1), (2.2), (2.3) in the algebraic form. Let us define an operator

$$F = e^{i\phi}|0\rangle\langle 0|, \tag{2.4}$$

where ϕ is an arbitrary real number. Then from (2.1), (2.2), (2.3) one has the following relations:

$$\begin{aligned} aF &= 0, & aF^* &= 0, & FF^* &= F^*F, \\ aa^* &= 1, & a^*a + FF^* &= 1. \end{aligned} \tag{2.5}$$

We call the algebra (2.5) the master field algebra. From equations (2.5) we get

$$(FF^*)^2 = FF^*$$

and the operator FF^* is an orthogonal projector.

Now let us recall the definition of algebra of functions $A_q = Fun(SU_q(2))$ on the quantum group $SU_q(2)$. The algebra A_q is the Hopf algebra with generators a, a^*, c, c^* satisfying the relations

$$\begin{aligned} ac^* &= qc^*a, & ac &= qca, & cc^* &= c^*c, \\ a^*a + cc^* &= 1, & aa^* + q^2cc^* &= 1 \end{aligned} \tag{2.6}$$

where $0 < |q| < 1$. One can get the relations (2.6) as the unitarity condition

$$gg^* = g^*g = 1 \tag{2.7}$$

for the matrix $g = (g_j^i)$ in the following canonical form

$$g = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}. \tag{2.8}$$

A_q is a Hopf algebra with the standard coproduct,

$$\begin{aligned}\Delta &: A_q \rightarrow A_q \otimes A_q \\ \Delta(g_j^i) &= \sum_{k=0,1} g_k^i \otimes g_j^k\end{aligned}$$

i.e.

$$\begin{aligned}\Delta(a) &= a \otimes a - qc^* \otimes c, \\ \Delta(c) &= c \otimes a + a^* \otimes c,\end{aligned}$$

with counit $\epsilon : A_q \rightarrow C$, $\epsilon(g_j^i) = \delta_j^i$, involution $*$ and antipode $S : A_q \rightarrow A_q$, $Sg = g^*$.

Taking $q = 0$ in (2.6) one gets the relations (2.5) if $F = c$. Therefore the master field algebra (2.5) is isomorphic to the algebra A_0 of functions on the quantum group $SU_q(2)$ for $q = 0$.

Let σ denote the flip automorphism of $A_q \otimes A_q$:

$$\sigma(x \otimes y) = y \otimes x$$

for any $x, y \in A_q$ and

$$A_q \otimes_{sym} A_q = \{z \in A_q \otimes A_q : \sigma(z) = z\}.$$

We say that x is the central element of the bialgebra A if $\Delta(x) \in A \otimes_{sym} A$. An element $x \in A_q = Fun(SU_q(2))$ is central if and only if x is a linear combination of characters.

The master field

$$\Phi = a + a^*$$

is the central element of the Hopf algebra A_q because

$$\Delta(\Phi) = a \otimes a + a^* \otimes a^* \in A_q \otimes_{sym} A_q.$$

Notice that for $q = 0$ one has more central elements in $Fun(SU_q(2))$ than for $|q| > 0$. For example a and a^* are central elements since

$$\Delta(a) = a \otimes a.$$

The bosonization of the quantum group $SU_q(2)$ was considered in [27]. If b and b^* are the standard creation and annihilation operators in the Bosonic Fock space,

$$[b, b^*] = 1, \quad b|0\rangle = 0,$$

then

$$a = \sqrt{\frac{1 - q^{2(N+1)}}{N+1}}b, \quad c = e^{i\phi}q^N \tag{2.9}$$

satisfies the relations (2.6). Here $N = b^*b$, ϕ is a real number. If $q \rightarrow 0$ one gets from (2.9)

$$a = \frac{1}{\sqrt{N+1}}b, \quad c = e^{i\phi}|0\rangle\langle 0|. \tag{2.10}$$

Therefore the master field takes the form

$$\Phi = b^* \frac{1}{\sqrt{N+1}} + \frac{1}{\sqrt{N+1}}b. \tag{2.11}$$

3 Canonical Master Field Algebra

The relations for the master field for $D=0$ and $g\phi^4$ interaction have the form [13, 14]

$$[\pi, \phi] = -i|0 \rangle \langle 0| \quad (3.1)$$

$$(i\pi + \frac{1}{2}\phi + g\phi^3)|0 \rangle = 0, \quad (3.2)$$

where ϕ and π are Hermitian

$$\phi^* = \phi, \quad \pi^* = \pi \quad (3.3)$$

One rewrites them as

$$[\pi, \phi] = -iP, \quad (3.4)$$

$$(i\pi + \frac{1}{2}\phi + g\phi^3)P = 0, \quad (3.5)$$

$$P^2 = P, \quad P^* = P. \quad (3.6)$$

Here we discuss only the case without interaction

$$(i\pi + \frac{1}{2}\phi)P = 0. \quad (3.7)$$

Let us define

$$a = \frac{1}{2}\phi + i\pi, \quad a^* = \frac{1}{2}\phi - i\pi. \quad (3.8)$$

Then equation (3.4) is equivalent to

$$[a, a^*] = P \quad (3.9)$$

and (3.7) is equivalent to

$$aP = 0. \quad (3.10)$$

If one has an irreducible representation of (3.4), (3.7) or (3.8), (3.9) such that P is projector on the cyclic vector $| \rangle$, then there is a relation

$$\frac{1}{4}\phi^2 + \pi^2 = 1 - \frac{1}{2}P, \quad (3.11)$$

or equivalently,

$$aa^* = 1. \quad (3.12)$$

Indeed, by acting equation (3.9) to the vector $|0 \rangle$ one gets $aa^*|0 \rangle = |0 \rangle$. Then $aa^*a^*|0 \rangle = (a^*a + P)a^*|0 \rangle = a^*|0 \rangle$, etc. Therefore for an irreducible representation with a cyclic vector one has algebras (3.4), (3.7), (3.11) and equivalently (3.9), (3.10), (3.12). Let us note that from (3.9) it follows that $P^* = P$, and from (3.9), (3.10), (3.12) it follows that $P^2 = P$.

The coproduct for ϕ, π, P elements reads

$$\begin{aligned} \Delta(\phi) &= \frac{1}{2}\phi \otimes \phi - 2\pi \otimes \pi \\ \Delta(\pi) &= \frac{1}{2}(\phi \otimes \pi + \pi \otimes \phi) \\ \Delta(P) &= P \otimes 1 + 1 \otimes P - P \otimes P \end{aligned} \quad (3.13)$$

These elements are central.

4 The Haar Measure on $SU_q(2)$ and the Wigner Semi-circle Distribution.

The quantum Haar measure h is an invariant state on the algebra A of functions on the quantum group, i.e. it satisfies the condition

$$h = (h \otimes id)\Delta = (id \otimes h)\Delta. \quad (4.1)$$

In particular by acting to the element g_j^i of the algebra A_q this equality reads

$$h(g_j^i) = \sum_{k=0,1} h(g_k^i)g_j^k = \sum_{k=0,1} g_k^i h(g_j^k) \quad (4.2)$$

The quantum Haar measure on compact quantum groups was constructed by Woronovicz [24] in the following way. One defines the convolution of two linear functionals ρ and χ on the Hopf algebra A as

$$(\rho * \chi)(f) = (\rho \otimes \chi)(\Delta(f)), \quad f \in A.$$

One denotes

$$\rho^{*k} = \rho * \rho * \dots * \rho$$

One proves that for any $f \in A$ there exists the limit

$$h_\rho(f) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \rho^{*k}(f) \quad (4.3)$$

that, for the faithful state, does not depend on the state ρ and defines the quantum Haar measure. One can interpret (4.3) as the central limit theorem in non-commutative probability theory.

By using the bosonization formula (2.9) for a and c the functional $h(f)$ can be written as

$$h(f) = \frac{Tr f e^{-\beta H}}{Tr e^{-\beta H}};$$

where

$$Tr f = \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} \langle n | f | n \rangle d\phi, \\ H = 2N,$$

and $|n\rangle$ are n -particle oscillator states, $N|n\rangle = n|n\rangle$. Therefore the quantum Haar functional is the Gibbsian state. In particular the partition function is

$$Z = Tr e^{-\beta N} = \frac{1}{1 - e^{-2\beta}}.$$

We denote

$$\int_{SU_q(2)} f d\mu = h(f) \quad (4.4)$$

Theory of representations of the quantum group $SU_q(2)$ [24, 28, 29] is similar to the theory of representations of the classical group $SU(2)$. An irreducible representation is

characterized by its dimension $2l + 1, l = 0, \frac{1}{2}, 1, \dots$. There exists an explicit construction of $(2l+1) \times (2l+1)$ matrix $W_{km}^l, k, m = -l, \dots, l$, such that

$$\Delta(W_{ij}) = \sum_k W_{ik} \otimes W_{kj}. \quad (4.5)$$

Operators W^l satisfy the following orthogonality relations

$$\langle (W_{mn}^j)^* W_{m'n'}^{j'} \rangle_q = \delta_{jj'} \delta_{mm'} \delta_{nn'} \frac{q^{-2m}}{[2j+1]_q}, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \quad (4.6)$$

The Hopf algebra A of polynomial on $SU_q(2)$ has an orthogonal decomposition (the quantum group analog of the Peter-Weyl theorem)

$$A = \bigoplus_{l \in N/2} W^l$$

with respect to the quantum Haar measure where W^l is spanned by matrix elements W_{mn}^l . An element f from A has the Fourier expansion

$$f = \sum_{l \in N/2} [2j+1]_q \text{Tr}_q(\tilde{f}^l W^l), \quad (4.7)$$

$$\tilde{f}_{mn}^l = \int_{SU_q(2)} f W_{mn}^{l*} d\mu.$$

For central functions one has a decomposition over characters

$$f(a + a^*) = \sum_{l \in N/2} \tilde{f}^l \chi_l(a + a^*) \quad (4.8)$$

The characters are related with W as

$$\chi_l = \sqrt{\frac{[2l+1]_q}{2l+1}} \sum_{n=-l, \dots, l} q^n W_{nn}^l \quad (4.9)$$

and there is an explicit formula

$$\chi_l(t) = \frac{\sin((l + \frac{1}{2})t)}{\sin(\frac{1}{2}t)} = 1 + 2 \cos t + \dots + 2 \cos lt; \quad (4.10)$$

where $t = 2 \arccos \frac{a+a^*}{2}$. Note that the explicit form (4.10) does not depend on the deformation parameter and coincides with the classical form. There is the recursive relation

$$\chi_{l+1/2}(t) + \chi_{l-1/2}(t) = \chi_{1/2}(t) \chi_l(t) \quad (4.11)$$

for all q . For $q = 0$ this relation gives

$$\chi_l(t)|0 \rangle = (a^*)^{2l}|0 \rangle. \quad (4.12)$$

The orthogonality condition for characters χ_l also has the same form as in the classical case

$$\int_{SU_q(2)} \chi_i^*(t) \chi_{i'}(t) d\mu = \delta_{ii'} = \frac{1}{\pi} \int_0^{2\pi} \chi_i^*(\tau) \chi_{i'}(\tau) (\sin(\frac{\tau}{2}))^2 d\tau \quad (4.13)$$

From (4.8) and (4.13) one notes that to perform the integration of a polynomial function of the central element $a + a^*$ over quantum group one can calculate the integral over classical group of the same polynomial,

$$\int_{SU_q(2)} f(t) d\mu = \frac{1}{\pi} \int_0^{2\pi} f(\tau) (\sin(\frac{\tau}{2}))^2 d\tau, \quad (4.14)$$

or

$$\int_{SU_q(2)} f(a + a^*) d\mu = \frac{1}{2\pi} \int_{-2}^2 f(\lambda) \sqrt{4 - \lambda^2} d\lambda. \quad (4.15)$$

One gets the Wigner semicircle distribution for any q . In particular for $q \rightarrow 0$ in the left hand side of (4.15) one has

$$\int_{SU_0(2)} (a + a^*)^k d\mu = \langle 0 | (a + a^*)^k | 0 \rangle \quad (4.16)$$

and therefore

$$\langle 0 | (a + a^*)^k | 0 \rangle = \frac{1}{2\pi} \int_{-2}^2 \lambda^k \sqrt{4 - \lambda^2} d\lambda. \quad (4.17)$$

that demonstrates the well-known Wigner distribution for the master field from quantum group point of view.

Note that for the case $q = 0$ the set of central functions is more large as compare with the case of arbitrary q . It is spanned by functions on a and a^* . For these central functions one can write down the special form of the Peter-Weyl decomposition.

The relation between the master field algebra and the quantum group $SU_q(2)$ permits to write immediately the coherent states for master field as well as for operators a and a^* . Coherent states for $SU_q(2)$ have the form [27]

$$\Psi(u) = \sum_{j \in N/2} \sqrt{(2j+1)[2j+1]_q} tr_q(W^{j*} T^j(u)) \quad (4.18)$$

Here u is an element of $SU(2)$,

$$g = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}$$

and

$$T^j(u) = (D_{mn}^j(u))$$

is a unitary representation of $SU(2)$ of spin j . From (4.18) and (4.6) it follows that if we introduce the kernel by

$$\mathcal{K}(u, u') = \int_{SU_q(2)} \bar{\Psi}(u) \Psi(u') d\mu \quad (4.19)$$

then this kernel satisfies the superposition relation

$$\int \mathcal{K}(u, u') \mathcal{K}(u', u'') du' = \mathcal{K}(u, u''), \quad (4.20)$$

where du stands for the Haar measure on $SU(2)$.

One can also introduce the coherent states for the master field $a + a^*$

$$\Psi_c(\alpha + \alpha^*) = \sum_{j \in N/2} (\chi_j(a + a^*))^* \chi_j(\alpha + \alpha^*) \quad (4.21)$$

Introducing the kernel

$$\mathcal{K}(\alpha + \alpha^*, \alpha' + \alpha'^*) = \int_{SU_q(2)} \bar{\Psi}_c(\alpha + \alpha^*) \Psi_c(\alpha' + \alpha'^*) d\mu \quad (4.22)$$

we get the superposition property

$$\frac{1}{\pi} \int_0^{2\pi} \mathcal{K}(\tau, \tau') \mathcal{K}(\tau', \tau'') (\sin(\frac{\tau'}{2}))^2 d\tau' = \mathcal{K}(\tau, \tau''), \quad (4.23)$$

$$\tau = 2 \arccos \frac{\alpha + \alpha^*}{2}$$

From (4.12) for $q = 0$ we have

$$\Psi_c(\alpha + \alpha^*) |0\rangle = \sum_{j \in N/2} \chi_j(\alpha + \alpha^*) (a^*)^{2j} |0\rangle \quad (4.24)$$

and

$$\mathcal{K}(\alpha + \alpha^*, \alpha' + \alpha'^*) = \sum_{j \in N/2} (\chi_j(\alpha + \alpha^*))^* \chi_j(\alpha' + \alpha'^*) \quad (4.25)$$

In conclusion, in this paper we discussed a relation between the simplest master field and quantum group $SU_q(2)$. It would be interesting to extend such a relation to more general master fields and quantum groups.

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