On the Structure of Classical and Quantum Flows

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In the framework of quantum probability, stochastic flows on manifolds and the interaction representation of quantum physics become unified under the notion of *Markov cocycle*. We prove a structure theorem for σ-weakly continuous Markov cocycles which shows that they are solutions of quantum stochastic differential equations on the largest *-subalgebra, contained in the domain of the generator of the Markov semigroup, canonically associated to the cocycle. The result is applied to prove that any Markov cocycle on the Clifford bundle of a compact Riemannian manifold, whose structure maps preserve the smooth sections and satisfy some natural compatibility conditions, uniquely determines a family of smooth vector fields and a connection, with the property that the cocycle itself is induced by the stochastic flow along the paths of the classical diffusion on the manifold, defined by these vector fields and by the Ito stochastic parallel transport associated to the connection. We use the language and techniques of quantum probability but, even when restricted to the classical case, our results seem to be new. © 1996 Academic Press, Inc.

1. Introduction

Let $(\Omega, \mathscr{F}, P)$ be the standard sample space of the d-dimensional Wiener process and let $(\mathscr{F}_{t]})$ and $(\mathscr{F}_{[t})$ denote respectively the past and future filtration generated by the brownian paths. Let $(M, \mathscr{B}, \mu)$ (or (M, μ) for short) be a measure space and, for each $p \in M$, let $X = \{ X(t, p, \omega), \ t \geqslant 0, X(p, 0) = p \}$ be a stochastic flow on M, i.e. a process such that for all $s, t \geqslant 0$

$$X(p, s + t, \omega) = X(X(p, s, \omega), t, \theta_s(\omega)) \qquad \text{a.e. in } \Omega, \qquad (1.1)$$

where $\theta_s(\omega)(t) := \omega(s + t) - \omega(s) \ (\omega \in \Omega)$. Assuming that the process preserves the null sets of μ, we define the family $(j_t, t \geqslant 0), j_t \colon L^\infty(M, \mu) \to L^\infty(M, \mu) \otimes \mathscr{B}(L^2(\Omega, \mathscr{F}, P)) = L^\infty(M, \mu; \mathscr{B}(L^2(\Omega, \mathscr{F}, \mathscr{P}))) \ t \geqslant 0$ of *-homomorphisms by

$$j_t(\phi)(p, \omega) := \phi(X(p, t, \omega)), \qquad \phi \in L^\infty(M, \mu), \qquad (1.2)$$

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where the tensor product is in the sense of Sakai (cf. [Sak, Definition 1.22.10]) and the right hand side of (1.2) is interpreted as a multiplication operator on the Hilbert space $L^2(M, \mu) \otimes L^2(\Omega, \mathscr{F}, P)$. Denoting $\hat{j}_t$ the trivial extension of j_t to $L^\infty(M, \mu) \otimes \mathscr{B}(L^2(\Omega, \mathscr{F}_{[t}, P))$, defined by

$$\hat{j}_t(\phi \otimes a_{[t}) := j_t(\phi) \otimes a_{[t}, \qquad \phi \in L^\infty(M, \mu), \ a_{[t} \in \mathscr{B}(L^2(\Omega, \mathscr{F}_{[t}, P)) \quad (1.3)$$

and identifying L^∞ with $L^\infty \otimes 1$, we can rephrase the flow equation (1.1) as

$$\hat{j}_{s+t}(\phi) = j_{s+t}(\phi) = \hat{j}_s(u_s^o(j_t(\phi))) = \hat{j}_s(u_s^o(\hat{j}_t(\phi))), \qquad (1.3a)$$

where u_t^o is the time shift induced by θ_t on $L^\infty(M, \mu) \otimes \mathscr{B}(L^2(\Omega, \mathscr{F}, P))$ (cf. (2.3c)).

Keeping in mind that u_t^o acts trivially on $L^\infty(M, \mu)$, equation (1.3) can be also rephrased as

$$\hat{j}_{s+t} = \hat{j}_s \circ u_s^o \circ \hat{j}_t \circ u_{-s}^o \qquad (1.4)$$

where u_{-s}^o denotes the left inverse of u_s^o.

If we replace the commutative algebra $L^\infty(M, \mu)$ by a von Neumann $\mathscr{A}_o$, of bounded operators on a separable Hilbert space, and the algebra $\mathscr{B}(L^2(\Omega, \mathscr{F}, \mathscr{P}))$ by an arbitrary von Neumann algebra $\mathscr{B}$, then the equations (1.3a), (1.4) make sense for any one parameter family (j_t) of normal *-homomorphisms $j_t \colon \mathscr{A}_o \to \mathscr{A}_o \otimes \mathscr{B}$ and we take them as the definition of a 1-(u_t^o)-*cocycle* (or *cocycle* for short).

Many of the results we are going to discuss in the following are valid also for C^*-algebras, but to fix the ideas we shall limit our discussion to the case in which $\mathscr{A}_o$ is a von Neumann algebra and j_t is normal.

Moreover we shall fix the von Neumann algebra $\mathscr{B}$ to be the algebra of all bounded operators on the space of all square integrable functionals of the increments of the standard Wiener process. It is well known that there is a canonical isomorphism between this space and the Fock space $\Gamma(L^2(\mathbf{R}_+))$ over the space $L^2(\mathbf{R}_+)$. On this algebra $\mathscr{B}$ there is a natural past filtration $(\mathscr{B}_{t]})$ and a localization $(\mathscr{B}_{[s, t]})$ which are described in detail in Section 2. Intuitively $\mathscr{B}_{t]}$ (resp. $\mathscr{B}_{[s, t]}$) can be thought as the algebra of all bounded operators on the subspace of square integrable functionals of the increments of the Wiener process corresponding to intervals contained in $[0, t]$ (resp. $[s, t]$).

The adaptedness (i.e. $\mathscr{F}_{t]}$-measurability) of the process $X(t, p, \omega)$ implies that

$$j_t(\mathscr{A}_o) \subseteq \mathscr{A}_o \otimes \mathscr{B}_{t]}.$$

This notion does not require the commutativity of $\mathscr{A}_o$ or $\mathscr{B}$ and, introducing the notations

$$\mathscr{A}_{[s,\,t]} := \mathscr{A}_o \otimes \mathscr{B}_{[s,\,t]}, \qquad \mathscr{A}_{t]} = \mathscr{A}_o \otimes \mathscr{B}_{t]}, \qquad \mathscr{A} = \mathscr{A}_o \otimes \mathscr{B},$$

it can be rephrased in terms of $\hat{\jmath}_t$, defined by (1.3a), as

$$\hat{\jmath}_t(\mathscr{A}_{[t,\,u]}) \subseteq \mathscr{A}_{u]} \qquad \forall u \geqslant t \geqslant 0. \tag{1.4a}$$

This property expresses the fact that $(\hat{\jmath}_t)$ is in some sense *localized* in the interval $[0, t]$.

These objects are natural perturbations of Markov conditional expectations in the sense explained in [Ac1]. A cocycle (j_t) with this property is called a *Markov cocycle* (cf. [Ac1]).

When M is a smooth compact Riemannian manifold with the standard volume measure it is well known [Elw], [Eme], [Kun] that the diffusion process associated with a family of smooth vector fields, defines a Markov cocycle. On the other hand any Markov semigroup on a finite set, and many such semigroups on the integers and diffusions in $\mathbb{R}^d$, can be realized as Feynman–Kac semigroups associated to Markov cocycles. For details we refer to ([Bi], [Mey1], [PaS], [Fag], [MoS]).

The notion of Markov cocycle also plays a fundamental role in quantum physics in view of the following considerations: suppose that (j_t) satisfies the algebraic relation (1.4) on the whole algebra $\mathscr{A}$, then the 1-parameter family (u_t), defined by

$$u_t := \hat{\jmath}_t \circ u_t^o, \qquad t \geqslant 0, \tag{1.5}$$

is a 1-parameter semigroup of endomorphisms. Conversely, given two such 1-parameter semigroups (u_t^o), (u_t) such that each u_t^o has an inverse, denoted u_{-t}^o, the 1-parameter family (j_t) defined by

$$j_t := u_t \circ u_{-t}^o, \qquad t \in \mathbf{R}_+, \tag{1.6}$$

is a cocycle in the sense that equation (1.4) is satisfied with $j_t = \hat{\jmath}_t$. Interpreting (u_t^o) as the *free evolution* of a quantum system and (u_t) as the *interacting evolution*, j_t becomes the *wave endomorphism at time t* and its limits as $t \to \pm \infty$ (when they exist) define respectively the backward and forward wave endomorphisms whose composition is the *scattering endomorphism*. When the cocycle is *inner*, i.e., when

$$j_t(x) = V_t^* x V_t, \qquad \forall x \in \mathscr{A}, \tag{1.7}$$

for some unitary or isometric operator V_t, then the cocycle condition (1.4) is implied by the condition

$$V_{s+t} = u_s^o(V_t)\, V_s$$

which defines a (right) *operator cocycle*; one speaks of an *operator Markov cocycle* if, for each $t \geqslant 0$, $V_t \in \mathscr{A}_{[o,\, t]}$. (c.f. Definition (3.5) below). In this case one recovers the notions (more familiar in physics) of *wave operator* and *scattering operator*.

The above considerations show that Markov cocycles are interesting mathematical objects and in recent years the following two problems have been investigated by several authors in the quantum probability literature:

Problem (I)   How to construct Markov cocycles?

Problem (II)   How to give an infinitesimal characterization of Markov cocycles?

Notice that, if the semigroup (u_s^o) is trivial (i.e. the identity), then the cocycle equation (1.4) reduces to the semigroup equation and, if the cocycle is implemented by a strongly continuous 1-parameter unitary semigroup (V_t), then both Problems (I) and (II) above are answered by the Stone theorem according to which

$$V_t = e^{itH}, \qquad \frac{d}{dt} j_t(x) = j_t(i[H, x]) \tag{1.8}$$

for some self-adjoint operator H (and x in the domain of the commutator $[H, \cdot]$).

Thus Problems (I) and (II) above constitute a quantum stochastic generalization of Stone theorem.

The first breakthrough concerning Problem (I) occurred when Hudson and Parthasarathy [HuPa 1] showed that the quantum stochastic calculus developed by them provides a powerful tool to construct inner Markov cocycles as solutions of quantum stochastic differential equations. This result was extended to general Markov cocycles by Evans and Hudson [EvHu 1] who showed that one can construct Markov cocycles on $\mathscr{A}_o$ by solving a quantum stochastic differential equation of the form

$$dj_t(x) = j_t(\theta^{ji}(x)) \, d\Lambda_{ij}(t) = j_t(dL_t(x)), \qquad x \in \mathscr{A}_o, \tag{1.9}$$

where the θ^{ji} are linear maps of $\mathscr{A}_o$ into itself, called *structure maps*, and satisfying some algebraic relations and the $\Lambda_{ij}(t)$ are the (creation, number, anhilation) processes introduced in [HuPa 1], [HuPa 2] (c.f. Section (2.) below).

When $dL_t(x) = i[H, x] \, dt$ equation (1.9) reduces to (1.8) and this leads to the notion of *stochastic derivation* [Ac Hu].

The early papers dealt with the case of bounded structure maps and a considerable literature has been subsequently devoted to extend the

construction of Markov cocyles to larger and larger classes of unbounded maps [Moh, FaSi].

The first nontrivial result concerning Problem (II), in the case of operator cocycles, is due to J. L. Journé [Jou] (see also [HuLi], [AcJouLi]). Journé also produced a counterexample showing that, in general, strong continuity of a cocycle is not sufficient to guarantee that it satisfies a stochastic differential equation. The algebraic analysis of Problem (II), developed in [AcHu] suggested that, under suitable regularity conditions, any Markov cocycle should satisfy an equation of the form (1.9) and this conjecture was confirmed, in [Br], in the case of bounded structure maps, i.e. in the case when the associated Markov semigroup is differentiable in norm. However, both in probability and in physics, the most interesting flows are only σ-weakly continuous and the associated structure maps, when they exist, are not bounded.

This is precisely the case we are going to consider in the present paper. More precisely, we wish to answer the following question: *when does a cocycle satisfy an equation of the form* (1.9) *with respect to some family* θ^{ji} *of structure maps*? The converse problem, which is quite simple in the bounded case, but deep in the unbounded case ([FaSi] [FaChe]) shall not be dealt with in this paper.

Our main results (c.f. Theorem (3.9) below) consists in showing that any injective, σ-weakly continuous Markov cocycle satisfies an equation of the form (1.9) on the largest *-algebra $\mathscr{B}_o$ contained in the domain of the generator of the Markov semigroup canonically associated to the cocycle via the quantum Feynman–Kac formula.

The second part of the present paper, starting from Section 4, contains an application of our representation theorem for Markov cocycles to the theory of stochastic parallel transports along the paths of a diffusion on a smooth Riemannian manifold. The problem that we consider here, which turns out to be intimately related to that of *imprimitivity systems* [AcMoh] goes in the direction of giving a quantum probabilistic characterization of the classical stochastic parallel transport. We begin by formulating this theory in the algebraic language of quantum probability: this leads to a Markov cocycle on the tensor bundle and, with this, to a hopefully natural notion of *quantum stochastic parallel transport*. A different notion of quantum stochastic parallel transport was proposed by D. Applebaum [Ap1, 2].

We then specialize our analysis to smooth Clifford section over the tangent bundle of the manifold, which can be equiped with a natural structure of von Neumann algebra. Any connection respecting the metric has a canonical extension to the sections of this bundle and also Ito stochastic parallel transport along the diffusion curves, has a natural extension as a Markov cocycle on this von Neumann algebra. It is then natural to ask

oneself: Does every (injective σ-continuous) Markov cocycle on the Clifford bundle can be realized as a stochastic parallel transport, along the paths of some diffusion on the manifold, with respect to a connection canonically associated to it?

We prove (Theorem (6.6)) that, if the cocycle is graded, has continuous trajectories (in the sense of our Definition (2.8)), and its structure maps preserve the smooth Clifford sections, this is indeed the case and, if moreover it is completely nondeterministic, then the connection and the structure maps are uniquely determined up to an additive derivation of the Clifford bundle, which is identically zero on the smooth functions. Notice that the diffusion on the manifold and the parallel transport *are both deduced* from the flow equation and not postulated ab initio.

2. NOTATIONS AND PRELIMINARIES

In the following all the Hilbert spaces considered are assumed to be complex and separable with inner product $\langle \cdot, \cdot \rangle$ linear in the second variable. For any Hilbert space H, we denote by $\Gamma(H)$ the symmetric (Boson) Fock space over H and $\mathscr{B}(H)$ the algebra of all bounded linear operators in H. For any $u \in H$, we denote by $e(u)$ the exponential vector in $\Gamma(H)$ associated with u:

$$u^{\otimes^o} := \Phi \quad \text{vacuum vector;} \qquad e(u) := \sum_{n \geq 0} \frac{1}{\sqrt{n!}} u^{\otimes^n}.$$

The family $\{e(u): u \in D^{(1)}\}$ is total for any dense linear subspace $D^{(1)}$ of H and linearly independent in $\Gamma(H)$. We denote by $\mathscr{E}(D^{(1)})$ its algebraic linear span. So linear operators may be defined densely on $\Gamma(H)$ by giving their action on $\mathscr{E}(D^{(1)})$. In particular when C is a bounded operator on H and u is an element of H, the second quantization $\Gamma(C)$ of C is determined uniquely by the relations

$$\Gamma(C) e(v) = e(Cv)$$

for all $v \in H$.

We fix two Hilbert spaces $\mathscr{H}_o$, $\mathscr{K}$. For $H = L^2(I, \mathscr{K})$ with $I = \mathbf{R}_+$, $[s, t)$ $(0 \leqslant s \leqslant t)$ we use the notations Γ_+, $\Gamma_{[s, t]}$ for $\Gamma(H)$ respectively and

$$\mathscr{H} = \mathscr{H}_o \otimes \Gamma_+, \qquad \mathscr{H}_{t]} = \mathscr{H}_o \otimes \Gamma_{[0, t]}, \qquad \mathscr{H}_{[t} = \Gamma_{[t, \infty)}.$$

The asymmetry in the definitions of $\mathscr{H}_{t]}$ and $\mathscr{H}_{[t}$ allows us to have *continuous tensor product property* of Γ_+: $\mathscr{H} = \mathscr{H}_{t]} \otimes \mathscr{H}_{[t}$. The Hilbert space

$\mathscr{H}_{t]}$ often will be identified with the subspace $\mathscr{H}_{t]} \otimes \Phi_{[t}$ of $\mathscr{H}$ where $\Phi_{[t}$ is the vacuum vector in $\mathscr{H}_{[t}$. Operators defined on a tensorial factor of $\mathscr{H}$ will be often identified with their canonical ampliations to the whole space and denoted by the same symbol.

We fix dense linear subspaces $\mathscr{D}_o \subseteq \mathscr{H}_o$, $D^{(1)} \subseteq L^2(\mathbf{R}_+, \mathscr{K})$, $\mathscr{K}_o \subseteq \mathscr{K}$ and denote by $\chi_{[s, t]}$ the characteristic function of the interval $[s, t]$.

The algebraic tensor product $\mathscr{D} \otimes \mathscr{E}(D^{(1)})$ is dense in $\mathscr{H}$. Following a widespread use, we often omit the symbol $\otimes$ when dealing with vectors in $\mathscr{H}_o \otimes \mathscr{E}(D^{(1)})$. For example we write $fe(u\chi_{[0, t)})\, e(\chi_{[t, \infty)})$ instead of $f \otimes e(u\chi_{[0, t)}) \otimes e(\chi_{[t, \infty)})$.

DEFINITION (2.1) A family $X \equiv \{X(t): t \geqslant 0\}$ of operators on $\mathscr{H}$ is called an *adapted operator process* with respect to $(\mathscr{D}_o, D^{(1)})$ (or *adapted*, in the following, since $\mathscr{D}_o$ and $D^{(1)}$ are fixed once for all) if for all $t \geqslant 0$, $f \in \mathscr{D}_o$, $u \in D^{(1)}$:

  (a) $\mathscr{D}(X(t)) \supseteq \mathscr{D}_o \otimes \mathscr{E}(D^{(1)})$

  (b) $X(t)\, fe(u\chi_{[0, t]}) \in \mathscr{H}_{t]}$ and $X(t)\, fe(u) = \{X(t)\, fe(u\chi_{[0, t)})\} e(u\chi_{[t, \infty)})$.

X is called *continuous*, (more precisely: strongly continuous on $\mathscr{D}_o \otimes \mathscr{E}(D^{(1)})$) if in addition, the map $t \to X(t)\, fe(u)$ from $\mathbf{R}_+$ into $\mathscr{H}$ is continuous for each $f \in \mathscr{D}_o$, $u \in D^{(1)}$. An adapted process is called *bounded*, *contractive*, *isometric*, *co-isometric* or *unitary* if the operators $X(t)$ have the corresponding property for every $t \geqslant 0$.

We fix an orthonormal basis $\{e_i, i \in S\}$ of $\mathscr{H}_o$, where S is a subset of the integers not containing zero and set using Dirac notation, $e_{ij} = |e_j\rangle \langle e_i|$: $i, j \in S$. Denote by $\bar{S}$ the set obtained by adjoining 0 to S, i.e. $\bar{S} = S \cup \{0\}$, and with respect to this basis we introduce the *integrator processes* $\{\Lambda_{ij}: i, j \in \bar{S}\}$ defined by

$$\Lambda_{ij}(t) = \begin{cases} \Lambda(\chi_{[0, t]} \otimes e_{ij}), & \text{if} \quad i, j \in S, \\ a(\chi_{[0, t]} \otimes e_i), & \text{if} \quad i \in S, \quad j = 0, \\ a^+(\chi_{[0, t]} \otimes e_j), & \text{if} \quad i = 0, \quad j \in S, \\ t1, & \text{if} \quad i = 0 = j. \end{cases} \tag{2.0}$$

In the following we shall use the same symbol to denote the vector $\chi_{[o, t]} \in L^2(\mathbf{R}_+)$ and the multiplication operator by the (bounded measurable) function $\chi_{[o, t]}$. The quantum Ito formula ([HuPa 2], [Ev]) can be expressed as

$$d\Lambda_{ij}\, d\Lambda_{kl} = \hat{\delta}_{il}\, d\Lambda_{kj} \tag{2.1}$$

for all $i, j, k, l \in \bar{S}$ where

$$\hat{\delta}_{il} = \begin{cases} 0, & \text{if} \quad l = 0 \text{ or } i = 0 \\ \delta_{il}, & \text{otherwise} \end{cases}$$

For $j \in S$ we shall use the notations

$$u^j(s) := \langle e_j, u(s) \rangle, \ u_j(s) := \overline{u^j(s)}; \qquad u_o(s) := u^o(s) := 1.$$

DEFINITION (2.2)   We set $D^{(1)} \equiv \{u \in L^2(\mathbf{R}_+ ; \mathscr{K}): u^j(\cdot) = 0$ for all but finitely many $j \in S\}$. For $u \in D^{(1)}$ denote by $N(u) := \{j \in \bar{S}; u^j(\cdot) \not\equiv 0\}$, so $\#N(u) < \infty$, i.e. $N(u)$ is the set of $\mathscr{H}_o$-valued functions whose image is contained in the linear span of a finite subset of the vectors) e_j). A family $L \equiv \{L^{ij}(s): i, j \in \bar{S}\}$ of adapted processes is said to be *square integrable* if each L^{ij} is adapted and for each $f \in \mathscr{D}$, $u \in D^{(1)}$ and $t \geqslant 0$

$$\|L\|_{f, u, t}^2 := \sum_{j \in \mathbf{N}(u)} \sum_{i \in \bar{S}} \int_0^t \|L^{ij}(s) fe(u)\|^2 \, dv_u(s) < \infty, \tag{2.2}$$

where

$$v_u(t) = \int_o^t (1 + \|u(s)\|^2) \, ds.$$

For a square integrable family L we shall use the notations

$$X_L(s, t) = \sum_{i, j \in \bar{S}} \int_s^t L^{ij}(\tau) \, d\Lambda_{ji}(\tau) = \int_s^t L(\tau) \, d\Lambda(\tau) \tag{2.2a}$$

for its stochastic integral. On the class of operators containing $\mathscr{D}_o \otimes \mathscr{E}(D^{(1)})$ in their domains, consider the family of seminorms

$$X \mapsto \|Xfe(u)\|, \qquad \text{resp.} \quad X \mapsto |\langle fe(u), Xge(v) \rangle|,$$

where $f \in \mathscr{D}$ and $u \in \mathscr{D}^{(1)}$. The topology generated these seminorms is called the topology of strong (resp. weak) convergence on $\mathscr{D}_o \otimes \mathscr{E}(D^{(1)})$. The inequality [Pa], [Me]

$$\|X_L(0, t) fe(u)\|^2 \leqslant 2^{v_u(t)} \|L\|_{f, u, t}^2, \tag{2.3}$$

where $\|L\|_{f, u, t}$ is defined by (2.2), shows that for each $t \geqslant 0$ the map $L \to X_L(o, t)$ from the class of adapted processes with $\mathscr{D}^{(1)} \otimes \mathscr{E}(D^{(1)})$ in their domain, to its stochastic integrals, is continuous in the topology defined by the seminorms $\| \cdot \|_{f, u, t}$ to the topology of strong (hence weak) convergence on $\mathscr{D}^{(1)} \otimes \mathscr{E}(D^{(1)})$.

We fix $\mathscr{A}_o$ to be a von Neumann subalgebra of $\mathscr{B}(\mathscr{H}_o)$ and $\mathscr{A}'_o$ be its commutant. Consider the family $\{A_{[s,\,t]} := \mathscr{A}_o \otimes \mathscr{B}_{[s,\,t]} 0 \leqslant s \leqslant t\}$ of local subalgebras of the von Neumann algebra $\mathscr{A} := \mathscr{A}_o \otimes \mathscr{B}$ and set $\mathscr{A}_{t]}$ for $\mathscr{A}_o \otimes \mathscr{B}_{[0,\,t]}$ and $\mathscr{A}_{[t}$ for $\mathscr{A}_o \otimes 1_{[0,\,t]} \otimes \mathscr{B}_{[t}$.

For our present purpose we restrict ourselve to the class of bounded family L_{ij} of adapted processes so that $L^{ij}(t) \in A_{t]}$ for each $t \geqslant 0$ and will extend the basic estimate (2.3). First we set for any vector ξ, in a Hilbert space $\mathscr{H}$,

$$\phi_\xi(x) = \langle \xi, x\xi \rangle, \qquad x \in \mathscr{B}(\mathscr{H})$$

Thus ϕ_ξ is a normal state on $\mathscr{B}(\mathscr{H})$. If $\xi = e(u)$ is an exponential vector, we write ϕ_u instead of $\phi_{e(u)}$. The fact that $\|L^{ij}(s)fe(u)\|^2 = \phi_f \otimes \phi_u(|L^{ij}(s)|^2)$ suggests the natural extension of seminorms in (2.3),

$$\|L\|^2_{\phi,\,u,\,t} := \sum_{j \in \mathbf{N}(u)} \sum_{i \in \overline{S}} \phi \otimes \phi_u(|L^{ij}(s)|^2)\,dv_u(s) \tag{2.3a}$$
$$\|X_L(o, t)\|^2_{\phi,\,u} := \phi \otimes \phi_u(|X_L(0, t)|^2),$$

where ϕ is any positive normal linear functional on $\mathscr{A}_o$ and all the other symbols are described as in (2.2). From (2.3) it follows that, for any process L such that for $t \geqslant 0$, $L(t) \in \mathscr{A}_{t]}$ and for any positive normal linear functional ϕ on $\mathscr{A}_o$), the following inequality holds:

$$\|X_L(0, t)\|^2_{\phi,\,u} = \phi \otimes \phi_u(|X_L(0, t)|^2) \leqslant \|L\|_{\phi,\,u,\,t}. \tag{2.3b}$$

Remark (2.3) In dealing with flows, the topology induced by the seminorms (2.3b) (resp. $X \mapsto |\phi \otimes \phi_u(X(0, t))|$) is sometimes more convenient (cf. the proof of Proposition (2.5) below). We shall call it the topology of strong (resp. weak) convergence on $\mathscr{A}_{o*} \otimes \widetilde{\mathscr{e}}(D_o^{(1)})$, where $\widetilde{\mathscr{e}}(D^{(1)})$ denotes the set of all vectors of the form ϕ_u, with $u \in D^{(1)}$. We shall also use the notation $D_{[s}^{(1)} = \{\chi_{[s}u: u \in D^{(1)}\}$.

Denote by θ_t the right shift on $L^2(\mathbf{R}_+, \mathscr{K})$, so that for all $t \geqslant 0$

$$(\theta_t u)(s) = \begin{cases} u(s-t), & \text{if } s \geqslant t \\ 0, & \text{if } 0 \leqslant s \leqslant t. \end{cases} \tag{2.3c}$$

θ_t is isometric with $\theta_t^* u(s) = u(s+t)$. Using the isomorphism $\mathscr{H}_o \otimes \mathscr{H} \sim \mathscr{H}_o \otimes \mathscr{H}_{[t} \otimes \mathscr{H}_{t]}$ we define the *time shift* as the 1-parameter normal semigroup of endomorphisms of $\mathscr{B}(\mathscr{H}) \equiv \mathscr{B}(\mathscr{H}_o) \otimes \mathscr{B}(\Gamma(L^2(\mathbf{R}_+, \mathscr{K})))$ characterized by the property

$$u_t^o(b_o \otimes b) = b_o \otimes \Gamma(\theta_t)\,b\Gamma(\theta_t^*)|_{\mathscr{H}_{t]}} \otimes 1_{t]},$$
$$b_o \in \mathscr{B}(\mathscr{H}_o), \quad b \in \mathscr{B}(\Gamma(L^2(\mathbf{R}_+, \mathscr{K}))), \quad t \geqslant 0.$$

The shift u_t^o is a normal, injective *-endomorphism and satisfies the following relations:

  (a)   For $s, t \geqslant 0$, $u_s^o u_t^o = u_{s+t}^o$.

  (b)   For $t \geqslant 0$, $u_t^o(\mathscr{A}) = \mathscr{A}_{[t}$.

The action of (u_t^o) can be naturally extended to any operator on $\mathscr{H}$ whose domain includes $\mathscr{D}_o \otimes \mathscr{E}(D^{(1)})$, in particular to stochastic integrals.

For a family $L = \{L^{ji}(t), i, j \in \overline{S}\}$ of square integrable process, the shifted process $L_s = \{L_s^{ji}(t), i, j \in \overline{S}\}$, $(s \geqslant 0)$ defined by

$$L_s^{ji}(t) = \begin{cases} u_s^o(L^{ji}(t-s)), & \text{if} \quad t \geqslant s \\ 0, & \text{if} \quad t < s \end{cases} \tag{2.4}$$

remains square integrable and moreover the identity

$$u_s^o \left(\int_0^t L^{ij} \, d\Lambda_{ij} \right) = \int_0^t u_s^o(L^{ij}) \, d\Lambda_{ij}^{(s)} \tag{2.4a}$$

is true whenever the stochastic integral of (L^{ij}) exists in the weak topology on $\mathscr{D}_o \otimes \mathscr{E}(D^{(1)})$.

Vacuum conditional expectations $\{E_{[s, t]}: \mathscr{A} \to \mathscr{A}_{[s, t]} 0 \leqslant s \leqslant t\}$ exist and are characterized by the property

$$E_{[s, t]}(a_o \otimes b_{[s, t]} \otimes b_{[s, t]}^c) = \langle e(0), b_{[s, t]^c}(0) \rangle \, a_o \otimes b_{[s, t]} \otimes 1_{[s, t]^c} \tag{2.5}$$

for all $a_o, \in \mathscr{A}_o b_{[s, t]} \in \mathscr{B}_{[s, t]}$, $b_{[s, t]^c} \in \mathscr{B}_{[s, t]^c}$ and where $[s, t]^c = \mathbf{R}_+ \backslash [s, t]$. They satisfy the projectivity condition

$$E_{[s, t]} E_{[s', t']} = E_{[s, t]}, \qquad \text{if} \quad [s, t] \subseteq [s', t'].$$

We also write $E_{s]}$ for $E_{[0, s]}$. As a consequence of (2.5), for all $A \in \mathscr{A}$ one has

$$E_{s]}[u_s^o(A)] = E_{o]}(A) \in \mathscr{A}_o. \tag{2.6}$$

DEFINITION (2.4)   A *stochastic process* on $\mathscr{A}_o$ is a family $j = \{j_t: \mathscr{A}_o \to \mathscr{A}, t \geqslant 0\}$ of *-homomorphisms with $j_o(x) = x \otimes 1$; (j_t) is called *adapted* if, for each $t \geqslant 0$, $j_t(x) \in \mathscr{A}_{t]}$; *conservative* if for each $t \geqslant 0$, $j_t(1) = 1$; *normal* if j_t is normal for each t; *σ-weakly continuous* if the map $(t, x) \mapsto j_t(x)$ is continuous for the σ-weak topology on $\mathscr{A}_o, \mathscr{A}$; *injective* if j_t is injective; *measurable* if for each $x \in \mathscr{A}_o$ the process $j(x) = \{j_t(x): t \geqslant 0\}$ is weakly measurable.

Let j be a normal stochastic process. We define $\hat{\jmath}_s$ as the unique normal *-homomorphism $\hat{\jmath}_s: \mathscr{A}_o \otimes 1_{[0, s]} \otimes \mathscr{B}_{[s, \infty)} \to \mathscr{A}$ characterized by

$$\hat{\jmath}_s(x \otimes 1) = j_s(x), \qquad \hat{\jmath}_s(1 \otimes a_{[s}) = j_s(1) \otimes a_{[s}, \ x \in \mathscr{A}_o, \ a_{[s} \in \mathscr{A}_{[s}. \quad (2.7)$$

Each $\hat{\jmath}_s$ can be extended, by the same action described as in (2.7), to the complex vector space $\mathscr{L}_o$ algebraically spanned by the elements of the form $x \otimes 1_{[0, s]} \otimes Y_{[s}$, where $x \in \mathscr{A}_o$, $Y_{[s}$ is an operator on $\mathscr{H}_{[s}$ with domain containing $\mathscr{E}(D_{[s}^{(1)})$. Moreover, if (X_s^α) is a net of adapted operator processes in $\mathscr{L}_0$ strongly (resp. weakly) convergent on $\mathscr{B}(\mathscr{H}_o)_* \otimes \tilde{\mathscr{E}}(D^{(1)})$ to a process (X_s) then, due to the identity

$$\varphi \otimes \varphi_u(\hat{\jmath}_s(X_s^{(\alpha)})) = \hat{\jmath}_s^*(\varphi \otimes \varphi_{\chi_{s]}u}) \otimes \varphi_{\chi_{[s}u}(X_s^{(\alpha)}), \quad (2.8)$$

also the net $(\hat{\jmath}_s(X_s^\alpha))$ also converges strongly (resp. weakly) on $\mathscr{B}(\mathscr{H}_o)_* \otimes \tilde{\mathscr{E}}(D^{(1)})$ to an adapted process which we denote by $(\hat{\jmath}_s(X_s))$. In particular $\hat{\jmath}_s)$ can be extended to stochastic integrals.

PROPOSITION (2.5) *Let (j_t) be a normal adapted stochastic process and let $L = \{L^{ji}(t), t \geqslant 0\}$ be square integrable bounded processes such that for each $j \in \bar{S}$ the process $\sum_{i \in \bar{S}} (L^{ji})^* L^{ji}$ is norm uniformly bounded on compacta and $\int_0^t L^{ji} d\Lambda_{ij} \in \mathscr{A}_{t]}$ for each $t \geqslant 0$. Then for each $i, j \in \bar{S}$, $L^{ji}(t) \in \mathscr{A}_{t]}$, for almost all $t \geqslant 0$ and for all $s, t \geqslant 0$,*

$$\hat{\jmath}_s\left(u_s^o\left(\sum_{i, j \in \bar{S}} \int_0^t L^{ji} d\Lambda_{ij}\right)\right) = \int_s^{s+t} \hat{\jmath}_s\left(u_s^o\left(\sum_{i, j \in \bar{S}} L^{ji}(\tau - s)\right) d\Lambda_{ij}(\tau)\right). \quad (2.9)$$

Proof. Fix $T \geqslant 0$, $j \in \bar{S}$ and choose c_j such that $\sum_{i \in \bar{S}} (L^{ij}(s))^* L^{ij}(s) \leqslant c_j$ for all $f \in \mathscr{D}_o$, $u \in D^{(1)}$, $0 \leqslant s \leqslant T$. It is known that one can choose a sequence of simple square integrable elements $L(s, n) \in \mathscr{A}_{s]}$, $n \geqslant 1$ such that $\sum_{i \in \bar{S}} (L^{ij}(s, n))^* L^{ij}(s, n) \leqslant c_j$ for all $f \in \mathscr{H}_o$, $0 \leqslant s \leqslant T$ and $\|L - L(n)\|_{f, u, s} \to 0$ for all $0 \leqslant s \leqslant T$. Moreover from the hypothesis that, for each $j \in \bar{S}$, the family $\sum_{i \in \bar{S}} (L^{ij})^* L^{ji}$ is uniformly bounded on $[0, T]$, it follows that, for any normal state ϕ on $\mathscr{A}_o$, $\|L - L^n)\|_{\phi, u, t}$ tends to zero, by dominated convergence, for any $t \leqslant T$. Because of (2.4a) we need to show that

$$\hat{\jmath}_s(X_{u_s^o(L)}(o, t)) = \hat{\jmath}_s\left(\int_0^t u_s^o(L) d\Lambda^{(s)}\right) = X_{\hat{\jmath}_s u_s^o(L)}(0, t)$$

$$= \int_0^t \hat{\jmath}_s(u_s^o(L)) d\Lambda^{(s)}. \quad (2.10)$$

The identity (2.10) is obviously true for step functions.

We shall prove that it is preserved under limits with respect to strong convergence on $\mathscr{B}(\mathscr{H}_o)_* \otimes \tilde{\mathscr{E}}(D^{(1)})$. To this goal notice that, in the notation (2.3a)

$$\|\hat{j}_s(X_{u_s^o(L)}(0,t)) - X_{\hat{j}_s u_s^o(L)}(0,t)\|_{\varphi,u} \leqslant \|\hat{j}_s X_{u_s^o(L)}(0,t) - X_{\hat{j}_s u_s^o(L_n)}(0,t)\|_{\varphi,u} \\ + \|X_{\hat{j}_s u_s^o(L_n)}(0,t) - X_{\hat{j}_s u_s^o(L)}(0,t)\|_{\varphi,u}. \tag{2.11}$$

Since the indentity (2.10) holds for step processes, the first term on the right-hand side of (2.11) is equal to

$$\|\hat{j}_s(X_{u_s^o(L-L_n)}(0,t))\|_{\varphi,u}. \tag{2.12}$$

Because of (2.3b), the second term on the right hand side of (2.11) is less than or equal to

$$\|\hat{j}_s u_s^o(L-L_n)\|_{\varphi,u,t}. \tag{2.13}$$

Now notice that, for any $a_o \in \mathscr{B}(\mathscr{H}_o)$ and $b_{[s} \in \mathscr{B}_{[s}$, one has

$$\|\hat{j}_s(a_o \otimes b_{[s})\|_{\varphi,u}^2 = \varphi \otimes \varphi_u(\hat{j}_s(a_o \otimes B_{[s})|^2) \\ = \varphi \otimes \varphi_{\chi_{[o,s]}u} \otimes \varphi_{\chi_{[s}u}(\hat{j}_s(|a_o|^2 \otimes |b_{[s}|^2) \\ = \hat{j}_s^*(\varphi \otimes \varphi_{\chi_{[o,s]}u}) \otimes \varphi_{\chi_{[s}u}(|a_o \otimes b_{[b}|^2),$$

where $\hat{j}_s^*(\phi \otimes \phi_{\chi_{[s}u})$ is a normal state on $\mathscr{A}_o$ due to the normality of the map $j_s: \mathscr{A}_o \to \mathscr{A}_o \otimes \mathscr{B}_{s]}$. By normality, the identity

$$\|\hat{j}_s(x)\|_{\varphi,u}^2 = \|x\|_{\hat{j}_s^*(\varphi \otimes \varphi_{\chi_{[s}}u),\chi_{[s}u}^2 \tag{2.14}$$

holds for any $x \in \mathscr{A}_o \otimes \mathscr{B}_{[s}$ and therefore also for any limit of such operators in the topology of strong convergence on $(\mathscr{A}_o)_* \otimes \tilde{\mathscr{E}}(D_{[s}^{(1)})$. (cf. the remarks after Lemma (2.2a)).

Since $u_s^o(L_t - L_t(n))$ belongs to the first type of operators and $X_{u_s^o(L-L(n))}(0,t)$ to the second, it follows that

$$\|\hat{j}_s(X_{u_s^o(L-L(n))}(0,t)\|_{\varphi,u}^2 = \|X_{u_{[L-L(n))}}(0,t)\|_{\hat{j}_s^*(\varphi \otimes \varphi_{\chi_s]}u),\chi_{[s}u}^2 \\ \leqslant \|u_s^0(L-L(n))\|_{\hat{j}_s^*(\varphi \otimes \varphi_{\chi_s]}u),\chi_{[s}u,t}^2 \\ = \|L-L(n)\|_{\hat{j}_s^*(\varphi \otimes \varphi_{\chi_s]}u),\theta_s^*\chi_{[s}u,t},$$

which tends to zero as $n \to \infty$. Similarly

$$\|\hat{j}_s u_s^o(L-L(n))\|_{\varphi,u,t}^2 = \|u_s^o(L-L(n))\|_{\hat{j}_s^*(\varphi \otimes \varphi_{\chi_s]}u),\chi_{[s}u,t}^2,$$

which again tends to zero as $n \to \infty$.

DEFINITION (2.6)   An operator process $X \equiv (X(t): t \geqslant 0)$ is called a *martingale* if it is a vacuum martingale, i.e., if for all $0 \leqslant s \leqslant t$

$$E_{s]}[X(t)] = X(s),$$

and a *regular* martingale if there is a Radon measure μ on $\mathbf{R}_+$ for which

$$\|[X(t) - X(s)]\psi\|^2 + \|[X(t)^* - X(s)^*]\psi\|^2 \leqslant \mu([s, t])\|\psi\|^2 \quad (2.15)$$

whenever $0 \leqslant s \leqslant$ and $\psi \in \Gamma_{[0, s]} \otimes \Phi_{[s}$.

The following Theorem was proved by Parthasarathy and Sinha [PaSi 1, PaSi 2]. The last statement is a remark in [Mey 2].

THEOREM (2.7)   *Let X be a bounded regular martingale on $\mathscr{H}$. Then there exists a family of bounded adapted processes $L^{ji}(t)$, $i, j \in \bar{S}$, $(i, j) \neq (0, 0)$ such that for each $t \geqslant 0$*

$$dX(t) = X_o + \sum_{(i, j) \neq (0, 0)} L^{ji}(t)\, d\Lambda_{ij}(t)$$

on $\mathscr{D}_o \otimes \mathscr{E}(D^{(1)})$. The measure μ in (2.15) can always be choosen to be absolutely continuous with respect to the Lebesgue measure.

For each $j \in \bar{S}$ the series $\sum_{i \in \bar{S}} (L^{ji})^* L^{ji}$ is convergent in the strong operator topology and such a family is unique modulo a set of Lebesgue measure zero. Moreover for each $j \in S$ the process $\sum_{i \in S} (L^{ji})^* L^{ji}$ is uniformly bounded on compact sets.

The quantum analogue of square and angle brackets for any two square integrable regular martingales X, Y are the adapted processes $[[X, Y]]$ and $\langle\!\langle X, Y\rangle\!\rangle$ defined respectively as (cf. [AcQu])

$$[[X, Y]](t) := \lim_{\mathrm{Max}(t_k - t_{k-1}) \to 0} \sum_k [X(t_{k+1}) - X(t_k)][Y(t_{k+1}) - Y(t_k)]$$

$$\langle\!\langle X, Y\rangle\!\rangle(t) := \lim_{\mathrm{Max}(t_k - t_{k-1}) \to 0}$$

$$\times \sum_k E_{t_k]}[[X(t_{k+1}) - X(t_k)][Y(t_{k+1}) - Y(t_k)]],$$

where the limit is to be interpreted in the sense of sesquilinear forms on $\mathscr{D}_o \otimes \mathscr{E}(D^{(1)})$. Its existence follows from the square integrability of the coefficients which appear in the stochastic representation of the regular

martingale $[Att]$. *In fact, in the notation* (2.1a), *the quantum Ito formula implies*

$$[[X_L, X_K]](t) = \int_0^t \sum_{k \in S} L^{ik}(s) K^{kj}(s) \, dA_{ij}(s) \tag{2.16}$$

$$\ll X_L, X_K \gg (t) = \int_0^t \sum_{k \in S} L^{0k}(s) K^{k0}(s) \, ds. \tag{2.17}$$

DEFINITION (2.8)   A regular martingale X is said to have *continuous trajectories*, if

$$[[X^*, X]] = \ll X^*, X \gg. \tag{2.18}$$

Since for a regular martingale X the process $\ll X^*, X \gg$ is absolutely continuous with respect to the Lebesgue measure, the above notion is the same as that introduced in $[AcQu]$. That this condition indeed extends the classical notion of *continuous trajectories* was proved in $[AcQu]$. The following proposition confirms the intuition behind the notion of Definition (2.8).

PROPOSITION (2.9)   *Let X be a regular martingale. Then it has continuous trajectories if and only if its stochastic representation is gauge free, i.e.,*

$$dX(t) = \sum_{k \in S} [L^{k0} \, dA_{k0}(t) + L^{0k} \, dA_{0k}(t)].$$

Proof.   Comparing (2.16) and (2.17) and using the independence of the basic processes one sees that condition (2.18) is equivalent to the vanishing of the martingale part in the representation (2.16), i.e., to the fact that for each $i, j \in S$, $\sum_{k \in S} (L^{ki}(s))^* L^{kj}(s) = 0$ almost everywhere. Thus $L^{ji}(t) = 0$ almost everywhere.

3. MARKOV SEMIGROUPS AND COCYCLES

DEFINITION (3.1)   A one parameter family of completely positive maps $P \equiv \{P^t, t \geqslant 0\}$ on a von Neumann algebra $\mathscr{A}_o$ is said to be a Markov semigroup if the following relations hold:

   (i) $P^o(x) = x$, $P^t(P^s(x)) = P^{s+t}(x)$, $s, t \geqslant 0$, $x \in \mathscr{A}_o$;

   (ii) $\|P^t\| \leqslant 1$, $t \geqslant 0$;

it is said to be conservative if

   (iii) $P^t(1) = 1$ for all $t \geqslant 0$.

A Markov semigroup P is said to be *σ-weakly continuous* if the map $(t, x) \in \mathbf{R} \times \mathscr{A}_o \to P^t(x)$ is jointly continuous in the σ-weak topology.

The generator $\mathscr{L}$ of a Markov semigroup (P^t) is characterized by the property

$$P^t(x) = x + \int_0^t P^s(\mathscr{L}(x)) \, ds, \qquad x \in \mathscr{D}(\mathscr{L}), \qquad (3.1)$$

where $\mathscr{D}(\mathscr{L})$ denotes the domain of $\mathscr{L}$.

From (3.1) we also have, for any $x \in \mathscr{D}(\mathscr{L})$ the inequality

$$\|P^t(x) - x\| \leqslant t \, \|\mathscr{L}(x)\|.$$

The following lemma is useful for relating the joint continuity property to an apparently weaker notion which one often meets in the literature on quantum Markov semigroups.

LEMMA (3.2)   *Let (P^t) be a Markov semigroup on a von Neumann algebra $\mathscr{A}_o$. The following statements are equivalent*:

(i)   *For each $x \in \mathscr{A}_o$ the map $t \to P^t(x)$ is continuous in the σ-weak operator topology and for each $t \geqslant 0$ the map $x \to P^t(x)$ is continuous in the σ-weak topology*.

(ii) (P^t) *is σ-weakly continuous*.

Proof.   Obviously (ii) implies (i). To prove the converse, first note that (i) implies that (P_*^t), the pre-dual semigroup of (P^t), acting on the Banach space $(\mathscr{A}_o)_*$ is continuous in the Banach space topology. Let $\mathscr{L}_*$ be the generator of (P_*^t). Thus for any $\rho \in \mathscr{D}(\mathscr{L}_*)$, the domain of $\mathscr{L}_*$, and any $x \in \mathscr{A}$, the following integral equation holds:

$$\langle P_*^t(\rho), x \rangle = \langle \theta, x \rangle + \int_0^t \langle P_*^s(\mathscr{L}_*(\rho), x)) \rangle \, ds. \qquad (3.1a)$$

This implies that, for any $\rho \in \mathscr{D}(\mathscr{L}_*)$, the map $(t, x) \to \langle \rho, P^t(x) \rangle$ is continuous. To prove this consider nets $x_n \in \mathscr{A}_o$, $t_n \in \mathbf{R}_+$ $(n \geqslant 1)$ such that $x_n \to x$, $t_n \to t$ and notice that

$$\lim_{n \to \infty} \langle \rho, P^{t_n}(x_n) \rangle = \lim_{n \to \infty} \langle \rho, x_n \rangle + \lim_{n \to \infty} \int_o^{t_n} \langle \mathscr{L}_*(\rho), P^s(x_n) \rangle \, ds.$$

By the uniform boundedness principle the integrand in the right hand side of the above expression is uniformly bounded, therefore, by dominated convergence theorem one can pass to the limit inside the integral sign and

this gives the right hand side of (3.1a) for any $\rho \in \mathscr{D}(\mathscr{L}_*)$. Since the domain is dense, a simple density argument completes the proof.

The following characterization of (P_t) is stronger than (3.1) and will be needed in the proof of Theorem (3.9) below.

LEMMA (3.3) *Let K^t, $t \geqslant 0$ be a uniformly bounded family of normal maps on $\mathscr{A}_0$ such that for each $x \in \mathscr{A}_0$ the map $t \to K^t(x)$ is continuous in the σ-weak topology and*

$$K^0 = 0, \qquad \frac{d}{dt} K^t = K^t \mathscr{L} \geqslant 0, \tag{3.2}$$

on $\mathscr{B}_0$, where $\mathscr{B}_0$ is a core for $\mathscr{L}$. Then $K^t = 0$ for all $t \geqslant 0$.

Proof. Define the bounded linear map R_λ, $(\lambda > 0)$ on $\mathscr{A}_0$ by

$$R_\lambda = \int_0^\infty e^{-t\lambda} K^t \, dt, \tag{3.3}$$

where in order to define the integral we have used the uniform boundedness and continuity property. A simple computation and (3.2) imply that $R_\lambda(\mathscr{L} - \lambda) = 0$ on $\mathscr{B}_0$. $\mathscr{B}_0$ being a core for $\mathscr{L}$, we also have $\overline{(\mathscr{L} - \lambda)(\mathscr{B}_0))} = \mathscr{A}_0$. Thus we obtain the result.

DEFINITION (3.4) A normal stochastic process (j_t) on $\mathscr{A}_o$ is said to be a *Markov cocycle* if it satisfies the cocycle equation

$$j_o(x) = x \otimes 1, \qquad j_{s+t}(x) = \hat{j}_t(u_t^o(j_s(x))), \tag{3.4}$$

for all $s, t \geqslant 0$ and $x \in \mathscr{A}_0$ where $(\hat{j}_t)$ is the extension of (j_t) defined by (2.7). It is said to be *σ-weakly continuous* if the map $(t, x) \to j_t(x)$ is continuous with respect to the σ-weak topology of the von Neumann algebras $\mathscr{A}_o$ and $\mathscr{A}$.

Notice that such a cocycle is automatically a Markov cocycle in the sense of (1.4a), in fact it has the stronger property that, for any $s \leqslant t$,

$$E_{t]} \hat{j}_s = \hat{j}_s E_{t]} = j_s E_{t]} \tag{3.5}$$

on $\mathscr{A}_o \otimes \mathscr{B}_{[s} = \mathscr{A}_{[s}$.

DEFINITION (3.5) A family $V \equiv \{V(t), t \geqslant 0\}$ of bounded operators on $\mathscr{H}_o \otimes \mathscr{B}(\Gamma(L^2(\mathbf{R}_+, \mathscr{K})))$ is called a (left) operator *cocycle* if

$$V(s+t) = V(t) \, u_t^o(V(s)), \qquad \forall s, t \geqslant 0, \tag{3.5a}$$

if moreover $V_t \in \mathscr{A}_{t]}$ for any $t \geqslant 0$, we speak of a *Markov operator cocycle*.

Given a strongly continuous unitary cocycle V, the family $j_t(x) := V(t) x V(t)^*$ defines a σ-weakly continuous Markov cocycle on $\mathcal{B}(\mathcal{H}_o)$.

PROPOSITION (3.6) *Let j_t, $t \geqslant 0$ be a Markov cocycle. Then the family $\{j_t(x), t \geqslant 0\}$ is commutative whenever $\mathcal{A}_o$ is commutative.*

Proof. Fix any two elements $x, y \in \mathcal{A}_o$. Because $\mathcal{A}_o$ is in the centre of $\mathcal{A}_o \otimes \mathcal{A}$ we have $x j_s(y) = j_s(y) x$. Now use the homomorphism property of u_t^o, j_t and the fact u_t^o acts trivially on $\mathcal{A}_o$ to obtain

$$j_t(x) \, \hat{j}_t(u_t^o(j_s(y))) = \hat{j}_t(u_t^o(j_s(y))) \, j_t(x)$$

The result follows from the above identity and the cocycle equation (3.4).

According to the quantum Feynman-Kac formula, to every (σ-weakly continuous) Markov cocycle (j_t) on $\mathcal{A}_o$, one can associate a (σ-weakly continuous) Markov semigroup P^t ($t \geqslant 0$) on $\mathcal{A}_o$, characterized by the identity

$$P^t(x) = E_{o]}[j_t(x)]$$

for all $t \geqslant 0$, $x \in \mathcal{A}_o$. Moreover the cocycle identity (3.4) and the localization property (3.5) imply that, for each $0 \leqslant s \leqslant t$,

$$E_{s]}[j_t(x)] = j_s(P^{t-s}(x)).$$

LEMMA (3.7) *Let $\{j_t : t \geqslant 0\}$ be a σ-weakly continuous Markov cocycle and let θ_o be the generator of the associated Markov semigroup $\{P^t : t \geqslant 0\}$, defined by (3.6). Then for each $x \in \mathcal{D}(\theta_o)$*

$$X(t, x) = j_t(x) - x - \int_0^t j_s(\theta_o(x)) \, ds \tag{3.6}$$

is a martingale. Moreover $X(t, x)$ is a regular martingale whenever x, x^*, x^*x, $xx^* \in \mathcal{D}(\theta_o)$.

Remark (3.8.) By the polarization identity, to assume that x, x^*, x^*x, $xx^* \in \mathcal{D}(\theta_o)$ whenever $x \in \mathcal{D}(\theta_o)$, is equivalent to assuming that $\mathcal{D}(\theta_o)$ is a *-algebra.

Proof. The martingale property follows from (3.1) and Lemma (3.7) because, for $0 \leqslant s \leqslant t$:

$$E_{s]}[X(t, x)] = j_s(P^{t-s}(x)) - x - \int_o^s j_r(\theta_o(x)) \, dr - \int_s^t j_s(P^{r-s}(\theta_o(x))) \, dr$$

$$= j_s \left\{ P^{t-s}(x) - \int_s^t P^{r-s}(\theta_o(x) \, dr \right\} - x - \int_o^s j_r(\theta_o(x)$$

$$= X(s, x). \tag{3.7}$$

To show that it is a regular martingale we first note that for any $\phi = \psi_{s]} \otimes \Phi_{s]} \in \mathscr{H}_{s]}$ we have

$$\|[X(t, x) - X(s, x)]\,\phi\|^2 \leqslant 2\,\|[j_t(x) - j_s(x)]\,\phi\|^2 + 2\,\left\|\int_s^t j_r(\theta_o(x))\,dr\,\phi\right\|^2. \tag{3.8}$$

Moreover, using the cocycle property, (3.5), (2.6), and (3.6), we find

$$\begin{aligned}
\|[j_t(x) &- j_s(x)]\,\phi\|^2 \\
&= \|\hat{j}_s(u_s^o(j_{t-s}(x) - x))\,\phi\|^2 \\
&= \langle \hat{j}_s(u_s^o((j_{t-s}(x) - x)^*\,(j_{t-s}(x) - x)))\,\phi, \phi\rangle \\
&= \langle j_s(P^{t-s}(x^*x) - P^{t-s}(x^*)\,x - x^*P^{t-s}(x) + x^*x))\,\phi, \phi\rangle \\
&\leqslant \|P^{t-s}(x^*x) - P^{t-s}(x^*)\,x - x^*P^{t-s}(x) + x^*x\|\,\|\phi\|^2 \\
&\leqslant \left\|\int_o^{t-s} [P^r(\theta_o(x^*x)) - P^r(\theta_o(x^*))\,x - x^*P^r(\theta_o(x))]\,dr\right\|\,\|\phi\|^2 \\
&\leqslant (t-s)[\|\theta_o(x^*x)\| + 2\,\|\theta_o(x^*)\|\,\|x\|]\,\|\phi\|^2. \tag{3.9}
\end{aligned}$$

From this we deduce that for any $\phi \in \mathscr{H}_{s]}$, any $t \geqslant s \geqslant 0$ and any X such that $x, x^*, x^*x \in \mathscr{D}(\theta_o)$, one has

$$\|[X(t, x) - X(s, x)]\,\phi\|^2 \leqslant (t-s)\,k(x)\,\|\phi\|^2, \tag{3.10}$$

where $k(x) := \{\|\theta_o(x^*x)\| \cdot \| \| + 2\|\theta_o(x^*)x\|\}$ is independent of $s \leqslant t$ for any $0 \leqslant s, t$. The analogue inequality for M^* follows from the identity $M(t, x)^* = X(t, x^*)$. Therefore $M(t, x)$ is a regular martingale.

The main result of the present paper is the following:

THEOREM (3.9) *Let (j_t) be a σ-weakly continuous adapted injective Markov cocycle on $\mathscr{A}_o$; let (P^t) $(t \geqslant 0)$ be the associated Markov semigroup on $\mathscr{A}_o$ and θ_o its generator. If $\mathscr{B}_o \subset \mathscr{D}(\theta_o)$ is * subalgebra of $\mathscr{A}_o$ then there exists a unique family θ^{ji}, $i, j \in S$ of linear maps from $\mathscr{B}_o$ into $\mathscr{A}_o$ such that, whenever $x \in \mathscr{B}_o$*

$$dj_t(x) = \sum_{i, j \in \bar{S}} j_t(\theta^{ji}(x))\,d\Lambda_{ij}(t) \tag{3.11}$$

on $\mathscr{H}_o \otimes \mathscr{E}(D^{(1)})$. Moreover, for $x, y \in \mathscr{B}_o$ the following relations hold:

$$\theta^{ji}(x)^* = \theta^{ij}(x^*) \tag{3.12}$$

$$\theta^{ji}(xy) = \theta^{ji}(x) \, y + x\theta^{ji}(y) + \sum_{k \in S} \theta^{ik}(x) \, \theta^{kj}(y). \tag{3.13}$$

Conversly, if the maps θ^{ij} satisfy (3.12) and (3.13), and if $\mathscr{B}_o$ is a core for θ_o then there exists at most one σ-weakly continuous stochastic process satisfying (3.11). In such a case the stochastic process is a Markov cocycle.

Proof. Given $x \in \mathscr{B}_o$, applying Theorem (2.7) to the regular martingale defined by (3.8) one has

$$j_t(x) = x + \sum_{i, j \in \bar{S}} \int_0^t L^{ji}(x, s) \, d\Lambda_{ij}(s), \tag{3.14}$$

where for each $j \in \bar{S}$ the series $\sum_i (L^{ji})^* \, L^{ji}$ is convergent in the strong operator topology and $L^{oo}(t, x) = j_t(\theta_o(x))$. Moreover such a family is unique modulo a set of Lebesgue measure zero. The uniqueness of the representation implies that $L^{ji}(x, t) \in \mathscr{A}_o \otimes \mathscr{B}(\Gamma(\mathbf{R}_+, \mathscr{K}))$. To see this fix an unitary element $U \in \mathscr{A}'_o$ and observe that $j_t(x) = U^* j_t(x) U$ hence by the uniqueness of the representation: $L^{ji}(x, t) = U^* L^{ji}(x, t) U$, i.e. $L^{ji}(x, t)$ belongs to $\mathscr{A}_o$. From the identity

$$u_r(j_t(x)) = x + \sum_{i, j \in \bar{S}} \int_0^t u_r^o(L^{ji}(x, s)) \, d\Lambda_{ij}(r + s)$$

we get, using the cocycle property and Proposition (2.5),

$$j_{r+t}(x) = j_r(x) + \sum_{i, j \in \bar{S}} \int_0^t \hat{j}_r(u_r^o(L^{ji}(x, s))) \, d\Lambda_{ij}(r + s). \tag{3.15}$$

In order to apply Proposition (2.5) we need to verify that for each $j \in \bar{S}$ the process $\sum_{i \in \bar{S}} L^{ji}(x, t)^* \, L^{ji}(x, t) \ (t \geqslant 0)$ is uniformly bounded on compact sets. First recall that the last part of Theorem (2.7) guarantees that this condition is satisfied for each $j \in S$. Since j_t is *-linear map we also have $(L^{ji}(x, t))^* = L^{ij}(x^*, t)$, $i, j \in \bar{S}$. Thus it is enough to verify the uniform boundedness on compact sets in the case $j = 0$. To this goal notice that, j_t being a *-homomorphism, the quantum Ito formula (2.1) and

$$dj_t(x^*x) = d(j_t(x^*) \, j_t(x)) = dj_t(x^*) \, j_t(x) + j_t(x^*) \, dj_t(x) + dj_t(x^*) \, dj_t(x)$$

implies the identity

$$j_t(\theta_0(x^*x)) = j_t(\theta_0(x^*))\, j_t(x) + j_t(x^*)\, j_t(\theta_0(x)) + \sum_{k \in S} L^{0k}(x, t)^*\, L^{0k}(x, t),$$

$$(3.16)$$

where for the second identity have used our hypothesis that $x^*x \in \mathcal{B}_0$ so that (3.14) holds for $j_t(x^*x)$ as well. Since each j_t is contractive, it follows by inspection from (3.16) that the sum $\sum_{k \in S} L^{0k}(x, t)^*\, L^{0k}(x, t)$ belongs to $\mathcal{A}_{t]}$ for each t and is uniformly bounded in norm as a function of t. The uniqueness of the representation, (3.14) and (3.16) implies that, for any $r, t \geq 0$ and for almost any $s \in [0, t]$,

$$\hat{j}_r(u_r^0(L^{ji}(x, s))) = L^{ji}(x, s + r),$$

for each fixed $r \geq 0$. By Fubini's theorem

$$\hat{j}_{t-s}(u_{t-s}(L^{ji}(x, s))) = L^{ji}(x, t) \qquad \text{a.e. } \{(s, t) : s \leq t\}. \qquad (3.16a)$$

Since the left hand side is a continuous function of t, also the right hand side is and this implies [Br] that there exists a version such that

$$\hat{j}_{t-s}(u_{t-s}(L^{ji}(x, s))) = L^{ji}(x, t) \qquad \text{for all} \quad t \geq s > 0. \qquad (3.16b)$$

Applying $E_{t-s]}$ to both sides of (3.16b) and using (2.6) gives

$$j_{t-s}(E_{o]}[L^{ji}(x, s)] = E_{t-s]}[L^{ji}(x, t)] \qquad \text{for all} \quad t \geq s > 0. \qquad (3.17)$$

Now our aim is to define $\theta^{ji}(x,)$ so that

$$j_t(\theta^{ji}(x,)) = L^{ji}(x, t) \qquad \text{for all} \quad t \geq 0. \qquad (3.18)$$

Since for each $i, j \in \bar{S}$, $x \in \mathcal{B}_0$ the family $L^{ji}(x, s)$, $s > 0$ is uniformly bounded, the same holds for the family $E_{o]}[L^{ji}(x, s)] \in \mathcal{A}_0$, $s > 0$. Since a norm bounded, closed set in $\mathcal{A}_0$ is compact in the σ-weak topology, we can extract a subsequence $s_n > 0$, such that $\lim_{s_n \to 0} E_{o]}[L^{ji}(x, s_n)] =: L^{ji}(x, 0) \in \mathcal{A}_o$ exists. Thus (3.18) follows from (3.17) once we pass to the limit $s \to 0$ because of our hypothesis that the map $(s, x) \to j_s(x)$ is σ-weakly continuous.

If j_t is injective, then $\theta^{ji}(x,))$ is uniquely determined by (3.18).

That it satisfies (3.12) and (3.13) is a simple consequence of the quantum Ito formula and of the uniqueness of the stochastic representation. This completes the proof of the first part of the theorem.

Now we show that, if $\mathcal{B}_o$ is a core for θ_o, then the maps θ^{ij} determine j_t uniquely via (3.11).

Let (j'_t) be an another normal stochastic process satisfying (3.11). The homomorphism property guarantees that each j'_t is contractive. Hence for each fixed $u, v \in M$ and $m, n \geqslant 0$ we can define the uniformly bounded family of normal maps $K^t_{(u, v, m, n)} : \mathscr{A}_o \to \mathscr{A}_o$ by

$$\langle f, K^t_{(u, v, m, n)}(x)\, g \rangle = \langle fu^{(m)}, [\, j_t - j'_t](x)\, gv^{(n)} \rangle, \qquad x \in \mathscr{A}_0,$$

for all $f, g \in \mathscr{H}_0$, where $u^{(m)} = 1/\sqrt{n}u^{\otimes m}$. Our aim is to show that for $m, n \geqslant 0$

$$K^t_{(u, v, m, n)}(x) = 0 \tag{3.19}$$

for all $u, v \in M$.

Since by our hypothesis $\mathscr{B}_0$ is core for $\mathscr{L}$, Lemma 3.3 says that it is enough to verify that each $K^t_{(u, v, m, n)}$ satisfies (3.2). For $m = 0 = n$ it is immediate from (3.11). Now we shall adapt an induction argument due to Fagnola [Fag] to verify (3.19) for arbitrary $m, n \geqslant 0$. From (3.11) and the analyticity of the exponential vectors, we also deduce the identities

$$\langle fu^{((m))}, [\, j_t - j'_t](x)\, gv^{((n))} \rangle$$
$$= \sum_{i, j \in \bar{S}} \int_0^t \langle fu^{(m_i)}, [\, j_s - j'_s](\theta^{ji}(x))\, gv^{((n_j))} \rangle\, u_i(s)\, v^j(s)\, ds, \tag{3.20}$$

where

$$n_i = \begin{cases} n-1, & \text{if} \quad i \in S, \\ n, & \text{if} \quad i = 0, \end{cases}$$

and similarly for m_i.

Assume that (3.19) holds for all $m, n \geqslant 0$ such that $m + n \leqslant k$ and fix any m, n such that $m + n = k + 1$. Thus, by induction hypothesis, it follows from (3.20) that $T^t_{(u, v, m, n)}$ satisfies (3.2). Hence Lemma 3.2 allows us to conclude that (3.19) holds for all m, n.

For the last statement, observe that for any fixed $s \geqslant 0$ the processes $k_t(x) = j_{s+t}(x)$ and $k'_t(x) = \hat{\jmath}_t(u^o_t(j_s(x)))$ satisfy (3.11) with the same initial value. Since $\mathscr{B}_o$ is a core, (3.11) admits a unique solution, thus the cocycle property follows. This completes the proof of Theorem (3.9).

4. Stochastic Parallel Transports on Clifford Bundles

Let (M, g) be a d-dimensional, oriented, smooth, compact Riemannian manifold with metric $g = \langle \cdot, \cdot \rangle$. By definition for each $p \in M$, there is a positive definite inner product g_p on the tangent space $T_p M$ to M at p.

Here we recall the standard construction of the smooth Clifford sections and various natural actions on it.

Denote by V the space of smooth vector fields on M and suppose that V is equipped with a covariant derivative ∇ which respects the metric i.e., for any smooth vector field $\zeta \in V$, ∇_ζ is a linear map on V satisfying the Leibnitz rule

$$\nabla_\zeta(fg) = f\,\nabla_\zeta(g) + \zeta(f)\,g \qquad (4.1)$$

and the so called *orthogonality relation*

$$\zeta\,\langle g, h\rangle = \langle \nabla_\zeta(g), h\rangle + \langle g, \nabla_\zeta(h)\rangle. \qquad (4.2)$$

We denote $T_n^m(M)$ the bundle of tensors of type (m, n) and $Tens(M) = \bigoplus_{(m, n)} T_n^m(M)$ the full tensor algebra. The connection ∇ lifts naturally to a derivative on the tensor algebra $Tens(V)$, denoted by the same symbol and characterized, by

$$\nabla_\zeta(g_1 \otimes g_2 \otimes \cdots \otimes g_m) = \nabla_\zeta(g_1) \otimes g_2 \otimes \cdots \otimes g_m + g_1 \otimes \nabla_\zeta(g_2)$$
$$\otimes \cdots \otimes g_m + \cdots + g_1 \otimes g_2 \otimes \cdots \otimes \nabla_\zeta(g_m).$$

Tens(V) also has an anti-involution θ ($\theta^2 = id$; $\theta(ab) = \theta(a)\,\theta(b)$) defined by

$$\theta(g_1 \otimes g_2 \otimes \cdots \otimes g_m) = g_m \otimes \cdots \otimes g_2 \otimes g_1,$$

which commutes with the connection, i.e., for any section $g \in Tens(V)$

$$\nabla_\zeta(\theta(g)) = \theta(\nabla_\zeta(g).$$

Let $\mathscr{I}$ be the two sided ideal generated by the sections $g \otimes h + h \otimes g - 2\langle g, h\rangle$. Then $\mathscr{I}$ is fixed by the anti-involution θ. Since the connection respects the metric it follows that the ideal $\mathscr{I}$ is preserved by ∇_ζ. Hence the connection on $Tens(V)$ pushes down to a connection on the quotient space denoted by $\mathscr{C}(V)$, whose fiber $((Tens(V))/\mathscr{I})_p$ at p is isomorphic to the universal Clifford algebra over V. This quotient is a *-algebra with involution $g^* = \theta(g)$ where g is any smooth section.

Since the connection commutes with θ we have for smooth sections g, h

$$\nabla_\zeta(g)^* = \nabla_\zeta(g^*), \qquad \nabla_\zeta(gh) = \nabla_\zeta(g)\,h + g\,\nabla_\zeta(h);$$

hence the covariant derivative defines a *-derivation on $Tens(V)$ and therefore on $\mathscr{C}(V)$. Since the Clifford algebras are either simple or the direct sum of isomorphic simple algebras, on each of them there is only one normalized invariant trace, which we denote by tr. We introduce a bilinear form on the fibers at p of the Clifford algebra by

$$\langle g, h\rangle(p) = tr_p(g(p)^*\,h(p)),$$

which extends the Euclidean structure on V. From (4.2) it follows that

$$\zeta \langle g, h \rangle = \langle \nabla_\zeta(g), h \rangle + \langle g, \nabla_\zeta(h) \rangle, \tag{4.3}$$

where g, h are smooth Clifford sections. We also introduce a bilinear form on the Clifford sections by setting

$$(g, h) = \int \langle g, h \rangle(p) \, d\mu(p),$$

where μ is the Riemannian measure on M. The completion $\mathscr{H}_o := L^2(\mathscr{C}(V))$ of the smooth sections is a Hilbert space and $\nabla_\zeta^* = \nabla_\zeta + div_\mu \zeta$, where $div_\mu \zeta$ is the divergence of ζ_k with respect to the volume measure μ.

Note also that $\langle g, h \rangle = 0$ whenever $g \in T_0^m(M)$, $h \in T_0^n(M)$ and $m \neq n$. We denote by P_n the orthogonal projection of $\mathscr{H}_0$ into the closed span of T_0^n. Thus $1 = \bigoplus_{0 \leq n \leq d} P_n$. For $g, h \in \mathscr{C}(V)$ we also note that

$$\|g \otimes h\|^2 = \int tr((g \otimes h)^* g \otimes h)) \, d\mu$$

$$= \int \langle g, g \rangle \langle h, h \rangle \, d\mu$$

$$\leq \sup\{\langle g, g \rangle(p), p \in M\} \, \|h\|^2;$$

hence left multiplication by an element of $\mathscr{C}(V)$ is a continuous operator on $L^2(\mathscr{C}(V))$ and for each element $g \in \mathscr{C}(V)$ we denote by $\lambda(g)$ the bounded extension of the linear operator $\lambda(g): h \to g \otimes h$ on $\mathscr{H}_o$ (left multiplication by g).

This is a *-representation of the Clifford algebra, i.e. $\lambda(gh) = \lambda(g) \lambda(h)$ and $\lambda(g)^* = \lambda(h^*)$. We shall denote $\mathscr{A}_o$ the von Neumann algebra generated by $\{\lambda(g), g \in \mathscr{C}(V)\}$. A linear map α on $\mathscr{A}_0$ will be called *graded* if $\alpha(P_n x P_m) = P_n \alpha(x) P_m$ for all $m, n \geq 0$.

5. Unitary Implementation of Parallel Transports

Let $\Gamma = (\Gamma_{jk}^i)$ be the affine connection on M associated to the covariant derivative ∇ and $T_n^m(p)$, $p \in M$ the bundle of tensors of type (m, n), which is the dual to $T_M^n(p)$ relative to the invariant bilinear form defined by the identification of $T_p M$ with its dual, induced by the metric g i.e., in a coordinate system where the metric tensor is the identity

$$(g, h) = g_{i_1 \cdots i_n}^{j_1 \cdots j_m} h_{j_1 \cdots j_m}^{i_1 \cdots i_n}, \tag{5.1}$$

where summation over repeated indices is understood. With the notation

$$(\Gamma_i g)_{i_1 \cdots i_n}^{j_1 \cdots j_m} := \Gamma_{ii_\nu}^k g_{i_1 \cdots i_{\nu-1} k i_{\nu+1} \cdots i_n}^{j_1 \cdots j_m} - \Gamma_{ik}^{j_\alpha} h_{i_1 \cdots i_n}^{j_1 \cdots j_{\alpha-1} k j_{\alpha+1} \cdots j_m},$$

we have

$$(\Gamma_\alpha g, h) = -(g, \Gamma_\alpha h), \qquad g \in T_n^m, \quad h \in T_m^n \tag{5.2}$$

and the covariant derivative $\nabla_i := \nabla_{\partial/\partial x_i}$ can be expressed as

$$\nabla_i = \partial_i - \Gamma_i.$$

Let $c(t)$, $t \geqslant 0$ be a smooth curve on M. A family of tensors $g = \{u(t) \in T_n^m(t), t \geqslant 0\}$ is said to be *parallel along the curve* $c(t)$ if it satisfies the ordinary differential equation

$$\dot{u} = (\Gamma_i u)\, \dot{c}^i, \qquad u(0) = u_o. \tag{5.3}$$

The smoothness of Γ implies that (5.3) uniquely determines a linear map $U_t(c)$ from $T_n^m(c(0))$ to $T_n^m(c(t))$ and it follows from (5.2) that it preserves the bilinear form (5.1).

Denote $\gamma^\zeta = \{\gamma_t^\zeta(\,\cdot\,), t \in \mathbf{R}\}$ the flow of diffeomorphisms on M determined infinitesimally by a complete smooth vector field ζ and τ_t^ζ the induced flow of automorphisms on T_n^m, defined by

$$\tau_t^\zeta(g)(p) = P_t(\gamma)^{-1} g(\gamma_t^\zeta(p)), \qquad p \in M.$$

Thus

$$\nabla_\zeta(g)(p) = \lim_{t \to 0} \frac{g(p) - \tau_t^\zeta(g(p))}{t}$$

for all smooth vector fields. The map τ_t^ζ is extended by linearity to the full tensor algebra. Since the connection respects the Riemannian metric

$$\langle \tau_t^\zeta(g), \tau_t^\zeta(h) \rangle(p) = \langle g, h \rangle(\gamma_t^\zeta(p)). \tag{5.3a}$$

Thus these maps preserve the two sided ideal $\mathscr{I}$ and push down to a canonical *-automorphism on the Clifford sections. We now introduce the strongly continuous one parameter group of unitary operators $V = \{V(t), t \geqslant 0\}$ on $\mathscr{H}_o$, given by the continuous extension of the following action on the smooth Clifford bundle

$$V(t) f = \left[\frac{d\mu(\gamma_t^\zeta)}{d\mu} \right]^{1/2} \tau_t^\zeta(f). \tag{5.4}$$

That $V(t)$ is an isometry on $\mathscr{C}(V)$ follows from the orthogonality relation (5.3a). It is also simple to observe that for all $f \in \mathscr{C}(V)$ we have

$$\lambda(\tau_t^\zeta(g)) = V(t)\, \lambda(g)\, V(t)^*. \tag{5.5}$$

Thus the natural extension of τ on $\mathscr{A}_o = \lambda(\mathscr{C}(V))''$ defined by $\tau_t^\zeta(x) = V(t) x V(t)^*$ is indeed a σ-weakly continuous 1-parameter automorphisms group on $\mathscr{A}_o$. In the following we shall often identify $\mathscr{C}(V)$ and $\lambda(\mathscr{C}(V))$.

Conversely let $\tau = \{\tau_t,\, t \in \mathbf{R}\}$ be a 1-parameter automorphisms group on $\mathscr{A}_o$ which preserves $\mathscr{C}(V)$ and the grading of the algebra. Since it preserves the smooth functions on M it follows [AbMaRa] that there exists a smooth vector field ζ such that $\tau_t(f)(p) = f(\gamma_t^\zeta(p))$ for all $f \in C^\infty(M)$.

It is natural to ask whether $\tau_t = \tau_t^\zeta$ for a suitable connection on V. Let δ be the generator of τ. It follows from the homomorphism property that δ is a linear derivation preserving smooth elements in V so that there exists a vector field ζ such that, for any two smooth sections f, g we have the relation

$$\langle \delta(f), g \rangle + \langle f, \delta(g) \rangle = \zeta \langle f, g \rangle.$$

It is evident that a single vector field (i.e. a single τ) carries too little information to determine a unique connection ∇, since it specifies how to transport vectors only along its integral curves. A more natural candidate is a family of such automorphisms τ^k $(1 \leq k \leq d)$, where d is the dimension of the manifold M, with the property that at each point $p \in M$ the set of vectors ζ_k $(1 \leq k \leq d)$, associated to the vector fields τ^k, generates the tangent space at p. Given such a family (ζ_k) $(1 \leq k \leq d)$, for any given smooth vector field ζ because of our hypothesis that the $\zeta_i(p)$ $(1 \leq i \leq d)$ generates $T^p(M)$, there exists a family of smooth maps ϕ_i $(1 \leq i \leq n)$ such that $\zeta(p) = \phi_i(p)\, \zeta_i(p)$.

Define the linear map $\nabla_\zeta : V \to V$ by the prescription

$$\nabla_\zeta(u)(p) = \phi_i(p)\, \delta_{\zeta_i}(u)(p). \tag{5.6}$$

It is simple to check that the right hand side of (5.6) depends only on $\zeta(p)$, and that it defines a unique connection ∇ on M such that $\nabla_{\zeta_i} = \delta_{\zeta_i}$.

6. Unitary Implementation of Stochastic Parallel Transports

The notion of stochastic parallel transport along a random curve was introduced by Ito [Ito] and subsequently studied by many authors [Dy], [DoGu], [Elw], [Eme], [Kun], [Mal], [Nel]. Here we briefly review this notion. For simplicity we consider a smooth, oriented, compact

manifold M of dimension d and, for each fixed $s \geqslant 0$, a classical stochastic differential equation in Stratonovich form

$$X(t) = p + \sum_k \int_s^{s+t} \zeta_k(X(\tau)) \circ dB_k^s(\tau), \tag{6.1}$$

where $\circ$ denotes Stratonovich differential, the ζ_k are smooth vector fields on the manifold M, $B_k^s(\tau) = B_k(s + \tau) - B_k(s)$, $B_k(s)$ are independent standard Brownian motions and $dB_o^s(\tau) = d\tau$. We denote by $X(p, t, \theta_s\omega)$ the unique solution of (6.1) The uniqueness of the solution implies that for all $s, t \geqslant 0$

$$X(p, t + s, \omega) = X(X(p, s, \omega), t, \theta_s\omega) \qquad \text{a.e.} \tag{6.2}$$

Define the family of *-homomorphism j_t, $t \geqslant 0$ from $L^\infty(M)$ into $L^\infty(M) \otimes L^\infty(\Omega, \mathscr{F}, P)$ by

$$j_t(\phi)(p, \omega) = \phi(X(p, t, \omega)). \tag{6.3}$$

Using the Wiener–Segal isomorphism to embed this algebra into the algebra of all bounded operators on $L^2(M, \mu) \otimes \Gamma(L^2(\mathbf{R}_+, \mathbf{C}^d))$ and recalling that, in this identification, the time shift u_s^o acts on $L^2(M) \otimes L^2(\Omega, \mathscr{F}, P; \mathbf{C}^d)$ as

$$u_s^o(F)(\omega) = F(\theta_s(\omega)),$$

we easily verify the cocycle property in the Wiener space, i.e.,

$$j_{s+t}(\phi)(p, \omega) = \phi(X(p, s + t, \omega) = \phi(X(X(p, s, \omega), t, \theta_s\omega))$$

$$= u_s^o(j_t(\phi))(X(p, s, \omega)) = j_s(u_s^o(j_t(\phi)))(p, \omega).$$

That j_t is a Markov cocycle on $L^\infty(M)$ follows from the adaptedness of the solution of (6.1). Equation (6.1) is a symbolic form for

$$j_t(\phi) = \phi + \sum_{1 \leqslant k \leqslant d} \int_0^t j_s(\zeta_k\phi) \, dB_k(s) + j_s(L(\phi)) \, ds,$$

where the generator L is $\zeta_0 + 1/2 \sum_{1 \leqslant k \leqslant d} \zeta_k\zeta_k$.

Our aim is to realize stochastic parallel transports as Markov cocycles on the Clifford-von Neumann algebra $\mathscr{A}_0$ described in Section (4) and show that they are implementable by unitary operator cocycles. Conversely as a consequence of the stochastic representation of σ-weakly continuous Markov cocycles we will show that any given Markov cocycle on $\mathscr{A}_0$ satisfying some natural additional hypothesis can be realized as stochastic

parallel transport with respect to a unique connection compatible with the metric.

Consider the T_n^m valued stochastic process $P(t, \omega)$ along the diffusion $X(p, t, \omega)$ $(t \geqslant 0)$ satisfying the Stratonovich stochastic differential equation

$$dP_t = (\Gamma_i P_t) \circ dX^i, \qquad P_o = 1. \qquad (6.4)$$

It is known [Eme] that the solution of (6.4) exists and is unique. Thus it determines a stochatic linear map $P_t(\omega)$ from $T_n^m(p)$ to $T_n^m(X(p, t, \omega))$.

The following property is a consequence of the uniqueness of the solution to (6.4):

$$P_{s+t}(p, \omega) = P_s(X(t, p, \omega), \theta_t \omega) \, P_t(p, \omega). \qquad (6.5)$$

That it preserves the bilinear form (5.1) is a simple consequence of Ito formula and (5.2). The fact that the map $p \to X(p, t)$ is a local diffeomorphism also guarantees that P_t is onto for almost all sample paths, thus invertible. Denote $j = (j_t)$ the stochastic linear maps on V defined by

$$j_t(v)(p) = (P_t)^{-1} v(X(p, t, \omega)).$$

The fact that the bilinear form is invariant under the parallel transport, implies the crucial identity for any $u \in T_n^m$ and $v \in T_m^n$

$$(u(X(p, t, \omega)), P_t v(p)) = (j_t(u)(p), v(p)). \qquad (6.6)$$

Moreover the process j_t satisfies the stochastic differential equation

$$d(u, j_t(v)) = (u, j_t(L(v))) \, dt + (u, j_t(\nabla_{\zeta_k}) v) \, dB_k(t), \qquad (6.7)$$

where $L(v) = \nabla_{\zeta_o}(v) + 1/2 \sum_{0 \leqslant k \leqslant n} \nabla_{\zeta_k} \nabla_{\zeta_k}(v)$.

When $M, \langle \cdot, \cdot \rangle$ is a Riemannian manifold and the connection respects the metric, we identify T_n^m with T_m^n using the bilinear form $(u, v) = \langle u, v \rangle$. It is simple to verify that j_t is a *-homomorphism on the full tensor algebra and that for any $u, v \in V$ we have

$$d\langle u, j_t(v) \rangle = \langle u, j_t(L(v)) \rangle \, dt + \langle u, j_t(\nabla_{\zeta_k} v) \rangle \, dB_k(t). \qquad (6.8)$$

Since P_t preserves the scalar products, one also has

$$\langle j_t(u), j_t(v) \rangle = j_t(\langle u, v \rangle) := \langle u, v \rangle(\gamma_g(\cdot)), \qquad (6.9)$$

where γ_t is the stochastic flow $\gamma_t(p) = X(p, t, \omega)$. Thus j_t pushes down to a canonical map on the Clifford sections and it is also clear from

(6.9) that for each $u \in \mathscr{C}(V)$, $j_t(u)$ defines a bounded operator on $\mathscr{H}_0 \otimes L^2(\Omega, \mathscr{F}, {}^n)$. That j_t satisfies the cocycle relation on $\mathscr{C}(V)$ follows from (6.5). Following [Sau] we define the strongly continuous unitary operators $V = \{V(t), t \geqslant 0\}$ to be the bounded extensions of the following action on $\mathscr{C}(V)$:

$$V(t) f = j_t(f) \left[\frac{d\mu \circ \gamma_t}{d\mu} \right]^{1/2}. \tag{6.10}$$

That $V(t)$ is an isometry on $\mathscr{C}(V)$ follows from (6.18). Following the arguments of [Sau] one checks that $V(t)$ satisfies the stochastic differential equation

$$dV(t) = \sum_{1 \leqslant k \leqslant n} V(t)(D_k \, dB_k(t) + (1/2 \, D_k^2 + D_o) \, dt) \tag{6.11}$$

on $\mathscr{C}(V) \otimes \mathscr{E}(M)$ where $D_k = \nabla_{\zeta k} + 1/2 \, \operatorname{div}_\mu \zeta_k$ $(0 \leqslant k \leqslant n)$. In particular $t \to V(t) f$ is strongly continuous for each $f \in \mathscr{C}(V)$. Thus the family $(V(t))$ of unitary operators is strongly continuous. Moreover since the operator $1/2 \sum_{1 \leqslant k \leqslant d} D_k^2 + D_0$ preserves $\mathscr{C}(V)$, it is a core and thus it follows by Proposition 4.4 in [Moh] that V is the unique strongly continuous operator Markov cocycle satisfying (6.11) on $\mathscr{C}(V)$. Now it is also routine to verify that for all $f \in \mathscr{C}(V)$

$$j_t(f) = V(t) \, \lambda(f) \, V(t)^*.$$

Thus the natural bounded extension of j_t to $\mathscr{A}_0$, i.e. $j_t(x) = V(t) \, x V(t)^*$, defines a σ-weakly continuous Markov cocycle.

Consider a family L^{ji}, $i, j \in \bar{S}$ of smooth bounded functions on M such that

$$L^{ji} + (L^{ij})^* + \sum_{k \in S} L^{ik}(L^{jk})^* = 0 \tag{6.12a}$$

$$L^{ji} + (L^{ij})^* + \sum_{k \in S} (L^{ki})^* \, L^{kj} = 0. \tag{6.12b}$$

Since the coefficients are bounded and satisfy (6.12b), the homomorphism property of j_t guarantees that there exists a co-isometric operator valued processes $U \equiv \{U(t), t \geqslant 0\}$ satisfying

$$dU(t) = U(t) \, j_t(L^{ji}) \, d\Lambda_{ij}(t), \qquad U(0) = I. \tag{6.13}$$

Now set $W(t) = U(t)V(t)$, $t \geqslant 0$. A simple application of the quantum Ito formula gives

$$dW(t) = W(t) \sum_{i, j \in \bar{S}} K^{ij} \, d\Lambda_{ij}(t) \tag{6.14}$$

on $\mathscr{C}(V) \otimes \mathscr{E}(M)$, where

$$K^{ji} = L^{ji} + Z^{ji} + \sum_{k \in S} k^{ki} Z^{jk}. \qquad (6.15)$$

Now we will make a more specific choice for L^{ji}, $i, j \in \overline{S}$. Choose

$$L^{ji} = \begin{cases} 0, & \text{if } i, j \in S \\ \phi_k, & \text{if } i \in S, \quad j = 0 \\ -\phi_k, & \text{if } i = 0, \quad j \in S \\ -1/2 \sum_{k \in S} \phi_k^2, & \text{if } i = 0 = j, \end{cases}$$

where $\phi_i = 1/2 \operatorname{div}_\mu \zeta_i$ and verify that

$$K^{ji} = \begin{cases} 0, & \text{if } i, j \in S \\ \nabla_{\zeta_i}, & \text{if } i \in S, \quad j = 0 \\ -\nabla_{\zeta_j}^*, & \text{if } i = 0, \quad j \in S \\ -1/2 \sum_{k \in S} \nabla_{\zeta_k}^* \nabla_{\zeta_k} + iH, & \text{if } i = 0 = j, \end{cases}$$

where $H = D_o + \sum_{k \in S} (\phi_k D_k + D_k \phi_k)$ is defined on $\mathscr{C}(V)$ and $H^* = -H$. So far the choice of ζ_k $(0 \leqslant k \leqslant n)$ were arbitrary. For any fixed vector field ζ'_o we now choose ζ_o such that $H = D_{\zeta'_o}$. The following equation (6.16) was discussed by D. Applebaum [Ap2].

PROPOSITION (6.1) *Let* ζ_k, $(0 \leqslant k \leqslant n)$ *be a family of smooth vector fields. Then there exists a unique family* $U = \{U(t), t \geqslant 0\}$ *of unitary operators such that*

$$dU(t) = \sum_{k \in S} U(t)[\nabla_{\zeta_k} dA_k^+(t) - \nabla_{\zeta_k}^* dA_k(t) + (1/2 \nabla_{\zeta_k}^* \nabla_{\zeta_k} + D_{\zeta_o}) dt] \qquad (6.16)$$

on $\mathscr{C}(V) \otimes \mathscr{E}(M)$.

Proof. That it exists and is co isometric follows from the preceeding paragraph. Since $L = 1/2 \nabla_{\zeta_k}^* \nabla_{\zeta_k} + D_{\zeta_o}$ is the generator of a contractive semigroup and $C(V)$ is invariant under the action of L, $\mathscr{C}(V)$ is a core for L. Hence by Proposition 4.4 in [Moh] we conclude that U is the unique solution of (6.16). To show the isometric property we consider, following [Jou], the time reflected cocycle $\tilde{U}(t) = \Gamma(R_t)U(t)^* \Gamma(R_t)^*$, where R_t is the *time reflection with respect to t*, defined by

$$R_t(u)(x) = \begin{cases} u(t-x), & \text{if } 0 \leqslant x \leqslant t, \\ u(x), & \text{otherwise.} \end{cases}$$

Note that $\tilde{U}(t)$ is weakly differentiable, and thus it admits a stochastic representation [Ac Jou Li], [Fag]. An easy computation shows that it has the stochastic representation

$$d\tilde{U}(t) = \sum_{k \in S} \tilde{U}(t)[-\nabla_{\zeta_k} dA_k^+(t) + \nabla_{\zeta_k}^* dA_k(t) + (1/2\, \nabla_{\zeta_k}^* \nabla_{\zeta_k} - D_{\zeta_o})\, dt]$$

on $\mathscr{C}(V) \otimes \mathscr{E}(M)$. Replacing the family ζ_k by $-\zeta_k$ in (6.16) we conclude from the first part that $\tilde{U}(t)$ is co-isometric. Thus $U(t)$ is unitary.

Notice that $\mathscr{A}_0 \otimes \mathscr{B}(\Gamma(L^2(\mathbf{R}_+)))$ inherits the grading of $\mathscr{A}_0$ and that the inner cocycle associated with $U(t)$ also respects this grading, i.e. $P_m j_t(x) P_n = j_t(P_m x P_n)$, such a cocycle will be called *graded*. The main result of the present section is that a σ-weakly continuous graded cocycle on $\mathscr{A}_0$ can be realized as stochastic parallel transport with respect to some connection on M. First we restrict ourselves to σ-weakly continuous Markov cocycles on $L^\infty(M, \mu)$.

DEFINITION (6.2)   Let (j_t), $(t \geqslant 0)$, be a σ-weakly continuous Markov cocycle on $L^\infty(M, \mu)$ and θ^{00} be the generator of the associated Markov semigroup P^t defined as in (3.6). (j_t) is said to be a diffusion if the following holds:

(a)   The domain of θ_{00} contains the smooth functions.

(b)   For each smooth ϕ the associated martingale $X(\phi, t)$, $t \geqslant 0$, defined by (3.6), has continuous trajectories in the sense of Definition (2.8), i.e. $[[X, X]] = \langle\langle X, X \rangle\rangle$

(j_t) is said to be *completely non deterministic* if there exists no non constant function ϕ such that $j_t(\phi) \in L^\infty(M, \mu)$ for all $t \geqslant 0$.

LEMMA (6.3)   *Let $\theta: C_o^\infty \to \mathscr{A}_o = L^\infty(M)$ be a linear derivation, i.e., for any two elements $\phi, \phi' \in C_o^\infty$*

$$\theta(\phi\phi') = \theta(\phi)\, \phi' + \phi\theta(\phi').$$

Then there exists a vector field ζ such that its local representative in a chart (O, χ) on M is

$$(x, \theta(\chi^1)(p), \theta(\chi^2)(p), ..., \theta(\chi^n)(p)), \qquad \chi(p) = x \in U',$$

where $\chi: U \to U' \subset \mathbf{R}^n$ and $\theta(\chi^k)$ are locally bounded measurable functions on U. Moreover the representation is independent of the chart.

Proof. The proof is an easy adaptation of the known characterization of smooth vector fields as derivations which preserve smooth functions. For a proof we refer to [AbMaRa].

PROPOSITION (6.4) *Let* j_t, $t \geqslant 0$ *be a diffusion in the sense of Definition* (6.2). *Then there exists a family* ζ_k, $(k \geqslant 0)$ *of locally bounded measurable vector fields on the manifold* M *such that*

$$dj_t(\phi) = \sum_{k \in S} j_t(\zeta_k(\phi)) \, dB_k(t) + j_t(1/2 \, \zeta_k \zeta_k(\phi) + \zeta_o(\phi)) \, dt \qquad (6.17)$$

on $\mathscr{C}(V) \otimes \mathscr{E}(M)$. *Moreover the following holds:*

(a) ζ_k *is smooth if each* θ^{io} *preserves the smooth elements.*

(b) *If* j_t *is completely non-deterministic, then for each point* $p \in M$ *the family* $\zeta_k(p)$ $(1 \leqslant k \leqslant n)$ *of vectors forms a basis for* $T_p(M)$, *the tangent space of* M *at* p.

Proof. Since smooth functions with compact support form an algebra under point-wise mutiplication the associated martingale $X(\phi, t)$ is regular. Hence ϕ is an element in the domain of each θ^{ji}, $i, j \in \bar{S}$. By our hypothesis $X(\phi, t)$ has continuous trajectories hence by Proposition (2.6) we have $\theta^{ji}(\phi) = 0$ for all $i, j \in S$. As a consequence of the structure relations (3.11) we have, for any $i \in S$,

$$\theta^{io}(\phi\phi') = \theta^{io}(\phi) \, \phi' + \phi\theta^{io}(\phi') \qquad (6.18)$$

$$\theta_o(\phi\phi') = \theta_o(\phi) \, \phi' + \phi\theta_o(\phi') + \sum_{i \in S} \theta^{oi}(\phi) \, \theta^{io}(\phi') \qquad (6.19)$$

for all $\phi, \phi' \in C_o^\infty(M)$. From (6.18) and Lemma (6.3) we conclude that there exist a family ζ_i $(i \in S)$ of bounded measurable vector fields such that $\theta^{i0}(\phi) = \zeta_i(\phi)$. Define the linear map D on $C_o^\infty(M)$ by

$$D(\phi) = \theta^{00}(\phi) - 1/2 \sum_{i \in S} \zeta_i(\zeta_i(\phi)).$$

Now because of (6.19) D is also a derivation hence, again by Lemma (6.3), there exists a bounded measurable vector field ζ_0 such that

$$\theta^{00}(\phi) = \zeta_0(\phi) + 1/2 \sum_{i \in S} \zeta_i(\zeta_i(\phi)).$$

(a) is trivial. To prove (b) it is enough to show that there is no non-constant smooth function ϕ such that

$$\zeta_k(\phi) = 0 \qquad \text{for all} \quad 1 \leqslant k \leqslant n. \qquad (6.20)$$

But for such a function (6.17) implies that $(j_t(f))$ is a deterministic operator process, i.e. a family of elements of $L^\infty(M, \mu)$, thus f is a constant by our hypothesis.

Remark (6.5)   It is not clear whether the θ^{i0}, $(i \in S)$ will preserve the smooth elements if θ^{00} does so. This technical point puts a hurdle to characterise the diffusions associated with smooth vector fields.

THEOREM (6.6)   *Let $\mathscr{A}_o = \mathscr{C}(V)''$. Let (j_t), $(t \geqslant 0)$ be an injective graded σ-weakly continuous Markov cocyle on $\mathscr{A}_0$ and suppose that the associated martingales $X(t, x)$ have continuous trajectories in the sense of Definition (2.5). Furthermore assume that the structure maps θ^{ji} $(i, j \in \bar{S})$ preserve the smooth Clifford sections and the associated diffusion is completely non deterministic. Then there exists a unique family $\{\zeta_k, 0 \leqslant k \leqslant n\}$ of smooth vector fields and a unique connection ∇ such that*

$$dj_t(f) = j_t(\nabla_{\zeta_k}(f)) \, dB_k(t) + j_t(L(f)) \, dt, \tag{6.21}$$

where

$$L = 1/2 \, \nabla_{\zeta_k} \nabla_{\zeta_k} + \nabla_{\zeta_o} + \delta \tag{6.22}$$

and δ is a derivation on $\mathscr{C}(V)$ such that $\delta(f) = 0$ for any smooth function f.

Proof.   We need to study the structure maps appearing in the representation theorem of the σ-weakly continuous Markov cocyle. It follows from the independence of the basic processes, applied to the stochastic representation of the processes $j_t(P_m x P_n) = P_m j_t(x) P_n$ that each θ^{ji} is graded map on $\mathscr{C}(V)$.

Since, by Proposition (2.9), the representation is gauge free i.e. $\theta^{ji} = 0$ for any $i, j \in S$, for each $i \in S$, θ^{io} is a graded derivation on $\mathscr{C}(V)$, in particular, its restriction to the smooth functions is induced by a smooth vector field $\zeta_i (1 \leqslant i \leqslant d)$. From the Clifford commutation relations $g \otimes h + h \otimes g = 2\langle g, h \rangle$ and the derivation property of θ_{io} it follows that

$$\langle \theta^{i0}(f), g \rangle + \langle f, \theta^{i0}(g) \rangle = \zeta_i(\langle f, g \rangle). \tag{6.23}$$

The process (j_t) being completely non-deterministic, the vectors $\zeta_i(p)$ $(1 \leqslant i \leqslant d)$ span $T_p(M)$ and thus for any fixed smooth vector field ζ, one has in a local chart (O, χ) $\zeta(p) = \phi_i(p) \zeta_i(p)$ for some smooth functions ϕ_i on O. Define

$$\nabla_\zeta = \phi_i \theta^{i0} \tag{6.24}$$

on V. Since the θ^{io} are globally defined, the right hand side is independent of the chart. It is evident from (6.23) that ∇ indeed defines a connection on M and it respects the metric.

Now, we define a linear map D on $\mathscr{C}(V)$ by

$$D(g) = \theta^{00}(g) - 1/2 \sum_{i \in S} \nabla_{\zeta_i} \nabla_{\zeta_i}(g)$$

and verify that D is also a derivation on $\mathscr{C}(V)$. Thus there exists a smooth vector field ζ_0 such that $D = \nabla_{\zeta 0} + \delta$ where δ is a derivation on $\mathscr{C}(V)$ such that $\delta(f) = 0$ for smooth function f.

Remark (6.7)   Note that the representation of a Markov cocycle given by Theorem (6.5), involves an unspecified derivation δ. When $\delta = 0$ we obtain Ito's parallel transport. The connection between Markov cocycles and a stochastic generalization of the imprimitivity theorem, shall be investigated elsewhere [AcMoh].

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