The Brownian Motion Generated by the Levy-Laplacian

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Abstract

An existence and uniqueness theorem for the the heat equation associated to the Levy-Laplacian is proved and a simple explicit formula is derived. The associated *Levy heat semigroup* is used to construct a classical Markov process called the *Levy Brownian motion*.

1 Introduction

Let $\mathcal{H}$ be a real Hilbert space, $\mathcal{D}$ a dense sub-space of $\mathcal{H}$ and let $\mathcal{L}(\mathcal{D})$ be a vector space of linear, not necessarily continuous operators from $\mathcal{D}$ to $\mathcal{H}$.

To every linear functional ψ on $\mathcal{L}(\mathcal{D})$ one can associate ([8], [2]) a (linear, constant coefficients) second order differential operator L_ψ, defined by:

$$(L_\psi f)(x) = \psi(f''(x)) \qquad\qquad x \in \mathcal{H}$$

L_ψ is a linear map from the space of real valued, twice differentiable functions on $\mathcal{H}$ (for some notion of derivative) whose second derivative at any point belongs to $\mathcal{L}(\mathcal{D})$, to the space of all real valued functions on $\mathcal{H}$. In particular, if $\mathcal{L}(\mathcal{D}) = L^1(\mathcal{H})$ is the space of all nuclear operators on $\mathcal{H}$ and ψ is the trace on $L^1(\mathcal{H})$, then L_ψ is the usual (classical) *Volterra Laplacian* Δ_V. If $\mathcal{H} = L^2([0,1])$ and $\mathcal{L}(\mathcal{D})$ is the sub-algebra of $\mathcal{B}(\mathcal{H})$ (= all the bounded operators on $\mathcal{H}$), generated by the compact operators and $L^\infty([0,1])$ (acting on $\mathcal{H}$ by pointwise multiplication), then every element of $\mathcal{L}(\mathcal{D})$ has the form

$$a = f + K$$

with $f \in L^\infty([0,1])$, K is a compact operator, and the linear functional which to $a = f + K \in \mathcal{L}(\mathcal{D})$ associates the integral of f in $[0,1]$, defines a second order operator Δ_L called *the Levy Laplacian* .

Analogously one introduces differental operators on spaces of vector valued functions and on spaces of functions defined on spaces which are not Hilbert spaces (the precise definitions of the notions used in the present note, shall be given below). In the paper [2] it was also described a multitude of functionals, on spaces of unbounded operators, corresponding to the series of the so-called *exotic Laplacians*, whose definition was proposed in [1] (the just defined Levy Laplacian, is the first element of this series). The interest to the study of the Levy Laplacian significantly grew after that, in the paper [1], it was proved that the Euclidean Yang–Mills equations on the n–dimensional Euclidean space, are equivalent to the Laplace equations, corresponding to the Levy Laplacian (in a space of matrix valued functions). In the paper [2] we also described the basic ideas for the use of the Fourier transform (FT) method for obtaining an explicit formula giving the solution of the heat equation, corresponding to the Levy Laplacian. In the present note we formulate a theorem of existence and uniqueness for this equation. The semi-group, which arises from this theorem, is called the *Levy semigroup* and is used to construct a Markov process, called the *Levy Brownian motion.*

2 Notations and terminology

Everywhere in the following we shall denote

$$\mathcal{H} = L^2(0,1) \qquad ; \qquad E \subseteq W_2^1[0,1] \subseteq L^2(0,1)$$

where $W_2^1[0,1]$ denotes the Sobolev space of absolutely continous functions with square integrable derivative and we shall suppose that on E it is given a topology making it a Frechet space [5] and for which the canonical embedding $E \to \mathcal{H}$ is continuous; E^* shall denote the dual space of E for this topology. Consequently we can, and we shall consider $\mathcal{H}$ as a vector subspace of E^*. Furthermore we shall suppose that the duality $\langle E^*, E\rangle$ agrees with the scalar product $(\ \cdot\ ,\ \cdot\ )$ in $\mathcal{H}$, i.e. that if $\varphi \in \mathcal{H}$, $\psi \in E$, then

$$< \varphi, \psi >_{E^*,E} = (\varphi, \psi)_{\mathcal{H}} \tag{1}$$

For every element φ in the space E, the symbol F_φ shall denote the linear functional on E, defined by the identity:

$$F_\varphi(g) = < g, \varphi > \qquad ; \qquad g \in E^* \tag{2}$$

$\mathcal{B}(E^*)$ shall denote the σ–algebra generated by all the functionals F_φ ($\varphi \in E$); $\mathcal{M}(E^*)$ the space of all countably additive signed measures on $\mathcal{B}(E^*)$. If

$\mu \in \mathcal{M}(E^*)$, then its Fourier transform $\hat{\mu}$ is a function on E, defined by the identity:

$$\hat{\mu}(x) = \int_{E^*} e^{i<y,x>} \mu(dy) \qquad ; \qquad x \in E \tag{3}$$

An orthonormal basis $e = (e_j)$ of $\mathcal{H}$ is called *uniformly (or equally) dense*, if for every $f \in L^\infty(0,1)$ the following identity holds [4]:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \int_0^1 f(t)(e_j(t))^2 dt = \int_0^1 f(t)dt \tag{4}$$

Such a basis is called *bounded* [4] if every function e_j is bounded and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} || e_j ||_\infty^2 < \infty \tag{5}$$

In the following $e = (e_j)$ shall denote a fixed uniformly dense, bounded basis of $\mathcal{H}$. Let, for each natural integer n , $P_{[1,n]}$, denote the orthogonal projector on the subspace of $\mathcal{H}$ generated by $e_1, ..., e_n$.

Definition 1 *Let $\mathcal{D}$ denote the subspace of the space $\mathcal{C}^2(E)$, of all real valued twice Frechet differentiable functions f on E [5], [6], such that the limit*

$$\lim_{n\to\infty} \frac{1}{n} Tr(P_{[1,n]} f''(x)) =: \Delta_L f(x) \tag{6}$$

exists for all $x \in E$. The map Δ_L, from $\mathcal{D} \subseteq \mathcal{C}^2(E)$ to the space of all functions on E, which to every function $f \in \mathcal{D}$ associates the function defined by (6), is called the Levy Laplacian on E.

One can prove [4], [7] that the right hand side of (6) does not depend on the choice of the uniformly dense, bounded, basis of $\mathcal{H}$. Let now S denote the shift with respect to the (e_j)–basis, i.e. the unique linear continuous (in fact isometric) map from $\mathcal{H}$ to $\mathcal{H}$ such that

$$Se_j = e_{j+1}$$

Consequently the adjoint map $S^* : E^* \to E^*$ is well defined. Finally we shall denote $\mathcal{M}(e)$ the vector space of all S- invariant measures on $\mathcal{B}(E^*)$. Clearly

$$\mathcal{M}(e) \subseteq \mathcal{M}(E^*) \tag{7}$$

We shall denote, moreover

$$\mathcal{M}_2(e) := \{\mu \in \mathcal{M}(e) \ : \ \int_{E^*} | < e_j, y >|^2 \ \mu(dy) < \infty \qquad ; \qquad \forall j \in \mathbf{N}\} \quad (8)$$

Notice that, if $\mu \in \mathcal{M}_2(e)$ is a probability measure, then using the notation

$$F_{ej}(x) =< x, e_j >$$

according to the ergodic theorem the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} | \ F_{e_j}(y) \ |^2 = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} |< y, e_j >|^2 = E^{\mu}_{\infty}\Big(| < y, \cdot >|^2\Big) \quad (9)$$

exists for μ–almost all $y \in E^*$ and E^{μ}_{∞}, on the right hand side of (9), denotes the conditional expectation on the fixed σ–algebra of the adjoint shift S^*. In the following we shall use the shorthand notation

$$E^{\mu}_{\infty}\Big(| < y, \cdot > |^2\Big) =: || \ y \ ||^2_{\mu} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} |< y, \cdot >|^2 \quad (10)$$

3 The heat semigroup

Theorem 1 *For any positive measure $\mu_o \in \mathcal{M}_2(e)$ and any $t \in \mathbf{R}_+$, define the measure $\mu(t)$ on E^* by*

$$\frac{\partial \mu(t)}{\partial \mu_o}(y) := e^{-t||y||^2_{\mu_o}} \quad ; \quad y \in E^* \quad (11)$$

Then $\mu(t) \in \mathcal{M}_2(e)$ for all t and the map $t \mapsto \mu(t)$ is the solution of the Cauchy problem for the Laplace equation

$$\frac{\partial}{\partial t}\hat{\mu}(t) = \Delta_L \hat{\mu}(t) \qquad ; \qquad \hat{\mu}(0) = \hat{\mu}_o \quad (12)$$

For the proof it is sufficient to make explicit Fourier transform in both sides of (2) and to apply (6). The possibility to exchange Δ_L with the integral of the Fourier Transform is proved in [Ro].

Remark (2.1) For each positive $\mu \in \mathcal{M}_2(e)$, let the symbol $\mathcal{M}_2(e,\mu)$, denote the space of measures in $\mathcal{M}_2(e)$, absolutely continuous with respect to μ. Then the semigroup (11) leaves $\mathcal{M}_2(e,\mu)$ invariant, i.e. for each $\nu \in \mathcal{M}_2(e,\mu)$, one has

$$e^{-t||y||^2_\mu}\nu \in \mathcal{M}_2(e,\mu) \tag{13}$$

Theorem 2 *There exists a unique 1-parameter semigroup* $\hat{P}^t : \mathcal{M}(e) \to \mathcal{M}(e)$ *such that for any* $\mu, \rho \in \mathcal{M}(e)$ *with* $\mu \equiv \rho$ *one has:*

$$\hat{P}^t\mu = e^{-t||\ \cdot\ ||^2_\rho}\mu \tag{14}$$

This is a corollary of Theorem 11 and Remark (2.1).

Proposition 1 *The subspace* $\mathcal{M}_2(e)$ *is a sub-algebra of the convolution algebra* $\mathcal{M}(E^*)$.

Proposition 1 implies that the image of the space $\mathcal{M}_2(e)$ under the Fourier transform is an (abelian) algebra under pointwise multiplication. In the following it shall be denoted with the symbol $\mathcal{B}$. Notice moreover that the Fourier transform $\mathcal{F}$ maps $\hat{P}^t$ in a semigroup of operators acting on $\mathcal{B}$:

$$\mathcal{F}^{-1}\hat{P}^t\mathcal{F} =: P^t : \mathcal{B} \to \mathcal{B}$$

4 Construction of the Levy Brownian Motion

In this Section we shall restrict to bounded measures on E.

Bochner Theorem implies that the Fourier transform maps, in a one-to-one way, the conus of positive measures in $\mathcal{M}_2(e)$ onto the conus of bounded positive definite functions in the algebra $\mathcal{B}$, which we shall denote with the symbol $\mathcal{B}_+$.

Since the semigroup $\hat{P}^t$ maps positive measures into positive, it follows that

$$\hat{P}^t(\mathcal{B}_+) \subseteq \mathcal{B}_+$$

The involution, given by the complex conjugation of a function, defines on $\mathcal{B}$ a structure of $*$-algebra. Moreover, the constant functions (on E) are contained in $\mathcal{B}$ since they are the Fourier transform of atomic measures concentrated in the origin (of E) , which surely belong to $\mathcal{M}_2(e)$. A Linear functional φ_o on $\mathcal{B}$, shall be called a *state* if it is positive on $\mathcal{B}_+$ and normalized, i.e.

$$\varphi_o(\mathcal{B}_+) = \mathbf{R}_+ = [0,\infty) \qquad ; \qquad \varphi_o(1) = 1$$

is a C^*-algebra with the *sup*-norm. Let $\tilde{E}$ denote its spectrum, so that $\mathcal{C}(\tilde{E})$ is a realization of $\mathcal{B}$ as the algebra of continuous functions on $\tilde{E}$.

Theorem 3 *For any state φ_o on $\mathcal{B}$, there exists a unique Markov process (L_t), with state space $\tilde{E}$, characterized by the property that, for each $t_1 < t_2 < \ldots < t_n \in \mathbf{R}_+$; $f_1, \ldots, f_n \in \mathcal{B}$ the following identity holds:*

$$E\Big(\tilde{f}_1(L(t_1)) \ldots \tilde{f}_n(L(t_n))\Big) = \varphi_o\Big(P^{t_1}(f_1 \cdot P^{t_2-t_1}(f_2 \cdot \ldots \cdot (f_{n-1} \cdot P^{t_n-t_{n-1}}(f_n) \ldots)\Big)$$

where $E(\cdot)$ denotes the expectation of the Markov process.

Definition 2 *The Markov process, defined by Theorem 3 is called the Levy Brownian motion with initial distirbution φ_o.*

Remark (3.3). One can clearly assume that $\tilde{E} \supseteq E$. It would be interesting to find a space containing E and *as small as possible* which could be taken as state space of the Levy Brownian motion. Choose now

$$E = W_2^1[0,1] \qquad ; \qquad e_n(t) = sin 2\pi nt \qquad ; \qquad n \in \mathbf{N}$$

Let (X_n) be a real stationary random process for which

$$P\{(X_n) \ : \ \sum_{n=1}^{\infty} \frac{x_n^2}{n^2} < +\infty\} = 1$$

and let $(\Omega, \mathcal{B}_\Omega, P)$ denote its sample space (unique up to isomorphism). The the map

$$\psi : \omega = (x_n) \in \Omega \to \sum_{n=1}^{\infty} x_n e_n \in E^*$$

is defined P-almost everywhere and

$$\mu := P \circ \psi^{-1} \in \mathcal{M}(e)$$

It is clear that each probability measure in $\mathcal{M}(e)$, can be choosen in this way. Notice that, if the process (X_n) is ergodic, then the function $y \in E^* \mapsto || \ y \ ||_\mu^2$, defined by (10) is constant μ–a.e. . To construct an example of a probability measure in $\mathcal{M}_2(e)$ which corresponds to a non ergodic process, consider the vector

$$a := \sum_{j=1}^{\infty} e_j \in E^*$$

And let μ denote the measure on $\mathcal{B}(e^*)$ obtained as image of the stadard Gaussian measure on $\mathbf{R}$ under the map

$$t \in \mathbf{R} \mapsto ta \in E^*$$

Then $\mu \in \mathcal{M}(e)$ since $S^* a = a$. Moreover for each $n \in \mathbf{N}$ one has

$$\frac{1}{n} \sum_{j=1}^{n} | F_{e_j}(y) |^2 = | F_{e_1}(y) |^2 \qquad ; \qquad \mu - \forall y \in E^*$$

because the F_{e_j} ($j \in \mathbf{N}$ coincide μ–a.e., since μ is concentrated on the muliples of the vector $a \in E^*$. Consequently

$$| F_{e_1}(y) |^2 = || y ||_\mu^2 \qquad ; \qquad \mu - \forall y \in E^*$$

and since the function F_{e_1} is non constant on the line $\mathbf{R} \cdot a$,which has $\mu - measure$ one, it follows that μ is not ergodic.

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