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MARKOVIAN KMS-STATES FOR ONE-DIMENSIONAL SPIN CHAINS

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We characterize a class of quantum Markov states in terms of a locality property of their modular automorphism group or, equivalently, of their φ-conditional expectations and we give an explicit description of the structure of these states. This study is meant as a starting point for the investigation of the structure of Markovian KMS-states of quantum spin chains as well as of multidimensional quantum spin lattices.

1. Introduction

Quantum Markov chains were introduced in Refs. 1 and 6 and since then several progresses have been made in their applications to physical models. In particular, starting from Ref. 20 a subclass of quantum Markov chains, also called *finitely correlated states*, was shown to coincide[14–16] with the so-called *valence bond states* introduced in the late '80s[7] as an affirmative example of the Hadane conjecture on antiferromagnetic Heisenberg models with integer spin. More recently the same class of states was shown to coincide with the class of the so-called *spin ladder models*[21] which possess the split property.[19] In another direction, the quantum Markov chains are currently used as trial states in Hartree–Fock approximations of solid state models, for example the Heisenberg model[20] or the fixed points of the *density matrix renormalization group* (DMRG).[17,23]

Several progresses have also been made in the problem of clarifying the mathematical structure of quantum Markov chains.[4,10,11] In particular, Park was able to compute explicitly their entropy,[22] Fannes, Nachtergaele and Werner clarified

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the ergodic structure of an important subclass of these states and gave necessary and sufficient conditions for a quantum Markov state to be pure[14–16] and Matsui[18] characterized them as zero energy states of nearest neighbor Hamiltonians on one-dimensional spin lattices.

This paper goes in some sense in the complementary direction with respect to the above-mentioned Matsui's result, i.e. we look for characterizations of quantum Markov chains as equilibrium states, defined in terms of the Kubo–Martin–Schwinger (KMS) condition. More precisely, following Ref. 4, we characterize a class of quantum Markov states in terms of a locality property of their modular automorphism group or, equivalently of their φ-conditional expectations. Starting from this property we are able to give a full structure theorem for the corresponding class of Markovian states.

We restrict our discussion to one-dimensional lattices, but our technique extends, with minor modifications, to multidimensional quantum spin lattices as well as to continuous time Markov processes. Thus the present results can therefore be considered as a starting point for a definition and a structure theory of quantum Markov fields.

2. Basic Notions and Notations

Throughout this paper $\mathcal{A}$ will be the C^*-algebra $\bigotimes_{n\in G} M_d$, obtained as infinite tensor product of the finite-dimensional matrix algebra M_d, $d \in \mathbb{N}$, $d \geq 2$, cf. Sec. 1.22 of Ref. 25. We consider only the cases $G = \mathbb{N}$ or $G = \mathbb{Z}$. Thus for all $n \in G$ there are injective unital $*$-homomorphisms $i_n : M_d \mapsto \mathcal{A}$ such that $i_n(b)$ and $i_m(b')$ commute for different $n, m \in G$. Denote for all $F \subset G$ by $\mathcal{A}_F = C^*(\{i_n(b) : n \in F, b \in M_d\})$ the C^*-subalgebra generated by $\{i_n(b) : n \in F,\ b \in M_d\}$. Further we denote $n] := \{\ldots, n\} \cap G$ and $[n, m] := \{n, \ldots, m\}$.

We follow Ref. 4 for the definition of Markov states on $\mathcal{A}$:

Definition 2.1. Let there be given a triple $(\mathcal{A}_\mathrm{i}, \mathcal{A}_\mathrm{b}, \mathcal{A}_\mathrm{o})$ (i = *inside*, o = *outside*, b = *boundary*) of commuting C^*-subalgebras of $\mathcal{A}$ (a *localization*) and denote $\mathcal{A}_\mathrm{bi} = C^*(\mathcal{A}_\mathrm{i} \cup \mathcal{A}_\mathrm{b})$ and similarly for $\mathcal{A}_\mathrm{obi}$, $\mathcal{A}_\mathrm{ob}$. A *quasi-conditional expectation* with respect to the triple $(\mathcal{A}_\mathrm{i}, \mathcal{A}_\mathrm{b}, \mathcal{A}_\mathrm{o})$ is a completely positive unit preserving map $\mathcal{E} : \mathcal{A}_\mathrm{obi} \to \mathcal{A}_\mathrm{ob}$ such that $\operatorname{Fix}\mathcal{E} \supseteq \mathcal{A}_\mathrm{o}$, i.e.

$$\mathcal{E}(a_\mathrm{o}) = a_\mathrm{o}, \qquad a_\mathrm{o} \in \mathcal{A}_\mathrm{o}\,. \tag{1}$$

An equivalent formulation gives the following:

Lemma 2.1. *$\mathcal{E}$ is a quasi-conditional expectation iff*

$$\mathcal{E}(a_\mathrm{o} a_\mathrm{b} a_\mathrm{i}) = a_\mathrm{o}\mathcal{E}(a_\mathrm{b} a_\mathrm{i}), \qquad a_\mathrm{i} \in \mathcal{A}_\mathrm{i}, a_\mathrm{b} \in \mathcal{A}_\mathrm{b}, a_\mathrm{o} \in \mathcal{A}_\mathrm{o}\,. \tag{2}$$

Proof. In fact, by polarization the equation

$$\mathcal{E}(a^* a) = \mathcal{E}(a^*)\mathcal{E}(a)$$

implies

$$\mathcal{E}(a^*y) = \mathcal{E}(a^*)\mathcal{E}(y), \qquad (y \in \mathcal{A}) \tag{3}$$

and the fact that $\mathcal{A}_o$ is a $*$-algebra completes the proof. $\square$

3. Markovian States on $\bigotimes_{\mathbb{N}} M_d$

First let $G = \mathbb{N}$. We call a state φ on $\mathcal{A}$ *locally faithful* if it is faithful on each $\mathcal{A}_{n]}$, $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$ there exists the *φ-conditional expectation* $\mathcal{E}^{\varphi}_{n+1,n]} : \mathcal{A}_{n+1]} \mapsto \mathcal{A}_{n]}$, defined as in Ref. 3. The following facts were proven in Ref. 4.

Theorem 3.1. *For a state φ on $\mathcal{A}$ which is locally faithful, the following statements are equivalent*:

(a) *For all $n \in \mathbb{N}$, the φ-conditional expectation $\mathcal{E}^{\varphi}_{n+1],n]}$ leaves the algebra $\mathcal{A}_{n-1]}$ pointwise invariant.*

(b) *For each $n \in \mathbb{N}$, there exists a completely positive unit preserving map $\mathcal{E}_{n+1],n]} : \mathcal{A}_{n+1]} \mapsto \mathcal{A}_{n]}$ such that*:

$$\varphi_{n]} \circ \mathcal{E}_{n+1],n]} = \varphi_{n+1]}\,, \tag{4}$$

where $\varphi_{n]}$ denotes the restriction of φ on $\mathcal{A}_{n]}$, and $\mathcal{A}_{n-1]} \subseteq \mathrm{Fix}\,\mathcal{E}_{n+1],n]}$.

(c) *For any $n \in \mathbb{N}$*

$$\sigma^{n+1]}_t|_{\mathcal{A}_{n-1]}} = \sigma^{n]}_t|_{\mathcal{A}_{n-1]}}\,, \tag{5}$$

where $\sigma^{n+1]}_t$, $\sigma^{n]}_t$ denote the modular automorphism groups associated to the pairs $(\mathcal{A}_{n+1]}, \varphi_{n+1]})$ and $(\mathcal{A}_{n]}, \varphi_{n]})$, respectively.

Therefore, we can make the following definition which also suits the states which are not locally faithful.

Definition 3.1. A state φ on $\mathcal{A}$ is called *Markovian* if for each $n \in \mathbb{N}$ it is invariant under a map $\mathcal{E}_{n+1],n]} : \mathcal{A}_{n+1]} \mapsto \mathcal{A}_{n]}$ which is a quasi-conditional expectation with respect to the localization $(\mathcal{A}_{n+1}, \mathcal{A}_n, \mathcal{A}_{n-1]})$.

If φ is a Markovian state on $\mathcal{A}$ and $n \in \mathbb{N}$, the limit of the ergodic averages of $\mathcal{E}_{n+1],n]}$ always exists and is a completely positive unit preserving map $\mathcal{E}^{\infty}_{n+1],n]} : \mathcal{A}_{n+1]} \mapsto \mathcal{A}_{n]}$. Moreover, $\mathcal{E}^{\infty}_{n+1],n]}$ projects onto

$$\mathrm{Fix}\,\mathcal{E}^{\infty}_{n+1],n]} = \mathrm{Fix}\,\mathcal{E}_{n+1],n]} \supseteq \mathcal{A}_{n-1]}$$

and

$$\varphi_{n]} \circ \mathcal{E}^{\infty}_{n+1],n]} = \varphi_{n+1]}\,.$$

Thus φ is Markovian with respect to the collection $(\mathcal{E}_{n+1],n]})_{n\in\mathbb{N}}$ iff it is Markovian with respect to $(\mathcal{E}^{\infty}_{n+1],n]})_{n\in\mathbb{N}}$. If φ is locally faithful, we know[3] that $\mathrm{Fix}\,\mathcal{E}_{n+1],n]}$ is a $*$-algebra and $\mathcal{E}^{\infty}_{n+1],n]}$ is an Umegaki conditional expectation. For $d = 2$, this is true for all such completely positive unit preserving maps, see Lemma 6.1. Suppose in addition that φ is stationary, i.e.

$$\varphi \circ \theta = \varphi ,$$

where θ is the right shift on $\mathcal{A}$. By translation invariance there is an Umegaki conditional expectation $\mathcal{E} : M_d \otimes M_d \to M_d$ with

$$\mathcal{E}^{\infty}_{n+1],n]}|_{\mathcal{A}_{[n,n+1]}} = i_n \circ \mathcal{E} \circ (i_n^{-1} \otimes i_{n+1}^{-1})|_{\mathcal{A}_{[n,n+1]}} \tag{6}$$

for all $n \in \mathbb{N}$.

We will follow this scheme, but drop the assumptions that φ is locally faithful and stationary. Just assume that φ is a Markovian state with the property that all $\mathcal{E}^{\infty}_{n+1],n]}$, $n \in \mathbb{N}$ satisfy (6) for some Umegaki conditional expectation $\mathcal{E}$, i.e. the fixed points of $\mathcal{E}^{\infty}_{n+1],n]}$ form a $*$-algebra. Let $\mathcal{B}_0$ be the range of $\mathcal{E}$ and let $P_1, \ldots, P_n$ be minimal projections in the center of $\mathcal{B}_0$ such that

$$\sum_j P_j = 1 \tag{7}$$

and $P_j\mathcal{B}_0 = P_j\mathcal{B}_0P_j$ is a factor. Then $\mathcal{B}_0$ can be realized as a matrix algebra on the space

$$\mathbb{C}^d =: \mathcal{H} = \bigoplus_j \mathcal{H}_j , \qquad \mathcal{H}_j = P_j\mathcal{H}$$

and $P_j\mathcal{B}_0 = P_j\mathcal{B}_0P_j$ is a subfactor of $\mathcal{B}(\mathcal{H}_j)$. Therefore, $\mathcal{H}_j = \mathcal{H}_{j0} \otimes \mathcal{H}_{j1}$ and

$$P_j\mathcal{B}_0P_j = P_j\mathcal{B}(\mathcal{H}_{j0}) \otimes 1_{\mathcal{H}_{j1}}P_j .$$

From this we see

$$\mathcal{B}_0' = \bigoplus_j P_j 1_{\mathcal{H}_{j0}} \otimes \mathcal{B}(\mathcal{H}_{j1})P_j$$

and

$$\mathcal{B}_0 \vee \mathcal{B}_0' = (\mathcal{B}_0 \cap \mathcal{B}_0')' = \bigoplus_j P_jM_dP_j .$$

Clearly, for each j, the following holds

$$P_jM_dP_j \cong P_j\mathcal{B}_0P_j \otimes P_j\mathcal{B}_0'P_j . \tag{8}$$

We will fix this identification in the following. The conditional expectation property of $\mathcal{E}^{\infty}_{n+1],n]}$ is reflected by the fact that $\mathcal{E}$ is a conditional expectation, if we identify $\mathcal{B}_0$ with $\mathcal{B}_0 \otimes 1 \subset M_d \otimes M_d$.

Lemma 3.1. *Any conditional expectation E from $M_d \otimes M_d$ onto $\mathcal{B}_0 \otimes 1 \sim \mathcal{B}_0$ has the form*

$$E(x) = \sum_j P_j\Phi_j(P_jxP_j)P_j , \tag{9}$$

where $\Phi_j : \mathcal{B}(\mathcal{H}_{j0}) \otimes \mathcal{B}(\mathcal{H}_{j1}) \otimes M_d \mapsto \mathcal{B}(\mathcal{H}_{j0}) \otimes 1_{j1}$ *is the Umegaki conditional expectation*

$$\Phi_j(b_{j0} \otimes b_{j1} \otimes b) = b_{j0}\phi_j(b_{j1} \otimes b) \otimes 1$$

for states ϕ_j *on* $\mathcal{B}(\mathcal{H}_{j1}) \otimes M_d$.

Proof. We know that $\mathcal{B}_0$ is mapped into itself by the conditional expectation E_P,

$$E_P(x) = \sum_j P_j x P_j, \qquad x \in M_d \otimes M_d \tag{10}$$

which is the unique conditional expectation onto $\mathcal{B}_0 \vee \mathcal{B}_0'$. Since $P_k \in \operatorname{Fix} E$, it follows that $P_k E(x) = E(P_k x P_k) = P_k E(P_k x P_k) P_k$ which implies by (7)

$$E(x) = \sum_j P_j E(P_j x P_j) P_j\,.$$

Denoting $E_j = E(P_j \cdot P_j)$, we see that E is a conditional expectation iff all the E_j's are conditional expectations from $\mathcal{B}(\mathcal{H}_j) \otimes M_d$ onto $\mathcal{B}(\mathcal{H}_{j0}) \otimes 1_{j1} \otimes 1_{M_d}$. Now we get from the conditional expectation property

$$E_j(a \otimes b \otimes c) = a \otimes 1 \otimes 1 E_j(1 \otimes b \otimes c) = E_j(1 \otimes b \otimes c) a \otimes 1 \otimes 1, \qquad a \in \mathcal{B}(\mathcal{H}_{j0})\,.$$

Since $E_j(1 \otimes b \otimes c)$ commutes with all $a \otimes 1 \otimes 1$, it must be a scalar. Therefore, there is a state φ_j on $\mathcal{B}(\mathcal{H}_{j1}) \otimes M_d$ with $E_j(1 \otimes b \otimes c) = \varphi_j(b \otimes c)$. This shows $E_j = \bar{\varphi}_j$ and completes the proof. □

Now we want to return to a Markovian state φ. It is standard to see[4] that

$$\begin{aligned}\varphi(b_0 \otimes \cdots \otimes b_n) &= \varphi(b_0 \otimes \cdots \otimes b_{n-1} \otimes \mathcal{E}(b_n \otimes 1)) \\ &= \varphi_0(\mathcal{E}(b_0 \otimes \mathcal{E}(b_1 \otimes \cdots \otimes \mathcal{E}(b_{n-1} \otimes \mathcal{E}(b_n \otimes 1)))))\,, \end{aligned} \tag{11}$$

i.e. φ is a quantum Markov chain in the sense of Refs. 1 and 6. Thus the knowledge of the initial state φ_0 (a state on M_d) and of $\mathcal{E}$ uniquely determines the structure of a Markov state.

Lemma 3.2. *If* φ *satisfies* (11), *then with the conditional expectation* E_P *defined by* (10):

$$\varphi(b_0 \otimes \cdots \otimes b_n) = \varphi(E_P(b_0) \otimes \cdots \otimes E_P(b_n))$$

and setting $\mathcal{E}_j(x) = P_j x P_j$:

$$\varphi(b_0 \otimes \cdots \otimes b_n) = \sum_{j_0,\ldots,j_n} \varphi(\mathcal{E}_{j_0}(b_0) \cdots \mathcal{E}_{j_n}(b_n))\,. \tag{12}$$

Proof. In the proof of Lemma 2.1 we saw that $\mathcal{E}(E_P(a)\otimes b) = \mathcal{E}(a\otimes b)$. This, together with (11), shows that

$$\begin{aligned}\varphi(E_P(b_0)\otimes\cdots\otimes E_P(b_n)) &= \varphi(\mathcal{E}(E_P(b_0)\otimes\cdots\otimes\mathcal{E}(E_P(b_{n-1})\otimes\mathcal{E}(E_P(b_n)\otimes 1))))\\ &= \varphi(\mathcal{E}(b_0\otimes\cdots\otimes\mathcal{E}(b_{n-1}\otimes\mathcal{E}(b_n\otimes 1))))\\ &= \varphi(b_0\otimes\cdots\otimes b_n)\,.\end{aligned}$$

Since $E_P = \sum_j \mathcal{E}_j$, this proves the second formula. □

Now we set

$$p_{j_0\ldots j_n} := \varphi(\mathcal{E}_{j_0}(1)\cdots\mathcal{E}_{j_n}(1)) = \varphi(P_{j_0}\otimes\cdots\otimes P_{j_n})\,.$$

Further, we define maps $\mathcal{E}_{jj'} : P_jM_dP_j\otimes P_{j'}\mathcal{B}_0P_{j'}\mapsto P_j\mathcal{B}_0P_j$ through

$$\mathcal{E}_{jj'}(a\otimes b) = \mathcal{E}(\mathcal{E}_j(a)\otimes\mathcal{E}_{j'}(b))\,.$$

Due to (12) and Markovianity, we obtain

$$\varphi(b_0\otimes\cdots\otimes b_n) = \sum_{j_0,\ldots,j_n,j_{n+1}}\varphi_0(\mathcal{E}_{j_0,j_1}(b_0\otimes\cdots\otimes\mathcal{E}_{j_n,j_{n+1}}(b_n\otimes P_{j_{n+1}})\cdots))\,.$$

First, we look at the center of $\mathcal{B}_0$. Set $\pi_{jj'} := \phi_j(P_j\otimes P_{j'})$ and $\pi_j := \varphi_0(P_j)$. Then one has

$$\varphi(P^{(0)}_{j_0}\cdots P^{(n)}_{j_n}P^{(n+1)}_{j_{n+1}}) = \varphi(P^0_{j_0}\ldots\mathcal{E}^\infty(P_{j_n}\otimes P_{j_{n+1}}))\,.$$

By definition,

$$\mathcal{E}^\infty(P_{j_n}\otimes P_{j_{n+1}}) = P_{j_n}\bar{\varphi}_{j_n}(P_{j_n}\otimes P_{j_{n+1}})P_{j_n} = \varphi_{j_n}(1\otimes P_{j_{n+1}})P_{j_n}\,.$$

We set $p_{jj'} = \phi_j(P_j\otimes P_{j'})$ and obtain

$$\begin{aligned}\varphi(P^{(0)}_{j_0}\cdots P^{(n)}_{j_n}P^{(n+1)}_{j_{n+1}}) &= \varphi(P^{(0)}_{j0}\cdots P^{(n)}_{j_n})\cdot\varphi_{j_n}(1\otimes P_{j_{n+1}})\\ &= \varphi(P^{(0)}_{j0}\cdots P^{(n)}_{j_n})p_{j_nj_{n+1}}\\ &\;\;\vdots\\ &= \varphi(P^{(0)}_{j_0})\cdot p_{j_0j_1}\cdots p_{j_nj_{n+1}}\,.\end{aligned}$$

Denoting $\pi_j = \varphi_0(P_j)$ we obtain

$$\varphi(P^{(0)}_{j_0}\cdots P^{(n+1)}_{j_{n+1}}) = \pi_{j_0}\cdot p_{j_0j_1}\cdots p_{j_nj_{n+1}}\,.$$

Corollary 3.1. *For all indices* $j_0,\ldots,j_n$

$$p_{j_0,\ldots,j_n} = \pi_{j_0}\cdot\pi_{j_0j_1}\cdots\pi_{j_{n-1}j_n}\,. \tag{13}$$

Proof. It is easy to see that

$$\mathcal{E}_{jj'}(P_j\otimes P_{j'}) = \mathcal{E}(P_j\otimes P_{j'}) = P_j\varphi_{jj'}(P_j\otimes P_{j'}) = P_j\pi_{jj'}\,.$$

A simple induction completes the proof. □

Remark 3.1. Observe that the right-hand side in (13) is equal to

$$\mathrm{Prob}(X_0 = j_0, X_1 = j_1, \dots, X_n = j_n, X_{n+1} = j_{n+1}),$$

where (X_n) is the homogeneous Markov process on $\{1, \dots, n\}$ (or equivalently, on the atoms of the center of $\mathrm{Fix}\,\mathcal{E}$) with initial distribution $P(X_0 = j) = \pi_j$ and transition probabilities $\pi_{jj'} = \phi_j(1 \otimes P_{j'})$. We denote the law of this process by μ.

Lemma 3.3. *For a fixed sequence $(j_n) =: \omega$, there exists a state φ_ω on*

$$\bigotimes_{n\in\mathbb{N}} P_{j_n} M_d P_{j_n} \cong \bigotimes_{n\in\mathbb{N}} P_{j_n}\mathcal{B}_0 P_{j_n} \otimes P_{j_n}\mathcal{B}'_0 P_{j_n}$$
$$\cong P_{j_0}\mathcal{B}_0 P_{j_0} \otimes \bigotimes_{n\in\mathbb{N}} P_{j_n}\mathcal{B}'_0 P_{j_n} \otimes P_{j_{n+1}}\mathcal{B}_0 P_{j_{n+1}} \tag{14}$$

such that, for all $n \in \mathbb{N}$, $b_0, \dots, b_n \in M_d$:

$$\varphi(\mathcal{E}_{j_0}(b_0) \otimes \cdots \otimes \mathcal{E}_{j_n}(b_n) \otimes P_{j_{n+1}} \otimes 1 \cdots)$$
$$= p_{j_0,\dots,j_n,j_{n+1}} \varphi_\omega(\mathcal{E}_{j_0}(b_0) \otimes \cdots \otimes \mathcal{E}_{j_n}(b_n) \otimes P_{j_{n+1}} \otimes \cdots). \tag{15}$$

Proof. Clearly, for all n there is a state φ_ω^n which fulfils (15) for all $b_0, \dots, b_n \in M_d$. We need only to prove compatibility. Observe

$$\begin{aligned}
&\varphi(\mathcal{E}_{j_0}(b_0) \otimes \cdots \otimes \mathcal{E}_{j_n}(b_n) \otimes P_{j_{n+1}} \otimes P_{j_{n+2}}) \\
&\quad = \varphi(\mathcal{E}_{j_0}(b_0) \otimes \cdots \otimes \mathcal{E}_{j_n}(b_n) \otimes \mathcal{E}(P_{j_{n+1}} \otimes P_{j_{n+2}}) \otimes \cdots) \\
&\quad = \varphi(\mathcal{E}_{j_0}(b_0) \otimes \cdots \otimes \mathcal{E}_{j_n}(b_n) \otimes P_{j_{n+1}} \otimes 1 \cdots)\phi_{j_{n+1}}(P_{j_{n+1}} \otimes P_{j_{n+2}}) \\
&\quad = \varphi(\mathcal{E}_{j_0}(b_0) \otimes \cdots \otimes \mathcal{E}_{j_n}(b_n) \otimes P_{j_{n+1}} \otimes 1 \cdots)\pi_{j_{n+1},j_{n+2}}.
\end{aligned}$$

Together with $p_{j_0,\dots,j_n,j_{n+1}} \pi_{j_{n+1}j_{n+2}} = p_{j_0,\dots,j_{n+1},j_{n+2}}$ this shows compatibility of the states (φ_ω^n) if $p_{j_0,\dots,j_{n+1},j_{n+2}} > 0$. But this is true for μ-a.a. $\omega = (j_n)$. □

Remark 3.2. Observe that (12) is not enough to construct the states φ_ω. We need in (15) at least one additional $P_{j_{n+1}}$ on the right. This reflects the quantum Markovian structure of φ.

For $\pi_{jj'} > 0$, we define the states $\eta_{jj'}$ on $P_j\mathcal{B}'_0 P_j \otimes P_{j'}\mathcal{B}_0 P_{j'}$,

$$\eta_{jj'}(b'_j \otimes b_{j'}) = \frac{\phi_j(b'_j \otimes b_{j'})}{\pi_{jj'}}, \qquad b_j \in P_j\mathcal{B}'_0 P_j,\, b_{j'} \in P_{j'}\mathcal{B}_0 P_{j'}.$$

Similarly, for $\pi_j > 0$, we define the states η_j by

$$\eta_j(b_j) = \frac{\varphi_0(b_j)}{\pi_j}, \qquad b_j \in P_j\mathcal{B}'_0 P_j.$$

Lemma 3.4. *In the representation* (14),

$$\varphi_\omega = \eta_{j_0} \otimes \bigotimes_{n\in\mathbb{N}} \eta_{j_n j_{n+1}} \tag{16}$$

for μ*-a.e.* $\omega = (j_n)$.

Proof. Fix $n \in \mathbb{N}$, $(j_n) = \omega$ such that $p_{j_0,\dots,j_{n+1}} > 0$ and $b_k \in P_{j_k}\mathcal{B}_0 P_{j_k}$, $b'_k \in P_{j_k}\mathcal{B}'_0 P_{j_k}$, $k = 0,\dots,n$. From (11) we obtain

$$\begin{aligned}
&p_{j_0,\dots,j_{n+1}}\varphi_\omega(b_0\otimes b'_0\otimes\cdots\otimes b_{n-1}\otimes b'_{n-1}\otimes b_n\otimes b'_n)\\
&\quad= \varphi(b_0\otimes b'_0\otimes\cdots\otimes b_{n-1}\otimes b'_{n-1}\otimes b_n\otimes b'_n\otimes P_{j_{n+1}})\\
&\quad= \varphi(b_0\otimes b'_0\otimes\cdots\otimes b_{n-1}\otimes b'_{n-1}\otimes \mathcal{E}(b_n\otimes b'_n\otimes P_{j_{n+1}}))\\
&\quad= \varphi(b_0\otimes b'_0\otimes\cdots\otimes b_{n-1}\otimes b'_{n-1}\otimes b_n\otimes 1)\varphi_{j_n,j_{n+1}}(b'_n\otimes P_{j_{n+1}})\\
&\quad= \varphi(b_0\otimes b'_0\otimes\cdots\otimes b_{n-1}\otimes b'_{n-1}\otimes b_n\otimes 1)\eta_{j_n,j_{n+1}}(b'_n\otimes P_{j_{n+1}})\pi_{j_n,j_{n+1}}\\
&\quad\;\;\vdots\\
&\quad= \eta_{j_0}(b_0)\eta_{j_0,j_1}(b'_0\otimes b_1)\dots\eta_{j_n,j_{n+1}}(b'_n\otimes 1)\pi_{j_0}\pi_{j_0,j_1}\pi_{j_n,j_{n+1}}\\
&\quad= p_{j_0,\dots,j_{n+1}}\eta_{j_0}(b_0)\eta_{j_0,j_1}(b'_0\otimes b_1)\dots\eta_{j_n,j_{n+1}}(b'_n\otimes 1)\,.
\end{aligned}$$

Thus, for all such ω, (16) is valid. But this set of ω has μ-measure 1 and the proof is complete. □

Summarizing, we obtain

Theorem 3.2. *Let* φ *be a Markovian state with stationary completely positive unit preserving maps* $\mathcal{E}_{n+1],n]}$ *related to* $\mathcal{E}$ *which is an Umegaki conditional expectation onto* $\mathcal{B}_0$ *with minimal central projections* $(P_j)_j$*. Define* π_j, $\pi_{jj'}$, η_j, $\eta_{j,j'}$ *as above. Further, let* μ *be the law of the classical Markov process with initial distribution* $(\pi_j)_j$ *and transition probabilities* $(\pi_{jj'})_{j,j'}$*. Then*

$$\varphi = \int_\oplus \mu(d\omega)\varphi_\omega \tag{17}$$

in the sense that for all $n \in \mathbb{N}$

$$\begin{aligned}
&\varphi(b_0\otimes\cdots\otimes b_n\otimes 1\otimes\cdots)\\
&\quad= \sum_{j_0,\dots,j_n,j_{n+1}} p_{j_0,\dots,j_{n+1}}\varphi_{j_0,\dots,j_{n+1},\dots}(\mathcal{E}_{j_0}(b_0)\otimes\cdots\otimes\mathcal{E}_{j_n}(b_n)\otimes P_{j_{n+1}}\otimes\cdots)\,. \qquad (18)
\end{aligned}$$

Moreover,

$$\varphi_{(j_0,\dots,j_n,\dots)} = \eta_{j_0}\bigotimes_{n=0}^{\infty}\eta_{j_n,j_{n+1}}\,. \tag{19}$$

Conversely, fix $\mathcal{B}_0$, a probability distribution $(\pi_j)_j$, transition probabilities $(p_{ij})_{ij}$, states η_j on $\mathcal{B}(\mathcal{H}_{j0})$ for each j with $\pi_j > 0$ and states $\eta_{j,j'}$ on $\mathcal{B}(\mathcal{H}_{j1} \otimes \mathcal{H}_{j'0})$ for all j, j' with $\pi_{jj'} > 0$. Then (17) *and* (19) *define a state φ on $\mathcal{A}$ which is Markovian. In the notations of Lemma* 3.1, *the structure of a corresponding Umegaki conditional expectation $\mathcal{E}$ is determined by*

$$\phi_j(b' \otimes b) = \phi_j(b' \otimes E_P(b)) . \tag{20}$$

(E_P is given by (10) *and*

$$\varphi_j(b' \otimes \mathcal{E}_{j'}(c \otimes c')) = \sum_{j''} \pi_{jj'} \eta_{jj'}(b' \otimes c) \pi_{j'j''} \eta_{j'j''}(c' \otimes 1) .)$$

Remark 3.3. Please note that the symbol $\int_\oplus$ is not a usual direct integral[8] because we need in (15) an additional $P_{j_{n+1}}$ to be present. In other words, it is like a "Markovian" direct integral.

Remark 3.4. One can easily apply the above technique also to the inhomogeneous case. In fact, we did something similar for the "fiber" state φ_ω.

Proof. The structure of Markovian states is determined by the above considerations. Thus we prove here only that states φ determined by (18) are Markovian. Clearly, this equation determines a state φ on $\mathcal{A}$. Due to (20) it is enough to check for μ-a.e. (j_n)

$$\begin{aligned} &\varphi((b_0 \otimes b_0') \otimes \cdots \otimes (b_n \otimes b_n') \otimes 1 \cdots) \\ &\quad = \varphi((b_0 \otimes b_0') \otimes \cdots \otimes \mathcal{E}((b_{n-1} \otimes b_{n-1}') \otimes (b_n \otimes b_n')) \otimes 1 \cdots) \end{aligned}$$

for all $b_l \in P_{j_l} \mathcal{B}_0 P_{j_l}$, $b_j' \in P_{j_l} \mathcal{B}_0' P_{j_l}$. Then

$$\begin{aligned} &\varphi((b_0 \otimes b_0') \otimes \cdots \otimes \mathcal{E}((b_{n-1} \otimes b_{n-1}') \otimes (b_n \otimes b_n')) \otimes 1 \cdots) \\ &\quad = \sum_j \varphi((b_0 \otimes b_0') \otimes \cdots \otimes b_{n-1} \otimes 1) \eta_{j_{n-1}j_n}(b_{n-1}' \otimes b_n) \eta_{j_n j}(b_n' \otimes P_j) \\ &\quad = \sum_j p_{j_0,\ldots,j_n,j} \eta_{j_0}(b_0) \eta_{j_0 j_1}(b_0' \otimes b_1) \cdots \eta_{j_{n-2}j_{n-1}}(b_{n-2}' \otimes b_{n-1}) \eta_{j_{n-1}j_n} \\ &\qquad \times (b_{n-1}' \otimes b_n) \eta_{j_n j}(b_n' \otimes P_j) \\ &\quad = \sum_j \varphi((b_0 \otimes b_0') \otimes \cdots \otimes (b_n \otimes b_n') \otimes P_j \otimes 1 \cdots) \\ &\quad = \varphi((b_0 \otimes b_0') \otimes \cdots \otimes (b_n \otimes b_n') \otimes 1 \cdots) . \end{aligned}$$

This completes the proof. □

We will shortly present some examples.

Example 3.1. Assume that $\mathcal{B}_0$ is maximal Abelian, i.e. $\mathcal{B}_0 = \bigoplus_j \mathbb{C}P_j$, all $\mathcal{H}_j \cong \mathbb{C}$. This leads to $\mathcal{A}_\omega = \mathbb{C}$ and φ_ω being trivial. But to each P_j there belongs a natural state $\psi_j : a \mapsto \operatorname{tr} P_j a$. So (18) translates into

$$\varphi(b_0 \otimes \cdots \otimes b_n) = \int \mu(d(j_n))\psi_{j_0}(b_0) \cdots \psi_{j_n}(b_n) .$$

Thus φ is a mixture of ordinary product states under a Markov process. Such states were considered in Ref. 5 and a limit theorem provided even a continuous time analogue.

Example 3.2. If $\mathcal{B}_0$ is a factor, the Markov process (X_n) is constant, i.e. μ is trivial. Thus $\varphi = \eta \otimes \bigotimes_{n=0}^{\infty} \tilde{\eta}$, where $\tilde{\eta}$ is a state on $\mathcal{B}_0' \otimes \mathcal{B}_0$. There are other recent works[11] suggesting one to think of such states as quantum Markov states. We want to note that such states also play a role in the construction of valence bond solid (VBS) states.[7,15]

In particular, if $\mathcal{B}_0 = \mathbb{C}1$ or $\mathcal{B}_0 = M_d$, the Markovian states are just ordinary product states.

Example 3.3. If $d = 2$, there are only three possible types of algebras $\mathcal{B}_0$: $\mathcal{B}_0 = \mathbb{C}1$, $\mathcal{B}_0 = M_d$ and maximal Abelian algebras. In view of Lemma 6.1, all Markovian states are either product states or come from a classical Markov process. This seems to be a partial reverse of the transfer matrix principle which relates to any classical spin system a quantum spin system of lower dimension.[24]

We can look for several criteria to be fulfilled for the state φ defined by (17) and (19) with the various ingredients. The following results are straightforward, we omit the proofs.

Lemma 3.5. *φ is locally faithful iff*

(a) *$\pi_j > 0$ for all j,*
(b) *for all j, j', it holds $\pi_{jj'} > 0$,*
(c) *for all j, the state η_j is faithful and*
(d) *For all j, j' the state $\eta_{jj'}$ is faithful.*

Remark 3.5. In fact, $\mathcal{E}^\infty$ is faithful iff (b) and (d) are satisfied.

Lemma 3.6. *φ is stationary iff*

(a) *μ is stationary, i.e. $\sum_j \pi_j \pi_{jj'} = \pi_{j'}$ and*
(b) *we have for all j'*

$$\sum_j \pi_j \pi_{jj'} \eta_{jj'}(P_j \otimes b) = \pi_{j'} \eta_{j'}(b), \qquad b \in P_{j'} \mathcal{B}_0 P_{j'} . \tag{21}$$

Consequently, the stationary Markovian state is unique iff the invariant distribution $(\pi_j)_j$ is unique.

Proof. Clearly, φ is stationary only if (X_n) is stationary. Moreover,

$$\begin{aligned}\varphi(1 \otimes (b_1 \otimes b_1') \otimes \cdots \otimes (b_n \otimes b_n') \otimes 1) \\ = \sum_j \varphi(P_j \otimes (b_1 \otimes b_1') \otimes \cdots \otimes (b_n \otimes b_n') \otimes 1 \\ = \sum_j \pi_j \pi_{j,j_1} \eta_{j,j_1}(1 \otimes b_1) \pi_{j_1,j_2} \eta_{j_1,j_2}(b_1' \otimes b_2) \cdots \sum_{j'} \pi_{j_n,j'} \eta_{j_n,j'}(b_n' \otimes 1)\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}\varphi((b_1 \otimes b_1') \otimes \cdots \otimes (b_n \otimes b_n') \otimes 1) \\ = \pi_{j_1} \eta_{j_1}(b_1) \pi_{j_1,j_2} \eta_{j_1,j_2}(b_1' \otimes b_2) \cdots \sum_{j'} \pi_{j_n,j'} \eta_{j_n,j'}(b_n' \otimes 1)\,.\end{aligned}$$

This shows (21). The proof of sufficiency is straightforward.

Moreover, if the invariant distribution $(\pi_j)_j$ is unique, then (21) determines all the ϕ_i^0 which are essential for determining the Markovian state. This completes the proof. □

An interesting problem seems to be whether a state φ defined by (17) and (19) is really leftinvariant under the original completely positive unit preserving maps $\mathcal{E}_{n+1],n]}$ or equivalently, $\mathcal{E}^{\infty}_{n+1],n]}$.

Theorem 3.3. *Suppose the completely positive unit preserving maps* $\mathcal{E}^{\infty}_{n+1],n]}$, $n \in \mathbb{N}$ *come from one completely positive unit preserving map for which* $\mathcal{E}$ *is an Umegaki conditional expectation given by* (9). *From this description, construct* π_{ij} *and* η_{ij}. *Then a state* φ *is invariant under all* $\mathcal{E}^{\infty}_{n+1],n]}$, $n \in \mathbb{N}$ *iff*

(a) *It has a representation* (17) *where* μ *is the law of the Markov process* $(X_n)\{(\pi_{ij}),\ (\pi_j)\}$.

(b) (X_n) *never visits points* j *for which there is a pair* (j', j'') *with* $\pi_{jj'}\pi_{j'j''} > 0$ *associated with and there exist* b, c, c' *such that*

$$\frac{\phi_j(b' \otimes P_{j'} c \otimes c' P_{j'})}{\pi_{jj'}} \neq \frac{\phi_j(b' \otimes P_{j'} c P_{j'})}{\pi_{jj'}} \frac{\phi_{j'}(c' \otimes P_{j''})}{\pi_{j'j''}}\,.$$

Proof. We already showed that conditions (17) and (19) are necessary for a state to be invariant. So, to prove the necessity of (b), we can go to the fibers and have to characterize the fiber states φ_ω, which are invariant under the maps $\mathcal{E}_n^{\infty,\omega}$ given by

$$\mathcal{E}_n^{\infty,\omega}(b_0 \otimes \cdots \otimes b_n \otimes b_{n+1} \otimes \cdots) = b_0 \otimes \cdots \otimes b_{n-1} \otimes \mathcal{E}^{\infty}_{j_n,j_{n+1}}(b_n \otimes b_{n+1}) \otimes \cdots\,,$$

where $\mathcal{E}^{\infty}_{j,j'}$ is defined by

$$\mathcal{E}^{\infty}_{j,j'}((b_j \otimes b_j') \otimes (b_{j'} \otimes b_{j'}')) = b_j \tilde{\phi}_{jj'}(b_j' \otimes b_{j'} \otimes b_{j'}')$$

and the state $\tilde{\phi}_{jj'}$ on $P_j\mathcal{B}'_0P_j \otimes P_{j'}M_dP_{j'}$ is given by

$$\tilde{\phi}_{jj'}(b' \otimes b) = \frac{\phi_j(b' \otimes b)}{\pi_{jj'}}.$$

We obtain for any invariant state φ for μ-a.a. fibers

$$\varphi_\omega((b_0 \otimes b'_0) \otimes \cdots \otimes \mathcal{E}^\infty_{j_{n-1},j_n}((b_{n-1} \otimes b'_{n-1}) \otimes (b_n \otimes b'_n)))$$
$$= \varphi_\omega((b_0 \otimes b'_0) \otimes \cdots \otimes \mathcal{E}^\infty_{j_{n-1},j_n}((b_{n-1} \otimes b'_{n-1}) \otimes \mathcal{E}^\infty_{j_n,j_{n+1}}((b_n \otimes b'_n) \otimes 1))).$$

It follows

$$\varphi_\omega((b_0 \otimes b'_0) \otimes \cdots \otimes b_{n-1} \otimes \cdots)(\tilde{\phi}_{j_{n-1},j_n}(b'_{n-1} \otimes (b_n \otimes b'_n))$$
$$- \tilde{\phi}_{j_{n-1},j_n}(b'_{n-1} \otimes b_n)\tilde{\phi}_{j_n,j_{n+1}}(b'_n \otimes P_{j_{n+1}})) = 0.$$

This shows that the term in parentheses has to vanish for μ-a.a. (j_n). This is equivalent to (b).

To prove sufficiency, it is enough to show that for μ-a.a. ω the states φ_ω are invariant under the maps $\mathcal{E}^{\infty,\omega}_n$, $n \in \mathbb{N}$, described above. By the calculations before Theorem 3.2, the following must hold

$$\eta_{jj'}(b' \otimes c) = \tilde{\phi}_{jj'}(b' \otimes c \otimes 1).$$

Now we find

$$\varphi_\omega(b_0 \otimes b'_0 \otimes \cdots \otimes \mathcal{E}^\infty_{j_{n-1},j_n}(b_{n-1} \otimes b'_{n-1} \otimes b_n \otimes b'_n))$$
$$= \varphi_\omega(b_0 \otimes b'_0 \otimes \cdots \otimes b_{n-1} \otimes 1)\tilde{\varphi}_{j_n,j_{n+1}}(b'_{n-1} \otimes b_n \otimes b'_n).$$

From the assumption we know that μ-a.a. $\omega = (j_n)$ fulfil for all $n \in \mathbb{N}$, $n \geq 1$

$$\tilde{\varphi}_{j_{n-1},j_n}(b' \otimes c \otimes c') = \tilde{\varphi}_{j_{n-1},j_n}(b' \otimes c \otimes 1)\tilde{\varphi}_{j_n,j_{n+1}}(c' \otimes 1 \otimes 1).$$

Thus we can conclude from the definition of φ_ω

$$\varphi_\omega(b_0 \otimes b'_0 \otimes \cdots \otimes \mathcal{E}^\infty_{j_{n-1},j_n}(b_{n-1} \otimes b'_{n-1} \otimes b_n \otimes b'_n))$$
$$= \varphi_\omega(b_0 \otimes b'_0 \otimes \cdots \otimes b_{n-1} \otimes 1)\tilde{\varphi}_{j_{n-1},j_n}(b'_{n-1} \otimes b_n \otimes 1)\tilde{\varphi}_{j_n,j_{n+1}}(b'_n \otimes 1 \otimes 1)$$
$$= \varphi_\omega(b_0 \otimes b'_0 \otimes \cdots \otimes b_{n-1} \otimes 1)\varphi_{j_{n-1},j_n}(b'_{n-1} \otimes b_n)\varphi_{j_n,j_{n+1}}(b'_n \otimes 1)$$
$$= \varphi^0_{j_0}(b_0)\varphi_{j_n,j_{n+1}}(b'_0 \otimes b_1) \ldots \varphi_{j_{n-2},j_{n-1}}(b'_{n-2} \otimes b_{n-1})\varphi_{j_{n-1},j_n}(b'_{n-1} \otimes b_n)$$
$$\times \varphi_{j_n,j_{n+1}}(b'_n \otimes 1)$$
$$= \varphi_\omega(b_0 \otimes b'_0 \otimes \cdots \otimes b_{n-1} \otimes b'_{n-1} \otimes b_n \otimes b'_n).$$

This completes the proof. $\square$

4. Relations to Potentials

In analogy to the classical case, we want to relate a Markovian state to a potential. Fix a locally faithful state φ on $\mathcal{A}$ (chosen as in the previous section). We define self-adjoint operators $(h_{n]})$, $h_{n]} \in \mathcal{A}_{n]}$ by

$$\rho_{n]} = e^{-h_{n]}}, \qquad n \in \mathbb{N} \tag{22}$$

if $\rho_{n]}$ is the density matrix of $\varphi_{n]}$. The following is known from Theorem 4.2 of Ref. 4.

Proposition 4.1. *For a locally faithful state φ on $\mathcal{A}$, the following statements are equivalent:*

(a) *φ is a Markovian state.*

(b) *For all $n \in \mathbb{N}$*

$$e^{-ith_{n]}} e^{ith_{n+1]}} \in \mathcal{A}_{[n,n+1]}, \qquad t \in \mathbb{R}. \tag{23}$$

(c) *For each $n \in \mathbb{N}$*

$$e^{-1/2h_{n]}} e^{1/2h_{n+1]}} \in \mathcal{A}_{[n,n+1]}.$$

Remark 4.1. In Ref. 4 a sequence $(h_{n]})_{n\in\mathbb{N}}$ satisfying (23) was called an *Ising potential.* Note that not all Ising potentials define a Markovian state because some compatibility conditions have to be fulfilled. Also the characterizations of Ising potential from Theorem 3.2 of Ref. 4 is too indirect for the construction of examples. Now, we want to go one step forward in this direction and look at an intrinsic property of $(h_{n]})$ assuring that φ is Markovian.

Proposition 4.2. *φ is a Markovian state if and only if for all $n \in \mathbb{N}$ there is some $h_n \in \mathcal{A}_n$ such that $h_{n+1]} - h_{n]} \in \mathcal{A}_{[n,n+1]}$, h_n commutes with $h_{n]}$ and $h_{n+1]}$ commutes with $h_{n]} - h_n$.*

Proof. Suppose that φ is Markovian. So there are the ingredients μ, η_j, $\eta_{jj'}$. We know from Lemma 3.5 that $p_{j_0,\ldots,j_n} > 0$ for all $j_0, \ldots, j_n$ and all η_j, $\eta_{jj'}$ are faithful. So we relate to the latter state potentials $h_j \in P_j\mathcal{B}_0P_j$, $h_{jj'} \in P_j\mathcal{B}_0'P_j \otimes P_{j'}\mathcal{B}_0P_{j'}$. Further, there are also (faithful) states

$$\tilde{\eta}_j(b) = \sum_{j'} \pi_{jj'}\eta_{jj'}(b \otimes P_{j'}), \qquad b \in P_j\mathcal{B}_0'P_j$$

with potentials $\tilde{h}_j \in P_j\mathcal{B}_0'P_j$. Then, using Theorem 3.2 we find by easy calculation

$$h_{n]} = \sum_{j_0,\ldots,j_n} \mathcal{E}_{j_0,\ldots,j_n}(-\ln(p_{j_0,\ldots,j_n})1 + i_0(h'_{j_0}) + i_{0,1}(h'_{j_0,j_1}) + \cdots + i_{n-1,n}(h'_{j_{n-1},j_n}) + i_n(\tilde{h}_{j_n}))$$

with

$$\mathcal{E}_{j_0,\ldots,j_n}(b) = P_{j_0} \otimes \cdots \otimes P_{j_n} b P_{j_0} \otimes \cdots \otimes P_{j_n}$$

and the short hand $i_{m,n} = i_m \otimes i_n$. Setting $h_n = \sum_j i_n(\tilde{h}_j)$ we derive easily the assertion.

Conversely, we obtain

$$e^{-ith_{n]}}e^{ith_{n+1]}} = e^{-ith_n}e^{-it(h_{n]}-h_n)}e^{ith_{n+1]}} = e^{-ith_n}e^{it((h_{n+1]}-h_{n]})+h_n)} \in \mathcal{A}_{[n,n+1]}\,.$$

By (23) this implies the Markovianity of φ and the proof is complete. □

Corollary 4.1. *Suppose φ is a locally faithful Markovian state on $\mathcal{A}$. Then it is a KMS state with respect to the one-parameter automorphism group σ_t*

$$\sigma_t(a) = \lim_{n\to\infty} e^{-ith_{[0,n]}}ae^{ith_{[0,n]}}, \qquad a \in \mathcal{A}, t \in \mathbb{R}\,.$$

In particular, φ has a faithful extension to $\pi_\varphi(\mathcal{A})''$, where π_φ is the GNS-representation of φ.

Proof. The first part follows from the remark after the proof of Theorem 4.2 of Ref. 4. The second part is an application of Corollary 5.3.9 of Ref. 9. □

5. Markovian States on $\bigotimes_{\mathbb{Z}} M_d$

Now we want to deal with Markovian states on the full chain. The definition now reads as

Definition 5.1. We call a state φ on $\mathcal{A}$ *Markovian state* if it is for all $k < n \in \mathbb{Z}$ invariant under a map $\mathcal{E}_{[k,n+1],[k,n]}\colon \mathcal{A}_{[k,n+1]} \mapsto \mathcal{A}_{[k,n]}$ which is a quasi-conditional expectation with respect to the localization $(\mathcal{A}_{n+1}, \mathcal{A}_n, \mathcal{A}_{[k,n-1]})$.

We will restrict to the case of locally faithful states. Again, the completely positive unit preserving maps should be stationary. A next problem is to construct a suitable Umegaki conditional expectation $\mathcal{E}^\infty\colon M_d \otimes M_d \mapsto M_d$ like in the case of a half-chain.

Lemma 5.1. *Suppose φ is locally faithful. Then for each $n \in \mathbb{Z}$ there exists an Umegaki conditional expectation $\mathcal{E}^{\infty_n}\colon M_d \otimes M_d \mapsto M_d$ such that every state $\bar{\varphi}$ invariant under all $\mathcal{E}^\varphi_{[k,n+1],[k,n]}$, $n > k \in \mathbb{Z}$ is invariant under the lifting of $\mathcal{E}^\infty_n$ to $\mathcal{A}_{n+1]}$ too. Furthermore, $\mathcal{A}_{n-1]} \otimes \mathrm{Fix}(\mathcal{E}^\infty_n) \subseteq \mathrm{Fix}(\mathcal{E}^\varphi_{[k,n+1],[k,n]})$ for all $\mathbb{Z} \ni k < n$.*

Proof. From Ref. 3 we get for all $k < n$ the Umegaki conditional expectation $\mathcal{E}^{\infty,k}$ which projects onto $\mathrm{Fix}\,\mathcal{E}^\varphi_{[k,n+1],[k,n]}$. Moreover, we know that $\mathcal{A}_{[k,n+1]}$ is mapped into $\mathcal{A}_{[k,n]}$ by $\mathcal{E}^{\infty,k-1}$. So $\mathcal{E}^{\infty,k-1}$ is another Umegaki conditional expectation leaving φ invariant. Therefore, the results of Ref. 3 imply that

$$\mathcal{E}^{\infty,k-1} \circ \mathcal{E}^{\infty,k} = \mathcal{E}^{\infty,k} \circ \mathcal{E}^{\infty,k-1} = \mathcal{E}^{\infty,k-1}\,.$$

Now, putting

$$\mathcal{E}^\infty_n = \lim_{k\to-\infty} \mathcal{E}^{\infty,k} \circ (i_n \otimes i_{n+1})$$

we obtain a completely positive unit preserving map which has all the announced properties. □

In the following, we will assume $\mathcal{E}_n^\infty = \mathcal{E} \circ (i_n \otimes i_{n+1})$ for all $n \in \mathbb{Z}$ and some Umegaki conditional expectation $\mathcal{E}$. We can also drop the assumption that φ is locally faithful. Clearly, we can repeat the whole computations from the half-line chain. Thus

Theorem 5.1. *Let φ be a Markovian state with stationary completely positive unit preserving maps $\mathcal{E}_n^\infty$ such that $\mathcal{E}$ is an Umegaki conditional expectation onto $\mathcal{B}_0$ with minimal central projections $(P_j)_j$. Define $\pi_{jj'}$, $\eta_{j,j'}$ as above. Further, let μ be the law of the classical Markov process (X_n) with transition probabilities $(p_{jj'})_{j,j'}$. Then* (17) *holds in the sense that*

$$\varphi(\cdots \otimes b_k \otimes \cdots \otimes b_n) = \int \mu(dj_n)\varphi_{(j_n)}(\cdots \otimes \mathcal{E}_{j_k}(b_k) \otimes \cdots \otimes \mathcal{E}_{j_n}(b_n) \otimes \cdots)\,, \quad (24)$$

where the states φ_ω on $\mathcal{A}_\omega = \bigotimes_{n\in\mathbb{Z}} \mathcal{B}(\mathcal{H}_{j_n})$ are determined as

$$\varphi_{j_n} = \bigotimes_{n\in\mathbb{Z}} \eta_{j_n,j_{n+1}}\,. \quad (25)$$

Conversely, fix $\mathcal{B}_0$, transition probabilities $(\pi_{jj'})_{jj'}$, and states $\eta_{j,j'}$ on $\mathcal{B}(\mathcal{H}_{j1} \otimes \mathcal{H}_{j'0})$ for all j, j' with $\pi_{jj'} > 0$. Then (17) *and* (25) *define a state φ on $\mathcal{A}$ which is Markovian. In the sense of Lemma* 3.1, *the structure of a corresponding Umegaki conditional expectation $\mathcal{E}$ is determined by* (20).

Remark 5.1. A close look at (24) and (25) shows that the structure of Markovian states is invariant under time reversal, mapping $\mathcal{A}_n$ into $\mathcal{A}_{-n}$. This happens in analogy to classical processes, for which the backward and forward Markov properties are equivalent. In the quantum case this is quite unexpected, as there is a considerable asymmetry in the definition of quasi-conditional expectations.

So it remains to look at the uniqueness of Markovian states in this more general framework.

Corollary 5.1. *Suppose $(\pi_{jj'})$ is aperiodic. Then all $\mathcal{E}$ Markovian states are stationary.*

If $(\pi_{jj'})$ has period p, any Markovian state is periodic and the dimension of the set of Markovian states is at most p.

Proof. It is straightforward, all results depend only on the possible $(\pi_{jj'})_{jj'}$ Markov processes. □

6. A Special Property for $d = 2$

There remains the following question. How much restrictive is the assumption that $\mathcal{E}$ is an Umegaki conditional expectation?

Lemma 6.1. *Suppose $\mathcal{E}$ is a completely positive unit preserving map on M_2 which is also a projection: $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$. Then it is an Umegaki conditional expectation, i.e. its range is a $*$-algebra.*

Proof. We assume the contrary. Clearly, $1 \in \mathrm{Fix}\,\mathcal{E}$. Moreover, there should be another $a \in M_2$ with $a = \mathcal{E}(a)$ not being a multiple of 1. Without loss of generality we may assume a to be self-adjoint. Thus it has two-point spectrum. Shifting a by a multiple of 1 and scaling appropriately we achieve that a is a projection. Say, $a = (1 + \sigma_x)/2$, where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices. Because we assumed that $\mathrm{Fix}\,\mathcal{E}$ is not a $*$-algebra there is yet another $b \in \mathrm{Fix}\,\mathcal{E}$ which is not in the linear hull of 1 and a. Again, we may assume that b is self-adjoint. Using linear algebra we may force $b = \sigma_y$, say. But $\sigma_x^2 = \sigma_y^2 = 1 \in \mathrm{Fix}\,\mathcal{E}$ and as Lemma 2.1, Eq. (3) shows

$$\sigma_z = i\sigma_x\sigma_y = i\mathcal{E}(\sigma_x)\mathcal{E}(\sigma_y) = i\mathcal{E}(\sigma_x\sigma_y) = \mathcal{E}(\sigma_z)\,.$$

Consequently, $\sigma_z \in \mathrm{Fix}\,\mathcal{E}$ which forces $\mathrm{Fix}\,\mathcal{E} = M_2$. This contradiction completes the proof. □

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