*Acta Math. Hungar.*
**63 (3)** (1994), 249–282.

# QUANTUM CENTRAL LIMIT THEOREMS FOR WEAKLY DEPENDENT MAPS. II

L. ACCARDI and Y. G. LU* (Roma)

## Introduction

In Part I ([21]), three central limit theorems have been stated: the first one (Theorem (1.3)) includes a law of large numbers and is a vanishing result; the second one (Theorem (1.4)) is a central limit theorem for processes with discrete parameter and the third result (Theorem (1.5)) is the extension of the second one to the case of continuous parameter. Our central limit theorems are *deformations* of the usual quantum central limit theorems, considered up to now, in three different ways:

i) The factor $\sigma(b, b')$ in the commutation relations

$$(*) \qquad j_t(b)j_s(b') = \sigma(t, s, b, b')j_s(b')j_t(b) + \varepsilon(t, s, b, b')$$

($b, b' \in B \subset \mathcal{B}$; B is the set of algebraic generators of $\mathcal{B}$) are not restricted to the values $\pm 1$.

ii) The factor $\varepsilon(t, s, b, b')$ in $(*)$ is not required to vanish identically.

iii) Independence is replaced by weak dependence.

First of all the states that we obtain in the limit are of *Gaussian type*, in the sense that their odd momenta vanish and the even ones are given by weighted sums of products of pair correlations. However, while the $\varepsilon$-factor simply produces a shift in the correlation function of the limiting state (cf. (1.17) of Part I); the $\sigma$-factor can give rise to more interesting phenomena. In fact, if all the $\sigma(b, b')$ are present, then the final expressions (1.18) or (1.24) in Part I differ from the usual expressions for the even momenta of a Gaussian state only for the presence of a combinatorial factor. However, if some of the $\sigma(b, b')$ are allowed to vanish, then the sums (1.18) and (1.24) will no longer be over *all* pair partitions, but only over a subset of them. Now it is well known that the notion of *free independence*, recently introduced by Voiculescu [16], leads naturally, by means of *free central limit theorems* to a notion of *free Gaussianity* characterized, in terms of momenta, precisely by the property that the summation in the expression of even momenta is

*On leave of absence from Beijing Normal University.

taken *not over all the pair partitions, but on a special sub-class of them* (cf. [15]).

The following conjectures are therefore natural:

I. By imposing that some of the factors $\sigma(b,b')$ vanish on some ordered pairs of generators, the expressions (1.18), (1.24) define free Gaussian states.

II. All the states with vanishing odd moments and with even moments given by weighted sums of products of pair correlations (*the sum not necessarily ranging over all the pair partitions, but only over a subset of them*) can be obtained by the present central limit theorems, by appropriately choosing the algebra $\mathcal{B}$ and the factors $\sigma$, $\varepsilon$.

The replacement of independence by weak dependence causes a new qualitative phenomenon: now the singletons give non-trivial contributions to the final state.

## §6. Proof of the main theorems

After the preliminaries in §2, §3, §4 and §5 (cf. [21]), we can prove our main theorems.

PROOF OF THEOREM (1.3). First we prove Theorem (1.3) for the counting measure, in this case by Lemma (3.7),

$$(6.1)\qquad \frac{1}{\nu([0,T))^{ak}}\int_{[0,T)^k} E\big(j_{t_1}(b_1)\dots j_{t_k}(b_k)\big)\,\nu(dt_1)\dots\nu(dt_k)=$$

$$=\frac{1}{N^{ak}}E\big(S_N(b_1)\cdots S_N(b_k)\big)=$$

$$=\frac{1}{N^{ak}}\sum_{p=1}^{n}\sum_{(S_1,\dots,S_p)\in\mathcal{P}_{k,p}}\sum_{t\in[S_1,\dots,S_p]_N}E\big(j_{t_1}(b_1)\cdots j_{t_k}(b_k)\big)=$$

$$=\frac{1}{N^{ak}}\sum_{p=1}^{k}\sum_{m=0}^{[k/2]}\sum_{1\leqq p_1<\dots<p_m\leqq k}\sum_{(q_1,\dots,q_m)}{}^{'}\sum_{(S_1,\dots,S_p)\in\mathcal{P}_{k,p}}$$

$$\sum_{\pi\in\mathcal{S}_k^{(S,p)}}\sum_{t\in I_N(S_1,\dots,S_p,\{p_h,q_h\}_{h=1}^m,\pi)}\prod_{h=1}^{m}\varepsilon\big(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h}\big)$$

$$\sigma(t_1,\dots,t_k,b_1,\dots,b_k)\cdot E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{n-2m}}}(b_{\pi(r_{n-2m})})\big)$$

where $t$ takes discrete values and $I_N(S_1,\dots,S_p,\{p_h,q_h\}_{h=1}^m,\pi)$ is defined by (3.19). Clearly the asymptotic behaviour of (6.1) is determined by the sum in $t$. By the boundedness of $\varepsilon$, $\sigma$ and formula (1.3b), we know that the right hand side of (6.1) is majorized by

$$\frac{1}{N^{ak}} C_1(b_1, \ldots, b_k) \cdot N^p. \tag{6.2}$$

Since $a > \frac{1}{2}$ or $a = \frac{1}{2}$ and $k$ odd, $p \leqq k/2$ implies that

$$\frac{1}{N^{ak}} \sum_{t \in [S_1, \ldots, S_p]_N} E\big(j_{t_1}(b_1) \cdots j_{t_k}(b_k)\big) \leqq C \cdot N^{p-ak} \longrightarrow 0 \tag{6.3}$$

we need only to consider the case $p > k/2$. For clarity we write the sum

$$\sum_{t \in I_N(S_1, \ldots, S_p, \{p_h, q_h\}_{h=1}^m, \pi)} \tag{6.4}$$

in the notation of Section (.3), i.e.

$$\sum_{\substack{t \in I_N(S_1, \ldots, S_p) \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)},\ h=1,\ldots,m}} = \sum^{(S,p)}_{\substack{1 \leqq t_1 \leqq \cdots \leqq t_k \leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)},\ h=1,\ldots,m}} \tag{6.5}$$

where we use the superscript $(S,p)$ to mean that the sum in $t$ runs over $[S_1, \ldots, S_p]_N$.

In the following by the boundedness of the $\sigma$-factors we can neglect the factor $\sigma(t_1, \ldots, t_k,\, b_1, \ldots, b_k)$. Now we introduce a procedure to eliminate, step by step the $\varepsilon$- factors, by repeated use of inequality (4.2).

i) Suppose that there exists a $h' \in \{1, \ldots, m\}$ such that

$$\{t_{\pi^{-1}(p_{h'})}, t_{\pi^{-1}(q_{h'})}\} \subset \{t_{r_1}, \ldots, t_{r_{k-2m}}\} \tag{6.6}$$

i.e., to the same pair of times there corresponds an $\varepsilon$-factor and also a product of two operators. For example, in the product

$$j_{t_1}(b_1) j_{t_2}(b_2) j_{t_1}(b_3) j_{t_2}(b_4) = \tag{6.7}$$

$$= \sigma(t_1, t_2, b_2, b_3) j_{t_1}(b_1 \cdot b_3) j_{t_2}(b_2 \cdot b_4) + \varepsilon(t_2, t_1, b_2, b_3) j_{t_1}(b_1) j_{t_2}(b_4)$$

the factor with $\varepsilon$ is of type i).

In this case by the uniform boundedness of $\varepsilon$-factors, we can neglect $\varepsilon(t_{\pi^{-1}(p_{h'})}, t_{\pi^{-1}(q_{h'})},\, b_{p_{h'}}, b_{q_{h'}})$. Therefore without loss of generality we can suppose that there is no $h' \in \{1,\dots,m\}$ such that (6.6) is valid.

ii) Suppose that there exists a $h' \in \{1,\dots,m\}$ such that

$$\{t_{\pi^{-1}(p_{h'})}, t_{\pi^{-1}(q_{h'})}\} \cap \{t_{r_1},\dots,t_{r_{k-2m}}\} = \emptyset \tag{6.8}$$

i.e., there is a pair of times which has produced an $\varepsilon$-factor and which is not in correspondence with any operator factor; e.g.

$$\varepsilon(t_1,t_2,b_1,b_2)j_{t_3}(b_3)$$

with $t_3 > t_1 > t_2$.

In this case, since $\varepsilon(\cdot,\cdot,b,b')$ is $s - L^1(\mathcal{C},dn)$ we know that the quantity

$$\frac{1}{N^{ak}} \sum_{\substack{1\leqq t_1\leqq\cdots\leqq t_k\leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)},\ h=1,\dots,m}}^{(S,p)} \left|\prod_{h=1}^{m} \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h})\right| \tag{6.9}$$

$$\left|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{k-2m}}}(b_{\pi(r_{k-2m})})\big)\right|$$

is dominated by

(6.10)

$$\frac{1}{N^{ak-1}} \sum_{\substack{1\leqq t_1\leqq\cdots\leqq \widehat{t_{p_{h'}}}\leqq\cdots\leqq\widehat{t_{q_{h'}}}\leqq\cdots\leqq t_k\leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)},\ h=1,\dots,m}}^{(S',p')} \cdot\frac{1}{N}\sum_{l_1=1}^{N}\sum_{l_2=l_1+1}^{N} \left|\varepsilon(l_2,l_1,b_{p_{h'}},b_{q_{h'}})\right|$$

$$\left|\prod_{\substack{1\leqq h\leqq m\\ h\neq h'}} \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h})\right| \cdot$$

$$\cdot\left|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{k-2m}}}(b_{\pi(r_{k-2m})})\big)\right| \leqq$$

$$\leqq M\cdot\frac{1}{N^{ak-1}} \sum_{\substack{1\leqq t_1\leqq\cdots\leqq \widehat{t_{p_{h'}}}\leqq\cdots\leqq\widehat{t_{q_{h'}}}\leqq\cdots\leqq t_k\leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)},\ h=1,\dots,m}}^{(S',p')}$$

$$\left|\prod_{\substack{1\leqq h\leqq m\\ h\neq h'}} \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h})\right| \cdot \left|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{k-2m}}}(b_{\pi(r_{k-2m})})\big)\right|$$

where $M$ is a constant and $(S',p')$ is defined as in (6.5) for the partition $(S'_1,\dots,S_{p'})$ obtained by taking away from the partition $(S_1,\dots,S_p)$ all the $t_{\pi^{-1}(p_{h'})}, t_{\pi^{-1}(q_{h'})}$ which satisfy (6.8). Suppose that there exist $r$ elements of $\{1,\dots,m\}$ such that (6.8) is valid, then by relabeling the indices we find that (6.9) is dominated by

(6.11)

$$\frac{1}{N^{ak-r}} \sum_{\substack{1\leqq t_1\leqq\cdots\leqq t_{k-2r}\leqq N\\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},\ h=1,\dots,m-r}}^{(S',p')} M^r\cdot\left|\prod_{h=1}^{m-r}\varepsilon\big(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h}\big)\right|$$

$$\left|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{k-2m}}}(b_{\pi(r_{k-2m})})\big)\right|.$$

iii) By step ii) the estimate of (6.9) is reduced to an estimate of expressions of the form (6.11), where for each $h=1,\dots,m-r$, one and only one of $\{t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)}\}$ is in $\{t_{r_1},\dots,t_{r_{k-2m}}\}$. Let us deal separately with the possible cases. If

(6.12a) $$t_{\pi^{-1}(p_1)}\notin\{t_{r_1},\dots,t_{r_{k-2m}}\}$$

and

(6.12b) $$t_{\pi^{-1}(q_1)}\in\{t_{r_1},\dots,t_{k_{n-2m}}\}$$

then (6.11) is majorized by

(6.13)

$$\frac{1}{N^{ak-r}} \sum_{\substack{1\leqq t_1\leqq\cdots\leqq \widehat{t_{\pi^{-1}(p_1)}}\leqq\cdots\leqq t_{k-2r}\leqq N\\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},\ h=2,\dots,m-r}}^{(S',p')} M^r\cdot\left|\prod_{h=2}^{m-r}\varepsilon\big(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h}\big)\right|$$

$$\left|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{k-2m}}}(b_{\pi(r_{k-2m})})\big)\right|\cdot\sum_{l=t_{\pi^{-1}(q_1)}+1}^{\infty}\left|\varepsilon(l,t_{\pi^{-1}(q_1)},b_{p_1},b_{q_1})\right|\leqq$$

$$\leqq\frac{1}{N^{ak-r}} \sum_{\substack{1\leqq t_1\leqq\cdots\leqq \widehat{t_{\pi^{-1}(p_1)}}\leqq\cdots\leqq t_{k-2r}\leqq N\\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},\ h=2,\dots,m-r}}^{(S',p')} M^{r+1}\,.$$

$$\cdot\left|\prod_{h=2}^{m-r}\varepsilon\big(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h}\big)\right|\left|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{n-2m}}}(b_{\pi(r_{n-2m})})\big)\right|.$$

Moreover since $t_{\pi^{-1}(q_1)}$ is equal to some $t_{r_j}$, if we omit the index $t_{\pi^{-1}(q_1)}$ form the right hand side of (6.13), nothing will change (since $t_{r_j}$ remains), i.e. (6.11) is majorized by

$$(6.14)\qquad \frac{1}{N^{ak-r}}\sum_{\substack{1\leqq t_1\leqq\cdots\leqq\widehat{t_{\pi^{-1}(p_1)}}\leqq\cdots\leqq\widehat{t_{\pi^{-1}(q_1)}}\leqq\cdots\leqq t_{k-2r}\leqq N\\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},\ h=2,\dots,m-r}}^{(S',p')} M^{r+1}\,\cdot$$

$$\cdot\left|\prod_{h=2}^{m-r}\varepsilon\big(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h}\big)\right|\left|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{k-2m}}}(b_{\pi(r_{k-2m})})\big)\right|.$$

Similarly, if

$$(6.15a)\qquad t_{\pi^{-1}(p_1)}\in\{t_{r_1},\dots,t_{r_{k-2m}}\}$$

and

$$(6.15b)\qquad t_{\pi^{-1}(q_1)}\notin\{t_{r_1},\dots,t_{r_{k-2m}}\}$$

then by the same argument, (6.11) is majorized by

(6.16)

$$\frac{1}{N^{ak-r}}\sum_{\substack{1\leqq t_1\leqq\cdots\leqq\widehat{t_{\pi^{-1}(q_1)}}\leqq\cdots\leqq t_{k-2r}\leqq N\\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},\ h=2,\dots,m-r}}^{(S',p')} M^{r}\cdot\left|\prod_{h=2}^{m-r}\varepsilon\big(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h}\big)\right|$$

$$\left|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{n-2m}}}(b_{\pi(r_{n-2m})})\big)\right|\cdot\sum_{l=1}^{t_{\pi^{-1}(q_1)}+1}\left|\varepsilon\big(t_{\pi^{-1}(p_1)},l,b_{p_1},b_{q_1}\big)\right|\leqq$$

$$\leqq\frac{1}{N^{ak-r}}\sum_{\substack{1\leqq t_1\leqq\widehat{t_{\pi^{-1}(p_1)}}\leqq\cdots\leqq\cdots\leqq\widehat{t_{\pi^{-1}(q_1)}}\leqq\cdots\leqq t_{k-2r}\leqq N\\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},\ h=2,\dots,m-r}}^{(S',p')} M^{r+1}\,\cdot$$

$$\cdot\left|\prod_{h=2}^{m-r}\varepsilon\big(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h}\big)\right|\left|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{k-2m}}}(b_{\pi(r_{k-2m})})\big)\right|.$$

Iterating the procedure we can eliminate the remaining pairs $(\{t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}\})$ for $h = 2, \ldots, m - r$. Then, relabeling the remaining indices, we find the following majorization of (6.11):

(6.17)

$$\frac{1}{N^{ak-r}} \sum_{1 \leqq t_1 \leqq \cdots \leqq t_{k-2r-(2m-2r)} \leqq N}^{(S',p')} M^m \cdot \left| E\big(j_{t_1}(b_{\pi(r_1)}) \cdots j_{t_{k-2m}}(b_{\pi(r_{k-2m})})\big) \right| =$$

$$= \frac{1}{N^{ak-r}} \sum_{1 \leqq t_1 \leqq \cdots \leqq t_{k-2m} \leqq N}^{(S',p')} M^m \cdot \left| E\big(j_{t_1}(b_{\pi(r_1)}) \cdots j_{t_{r_{k-2m}}}(b_{\pi(k-2m)})\big) \right|$$

where $r_1, \ldots, r_{k-2m}$ are defined by (3.9a). By the definition of $r$, after (6.10), one has $r \leqq m$ and in any case $ak - r \geqq ak - m = \delta(n - 2m)$, where $\delta := \frac{ak-m}{k-2m}$, so that:

(1) if $a > \frac{1}{2}$ then $\delta > \frac{1}{2}$;

(2) if $a = \frac{1}{2}$ and $k$ is odd, then $k - 2m$ is odd and $\delta = \frac{1}{2}$.

In case (1), the right hand side of (6.17) tends to zero because of Corollary (2.2), with $\delta$ replacing $a$ and $k - 2m$ replacing $k$. In case (2), since the time indices in the sum are in increasing order, we can apply Lemma (2.1). Moreover, since $k - 2m$ is odd and the different times $t_j$ are only $p'$, one can only have:

$$p' < \frac{k-2m}{2}, \quad \text{or} \quad p' > \frac{k-2m}{2}.$$

The former case corresponds to $ak > p$ in Lemma (2.1); in the latter case, there must be at least a singleton (i.e. some $t_j$ not equal to any $t_h$ for $j \neq \neq h$). Therefore we can apply condition (ii) of Lemma (2.1). Thus, in case (2) effectively (6.17) tends to zero. Therefore in both cases we conclude that (6.17) tends to zero as $N \to \infty$ and this ends the proof for the discrete case.

If $\nu$ is the Lebesgue measure, the situation is simpler because one needs only to consider the case $p = k$ (i.e. all singletons) since the Lebesgue measure on $\mathbf{R}^k$ is equal to zero on the subsets of those $(t_1, \ldots, t_k)$ such that for some indices $1 \leqq j \neq h \leqq k$, $t_j = t_h$. This implies that both cases i) and iii) in the proof of Theorem (1.3) correspond to sets of measure zero, and therefore we need only to consider the situation ii) in the proof of Theorem

(1.3) and replace the sum

$$\sum_{\substack{1\leqq t_1\leqq\cdots\leqq t_k\leqq N\\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},\ h=1,\dots,m}}^{(S,p)} \tag{6.18a}$$

by the integral

$$\int_{\substack{1\leqq t_1\leqq\cdots\leqq t_k\leqq N\\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},\ h=1,\dots,m}} dt_1\cdots dt_k. \tag{6.18b}$$

The arguments of the discrete case are easily adapted to this case.

Theorem (1.3) shows that, in the following, it will be sufficient to study the limits

$$\lim_{N\to\infty}\frac{1}{N^n}E(S_N(b_1)\cdots S_N(b_{2n})) \tag{6.18}$$

and

$$\lim_{T\to\infty}\frac{1}{T^n}E\left(\int_{[0,T)^{2n}} j_{t_1}(b_1)\cdots j_{t_{2n}}(b_{2n})\right)dt_1\dots dt_{2n} \tag{6.19}$$

for each $n\in\mathbf{N}$ and $b_1,\dots,b_{2n}\in B$. Our conclusions are stated in Theorems (1.4), (1.5). The proof is rather long therefore we first prove some lemmata.

Applying Lemma (4.2) we find that

$$\frac{1}{N^n}E(S_N(b_1)\cdots S_N(b_{2n}))= \tag{6.20}$$

$$=\frac{1}{N^n}\sum_{p=1}^{2n}\sum_{m=0}^{n}\sum_{1\leqq p_1<\cdots<p_m\leqq 2n}\sideset{}{'}\sum_{(q_1,\dots,q_m)}\sum_{(S_1,\dots,S_p)\in\mathcal{P}_{2n,p}}$$

$$\sum_{\pi\in\mathcal{S}_{2n}^{(S,p)}}\sum_{t\in I_N(S_1,\dots,S_p,\{p_h,q_h\}_{h=1}^m,\pi)}\prod_{h=1}^{m}\varepsilon\big(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h}\big)\cdot$$

$$\cdot\sigma(t_1,\dots,t_{2n},b_1,\dots,b_{2n})\cdot E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big).$$

Because of Theorem (1.3), the limit of the right hand side of (6.20) is equal to the limit (as $N \to \infty$) of

$$\frac{1}{N^n} \sum_{p=n}^{2n} \sum_{m=0}^{n} \sum_{1 \leqq p_1 < \dots < p_m \leqq 2n} \sum_{(q_1,\dots,q_m)}' \sum_{(S_1,\dots,S_p) \in \mathcal{P}_{2n,p}} \sum_{\pi \in \mathcal{S}_{2n}^{(S,p)}} \tag{6.21}$$

$$\sum_{\substack{1 \leqq t_1 \leqq \dots \leqq t_{2n} \leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)},\ h=1,\dots,m}}^{(S,p)} \prod_{h=1}^{m} \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h})$$

$$\sigma(t_1,\dots,t_{2n},b_1,\dots,b_{2n}) \cdot E\big(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big).$$

In the following for each $N \in \mathbf{N}$ and each fixed $p = n,\dots,2n$, $m = 0,1,\dots,n$, $1 \leqq p_1 < \dots < p_m \leqq 2n$, $1 \leqq q_1,\dots,q_m \leqq 2n$ satisfying (3.8a), (3.8b), (3.8c), $(S_1,\dots,S_p) \in \mathcal{P}_{2n,p}$, $\pi \in \mathcal{S}_{2n}^{(S,p)}$, we denote

$$\Delta_N(2n,(S_1,\dots,S_p),\pi) := \tag{6.22}$$

$$:= \frac{1}{N^n} \sum_{\substack{1 \leqq t_1 \leqq \dots \leqq t_{2n} \leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)},\ h=1,\dots,m}}^{(S,p)} \prod_{h=1}^{m} \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h})$$

$$\sigma(t_1,\dots,t_{2n},b_1,\dots,b_{2n}) \cdot E\big(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big).$$

The first step in dealing with $\Delta_N(2n,(S_1,\dots,S_p),\pi)$ is:

LEMMA (6.1). *For $(S_1,\dots,S_p) \in \mathcal{P}_{2n,p}$, if there exists an $h' \in \{1,\dots,m\}$ satisfying* (6.6) *(replacing $k$ by $2n$), then*

$$\lim_{N\to\infty} \Delta_N(2n,(S_1,\dots,S_p),\pi) = 0. \tag{6.23}$$

PROOF. Without loss of generality, by the boundedness of the $\sigma$-factors, we can neglect the factor $\sigma(t_1,\dots,t_{2n},b_1,\dots,b_{2n})$. We shall also suppose that $h' = 1$.

By the same arguments of ii) and iii) in the proof of Theorem (1.3) and in the same notations, we find that

$$\Big|\Delta_N(2n,(S_1,\dots,S_p),\pi)\Big| \leqq \tag{6.24}$$

$$\leqq \frac{1}{N^{n-r}} \sum_{\substack{1\leqq t_1\leqq\cdots\leqq t_{2n-2(m-1)}\leqq N \\ t_{\pi^{-1}(p_1)}>t_{\pi^{-1}(q_1)}}}^{(S',p')} M^{m-1}\cdot\left|\varepsilon\big(t_{\pi^{-1}(p_1)},t_{\pi^{-1}(q_1)},b_{p_1},b_{q_1}\big)\right|$$

$$\left|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big)\right|$$

where $r \leqq m-1$. Condition (6.6) implies that in the sum

$$\sum_{\substack{1\leqq t_1\leqq\cdots\leqq t_{2n-2(m-1)}\leqq N \\ t_{\pi^{-1}(p_1)}>t_{\pi^{-1}(q_1)}}}^{(S',p')}$$

there are at most $2(n-m)$ free indices, therefore, by relabeling the indices, the right hand side of (6.24) can be rewritten as

$$\frac{1}{N^{n-r}} \sum_{\substack{1\leqq t_1\leqq\cdots\leqq t_{2n-2m}\leqq N \\ t_{\pi^{-1}(p_1)}>t_{\pi^{-1}(q_1)}}}^{(S',p')} M^{m-1}\cdot\left|\varepsilon\big(t_{\pi^{-1}(p_1)},t_{\pi^{-1}(q_1)},b_{p_1},b_{q_1}\big)\right| \tag{6.25}$$

$$\left|E\big(j_{t_1}(b_{\pi(r_1)})\cdots j_{t_{2n-2m}}(b_{\pi(r_{2n-2m})})\big)\right|.$$

By the boundedness of the factor $\varepsilon$, (6.25) is majorized by

$$\frac{1}{N^{n-r}} \sum_{1\leqq t_1\leqq\cdots\leqq t_{2n-2m}\leqq N}^{(S',p')} M^{m}\cdot\left|E\big(j_{t_1}(b_{\pi(r_1)})\cdots j_{t_{2n-2m}}(b_{\pi(r_{2n-2m})})\big)\right|. \tag{6.26}$$

If $m=n$, then (6.26) is $O(\frac{1}{N})$ which tends to zero; if $m<n$, then since $r \leqq m-1$, it follows that $n-r \geqq n-m+1 = \delta(2n-2m)$ with $\frac{1}{2}+\frac{1}{2(n-m)} =: \delta > \frac{1}{2}$, therefore the statement follows.

The second step is:

LEMMA (6.2). *For* $(S_1,\dots,S_p)\in\mathcal{P}_{2n,p}$ *and* $\pi\in\mathcal{S}_{2n}^{(S,p)}$, *if there exists an* $h'\in\{1,\dots,m\}$ *satisfying* (6.12a) *and* (6.12b) *or* (6.15a) *and* (6.15b), *then*

$$\lim_{N\to\infty}\Delta_N(2n,(S_1,\dots,S_p),\pi)=0. \tag{6.27}$$

PROOF. By the same arguments as in the proof of Lemma (6.3) and the formula (6.13), one obtains (6.26) again.

Before the third step, we prove the following:

LEMMA (6.3). *For each $k \in \mathbf{N}$, $p = 1, \dots, k$, $b_1, \dots, b_k \in B$, if the limit*

$$\lim_{N\to 0} \frac{1}{N} \sum_{h=1}^{N} E\big(j_h(b \cdot b')\big) \tag{6.28}$$

*exists and $E(\cdot, \cdot, b, b')$ is $s - L^1(C, dn)$ for each $b, b' \in B$, then the quantity*

$$W_{k-1} := \frac{1}{N^{ak}} \sum_{t \in I_N(S_1,\dots,S_p)} \Big| E\big(j_{t_1}(b_1) \cdots j_{t_k}(b_k)\big) \Big| \tag{6.29}$$

*with $a \geqq \frac{1}{2}$ is bounded.*

PROOF. By Theorem (1.3) we can suppose that $p \geqq n$. Now we prove this lemma by induction. It is clear that one needs only to prove (6.29) in the case of $a = \frac{1}{2}$ and $k \geqq 2$.

If $k = 2$, (6.29) becomes

$$\frac{1}{N} \sum_{h=1}^{N} \Big| E\big(j_h(b_1 b_2)\big) \Big| + \frac{1}{N} \sum_{1 \leqq h < k \leqq N}^{N} \Big| E\big(j_h(b_1) j_k(b_2)\big) \Big| + \tag{6.29a}$$

$$+ \frac{1}{N} \sum_{1 \leqq k < h \leqq N}^{N} \Big| E\big(j_h(b_1) j_k(b_2)\big) \Big|.$$

Applying the commutation relation (1.1) to the third term of (6.29a) and by the boundedness of the $\sigma$ and $\varepsilon$-factors, (6.29a) is dominated by

$$\frac{1}{N} \sum_{h=1}^{N} \Big| E\big(j_h(b_1 b_2)\big) \Big| + \frac{1}{N} \sum_{1 \leqq h < k \leqq N}^{N} \Big| E\big(j_h(b_1) j_k(b_2)\big) \Big| + \tag{6.30}$$

$$+ \frac{1}{N} \sum_{1 \leqq h < k \leqq N}^{N} \Big| E\big(j_h(b_2) j_k(b_1)\big) \Big| + M$$

where $M$ is $\frac{1}{N}$ times the sum of $N$ $\varepsilon$-factors. Since the map $E$ is FP mixing, one finds that, for $N$ big enough, (6.30) is majorized by

(6.31)

$$\frac{1}{N} \sum_{\substack{1 \leqq h < k \leqq N \\ k \leqq h + d_N}}^{N} \Big| E\big(j_h(b_1) j_k(b_2)\big) \Big| + \frac{1}{N} \sum_{\substack{1 \leqq h < k \leqq N \\ h \leqq k + d_N}}^{N} \Big| E\big(j_h(b_2) j_k(b_1)\big) \Big| + 2M + 1$$

and the right hand side of (6.31) is bounded since $E(\cdot,\cdot,b,b')$ is $s-L^1(C,dn)$ for each $b,b'\in B$. Thus we obtain a bound of $W_1$, denoted by $|W|_1$.

Suppose that for each $k\leqq n$, we obtain the bound $|W|_k$ on $W_k$ and without loss of generality, assume that $|W|_1\leqq\cdots\leqq|W|_{n-1}$. Then

$$W_n\leqq\frac{1}{N^{\frac{n+1}{2}}}\sum_{\substack{t\in I_N(S_1,\ldots,S_p)\\ t_{S_p}\leqq t_{S_{p-1}}+d_N}}\Big|E\big(j_{t_{S_1}}(b_{S_1})\cdots j_{t_{S_p}}(b_{S_p})\big)\Big|+$$

$$+\frac{1}{N^{\frac{n+1}{2}}}\sum_{\substack{t\in I_N(S_1,\ldots,S_p)\\ t_{S_p}>t_{S_{p-1}}+d_N}}\Big|E\big(j_{t_{S_1}}(b_{S_1})\cdots j_{t_{S_{p-1}}}(b_{S_{p-1}})\big)\Big|\cdot\Big|E\big(j_{S_p}(b_{S_p})\big)\Big|+$$

$$+\delta_N\cdot N^p\cdot C$$

where $C$ is a constant. For $N$ large enough, one has $\delta_N\cdot N^p\cdot C\leqq1$ and so by the induction assumption,

(6.32)
$$W_n\leqq\frac{1}{N^{\frac{n+1}{2}}}\sum_{\substack{t\in I_N(S_1,\ldots,S_p)\\ t_{S_p}\leqq t_{S_{p-1}}+d_N}}\Big|E\big(j_{t_{S_1}}(b_{S_1})\cdots j_{t_{S_p}}(b_{S_p})\big)\Big|+|W|_{n-1}\cdot M+1.$$

Repeating the same arguments to (6.32), we find that

(6.33)
$$W_n\leqq\frac{1}{N^{n+1}}\sum_{\substack{t\in I_N(S_1,\ldots,S_p)\\ t_{S_p}\leqq t_{S_{p-1}}+d_N,t_{S_{p-1}}\leqq t_{S_{p-2}}+d_N}}\Big|E\big(j_{t_{S_1}}(b_{S_1})\cdots j_{t_{S_p}}(b_{S_p})\big)\Big|+$$

$$+\big(|W|_{n-1}+|W|_{n-2}\big)\cdot M+2\leqq$$

$$\leqq\cdots\leqq$$

$$\leqq\frac{1}{N^{\frac{n+1}{2}}}\sum_{\substack{t\in I_N(S_1,\ldots,S_p)\\ t_{S_h}\leqq t_{S_{h-1}}+d_N,\ h=2,\ldots,p}}\Big|E\big(j_{t_{S_1}}(b_{S_1})\cdots j_{t_{S_p}}(b_{S_p})\big)\Big|+$$

$$+(|W|_2+\cdots+|W|_{n-1})\cdot M+n-1\leqq(|W|_1+\cdots+|W|_{n-1})\cdot M+n.$$

Putting

(6.34) $$|W|_n:=(|W|_1+\cdots+|W|_{n-1})\cdot M+n$$

we finish the proof.

Lemmata (6.1) and (6.2) show that in order to consider the limit of $\Delta_N(2n,(S_1,\ldots,S_p),\pi)$ we need only to consider those $(S_1,\ldots,S_p)$ and $\pi \in$ $\in \mathcal{S}_{2n}^{(S,p)}$ in which

$$\{t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}\}_{h=1}^{m} \cap \{t_{r_1},\ldots,t_{r_{2n-2m}}\} = \emptyset. \tag{6.35}$$

Moreover (third step)

LEMMA (6.4). *For $(S_1,\ldots,S_p) \in \mathcal{P}_{2n,p}$ and $\pi \in \mathcal{S}_{2n}^{(S,p)}$, if the cardinality of the index set $\{t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}\}_{h=1}^{m}$ is not equal to $2m$, then $\Delta_N(2n,(S_1,\ldots,S_p),\pi)$ tends to zero as $N\to\infty$.*

REMARK. Lemma (6.4) shows that one needs only to consider the case when all the $\{\pi^{-1}(p_h),\pi^{-1}(q_h)\}_{h=1}^{m}$ are singletons of the partition $(S_1,\ldots,S_p)$.

PROOF. By the same arguments as in ii) of the proof of Theorem (1.3) and relabeling the indices, one has

$$\left|\Delta_N(2n,(S_1,\ldots,S_p),\pi)\right| \leqq \tag{6.36}$$

$$\leqq \frac{1}{N^{n-m}} \sum_{1\leqq t_1\leqq\cdots\leqq t_{2n-2m}\leqq N}^{(S',p')} \left|E\big(j_{t_1}(b_{\pi(r_1)})\cdots j_{t_{2n-2m}}(b_{\pi(r_{2n-2m})})\big)\right|$$

$$\cdot\frac{1}{N^m} \sum_{1\leqq h_1<k_1,\ldots,h_m<k_m\leqq N} \left|\varepsilon(k_1,h_1,b_1',b_1'')\cdots\varepsilon(k_m,h_m,b_m',b_m'')\right|.$$

Applying Lemma (6.3) we find a majorization of the right hand side of (6.36):

$$\frac{C}{N^m}\cdot \sum_{1\leqq h_1<k_1,\ldots,h_m<k_m\leqq N} \left|\varepsilon(k_1,h_1,b_1',b_1'')\cdots\varepsilon(k_m,h_m,b_m',b_m'')\right| \tag{6.37}$$

where $C$ is a constant and $\{b_h',b_h''\}_{h=1}^{m} \subset \{b_h\}_{h=1}^{2n}$. Thus, if the cardinality of the $\{h_j,k_j\}_{j=1}^{m}$ is not $2m$, the statement follows from Lemma (3.4).

REMARK. In the following, for each fixed "good" partition-permutation pair $(S_1,\ldots,S_p)$ and $\pi\in\mathcal{S}_{2n}^{(S,p)}$ ("good" means that $\{\pi^{-1}(p_h),\pi^{-1}(q_h)\}_{h=1}^{m}$ are singletons) we prove that one needs only to consider those terms in which the indices $t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}$ are consecutive for each $h=1,\ldots,m$ (in full analogy with what happens in the weak coupling limit). This is our fourth step. First we prove the following result:

LEMMA (6.5). *For each* $n \in \mathbf{N}$, $1 \leqq p \leqq 2n$, $(S_1, \ldots, S_p) \in \mathcal{P}_{2n,p}$ *and for each* $b_1, \ldots, b_{2n} \in B$,

$$\lim_{N \to 0} \frac{1}{N^n} \sum_{\substack{1 \leqq t_1 < \cdots < t_p \leqq N \\ t_1 \leqq d_N}} E\big(j_{t_1}(b_{S_1}) \cdots j_{t_p}(b_{S_p})\big) = 0. \tag{6.38}$$

PROOF.

$$\frac{1}{N^n} \sum_{\substack{1 \leqq t_1 < \cdots < t_p \leqq N \\ t_1 \leqq d_N}} \Big| E\big(j_{t_1}(b_{S_1}) \cdots j_{t_p}(b_{S_p})\big) \Big| = \tag{6.39}$$

$$= \frac{1}{N^n} \Big( \sum_{\substack{1 \leqq t_1 < \cdots < t_p \leqq N \\ t_1 \leqq d_N, t_2 \leqq d_N + t_1}} + \sum_{\substack{1 \leqq t_1 < \cdots < t_p \leqq N \\ t_1 \leqq d_N, t_2 > d_N + t_1}} \Big) \Big| E\big(j_{t_1}(b_{S_1}) \cdots j_{t_p}(b_{S_p})\big) \Big|.$$

Applying the FP mixing property to the second term of the right hand side of (6.39), one finds that (6.39) is majorized by

$$\frac{1}{N^n} \sum_{\substack{1 \leqq t_1 < \cdots < t_p \leqq N \\ t_1 \leqq d_N, t_2 \leqq d_N + t_1}} \Big| E\big(j_{t_1}(b_{S_1}) \cdots j_{t_p}(b_{S_p})\big) \Big| + \tag{6.40}$$

$$+ \frac{1}{N^n} \sum_{\substack{1 \leqq t_1 < \cdots < t_p \leqq N \\ t_1 \leqq d_N, t_2 > d_N + t_1}} \Big| E\big(j_{t_1}(b_{S_1})\big) \Big| \cdot \Big| E\big(j_{t_2}(b_{S_2}) \cdots j_{t_p}(b_{S_p})\big) \Big| + \delta_N \cdot N^{p-n}.$$

Now let us analyze the three terms of (6.40):

— the third term of (6.40) tends clearly to zero;

— if $S_1$ is a singleton, the second term of (6.40) is equal to zero;

— if $S_1$ is not singleton then $|S_2| + \cdots + |S_p| \leqq 2n - 2$, and by Lemma (6.3),

$$\frac{1}{N^{n-1}} \sum_{1 \leqq t_2 < \cdots < t_p \leqq N} \Big| E\big(j_{t_2}(b_{S_2}) \cdots j_{t_p}(b_{S_p})\big) \Big|$$

is bounded, therefore the second term of (6.40) is dominated by

$$C \cdot \frac{1}{N} \sum_{h=1}^{d_N} \Big| E\big(j_h(b_{S_1})\big) \Big|$$

for some constant $C$. This clearly tends to zero as $N \to \infty$ because of (1.5a). Thus we have only to prove that the first term of (6.40) tends to zero as $N \to \infty$.

Iterating the above arguments, i.e. splitting the sum with respect to $t_3$, and then $t_4$, etc., we find that

$$\lim_{N\to\infty} \frac{1}{N^n} \sum_{\substack{1 \leqq t_1 < \cdots < t_p \leqq N \\ t_1 \leqq d_N}} \left| E\big(j_{t_1}(b_{S_1}) \cdots j_{t_p}(b_{S_p})\big) \right| = \tag{6.41}$$

$$= \lim_{N\to\infty} \frac{1}{N^n} \sum_{\substack{1 \leqq t_1 < \cdots < t_p \leqq N \\ t_1 \leqq d_N, t_2 \leqq d_N + t_1, \ldots, t_p \leqq d_N + t_{p-1}}} \left| E\big(j_{t_1}(b_{S_1}) \cdots j_{t_p}(b_{S_p})\big) \right|.$$

And the statement follows again from (1.5a) since the sum on the right hand side of (6.41) is of order less than $O(d_N^p)$.

Now we can make our fourth step:

LEMMA (6.6). *For each $n \in \mathbf{N}$, $1 \leqq p \leqq 2n$, let $(S_1, \ldots, S_p) \in \mathcal{P}_{2n,p}$ and $\pi \in \mathcal{S}_{2n}^{(S,p)}$ be good in the sense of the remark preceding Lemma (6.5). Then denoting*

$$\sum_{\substack{1 \leqq t_1 \leqq \cdots \leqq t_{2n} \leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)}, h=1,\ldots,m}}^{0} \tag{6.42}$$

*the subsum of*

$$\sum_{\substack{1 \leqq t_1 \leqq \cdots \leqq t_{2n} \leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)}, h=1,\ldots,m}}^{(S,p)}$$

*extended to all $1 \leqq t_1 \leqq \cdots \leqq t_{2n} \leqq N$ for which there exists an $h = 1, \ldots, m$ such that $t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}$ are not consecutive indices, and denoting*

$$\Delta_N^0(2n, (S_1, \ldots, S_p), \pi) := \tag{6.43}$$

$$:= \frac{1}{N^n} \sum_{\substack{1 \leqq t_1 \leqq \cdots \leqq t_{2n} \leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)}, h=1,\ldots,m}}^{0} \prod_{h=1}^{m} \left| \varepsilon\big(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h}\big) \right|$$

$$\left|E\left(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\right)\right|,$$

$\Delta_N^0(2n,(S_1,\dots,S_p),\pi)$ *tends to zero as* $N\to\infty$.

PROOF. Let $(S_1,\dots,S_p)\in\mathcal{P}_{2n,p}$ and $\pi\in\mathcal{S}_{2n}^{(S,p)}$ be good. Without loss of generality, we can assume that $t_{\pi^{-1}(p_1)},t_{\pi^{-1}(q_1)}$ are *not* consecutive indices. Denote

$$l:=t_{\pi^{-1}(p_1)},\quad k:=t_{\pi^{-1}(q_1)}$$

and let $l_1$ be the smallest $t_h$ greater than $l$.

Since $l,k$ are not consecutive indices, one has $l_1<k$. With these notations (6.43) is majorized by

$$\frac{1}{N^n}\sum\cdots\sum_{l=1}^{N}\sum_{l_1=l+1}^{N}\cdots\sum_{k=l_1+1}^{N}\cdots\sum\left|\varepsilon(l,k,b_{p_1},b_{q_1})\right| \tag{6.44}$$

$$\prod_{h=2}^{m}\left|\varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})\right|\left|E\left(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\right)\right|.$$

Notice that for each $X(l,\dots,k)$, since both $l$ and $k$ are singletons, we can exchange the $l_1$-summation with all those preceding the $k$-summation. That is, by our assumptions on $l,k,l_1$, one has

$$\sum_{l=1}^{N}\sum_{l_1=l+1}^{N}\cdots\sum_{k=l_1+1}^{N}X(l,\dots,k)=\sum_{l_1=1}^{N}\sum_{l=1}^{l_1-1}\cdots\sum_{k=l_1+1}^{N}X(l,\dots,k) \tag{6.45}$$

$$=\sum_{l_1=1}^{N}\cdots\sum_{l=1}^{l_1-1}\sum_{k=l_1+1}^{N}X(l,\dots,k).$$

Therefore (6.44) is less than or equal to

(6.46)

$$\frac{1}{N^{n-1}}\sum_{\substack{1\leqq t_1\leqq\cdots\widehat{t_{\pi^{-1}(p_1)}}\leqq\cdots\widehat{t_{\pi^{-1}(q_1)}}\leqq\cdots\leqq t_{2n}\leqq N\\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},h=2,\dots,m}}\prod_{h=2}^{m}\left|\varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})\right|$$

$$\left|E\left(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\right)\right|\frac{1}{N}\sum_{l=1}^{l_1-1}\sum_{k=l_1+1}^{N}\left|\varepsilon(l,k,b_{p_1},b_{q_1})\right|.$$

For each $K \in \mathbf{N}$, split the sum

$$\sum_{\substack{1 \leqq t_1 \leqq \cdots \widehat{t_{\pi^{-1}(p_1)}} \leqq \cdots \widehat{t_{\pi^{-1}(q_1)}} \leqq \cdots \leqq t_{2n} \leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)}, h=2,\ldots,m}}$$

into two parts

(6.47)

$$\sum_{\substack{1 \leqq t_1 \leqq \cdots \widehat{t_{\pi^{-1}(p_1)}} \leqq \cdots \widehat{t_{\pi^{-1}(q_1)}} \leqq \cdots \leqq t_{2n} \leqq N \\ l_1 \leqq K, t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)}, h=2,\ldots,m}} + \sum_{\substack{1 \leqq t_1 \leqq \cdots \widehat{t_{\pi^{-1}(p_1)}} \leqq \cdots \widehat{t_{\pi^{-1}(q_1)}} \leqq \cdots \leqq t_{2n} \leqq N \\ l_1 > K, t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)}, h=2,\ldots,m}}$$

Since $\varepsilon(\cdot,\cdot,b_{p_1},b_{q_1})$ is $s - L^1(\mathcal{C}, dn)$, for each $\eta > 0$, we can choose $K$ satisfying

$$(6.48) \qquad \sum_{k=K+1}^{N} \left|\varepsilon(l,k,b_{p_1},b_{q_1})\right| < \eta.$$

By the same arguments as in i), ii) and iii) of the proof of Theorem (1.3) and relabeling the indices, using (6.47) and (6.48), we see that (6.46) is dominated by

$$(6.49) \qquad C \cdot \eta + \frac{1}{N^{n-1}} \sum_{\substack{1 \leqq t_1 \leqq \cdots \widehat{t_{\pi^{-1}(p_1)}} \leqq \cdots \widehat{t_{\pi^{-1}(q_1)}} \leqq \cdots \leqq t_{2n} \leqq N \\ l_1 \leqq K, t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)}, h=2,\ldots,m}}$$

$$\prod_{h=2}^{m} \left|\varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h})\right| \left|E\big(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big)\right|$$

$$\frac{1}{N} \sum_{l=1}^{l_1-1} \sum_{k=l_1+1}^{N} \left|\varepsilon(l,k,b_{p_1},b_{q_1})\right| \leqq C\eta \;+\; C_1 \frac{K}{N}$$

where, $C, C_1$ are constants. This ends the proof.

In the continuous case, the situation is much easier. In fact, by the same considerations as before (6.18), one needs only to consider the analogue of Lemma (6.6) in the case of $p = 2n$. Replacing again (6.18a) by (6.18b), in this case we are led to estimate the integral

(6.50)

$$\Delta'_T(2n) := \frac{1}{T^k} \int_{\substack{0 \leqq t_1 \leqq \cdots \leqq t_{2n} < T \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)}, h=1,\ldots,m}} \prod_{h=1}^{m} \left| \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h}) \right|$$

$$\left| E\left(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\right) \right| dt_1 \cdots dt_{2n}$$

which corresponds to the case in which all the blocks of the partitions are singletons. Moreover the partitions $(S_1, \ldots, S_{2n})$ are good in the sense of the remark preceding Lemma (6.5). Denoting, in analogy with (6.42)

(6.51) $$\Sigma_{T,\pi,\{p_h,q_h\}_{h=1}^m} := \left\{ (t_1, \ldots, t_{2n}) \in [0,T)^{2n} : \right.$$

$$t_1 < t_2 < \cdots < t_{2n};\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)}, h = 1, \ldots, m \text{ and there exists a}$$

$$\left. h \in \{1, \ldots, m\} \text{ such that } t_{\pi^{-1}(p_h)} \text{ and } t_{\pi^{-1}(q_h)} \text{ are not consecutive} \right\}$$

and

(6.52) $$\Delta^0_T(2n) := \frac{1}{T^k} \int_{\Sigma_{T,\pi,\{p_h,q_h\}_{h=1}^m}} \prod_{h=1}^{m} \left| \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h}) \right|$$

$$\left| E\left(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\right) \right| dt_1 \cdots dt_{2n}$$

one has the following

LEMMA (6.7). *$\Delta^0_T(2n)$ defined by (6.52) tends to zero as $T \to \infty$.*

PROOF. The proof is similar to that of Lemma (6.6) (just replace the sums there by integrals).

Now let us come back to the discrete situation. Since the indices

$$\{t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}\}_{h=1}^m$$

are all singletons, we know that $p \geqq 2m$ and the index set $\{r_1, \ldots, r_{2n-2m}\}$ is divided into $p - 2m$ sets, hence from Theorem (1.3) we know that $p - 2m$ should be greater than or equal to $n - m$, i.e. $m \leqq p - n$. Now we deal with the property of the partition $(S_1, \ldots, S_p)$ and show that for each $h = 1, \ldots, m$, $S_h$ has at most 2 elements. This is our fifth step.

As a first generalization of Lemma (2.1) we prove the following result:

LEMMA (6.8). *For each $n \in \mathbf{N}$, $p \geqq n$, $(S_1, \ldots, S_p) \in \mathcal{P}_{2n,p}$ and $b_1, \ldots, b_{2n} \in B$, if there exists an $h \in \{1, \ldots, p\}$ such that the cardinality of the set $S_h$ is greater than or equal to 3, then the limit of the quantity*

$$(6.53)\qquad \Delta_N(n,p) := \frac{1}{N^n} \sum_{t\in I_N(S_1,\dots,S_p)} \Big| E\big(j_{t_1}(b_{S_1})\cdots j_{t_p}(b_{S_p})\big)\Big|$$

*is zero.*

PROOF. We shall prove that $\Delta_N(n,p)$ tends to zero by induction.

For the case of $n = 1$, there is nothing to prove. Suppose that the conclusion of the lemma is true for all integers less than or equal to $n$, and let us consider the situation for $n+1$. By the FP mixing property of $E$ we have that

$$(6.54)\qquad \Delta_N(n+1,p) = \frac{1}{N^{n+1}} \sum_{\substack{t\in I_N(S_1,\dots,S_p)\\ t_p\leqq t_{p-1}+d_N}} \Big| E\big(j_{t_1}(b_{S_1})\cdots j_{t_p}(b_{S_p})\big)\Big| +$$

$$+\frac{1}{N^{n+1}} \sum_{t\in I_N(S_1,\dots,S_{p-1})} \sum_{t_p>t_{p-1}+d_N} \Big| E\big(j_{t_1}(b_{S_1})\cdots j_{t_{p-1}}(b_{S_{p-1}})\big)\Big| \cdot$$

$$\cdot\Big| E\big(j_{t_p}(b_{S_p})\big)\Big| + \delta_N\cdot N^{p-n-1}.$$

It is clear that the third term on the right hand side of (6.54) tends to zero as $N\to 0$. Moreover the second term on the right hand side of (6.54) is equal to zero if $|S_p| = 1$. If $|S_p| = 2$ then that term is equal to

$$(6.55)\qquad \frac{1}{N^n} \sum_{t\in I_N(S_1,\dots,S_{p-1})} \Big| E\big(j_{t_1}(b_{S_1})\cdots j_{t_{p-1}}(b_{S_{p-1}})\big)\Big| \cdot \frac{1}{N} \sum_{t_p>t_{p-1}+d_N} \Big| E\big(j_{t_p}(b_{S_p})\big)\Big|$$

$$= \Delta_N(n,p-1)\cdot \frac{1}{N} \sum_{t_p>t_{p-1}+d_N} \Big| E\big(j_{t_p}(b_{S_p})\big)\Big|.$$

Notice that $\frac{1}{N}\sum_{t_p>t_{p-1}+d_N}\Big|E\big(j_{t_p}(b_{S_p})\big)\Big|$ is bounded. By assumption there exists an $h\in\{1,\dots,p-1\}$ such that $|S_h|\geqq 3$, so the induction assumption implies that (6.55) tends to zero. If $|S_p|\geqq 3$ then $(S_1,\dots,S_{p-1})$ is a partition of at most $2(n+1)-3$ elements, so we can apply Corollary (2.2) with $k\leqq$ $\leqq 2(n+1)-3$ and $ak = n = \frac{1}{2}\big(2(n+1)-2\big) > \frac{1}{2}k$. Thus we conclude again that (6.55) tends to zero. These arguments show that the limit of

$\Delta_N(n+1,p)$ is equal to the limit of the first term on the right hand side of (6.54). We write that term in the form

$$\frac{1}{N^{n+1}} \sum_{\substack{t\in I_N(S_1,\dots,S_{p-1})\\ t_p\leqq t_{p-1}+d_N,\,t_{p-1}\leqq t_{p-2}+d_N}} \left|E\big(j_{t_1}(b_{S_1})\cdots j_{t_p}(b_{S_p})\big)\right| + \tag{6.56}$$

$$+\frac{1}{N^n}\sum_{t\in I_N(S_1,\dots,S_{p-2})} \left|E\big(j_{t_1}(b_{S_1})\cdots j_{t_{p-2}}(b_{S_{p-2}})\big)\right|\cdot$$

$$\cdot\frac{1}{N}\sum_{t_{p-1}>t_{p-2}+d_N,\ t_p\leqq t_{p-1}+d_N} \left|E\big(j_{t_{p-1}}(b_{S_{p-1}})j_{t_p}(b_{S_p})\big)\right| + \delta_N\cdot N^{p-n-1}.$$

It is clear that the third term on the right hand side of (6.56) tends to zero as $N\to 0$. Moreover in the second term on the right hand side of (6.56):

— if $|S_{p-1}|=|S_p|=1$, then since $E(j.(b)j.(b'))$ is in $s-L^2(C,dn)$, one has that

$$\frac{1}{N}\sum_{t_{p-1}>t_{p-2}+d_N,\ t_p\leqq t_{p-1}+d_N} \left|E\big(j_{t_{p-1}}(b_{S_{p-1}})j_{t_p}(b_{S_p})\big)\right|$$

is bounded. By assumption there exists an $h\in\{1,\dots,p-2\}$ such that $|S_h|\geqq 3$, so by the induction assumption this term tends to zero.

— if $|S_{p-1}|+|S_p|>2$, then by Lemma (6.5) with $k\leqq 2n-3$ (so that $n-\frac{1}{2}>\frac{k}{2}$), the quantity

$$\frac{1}{N^{n-\frac{1}{2}}}\sum_{t\in I_N(S_1,\dots,S_{p-2})} \left|E\big(j_{t_1}(b_{S_1})\cdots j_{t_{p-2}}(b_{S_{p-2}})\big)\right| \tag{6.57}$$

is bounded and because of (1.5a),

(6.58)
$$\frac{1}{N^{1+\frac{1}{2}}}\sum_{t_{p-1}>t_{p-2}+d_N,\ t_p\leqq t_{p-1}+d_N} \cdot\left|E\big(j_{t_{p-1}}(b_{S_{p-1}})j_{t_p}(b_{S_p})\big)\right| = C\cdot\frac{d_N}{N^{1/2}}\longrightarrow 0.$$

Therefore the second term on the right hand side of (6.56) tends to zero as $N\to 0$.

Summing up we have proved that the limit of $\Delta_N(n+1,p)$ is equal to

$$\lim_{N\to\infty}\frac{1}{N^{n+1}} \sum_{\substack{t\in I_N(S_1,\dots,S_{p-1})\\ t_p\leqq t_{p-1}+d_N,\,t_{p-1}\leqq t_{p-2}+d_N}} \left|E\big(j_{t_1}(b_{S_1})\cdots j_{t_p}(b_{S_p})\big)\right|. \tag{6.59}$$

Iterating the above argument one finishes the proof.

Applying Lemma (6.2) to $\Delta_N(2n,(S_1,\dots,S_p),\pi)$ we conclude that one needs only to consider the case of $|S_h| \leqq 2$ for each $h = 1,\dots,p$. Moreover Lemma (6.1), Lemma (6.2) and Lemma (6.4) show that we can restrict ourselves to partitions $(S_1,\dots,S_p)$ and permutations $\pi$ such that the indices $t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}$ $(h = 1,\dots,m)$ defined in Lemma (2.6) are singletons of the partition (the other ones give zero contribution). If we omit these singletons, we obtain a subpartition $(S'_1,\dots,S'_{p'})$ of $(S_1,\dots,S_p)$. Thus $p' = p - 2m$ and $|S'_h| \leqq 2$ for each $h = 1,\dots,p' = p - 2m$. In the following we shall restrict ourselves to this case.

The sixth step is to show that one needs to consider only partitions with the property that between any two nonsingleton blocks there is an even number of singletons. A corollary of the following lemma is that the total number of singletons is even.

LEMMA (6.9). *With the same notations and assumptions as in Lemma (6.8), if the partition $(S_1,\dots,S_p)$ is such that either there exist $1 < i < j < < 2n$ with the properties:*

*— $S_i$ and $S_j$ are not singleton,*

*— for each $i < h < j$, $S_h$ is a singleton,*

*— $j - i - 1$ is odd,*

*or the partition either begins or ends with an odd number of singletons, then the limit of $\Delta_N(n,p)$, as $N \to \infty$, is zero.*

PROOF. The proof is similar to that of Lemma (6.8).

For the case of $n = 1$, nothing has to be proved. Suppose that the conclusion of the lemma is true for all integers less than or equal to $n$, and consider the situation for $n + 1$.

By the FP mixing property of $E$ we have

$$(6.60)\qquad \Delta_N(n+1,p) = \frac{1}{N^{n+1}} \sum_{\substack{t\in I_N(S_1,\dots,S_{p-1})\\ t_p\leqq t_{p-1}+d_N}} \Big| E\big(j_{t_1}(b_{S_1})\cdots j_{t_p}(b_{S_p})\big)\Big| +$$

$$+\frac{1}{N^{n+1}} \sum_{t\in I_N(S_1,\dots,S_{p-1})} \sum_{t_p>t_{p-1}+d_N} \Big| E\big(j_{t_1}(b_{S_1})\cdots j_{t_{p-1}}(b_{S_{p-1}})\big)\Big| \cdot$$

$$\cdot \Big| E\big(j_{t_p}(b_{S_p})\big)\Big| + \delta_N \cdot N^{p-n-1}.$$

It is clear that the third term on the right hand side of (6.60) tends to zero as $N \to 0$ therefore we can neglect it and consider only the first two terms. We shall distinguish several situations:

The first term of (6.60) is equal to

$$(6.61)\qquad \frac{1}{N^{n+1}}\sum_{\substack{t\in I_N(S_1,\dots,S_{p-2})\\ t_p\leqq t_{p-1}+d_N,t_{p-1}\leqq t_{p-2}+d_N}}\left|E\big(j_{t_1}(b_{S_1})\cdots j_{t_p}(b_{S_p})\big)\right|+$$

$$+\frac{1}{N^{n-\frac{1}{2}}}\sum_{t\in I_N(S_1,\dots,S_{p-2})}\left|E\big(j_{t_1}(b_{S_1})\cdots j_{t_{p-2}}(b_{S_{p-2}})\big)\right|\cdot$$

$$\cdot\frac{1}{N^{1+\frac{1}{2}}}\sum_{t_{p-1}>t_{p-2}+d_N,t_p\leqq t_{p-1}+d_N}\left|E\big(j_{t_{p-1}}(b_{S_{p-1}})j_{t_p}(b_{S_p})\big)\right|+\delta_N\cdot N^{p-n-1}.$$

The third term of (6.61) obviously tends to zero and by the same argument as in the proof of Lemma (6.8), the second term of (6.61) also tends to zero. Iterating $(p-1)$-times the above arguments, one reduces the first term of (6.61) to a term of order $d_N^p/N^{n-\frac{1}{2}}$ which tends to zero by (1.5a). So the first term of (6.60) always tends to zero.

If $S_p$ is a singleton, then the second term is equal to zero. If $S_p$ is not a singleton, i.e. $|S_p|=2$ then the second term of (6.60) is dominated by

$$C\cdot\frac{1}{N^n}\sum_{t\in I_N(S_1,\dots,S_{p-1})}\left|E\big(j_{t_1}(b_{S_1})\cdots j_{t_{p-1}}(b_{S_{p-1}})\big)\right|$$

for some constant $C$ and surely the subpartition $(S_1,\dots,S_{p-1})$ satisfies the condition of this lemma. Therefore the limit is zero by the induction assumption and this ends the proof.

Applying Lemma (6.9) to $\Delta_N(2n,(S_1,\dots,S_p),\pi)$, we conclude that one needs only to consider the situation in which each set of the subpartition $(S_1',\dots,S_{p-2m}')$ has one or two elements; moreover between any two nonsingleton blocks, there is an even number of singletons (may be zero).

Summing up the results obtained up to now we conclude that for each $p=n,\dots,2n$ (cardinality of the partition), $m=0,1,\dots,p-n$ (number of $\varepsilon$-factors) and $1\leqq p_1<\dots<p_m\leqq 2n$, $1\leqq q_1,\dots,q_m\leqq 2n$ (the indices of those operators that are coupled by an $\varepsilon$-factor) satisfying (3.8a), (3.8b), (3.8c), one needs only to consider those partitions $(S_1,\dots,S_p)$ in which

$$(6.62)\qquad |S_h|\leqq 2,\quad \text{for each}\quad h=1,\dots,p$$

and there exist at least $2m$ singletons; and only those permutations $\pi\in\in\mathcal{S}_{2n}^{(S,p)}$ for which

a) $\{\pi^{-1}(p_h),\pi^{-1}(q_h)\}_{h=1}^m$ are singletons;

b) if we denote $(S'_1, \ldots, S'_{p-2m})$ the subpartition obtained by omitting $\{\pi^{-1}(p_h), \pi^{-1}(q_h)\}_{h=1}^m$ from $(S_1, \ldots, S_p)$, then in this subpartition, between any two nonsingleton sets there exists an even number of singletons.

REMARK. Notice that if $p = 2n$, the partition $(S_1, \ldots, S_{2n})$ satisfies (6.62).

In the following we shall denote all $(S_1, \ldots, S_p)$ which satisfy this condition by $\mathcal{P}^0_{2n,p}$ and for each fixed partition $(S_1, \ldots, S_p)$ , by $\mathcal{S}_{2n}^{(S,p,\{p_h,q_h\}_{h=1}^m)}$ the set of all $\pi \in \mathcal{S}_{2n}^{(S,p)}$ satisfying a), b) above.

For each fixed $(S_1, \ldots, S_p) \in \mathcal{P}^0_{2n,p}$ and $\pi \in \mathcal{S}_{2n}^{(S,p,\{p_h,q_h\}_{h=1}^m)}$ , our seventh step concerns $t \in I_N(S_1, \ldots, S_p)$: we shall prove that if $S'_i$ is the $2h-1$-st singleton and $S'_j$ is the $2h$-th singleton and there exists some index $l$ of some $\varepsilon$-factor such that $t_{S'_i} < t_l < t_{S'_j}$, then we get limit zero. More precisely:

LEMMA (6.10). *Denote* $\sum_{(S,p,\varepsilon)}$ *the sub-sum of*

$$\sum_{\substack{1 \leqq t_1 \leqq \cdots \leqq t_{2n} \leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)},\ h=1,\ldots,m}}^{(S,p)} - \sum_{\substack{1 \leqq t_1 \leqq \cdots \leqq t_{2n} \leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)},\ h=1,\ldots,m}}^{0} \tag{6.63}$$

*with the property that there exist* $1 \leqq i < j \leqq p'$ *such that:*

— $S'_i$ *is the* $2h-1$*-st singleton and* $S'_j$ *is the* $2h$*-th singleton,*

— *there exists some index* $t_l$ *of some* $\varepsilon$*-factor such that* $t_{S'_i} < t_l < t_{S'_j}$,

*then*

$$\frac{1}{N^n} \sum_{(S,p,\varepsilon)} \prod_{h=1}^m \left| \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h}) \right| \tag{6.64}$$

$$\left| E\left(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\right) \right|$$

*converges to zero as* $N \to \infty$.

PROOF. Since the total number of singletons is even, for each term of (6.64) one can find an integer $m'$ such that there are exactly $2m'$ singletons in the sub-partition $(S'_1, \ldots, S'_{p-2m})$, defined in b) above. We label them by $i_1, \ldots, i_{2m'}$. By the assumption of this lemma, we know that there exists an $x = 1, \ldots, m'$ such that between $t_{S'_{i_{2x-1}}}$ and $t_{S'_{i_{2x}}}$ there is an index of some $\varepsilon$-factor. Since the indices of the $\varepsilon$-factors are consecutive (by Lemma (6.6)), it follows that between $t_{S'_{i_{2x-1}}}$ and $t_{S'_{i_{2x}}}$ there is an even number of indices of $\varepsilon$-factors.

Let us split (6.64) into two parts:

$$(6.65)\qquad \frac{1}{N^n}\Bigg( \sum_{\substack{(S,p,\varepsilon)\\ t_{S'_{i_{2x-1}}} > t_{S'_{i_{2x-1}-1}} + d_N,\ \text{and}\ t_{S'_{i_{2x}}} + d_N < t_{S'_{i_{2x}+1}}}} + \sum_{\substack{(S,p,\varepsilon)\\ t_{S'_{i_{2x-1}}} \leqq t_{S'_{i_{2x-1}-1}} + d_N,\ \text{or}\ t_{S'_{i_{2x}}} + d_N \leqq t_{S'_{i_{2x}+1}}}} \Bigg)$$

$$\prod_{h=1}^{m} \left|\varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h})\right| \cdot \left| E\big(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big)\right|.$$

By the same argument as in the proof of Theorem (1.3) it follows that the second term of (6.65) goes to zero as $N \to \infty$. Moreover, the same arguments as in the proof of Lemma (6.1) imply that the first term of (6.65) is majorized by

$$(6.66)\qquad C \cdot \frac{1}{N^2} \sum_{1 \leqq l_1 < h < k < l_2 \leqq N} \left| E\big(j_{l_1}(b'_1) j_{l_2}(b'_2)\big) \cdot \varepsilon(k, h, b, b')\right|$$

where $C$ is a constant and $b, b', b_1, b_2 \in B$, therefore the statement follows from (4.2a).

In the continuous case, for a fixed partition $(S_1, \ldots, S_{2n})$, a fixed permutation $\pi$, a fixed $m$, and a fixed set $p_1 < \cdots < p_m$, $q_1 < \cdots < q_m$, denote

$$(6.67)\quad \Omega_c(\pi, \{p_h, q_h\}_{h=1}^m) := \big\{(t_1, \ldots, t_{2n}) \in [0,T)^{2n} :\ \text{(i)}\ t_1 < \cdots < t_{2n};$$

$$\text{(ii)}\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)}\ \text{and they are consecutive for any}\ h = 1, \ldots, m;$$

$$\text{(iii) the indices}\ t_{S'_{2h}},\ t_{S'_{2h+1}}\ \text{are consecutive for each}\ h = 1, \ldots, n-m\big\}.$$

Recall that we have already shown that, in the continuous case, we need only to consider the case in which $p = 2n$, i.e. only those partitions which are entirely made up of singletons.

LEMMA (6.11). *As $T \to \infty$ the limit of $\Delta_T(2n, (S_1, \ldots, S_{2n}), \pi)$ is equal to the limit of the quantity*

$$(6.68)\qquad \frac{1}{T^n} \int_{\Omega(\pi, \{p_h, q_h\}_{h=1}^m)} dt_1 \cdots dt_{2n} \prod_{h=1}^{m} \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h})$$

$$\Big|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big)\Big|.$$

PROOF. The same as in Lemma (6.10).

Summing up our conclusions:

— In the discrete case, one needs only to consider the partitions $(S_1,\ldots,S_p)\in\mathcal{P}^0_{2n,p}$ and the permutations $\pi\in\mathcal{S}_{2n}^{(S,p,\{p_h,q_h\}_{h=1}^m)}$ (cf. (6.62) and below). In this situation,

$$\lim_{N\to\infty}\Delta_N(2n,(S_1,\ldots,S_p),\pi)= \tag{6.69}$$

$$=\lim_{N\to\infty}\frac{1}{N^n}\sum_{t\in\Omega_d(\pi,\{p_h,q_h\}_{h=1}^m)}\prod_{h=1}^m\varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})$$

$$\Big|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big)\Big|$$

where

$$\Omega_d(\pi,\{p_h,q_h\}_{h=1}^m):= \tag{6.70}$$

$$:=\big\{t\in I_N(S_1,\ldots,S_p):\ \{t_{\pi^{-1}(p_h)}\},\{t_{\pi^{-1}(q_h)}\}\text{ are singletons, } h=1,\ldots,m;$$

$$t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)}\text{ and they are consecutive for any } h=1,\ldots,m;$$

$$t_{S'_{2i_l}}\text{ is the consecutive index of } t_{S'_{2i_l-1}},\ l=1,\ldots,m'\text{ and}$$

$$\{S'_{2i_l},S'_{2i_l-1}\}_{l=1}^{m'}\text{ are singletons}\big\}$$

— In the continuous case, the limit of $\Delta_T(2n,(S_1,\ldots,S_{2n}),\pi)$ is given by (6.68).

Moreover for each $p=n,\ldots,2n$ and $(S_1,\ldots,S_p)$, the above conclusions show that we need only to consider the term in which there exist $2(m+m')$ singletons and $p-2(m+m')$ nonsingletons. Since each nonsingleton set must have cardinality 2, we have the following relation

$$2n=2(m+m')+2(p-2(m+m'))$$

i.e.

$$p=n+m+m'.$$

This shows that we have $p-2(m+m')=p-2(p-n)=2n-p$ nonsingletons and each of them is of cardinality 2. Recall that in the product map case one needs only to consider the case $p=n$ and for each partition $(S_1,\ldots,S_p)$, $|S_1|=\cdots=|S_p|=2$. Therefore the number of nonsingleton sets is an invariant which depends on the map $E$ and the commutation relations.

Now we pass to the eighth step of the proof:

LEMMA (6.12). *For each $n \in \mathbf{N}$, $p = n, \ldots, 2n$ and $(S_1, \ldots, S_p) \in \mathcal{P}_{2n,p}$ if*

*— $|S_h| \leqq 2, \quad h = 1, \ldots, p$;*

*— between each two nonsingletons there exists an even number of singletons.*

*Then for $b_1, \ldots, b_{2n} \in B$, the limit of*

$$\frac{1}{N^n} \sum_{1 \leqq t_1 < \cdots < t_p \leqq N} \left| E\big(j_{t_1}(b_{S_1}) \cdots j_{t_p}(b_{S_p})\big) - E^c\big(j_{t_1}(b_{S_1}) \cdots j_{t_p}(b_{S_p})\big) \right| \tag{6.71}$$

*is equal to zero, where*

$$\begin{aligned} E^c\big(j_{t_1}(b_{S_1}) \cdots j_{t_p}(b_{S_p})\big) &:= E\big(j_{t_1}(b_{S_1})\big) \cdots E\big(j_{t_{i_1}}(b_{S_{i_1}}) j_{t_{i_1+1}}(b_{S_{i_1+1}})\big) \\ &\cdots E\big(j_{t_{i_{m'}}}(b_{S_{i_{m'}}}) j_{t_{i_{m'}+1}}(b_{S_{i_{m'}+1}}) \cdots E\big(j_{t_p}(b_{S_p})\big) \end{aligned} \tag{6.72}$$

*and $\{i_h\}_{h=1}^{m'} \subset \{1, \ldots, p\}$ with $i_1 < \cdots < i_{m'}$, $\{S_{i_h}, S_{i_h+1}\}_{h=1}^{m'}$ are all singletons.*

PROOF. We apply induction on $n$. For $n = 1$, (6.71) is identically equal to zero. Supposing that (6.71) tends to zero for each integer less than or equal to $n$, let us see the situation for $n + 1$.

If $|S_p| = 2$, we split (6.71) into two parts:

$$\frac{1}{N^{n+1}} \Bigg( \sum_{\substack{1 \leqq t_1 < \cdots < t_p \leqq N \\ t_p > t_{p-1} + d_N}} + \sum_{\substack{1 \leqq t_1 < \cdots < t_p \leqq N \\ t_p \leqq t_{p-1} + d_N}} \Bigg) \left| E\big(j_{t_1}(b_{S_1}) \cdots j_{t_p}(b_{S_p})\big) - E^c\big(j_{t_1}(b_{S_1}) \cdots j_{t_p}(b_{S_p})\big) \right|. \tag{6.73}$$

The same arguments as in the proof of Theorem (1.3) imply that the second term of (6.73) converges to zero as $N \to 0$. From the FP mixing property of $E$, one knows that the first term of (6.73) is dominated by

$$\frac{1}{N^n} \sum_{1 \leqq t_1 < \cdots < t_{p-1} \leqq N} \Big| E\big(j_{t_1}(b_{S_1}) \cdots j_{t_{p-1}}(b_{S_{p-1}})\big) - E^c\big(j_{t_1}(b_{S_1}) \cdots j_{t_{p-1}}(b_{S_{p-1}})\big) \Big| \cdot \frac{1}{N} \sum_{1 \leqq t_p \leqq N} \Big| E\big(j_{t_p}(b_{S_p})\big) \Big| \tag{6.74}$$

where $(n+1)-1=n \leqq p-1$. The second sum of (6.73) is bounded and the induction assumption implies that the first sum of (6.73) goes to zero. Therefore (6.71) tends to zero.

On the other hand, if $|S_p|=1$, then $|S_{p-1}|=1$ and the proof that (6.71) goes to zero is almost the same as before, the only difference being that one has to split (6.71) into two parts between the indices $t_{p-2}$ and $t_{p-1}$.

As an improvement of Lemma (6.14), one has

LEMMA (6.13). *For each $n \in \mathbf{N}$, $p=2,\dots,2n$ and $(S_1,\dots,S_p) \in \in \mathcal{P}^0_{2n,p}$, $\pi \in \mathcal{S}_{2n}^{(S,p,\{p_h,q_h\}_{h=1}^m)}$, the quantity*

$$(6.75)\qquad \frac{1}{N^n}\sum_{t\in\Omega_d(\pi,\{p_h,q_h\}_{h=1}^m)}\prod_{h=1}^m \varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})$$

$$\Big|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big)-$$

$$-E^c\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big)\Big|$$

*tends to zero as $N\to\infty$.*

PROOF. The lemma is an application of Lemma (6.14) and the property that $\varepsilon(\cdot,\cdot,b_l,b_l')$ is in $s-L^1(\mathcal{C},dn)$ for each $l=1,\dots,m$.

The continuous analogue of Lemma (6.13) is

LEMMA (6.14). *For each $n\in\mathbf{N}$,*

$$(6.76)\qquad \frac{1}{T^n}\int_{\Omega_c(\pi,\{p_h,q_h\}_{h=1}^m)} dt_1\cdots dt_{2n}\prod_{h=1}^m\big|\varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})\big|$$

$$\Big|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big)-$$

$$-E^c\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big)\Big|$$

*tends to zero as $T\to\infty$, where $E^c$ has the same meaning as in* (6.72).

PROOF OF THEOREMS (1.4) AND (1.5). Applying Lemmata (5.2), (5.4) to

$$(6.77)\qquad \frac{1}{N^n}\sum_{t\in\Omega_d(\pi,\{p_h,q_h\}_{h=1}^m)}\prod_{h=1}^m \varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})$$

$$E^c\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big)$$

and

$$(6.78)\qquad \frac{1}{T^n}\int_{\Omega_c(\pi,\{p_h,q_h\}_{h=1}^m)} dt_1\cdots dt_{2n}\prod_{h=1}^m\big|\varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})\big|$$

$$E^c\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big)$$

respectively, one has the following

THEOREM (6.15). *In the discrete case, suppose that for each $b,b'\in B$,*
(i) $\sigma(\cdot,\cdot,b,b')=\sigma(b,b')$,
(ii) $E(\cdot,\cdot,b,b')$, $\varepsilon(\cdot,\cdot,b,b')$ *are in* $S-L^1(\mathcal{C},dn)$ *in the sense of Definition* (4.1),
(iii) *the limit*

$$(6.79)\qquad \lim_{N\to\infty}\frac{1}{N}\sum_{h=1}^N E\big(j_h(b\cdot b')\big):=C(bb')$$

*exists.*

*Then for each $n\in\mathbf{N}$ and $b_1,\ldots,b_{2n}\in B$, the central limit* (6.18) *exists and if we denote*

$$(6.80)\qquad f(b,b'):=\sum_{k=h+1}^\infty\varepsilon(k,h,b,b')$$

*and*

$$(6.81)\qquad F(b,b'):=\sum_{k=h+1}^\infty E\big(j_h(b)j_k(b')\big)$$

*then the limit* (6.18) *is equal to*

(6.82)

$$\frac{1}{n!}\sum_{p=n}^{2n}\sum_{m=0}^{p/2\wedge(p-n)}\sum_{1\le p_1<\cdots<p_m\le 2n}\sum_{(q_1,\ldots,q_m)}'\sum_{(S_1,\ldots,S_p)\in\mathcal{P}^0_{2n,p}}\sum_{\pi\in\mathcal{S}_{2n}^{(S,p,\{p_h,q_h\}_{h=1}^m)}}$$

$$\sigma(\pi,b_1,\ldots,b_{2n})\cdot\prod_{h=1}^m f(b_{p_h},b_{q_h})C(b_{\pi(S_1')})\cdots F(b_{\pi(S'_{i_1})},b_{\pi(S'_{i_1+1})})\cdots$$

$$\cdots F(b_{\pi(S'_{i_{p-n-m}})}, b_{\pi(S'_{i_{p-n-m}+1})}) \cdots C(b_{\pi(S'_{p-2m})})$$

*where* $\{S'_{i_l}, S'_{i_l+1}\}_{l=1}^{p-n-m}$ *are the singletons of* $(S'_1, \ldots, S'_{p-2m})$ *and* $\sigma(b_1, \ldots, b_{2n})$ *is a product of some* $\sigma(b_k, b_h)$ *with* $1 \leqq k < h \leqq 2n$.

The continuous analogue of Theorem (6.15) is

THEOREM (6.16). *In the continuous case, suppose that for each* $b, b' \in \in B$,

(i) $\sigma(\cdot, \cdot, b, b') = \sigma(b, b')$, *and*

(ii $E(\cdot, \cdot, b, b')$, $\varepsilon(\cdot, \cdot, b, b')$ *are in* $S - L^1(C, dt)$ *in the sense of Definition* (4.1).

*Then for each* $n \in \mathbf{N}$ *and* $b_1, \ldots, b_{2n} \in B$, *the central limit* (6.19) *exists and moreover if we denote*

$$f(b, b') := \int_{[h,\infty)} ds \varepsilon(s, h, b, b') \tag{6.83}$$

*and*

$$F(b, b') := \int_{[h,\infty)} ds E\big(j_h(b) j_s(b')\big) \tag{6.84}$$

*then limit* (6.19) *is equal to*

$$\frac{1}{n!} \sum_{1 \leqq p_1 < \cdots < p_m \leqq 2n} \sideset{}{'}\sum_{(q_1, \ldots, q_m)} \sum_{\pi \in \mathcal{S}_{2n}} \sigma(\pi, b_1, \ldots, b_{2n}) \cdot \prod_{h=1}^{m} f(b_{p_h}, b_{q_h}) \tag{6.85}$$

$$F(b_{\pi(r_1)}, b_{\pi(r_2)}) \cdots F(b_{\pi(r_{2n-2m-1})}, b_{\pi(r_{2n-2m})})$$

*where,* $\sigma(b_1, \ldots, b_{2n})$ *is a product of some* $\sigma(b_k, b_h)$ *with* $1 \leqq k < h \leqq 2n$.

Now let us consider some special situations in the discrete case.

First of all consider the term with $m = 0$ in (6.82). In this case, we have no $\varepsilon$-factors and (6.82) has the form

$$\frac{1}{n!} \sum_{p=n}^{2n} \sum_{(S_1, \ldots, S_p) \in \mathcal{P}^0_{2n,p}} \sum_{\pi \in \mathcal{S}_{2n}^{(S,p)}} \tag{6.86}$$

$$\sigma(\pi, b_1, \ldots, b_{2n}) \cdot C(b_{\pi(S_1)}) \cdots F(b_{\pi(S_{i_1})}, b_{\pi(S_{i_1+1})}) \cdots$$

$$\cdots F(b_{\pi(S_{i_{p-n}})}, b_{\pi(S_{i_{p-n}+1})}) \cdots C(b_{\pi(S_p)})$$

where

(6.87) $\mathcal{P}^0_{2n,p} := \{(S_1, \ldots, S_p) \in \mathcal{P}_{2n,p} :$ there exist $2p-2n$ singletons and

between two nonsingletons there exists an even number of singletons$\}$.

We write (6.86) as a sum of two terms:

$$(6.86a)\quad \frac{1}{n!} \sum_{(S_1,\ldots,S_n)\in\mathcal{P}^0_{2n,n}} \sum_{\pi\in\mathcal{S}^{(S,n)}_{2n}} \sigma(\pi, b_1, \ldots, b_{2n}) \cdot C(b_{\pi(S_1)}) \cdots C(b_{\pi(S_n)}) +$$

$$+\frac{1}{n!} \sum_{p=n+1}^{2n} \sum_{(S_1,\ldots,S_p)\in\mathcal{P}^0_{2n,p}} \sum_{\pi\in\mathcal{S}^{(S,p)}_{2n}}$$

$$\sigma(\pi, b_1, \ldots, b_{2n}) \cdot C(b_{\pi(S_1)}) \cdots F(b_{\pi(S_{i_1})}, b_{\pi(S_{i_1}+1)}) \cdots$$

$$\cdots F(b_{\pi(S_{i_{p-n}})}, b_{\pi(S_{i_{p-n}}+1)}) \cdots C(b_{\pi(S_p)}).$$

Notice that in the first term of (6.86a), all partitions are pair partitions (without singletons), therefore, each $\pi \in \mathcal{S}^{(S,n)}_{2n}$ exchanges the pairs, i.e. it is equivalent to an $n$-permutation. More precisely, for each fixed $(S_1, \ldots, S_n) \in \mathcal{P}^0_{2n,n}$ with $S_h = \{l_h, k_h\}$ and $l_h < k_h$ for any $h = 1, \ldots, n$ and for each $\pi \in \mathcal{S}^{(S,n)}_{2n}$, define a transformation $\sigma$ by

$$(6.88)\qquad l_{\sigma(h)} := \pi(l_h), \quad k_{\sigma(h)} := \pi(k_h), \quad h = 1, \ldots, n.$$

It is easy to check that the first term of (6.86a) is equal to

$$(6.89)\qquad \frac{1}{n!} \sum_{(S_1,\ldots,S_n)\in\mathcal{P}^0_{2n,n}} \sum_{\pi\in\mathcal{S}_n} \sigma(\pi, b_1, \ldots, b_{2n}) \cdot C(b_{S_{\pi(1)}}) \cdots C(b_{S_{\pi(n)}});$$

this gives the result when $E$ is a product map and the $\varepsilon$ factor is zero.

On the other hand, if we introduce the notation

$$(6.90)\qquad C(\pi, S, n, b_i, b_j) := C(b_{\pi(i)} b_{\pi(j)})$$

then we can rewrite (6.89) as

$$(6.91)\quad \frac{1}{n!} \sum_{p.p.} \sum_{\pi\in\mathcal{S}_n} \sigma(\pi, b_1, \ldots, b_{2n}) \cdot C(\pi, S, n, b_{l_1}, b_{k_1}) \cdots C(\pi, S, n, b_{l_n}, b_{k_n})$$

where $\sum_{p.p.}$ defined in [1] is the sum over all the pair partitions of $\{1,\ldots,2n\}$, i.e. all the $\{l_h,k_h\}_{h=1}^m=\{1,\ldots,2n\}$ such that

$$l_h<k_h,\quad h=1,\ldots,n \tag{6.91a}$$

and

$$l_1<\cdots<l_n. \tag{6.91b}$$

More generally, considering also the second term in (6.86a), if we define

(6.92)

$$C(\pi,S,p,b_i,b_j):=\begin{cases} C(b_{\pi(i)}b_{\pi(j)})\,, & \text{if } i,j \text{ are in the same } S_h\\ F(b_{\pi(i)},b_{\pi(j)})\,, & \text{if } i,j \text{ are not in the same } S_h,\end{cases}$$

and notice that in (6.86a), the $\{S_{i_h},S_{i_h+1}\}_{h=1}^{p-n}$ are singletons, then (6.86a) is equal to

$$\frac{1}{n!}\sum_{p=n}^{2n}\sum_{(S_1,\ldots,S_p)\in\mathcal{P}^0_{2n,p}}\sum_{\pi\in\mathcal{S}^{(S,p)}_{2n}}\sigma(\pi,b_1,\ldots,b_{2n})\cdot C(\sigma,S,p,b_{l_1},b_{k_1})\cdots$$
$$\cdots C(\sigma,S,p,b_{l_n},b_{k_n}) \tag{6.93}$$

where

$$\{l_1,k_1\}=S_1,\ldots,l_{i_1}=S_{i_1},k_{i_1}=S_{i_1+1},\ldots,l_{i_{p-n}}=$$
$$=S_{i_{p-n}},k_{i_{p-n}}=S_{i_{p-n}+1},\ldots,\{l_n,k_n\}=S_p \tag{6.94}$$

In the general case, i.e. when $m$ needs not to be zero, we have

THEOREM (6.17). *Define $C(\pi,S,p,b_i,b_j)$ to be equal to $C(b_{\pi(i)}b_{\pi(j)})$ if $i$, $j$ are in the same block; equal to $F(b_{\pi(i)},b_{\pi(j)})$ if $i$, $j$ are in two different blocks and $\pi$ leaves $i$ and $j$ fixed; equal to $F(b_{\pi(i)},b_{\pi(j)})+f(b_i,b_j)$ if $i$, $j$ are in two different blocks and $\pi$ permutes $i$ and $j$. Then* (6.82) *is equal to*

$$\frac{1}{n!}\sum_{p=n}^{2n}\sum_{(S_1,\ldots,S_p)\in\mathcal{P}^0_{2n,p}}\sum_{\pi\in\mathcal{S}^{(S,p)}_{2n}} \tag{6.95}$$
$$\sigma(\pi,b_1,\ldots,b_{2n})\cdot C(\pi,S,p,b_{l_1},b_{k_1})\cdots C(\pi,S,p,b_{l_n},b_{k_n}).$$

PROOF. Since in the product

$$C(\sigma,S,p,b_{l_1},b_{k_1})\cdots C(\sigma,S,p,b_{l_n},b_{k_n}) \tag{6.96}$$

some terms have the form $F+f$, we can expand the product of sums into sums of products. Suppose that in the expansion there are $m$ factors $f$, then by the definition of $C(\pi,S,p,b_i,b_j)$ we know that $m$ can be equal to $0,1,\dots,p-n$. Moreover, that definition shows that each factor $f$ corresponds to two singletons, therefore, $m \leqq p/2$.

For each fixed $m=0,1,\dots,p/2\wedge(p-n)$, we label the $f$ factors by the indices $\{p_h,q_h\}_{h=1}^m$. Then $\{p_h,q_h\}_{h=1}^m$ are singletons and it follows that we cannot choose all partitions in $\mathcal{P}_{2n,p}$ but only in $\mathcal{P}_{2n,p}^0$; moreover $\pi\in\mathcal{S}_{2n}^{(S,p)}$ should satisfy $\pi^{-1}(p_h)>\pi^{-1}(q_h)$ for all $h=1,\dots,m$. This ends the proof.

Similarly, in continuous case, we have

THEOREM (6.18). *Defining*

$$C(\pi,b_i,b_j):=\begin{cases} F(b_{\pi(i)},b_{\pi(j)})\ , & \text{if } \pi:\ \binom{\cdots i\cdots j\cdots}{\cdots i\cdots j\cdots} \\ F(b_{\pi(i)},b_{\pi(j)})+f(b_i,b_j)\ , & \text{if } \pi:\ \binom{\cdots j\cdots i\cdots}{\cdots i\cdots j\cdots}, \end{cases} \tag{6.97}$$

(6.85) *is equal to*

$$\frac{1}{n!}\sum_{\pi\in\mathcal{S}_{2n}^{(S,p)}}\sigma(b_1,\dots,b_{2n})\cdot C(\pi,b_{l_1},b_{k_1})\cdots C(\pi,b_{l_n},b_{k_n}). \tag{6.98}$$

From Theorem (6.17) (resp. Theorem (6.18)) one can finish the proof of Theorem (1.4) (resp. Theorem (1.5)). In fact, by expanding $C_0$ to $C+G$, the expression (1.18) is equal to

$$\frac{1}{n!}\sum_{p.p.}\sigma(i_1,j_1,\dots,i_n,j_n)\sum_{l=0}\sum_{1\leqq r_1<\dots<r_l\leqq n}C(b_{i_{\pi(1)}}b_{j_{\pi(1)}})\cdots$$
$$\cdots G(b_{i_{\pi(r_1)}}b_{j_{\pi(r_1)}})\cdots G(b_{i_{\pi(r_l)}}b_{j_{\pi(r_l)}})\cdots C(b_{i_{\pi(n)}}b_{j_{\pi(n)}}). \tag{6.99}$$

Now regard the pair partition

$$\{i_1,j_1,\dots,i_{r_1},j_{r_1},\dots,i_{r_l},j_{r_l},\dots,i_n,j_n\}$$

as partition $(S_1,\dots,S_{n+l})$ with singletons $i_{r_1},j_{r_1},\dots,i_{r_l},j_{r_l}$; and put $n+l=p$. Then (6.99) becomes

$$\frac{1}{n!}\sum_{p=n}^{2n}\sum_{(S_1,\dots,S_p)\in\mathcal{P}_{2n,p}^0}\sum_{\pi\in\mathcal{S}_n}C(b_{i_{\pi(1)}}b_{j_{\pi(1)}})\cdots G(b_{i_{\pi(r_1)}}b_{j_{\pi(r_1)}})\cdots \tag{6.100}$$

$$\cdots G(b_{i_{\pi(r_l)}} b_{j_{\pi(r_l)}}) \cdots C(b_{i_{\pi(n)}} b_{j_{\pi(n)}}).$$

Notice that $G(b_{i_{\pi(r_d)}} b_{j_{\pi(r_d)}})$, $d = 1, \dots, l$ is a sum of two terms which correspond to an exchange of two singletons. Since the exchanges act on different pairs of singletons, they commute each other. Therefore, combining the exchanges and $\pi \in \mathcal{S}_n$ to a new permutation $\pi'$, then $\pi' \in \mathcal{S}_{2n}^{(S,p)}$ and which exchanges pairs (non-singleton block or two singletons) and keeps the order of elements *in* each non-singleton block; acts as a 2-exchange on two singletons. This shows that (6.100) is equal to (6.95).

The arguments in the continuous case are similar.

## References

[1] L. Accardi and A. Bach, Quantum central limit theorems for strongly mixing random variables, *Z. W. V. G.*, **68**(1985), 393–402.

[2] L. Accardi and A. Bach, The harmonic oscillator as quantum central limit of Bernoulli processes. Accepted by *Prob. Th. and Rel. Fields.*

[3] L. Accardi, A. Frigerio and Y. G. Lu, The weak coupling limit as a quantum functional central limit, *Commun. Math. Phys.*, **131**(1990), 537–570.

[4] L. Accardi and Y. G. Lu, The number process as the low density limit of Hamiltonian model, *Commun. Math. Phys.*, **141**(1991), 9–39.

[5] D. E. Evans and J. T. Lewis, Some semigroup of completely positive maps on the CCR algebra, *J. Funct. Anal.*, **26**(1977), 369–377.

[6] D. E. Evans, Completely positive quasi-free maps on the CAR algebra, *Commun. Math. Phys.*, **70**(1979), 53–68.

[7] F. Fagnola, A Lèvy theorem for "free" noises, preprint.

[8] M. Fannes and M. J. Quaegebeur, Infinitely divisible completely positive mappings, *Public. RIMS Kyoto,* **19**(1983), 469.

[9] N. Giri and W. von Waldenfels, An algebraic version of the central limit theorem, *Z. W. V. G.*, **42**(1978), 129–134.

[10a] D. Goderis, A. Verbeure and P. Vets, Non-commutative central limits, *Prob. Theory Rel. Fields,* **82**(1989), 527–544.

[10b] D. Goderis, A. Verbeure and P. Vets, Dynamics of fluctuations for quantum lattice systems, *Commun. Math. Phys.*, **128**(1990), 533–549.

[10c] D. Goderis, A. Verbeure and P. Vets, Quantum central limit and coarse graining, preprint.

[10d] D. Goderis, A. Verbeure and P. Vets, Theory of quantum fluctuations and the Onsager relations, *J. Statist. Phys.*, **56**(1989), 721–746.

[10e] D. Goderis, A. Verbeure and P. Vets, Theory of fluctuations and small oscillations for quantum lattice systems, *J. Math. Phys.*, **29**(1988), 2581.

[10f] D. Goderis and P. Vets, Central limit theorem for mixing quantum systems and the CCR-algebra of fluctuations, *Commun. Math. Phys.*, **122**(1989), 249–265.

[11] A. A. Cockroft, S. P. Gudder and R. L. Hudson., A quantum-mechanical central limit theorem, *J. Multivariate Anal.*, **7**(1977), 125–148.

[12] A. A. Cockroft and R. L. Hudson, Quantum mechanical Wiener process, *J. Multivariate Anal.*, **7**(1977) 107–124.

[13] D. Petz, *An Invitation to the $C^*$-Algebra of the Canonical Commutation Relation*, Leuven University Press (1990).

[14] J. Quaegebeur, A non commutative central limit theorem for CCR-algebras, *J. Funct. Anal.*, **57**(1984).

[15] R. Speicher, A new example of 'independence' and 'white noise', *Prob. Th. Rel. Fields*, **84**(1990) 141–159.

[16] D. Voiculescu, Limit laws for random matrices and free products. Preprint IHES.

[17] W. von Waldenfels, An algebraic central limit theorem in the anticommuting case, *Z. W. V. G.*, **42**(1978), 135–140.

[18] L. Accardi, An outline of quantum probability, accepted by *USPEHI AN SSSR.*

[19] G. C. Hegerfeldt, Noncommutative analogues of probabilitic notions and results, *J. Funct. Anal.*, **64**(1985), 436–456.

[20] B. Demoen, P. Vanheuverzwijn and A. Verbeure, Completely positive maps on the CCR–algebra, *Lett. Math. Phys.*, **2**(1977), 161–166.

[21] L. Accardi and Y. G. Lu, Quantum central limit theorems for weakly dependent maps, *Acta Math. Hungar.*

[22] W. von Waldenfels and N. Giri, An algebraic version of the central limit theorem, *Z.W.V.G.*, **42**(1978), 129–134.

*(Received September 18, 1990; revised November 12, 1991)*

CENTRO MATEMATICO V. VOLTERRA
DIPARTIMENTO DI MATEMATICA
UNIVERSITA' DI ROMA II
ROMA
ITALIA