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QUANTUM CENTRAL LIMIT THEOREMS FOR WEAKLY DEPENDENT MAPS I

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§0. Introduction

Recall [3] that a *stochastic process* over a *-algebra $\mathcal{B}$, indexed by a set T, is a triple

$$\{\mathcal{A}, \varphi, (j_t)_{t\in T}\} \tag{0.1}$$

where $\mathcal{A}$ is a *-algebra (unless otherwise specified, all algebras in the present paper are complex, associative, with identity); φ a state on $\mathcal{A}$; and $j_t : \mathcal{B} \to \to \mathcal{A}$ a *-homomorphism. Every classical stochastic process (X_t) $(t \in T)$, from a probability space $(\Omega, \mathcal{F}, P)$ to a state space $(S, \mathcal{O})$ (a measurable space) naturally defines a structure as described above by choosing

$$\mathcal{A} = L^\infty(\Omega, \mathcal{F}, P)\ ;\ \mathcal{B} = L^\infty(S, \mathcal{B}),$$
$$j_t : f \in L^\infty(S, \mathcal{B}) \to j_t(f) := f \circ X_t \in L^\infty(\Omega, \mathcal{F}, P)$$

and φ to be the integral with respect to the P-measure. Conversely, every triple of the form (0.1) with $\mathcal{A}$ and $\mathcal{B}$ abelian, determines a (unique up to isomorphism) classical stochastic process.

Now let T be a subset of the natural integers. The classical law of large numbers (resp. central limit theorem) studies the asymptotic behaviour (for $N \to \infty$) of the normalized sums

$$\frac{1}{N}\sum_{j=1}^{N} f(X_j) \quad \left(\text{resp. } \frac{1}{\sqrt{N}}\sum_{j=1}^{N}[f(X_j) - \overline{f}(X_j)]\right)$$

where $f \in L^\infty(S, \mathcal{O})$ and

$$\overline{f}(X_j) = \int_\Omega f(X_j)\, dP.$$

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In the present algebraic context the analogue of these sums are

$$\frac{1}{N}\sum_{h=1}^{N} j_h(b) \quad \left(\text{resp. } \frac{1}{\sqrt{N}}\sum_{h=1}^{N}\left[j_h(b)-\varphi(j_h(b))\right]\right)$$

where $b \in \mathcal{B}$ and the study of the asymptotics of these sums, for $N \to \infty$, is the object of the *algebraic (or quantum) laws of large numbers (resp. central limit theorem).*

In the paper by Giri and von Waldenfels [9] the first quantum central limit theorems for independent random variables was proved under the assumption that, for $h \neq k$ the algebras $j_h(\mathcal{B})$ and $j_k(\mathcal{B})$ commute. Previous results, by [11], [12] even if phrased in a quantum mechanical language are essentially classic in nature. In von Waldenfels [17] this result was extended to the case in which $\mathcal{B}$ is a $\mathbf{Z}_2$-graded algebra, the j_k are graded homomorphisms, and for $h \neq k$ the odd elements of $j_h(\mathcal{B})$ and $j_k(\mathcal{B})$ anticommute.

In these papers it was shown that, like in the classical quantum central limit theorem the limit distributions are Gaussian measures, in the quantum case the limit states are the quantum analogues of the Gaussian measures, i.e. the quasi-free states arising naturally in quantum field theory (cf. [13]). It was also shown that the usual Heisenberg commutation relation in unbounded form (or anticommutation, in the Fermi case) arise naturally from the quantum central limit theorems (cf. [18] for a simple proof). A proof of the Giri–von Waldenfels result, using cumulants techniques and an elegant noncommutative calculus of formal power series is due to Hegerfeldt [19].

Fannes and Quaegebeur [8] and, in a different context, Accardi and Bach [1] extended the central limit theorem to maps. If one starts from product maps on the CCR or the CAR algebra, the limit maps are the quasi-free maps introduced by Demoen, Vanheuverzwijn, Verbeure [20] and Evans, Lewis [5,6].

Motivated by the goal of extending the central limit theorem to quantum Markov chains, Accardi and Bach [1] extended the central limit theorem to non-independent random variables (i.e. to states φ which do not factorize on products of the form $a_{k_1} \cdot a_{k_2} \cdot \ldots \cdot a_{k_n}$ with $k_1 < k_2 < \ldots < k_n$ and $a_{k_m} \in$ $\in j_{k_m}(\mathcal{B})$) (for more details on this, cf. the remarks preceding Definition (1.2) below, where the basic strategy of [1] is outlined). In the present paper we take up the method of [1] and extend it to include the case in which the algebras $j_k(\mathcal{B})$ are not assumed to simply commute or anticommute, but to satisfy a more general commutation relation of the form

$$(0.2) \qquad j_h(b)j_k(b') = \sigma_{h,k}(b,b')j_k(b')j_h(b) + \varepsilon_{h,k}(b,b')\,; \qquad h > k.$$

The first deduction of the CCR *in bounded form* from a quantum central limit theorem was given in [2], where the quantum harmonic oscillator was shown to be central limit of quantum Bernoulli processes.

In a series of papers starting from 1988, Goderis, Verbeure and Vets have deduced the CCR in bounded form from quantum central limit theorems in much more general conditions and with a new technique which allows only L^1-decay of correlations in the dependent case. Moreover in their techniques, the order structure of the index set is not relevant, hence their results include the case of a multidimensional index set (e.g. $\mathbf{Z}^d$). On the other hand, for these techniques, the commutativity of random variables localized on different sites of the lattice seems to be essential, while the consideration of the very general commutation relation (0.2) is a main gool of the present paper. Under such general commutation relation our results are new even in the product (i.e. totally independent) case.

In order to appreciate the generality of the commutation relations (0.2), let us examine some particular cases.

EXAMPLE 1. $\varepsilon_{h,k} \equiv 0$ and $\sigma_{h,k} \equiv +1$ for all $h, k \in \mathbf{N}$. This is the commuting case considered by Giri and von Waldenfels [9], and also in the papers by Goderis, Verbeure, Vets [10].

EXAMPLE 2. $\mathcal{B}$ is $\mathbf{Z}_2$-graded, $\varepsilon_{h,k} \equiv 0$ and $\sigma_{h,k} \equiv -1$ on odd elements. This is the anticommuting case of von Waldenfels [17].

EXAMPLE 3. Let H be a pre-Hilbert space with scalar product $< \cdot, \cdot >$; $\mathcal{A} = W(H)$ is the Weyl C^*-algebra of the canonical commutation relations over H with symplectic form $\operatorname{Im}\langle f, f' \rangle \quad (f, f' \in H)$. It is then given a family of pre-Hilbert subspaces $H_k \subseteqq H$ (not necessarily mutually orthogonal) such that each H_k is isomorphic to a single pre-Hilbert space H_0. Fix such an isomorphism $J_k : H_0 \to H_k$ and let $\mathcal{B} = W(H_0)$ be the Weyl C^*-algebra over H_0; for each $k \in \mathbf{N}$ define

$$j_k(W(f_0)) = W(J_k f_0) \; ; \; f_0 \in H_0 \; .$$

Then (0.2) holds with $\varepsilon_{h,k} \equiv 0$, $B = \{W(f_0) : f_0 \in H_0\}$, and

$$\sigma_{h,k}(W(f_0),\ W(g_0)) = \exp 2i \operatorname{Im}\langle J_h f_0,\ J_k g_0 \rangle \; .$$

EXAMPLE 4. Let $H, (H_k), H_0, W(H_0), W(H)$ be as in Example 3 above. Suppose that both $W(H)$ and $W(H_0)$ act on Hilbert spaces $\mathcal{H}, \mathcal{H}_0$ respectively so that the field operators exist and admit a common invariant dense domain $\mathcal{D}$ (resp. $\mathcal{D}_0$). Let $\mathcal{A}$ (resp. $\mathcal{B}$) denote the *-algebra of the polynomials in the fields, defined on the invariant domain $\mathcal{D}$ (resp. $\mathcal{D}_0$). Then if $A(f)$, $A^+(g)$ (resp. $A_0(f_0)$, $A^+(g_0)$ $(f, g \in H,\ f_0, g_0 \in H_0)$) denote the annihilation and creation operators in $\mathcal{A}$ (resp. $\mathcal{B}$), then the maps

$$j_k(A_0(f_0)) := A_0(J_k f_0); \quad j_k(A_0^+(f_0)) := A_0^+(J_k f_0)$$

define embeddings $j_k : \mathcal{B} \to \mathcal{A}$. If

$$B = \{A_0(f_0)),\ A_0^+(g_0)\ :\ f_0, g_0 \in H_0\}$$

then (0.2) holds with $\sigma_{h,k} \equiv 1$ and

$$\varepsilon_{h,k}(A_0(f_0),\ A_0^+(g_0)) = \langle J_h f_0, J_k g_0 \rangle,$$
$$\varepsilon_{h,k}(A_0^+(g_0),\ A_0(f_0)) = -\langle J_k g_0, J_h f_0 \rangle,$$
$$\varepsilon_{h,k}(A_0(f_0), A_0(g_0)) = \varepsilon_{h,k}(A_0^+(f_0),\ A_0^+(g_0)) = 0.$$

EXAMPLE 5. Example 4 can be modified in a obvious way to obtain the Fermion case.

EXAMPLE 6. Let $H, (H_k), (J_k), H_0$ be as in Example 4 above and $\mathcal{F}(\mathcal{F}_0)$ be the full Fock space (i.e. the tensor algebra) over $H(H_0)$ and let, for $f, g \in$ $\in H$ $l(g)$, $l^*(f)$ denote the free annihilation and creation operators (defined as in [16], cf. also [15] or [7]). Similarly one defines $l_0(f_0), l^*(g_0)$ $(f_0, g_0 \in$ $\in H_0)$. Let $\mathcal{A} \subseteqq \mathcal{B}(\mathcal{F})$ $(\mathcal{B} \subseteqq \mathcal{B}(\mathcal{F}_0))$ denote the algebra generated by the family

$$\{l^*(f), l(g) : f, g \in H\}$$

(resp. $\{l_0^*(f_0), l_0(g_0) : f_0, g_0 \in H_0\}$). Then for each $k \in \mathbf{N}$, the maps

$$j_k(l_0(f_0)) = l(J_k f_0)\ ;\ j_k(l_0^*(g_0)) = l^*(J_k g_0)$$

define embeddings $j_k : \mathcal{B} \to \mathcal{A}$. If

$$B = \{l_0(f_0), l^*(g_0) : f_0, g_0 \in H_0\}$$

then (0.2) holds with $\sigma_{h,k} \equiv 0$ and

$$\varepsilon_{h,k}(l_0(f_0), l_0^*(g_0) = \langle J_h f_0, J_k g_0 \rangle\ .$$

The above examples show that the variety of situations that can be covered by our results is very wide.

REMARK. The paper has been split into two parts: in Part I all the preliminary estimates are established; in Part II these estimates are put together to obtain the main results, i.e. the three theorems stated at the end of Section §1.

§1. Notations, definitions and statement of the main results

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be associative algebras, assume that $\mathcal{A}$ and $\mathcal{C}$ have an identity denoted, when no confusion can arise, by the same symbol 1. Let B be a subset of $\mathcal{B}$, usually it will be a subset of generators of the algebra. Let for each $t \in \mathbf{R}_+$ be given a homorphism $j_t : \mathcal{B} \to \mathcal{A}$, such that for each $t, s \in \in \mathbf{R}_+$, $t \neq s$ and $b, b' \in B$, there exist two scalars $\sigma(t,s,b,b')$ and $\varepsilon(t,s,b,b')$ satisfying

$$j_t(b)j_s(b') = \sigma(t,s,b,b')j_s(b')j_t(b) + \varepsilon(t,s,b,b'). \tag{1.1}$$

Notice that if $\mathcal{A}$, $\mathcal{B}$ are *-algebras and E is a state on $\mathcal{A}$, then the triple $\{\mathcal{A}, (j_t), E\}$ is a stochastic process over $\mathcal{B}$ in the sense of [3].

Following the notations of [1], we denote $\mathcal{S}_p$ the family of all p-permutations and $\mathcal{P}_{k,p}$ the family of all ordered partitions $(S_1, \ldots, S_p)$ of the set $\{1, \ldots, k\}$ into exactly p non-empty subsets ($k \in \mathbf{N}$ and $p \leqq k$). The partition $(S_1, \ldots, S_p)$ is ordered with order "$<$" in the following sense: $S_i < S_j$ if and only if $\min\{r : r \in S_i\} < \min\{r : r \in S_j\}$ and each set S_j has the natural order. If some S_h has only one elements, we shall call it a *singleton.*

For each $(S_1, \ldots, S_p) \in \mathcal{P}_{k,p}$ and $T \in \mathbf{R}_+$, denote $[S_1, \ldots, S_p]_T$ the set of all k-tuples $(t_1, \ldots, t_k) \in [0,T)^k$ such that

(i) for each $j = 1, 2, \ldots, p$ and $i, i' \in S_j$, we have $t_i = t_{i'}$;

(ii) for each $j, j' = 1, 2, \ldots, p$, $j \neq j'$, $i \in S_j$ and $i' \in S_{j'}$, we have $t_i \neq t_{i'}$.

The elements of $[S_1, \ldots, S_p]_T$ can be identified to the functions t from $\{1, \ldots k\}$ to $[0,T)$ which are constant on the elements of the partition $(S_1, \ldots, S_p)$ and which take exactly p different values.

Similarly for each $N \in \mathbf{N}$, we denote by $[S_1, \ldots, S_p]_N$ the set of all maps $\alpha : \{1, \ldots, k\} \to \{1, \ldots, N\}$ such that

(i) for each $j = 1, \ldots, p$ and $i, i' \in S_j$, we have $\alpha(i) = \alpha(i')$;

(ii) for each $j, j' = 1, 2, \ldots, p$, $j \neq j'$, $i \in S_j$ and $i' \in S_j$, we have $\alpha(i) \neq \alpha(i')$.

Throughout the paper, we shall denote by ν either the Lebesgue measure on $\mathbf{R}$ or the counting measure on $\mathbf{Z}$, both characterized by translation invariance and

$$\nu([0,T)) = T \ ; \qquad T \in \mathbf{R} \text{ or } \mathbf{N}.$$

We shall use the notations

$$S_T(b) = \int_{[0,T)} j_s(b)\nu(ds) \quad ; \quad T \in \mathbf{R}_+ \tag{1.2}$$

so that, if ν is the counting measure

$$S_N(b) = \sum_{k=1}^{N} j_k(b) \quad ; \quad N \in \mathbf{N} \tag{1.2a}$$

for each $b \in \mathcal{B}$. Moreover, we assume that

(i) on $\mathcal{C}$, there is a semi-norm $|\cdot|$ and it is given a map $E : \mathcal{A} \to \mathcal{C}$ with property

$$E(1) = 1; \tag{1.3a}$$

(ii) for each $k \in \mathbf{N}$, $b_1, \dots, b_k \in B := \{b : E(j_t(b)) = 0$ for each $t \in \in \mathbf{R}_+\}$, there exists a positive constant $C(b_1, \dots, b_k) \in \mathbf{R}_+$, such that, for each $p \leqq k, (S_1, \dots, S_p) \in \mathcal{P}_{k,p}$, $(s_1, \dots, s_p) \in \mathbf{R}_+^p$

$$|E(j_{s_1}(b_{S_1}) \dots j_{s_p}(b_{S_p}))| \leqq C(b_1, \dots, b_k) \tag{1.3b}$$

where and in the following, for $S_j = \{i_1, \dots, i_r\}$, we use the notations

$$b_{S_j} = b_{i_1} \cdots b_{i_r} \tag{1.4}$$

DEFINITION 1.1. We call $E : \mathcal{A} \to \mathcal{C}$ an *FP-mixing map* (FP meaning "faster than polynomial") if there exist two functions $d, \delta : \mathbf{R}_+ \to \mathbf{R}_+$, (resp. d, $\delta : \mathbf{N} \to \mathbf{N}$) satisfying

(i) for each $q > 0$

$$d_T \to \infty, \quad \frac{d_T}{T^q} \to 0, \text{ as } T \to \infty \tag{1.5a}$$

i.e. d_T tends to infinity more slowly than any power of T;

(ii) for each $q > 0$,

$$\delta_T \cdot T^q \to 0 \quad \text{as } T \to \infty \tag{1.5b}$$

i.e. δ_T tends to zero more rapidly than any polynomial fuction.

(iii) for each $k \in \mathbf{N}$, $x \in \mathbf{R}$ (resp. $x \in \mathbf{N}$), $b_1, \dots, b_k \in B$, one has

$$|E(M_x N_{x+d_T}) - E(M_x)E(N_{x+d_T})| \leqq C(b_1, \dots, b_k)\delta_T \tag{1.6}$$

where the constants $C(b_1, \dots, b_k)$ can be taken equal to those in (1.3b) and

$$M_x := j_{s_1}(b_{S_1}) \dots j_{s_q}(b_{S_q}), \tag{1.7}$$

$$N_{x+d_T} := j_{s_{q+1}}(b_{S_{q+1}}) \dots j_{s_p}(b_{S_p}) \tag{1.8}$$

with $q \leqq p = 1, \ldots, k$, $(S_1, \ldots, S_p) \in \mathcal{P}_{k,p}$ and $(s_1, \ldots, s_p) \in \mathbf{R}_+^p$ (resp. $(k_1, \ldots, k_p) \in \mathbf{N}^p$), such that $s_1 < \ldots < s_p$ (resp. $k_1 < \ldots < k_p$) and

$$s_j \leqq x, \qquad j = 1, \ldots, q, \tag{1.9}$$

$$s_j \geqq x + d_T, \qquad j = q+1, \ldots, p, \tag{1.10}$$

i.e. the correlations between observables which are localized in intervals $I, J \subset \mathbf{R}_+$ whose distance is greater than d_T decay at a rate which is faster than δ_T.

In the following, E will always denote an FP-mixing map satisfying (1.3a) and (1.3b). Since any state with exponential decay of the correlations is FP-mixing and since it is known that ergodic quantum Markov chains are exponentially mixing (cf. [2]), a Corollary of our results is that the central limit theorem holds for ergodic quantum Markov chains on countable tensor products of matrix algebras.

The basic idea of the proofs is the same as in [1], i.e. a quantum generalization of Bernstein's method to prove the central limit theorem for weakly dependent random variables. The idea is that, if the correlations decay sufficiently fast (conditions (1.6) and (1.5b)), then the blocks of random variables which are separated by a gap of length d_T become asymptotically independent. Moreover condition (1.5a) implies that, neglecting blocks of length d_T, we make an error which becomes negligible in the limit.

The present paper extends the results of [1] and corrects two errors in that paper: one, noted by von Waldenfels, is that in the formula (1.3) of [1] a combinatorial factor $(1/p!)$ was omitted. The other, noted by Verbeure, is that in the expression of the correlation function in Theorem (1.1) of [1], the term arising from the fact that the correlations at different times do not vanish (the term F of formula (1.16) of the present paper), was omitted due to an error in the proof of Lemma (2.2) of [1].

We are grateful to the above mentioned authors for pointing out these errors. The results of the present paper show however that the technique of the proof, developed in [1], was correct and applicable to a much more general situation, like the present one.

In the proofs we have tried to understand the analogies between the techniques used in the present paper and those developed by the authors to deal with the weak coupling and low density problems (cf. [3], [4] and the Remark (6.6a) in the following).

Since the proofs are long and technical, we formulate here the main results. In order to do that we need the following:

Definition 1.2. We say that $f : \mathbf{R}_+^2 \to \mathcal{C}$ is $\mathbf{s} - \mathbf{L}^1(\mathcal{C}, \mathrm{d}\nu)$ if it is bounded and for each $s \in \mathbf{R}_+$, $f(\cdot, s) \in L^1([s, \infty), d\nu, \mathcal{C})$, the functions

$$s \longmapsto \int_{[s,\infty)} f(t,s)\, \nu(dt)\,, \qquad t \longmapsto \int_{[0,t)} f(t,s)\, \nu(ds) \tag{1.11a}$$

are bounded; the limit

$$\lim_{T\to\infty} \int_{[s,T)} f(t,s)\ \nu(dt) \tag{1.11b}$$

is uniform in s.

Moreover if the first integral of (1.11a) is not only bounded but also independent of s, we say that $f(\cdot,\cdot)$ is $\mathbf{S}-\mathbf{L}^1(\mathcal{C},\mathbf{d}\nu)$.

In particular, if ν is the counting (or the Lebesgue) measure, we denote $s-L^1(\mathcal{C},d\nu)$ by $s-L^1(\mathcal{C},dn)$ $(s-L^1(\mathcal{C},dt))$ and the same for $S-L^1(\mathcal{C},d\nu)$.

REMARK. If there exists an $\{f(k)\}_{k=1}^{\infty}\subset\mathbf{R}_+$ satisfying

(i) $|\varepsilon(h,r)|\leqq f(h-r)$ for each $r<h\leqq N$;

(ii) the series $\sum_{k=1}^{\infty} f(k)$ converges;

then, ε is $S-L^1(\mathcal{C},dn)$.

The meaning of this assumption is best understood by looking at (1.1) in the particular case in which $\sigma(t,s,b,b')=1$. In this case we immediately recognize that the condition $\varepsilon\in L^1(\mathcal{C},dn)$ is a condition of asymptotic abelianness, i.e. if s and t are very far apart, then $j_t(b)$ and $j_s(b')$ *almost commute.*

THEOREM 1.3. *Let E be an FP-mixing map and let $B\subset\mathcal{B}$ be a set of elements satisfying the commutation relation* (1.1) *and the mean zero condition*

$$E(j_t(b))=0\ ,\qquad \forall\ t\in T,\ b\in B.$$

If for each $b,b'\in B$, $\varepsilon(\cdot,\cdot,b,b')$ is $s-L^1(\mathcal{C},d\nu)$ and $\sigma(\cdot,\cdot,b,b')$ is bounded then, for each $b_1,\dots,b_k\in B$ and $a>\frac{1}{2}$ or $a=\frac{1}{2}$ and k odd,

$$\lim_{T\to\infty}\frac{1}{\nu([0,T))^{ak}}\int_{[0,T)^k} E(j_{t_1}(b_1)\dots j_{t_k}(b_k))\nu(dt_1)\dots\nu(dt_k)=0. \tag{1.12}$$

REMARK. If E is a stationary state, i.e. $E(j_t(b))=E_0(b)$ on B, independent of t, for some state E_0 and $a=1$, (1.12) is simply the *law of large numbers.* If $a=\frac{1}{2}$ and k is odd, (1.12) is the *first half* of the central limit theorem, i.e. the vanishing of odd moments for mean zero Gaussian state.

Moreover

THEOREM 1.4. *In the assumptions of Theorem* 1.3, *suppose that $T\subset\subset\mathbf{N}$ and that for each $b,b'\in B$,*

(i) $\sigma(\cdot,\cdot,b,b')=\sigma(b,b')$ *(i.e. $\sigma(h,k,b,b')$ does not depent on h,k).*

(ii) $E(\cdot,\cdot,b,b')$, $\varepsilon(\cdot,\cdot,b,b')$ *are in* $S - L^1(\mathcal{C},dn)$ *in the sense of Definition* 1.1.

(iii) *The limit*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{h=1}^{N}E(j_h(b\cdot b'))=:C(bb') \tag{1.13}$$

exists.

Then for each $n\in\mathbf{N}$ *and* $b_1,\dots,b_{2n}\in B$, *the central limit*

$$\lim_{N\to\infty}\frac{1}{N^n}E(S_N(b_1)\dots S_N(b_{2n})) \tag{1.14}$$

exists and if we denote

$$f(b,b'):=\sum_{k=h+1}^{\infty}\varepsilon(k,h,b,b'), \tag{1.15}$$

$$F(b,b'):=\sum_{k=h+1}^{\infty}E\big(j_h(b)j_k(b')\big), \tag{1.16}$$

$$C_0(b,b'):=C(bb')+F(b,b')+F(b',b)+f(b,b'), \tag{1.17}$$

then the limit (1.14) *is equal to*

$$\frac{1}{n!}\sum_{p.p.}\sum_{\pi\in\mathcal{S}_n}\sigma(i_1,j_1,\dots,i_n,j_n;b_1,\dots,b_{2n})\times \tag{1.18}$$
$$\times C_0(b_{i_{\pi(1)}},b_{j_{\pi(1)}})\cdots C_0(b_{i_{\pi(n)}},b_{j_{\pi(n)}})$$

where, as usual $\sum_{p.p.}$ *means the sum over all ordered pairs partition of* $\{1,\dots,2n\}$, *i.e. all pairs* $\{i_1,j_1,\dots,i_n,j_n\}$ *such that*

$$\{i_1,j_1,\dots,i_n,j_n\}=\{1,\dots,2n\}, \tag{1.19a}$$

$$i_h<j_h,\quad \textit{for any } h=1,\dots,n, \tag{1.19b}$$

$$j_1<j_2<\dots<j_n \tag{1.19c}$$

and the $\sigma(i_1,j_1,\dots,i_n,j_n;b_1,\dots,b_{2n})$ *is a product of* σ*-factors.*

In the continuous analogue of Theorem 1.4 a qualitatively new phenomenon arises.

THEOREM 1.5. *In the continuous case, with the assumptions of Theorem* 1.3, *assume that for each* $b,b'\in B$,

(i) $\sigma(\cdot,\cdot,b,b')=\sigma(b,b')$,

(ii) $E(\cdot,\cdot,b,b')$, $\varepsilon(\cdot,\cdot,b,b')$ *are in* $S-L^1(C,dt)$ *in the sense of Definition* 1.1.

Then for each $n\in\mathbf{N}$ *and* $b_1,\dots,b_{2n}\in B$, *the central limit*

$$\lim_{T\to\infty}\frac{1}{\nu([0,T))^n}\int_{[0,T)^{2n}}E(j_{t_1}(b_1)\dots j_{t_{2n}}(b_{2n}))dt_1\dots dt_{2n} \tag{1.20}$$

exists. Moreover if we denote

$$f(b,b'):=\int_{[h,\infty)}ds\varepsilon(s,h,b,b'), \tag{1.21}$$

$$F(b,b'):=\int_{[h,\infty)}dsE\big(j_h(b)j_s(b')\big) \tag{1.22}$$

and

$$C_0(b,b'):=F(b,b')+F(b',b)+f(b,b') \tag{1.23}$$

then the limit (1.20) *is equal to*

(1.24)

$$\frac{1}{n!}\sum_{p.p.}\sigma(i_1,j_1,\dots,i_n,j_n;b_1,\dots,b_{2n})\times C_0(b_{i_{\pi(1)}},b_{j_{\pi(1)}})\cdots C_0(b_{i_{\pi(n)}},b_{j_{\pi(n)}})$$

REMARK. Notice that in the continuous case there is no analogue of condition (iii) in Theorem 1.4. This is because this condition is on products of pairs and we shall show that in the continuous case, only the partitions made up entirely of singletons survive in the limit.

§2. Some technical lemmata

In this section we introduce some notations and prove some lemmata needed in the following sections.

LEMMA 2.1. *Let* E *be as specified in Section* 1 *and let* $a\geqq\frac{1}{2}$, $p\leqq k\in\in\mathbf{N}$, $b_1,\dots,b_k\in B$, $(S_1,\dots,S_p)\in\mathcal{P}_{k,p}$. *Assume that either of the following conditions is satisfied:*

(i) $ak>p$;

(ii) $(S_1,\dots,S_p)$ *contains exactly* q *singletons with* $q\geqq 1$, $a>\frac{1}{2}$ *or* $a=\frac{1}{2}$ *and* k *odd.*

Then

(2.1)

$$\lim_{T\to\infty}\frac{1}{\nu([0,T))^{ak}}\int\limits_{0\leqq s_1<\ldots<s_p\leqq T}|E(j_{s_1}(b_{S_1})\ldots j_{s_p}(b_{S_p}))|\cdot\nu(ds_1)\ldots\nu(ds_p)=0.$$

PROOF. (i) If $ak>p$, then by (1.6),

$$(2.2)\quad \frac{1}{\nu([0,T))^{ak}}\int\limits_{0\leqq s_1<\ldots<s_p\leqq T}|E(j_{s_1}(b_{S_1})\ldots j_{s_p}(b_{S_p}))|\nu(ds_1)\ldots\nu(ds_p)\leqq$$
$$\leqq\frac{1}{\nu([0,T))^{ak}}C(b_1,\ldots,b_k)\nu([0,T))^p\to 0$$

(ii) Assuming that $(S_1,\ldots,S_p)$ contains exactly q singletons which correspond to the indices $j_1<\ldots<j_q$, we define the set

$$(2.3)\quad A(T,d_T,p,\{j_n\}_{n=1}^q):=\Big\{(s_1,\ldots,s_p)\in[0,T)^p:s_1<\ldots<s_p$$
$$\text{and for each } r=1,\ldots,q,\ \text{either } s_{j_r}-s_{j_r-1}\leqq d_T \text{ or } s_{j_r+1}-s_{j_r}\leqq d_T\Big\}$$

denoting ν^p the product measure $(\otimes\nu)^p$, then, for each $\{j_n\}_{n=1}^q$, the quantity $\nu^p\left(A\left(T,d_T,p,\{j_n\}_{n=1}^q\right)\right)$ can be written as

$$(2.4)\quad \int_0^T\nu(ds_1)\int_{s_1}^T\nu(ds_2)\ldots\int_{s_n-1}^T\nu(ds_n)\chi^{\varepsilon(1)}_{[s_{j_1-1},s_{j_1-1}+d_T)}(s_{j_1})\times$$
$$\times\chi^{\varepsilon(2)}_{[s_{j_1},s_{j_1}+d_T)}(s_{j_1+1})\times$$
$$\times\ldots\times\chi^{\varepsilon(2q-1)}_{[s_{j_q-1},s_{j_q-1}+d_T)}(s_{j_q})\times\chi^{\varepsilon(2q)}_{[s_{j_q},s_{j_q}+d_T)}(s_{j_q+1})$$

where $\varepsilon\in\{0,1\}^{2q}$ is determined uniquely by the rule

$$(2.5a)\quad \varepsilon(2r-1)=\begin{cases}0\,, & \text{if } s_{j_r}-s_{j_r-1}>d_T;\\ 1\,, & \text{if } s_{j_r}-s_{j_r-1}\leqq d_T;\end{cases}$$

$$(2.5b)\quad \varepsilon(2r)=\begin{cases}0\,, & \text{if } s_{j_r+1}-s_{j_r}>d_T;\\ 1\,, & \text{if } s_{j_r+1}-s_{j_r}\leqq d_T\end{cases}$$

and where, by definition, for any set I, $\chi_I^0=1$; $\chi_I^1=\chi_I$. From the definition of $A(T,d_T,p,\{j_h\}_{h=1}^q)$, it follows that $\varepsilon(2n-1)+\varepsilon(2n)\geqq 1$ for each $n=$ $=1,\ldots,q$. Notice that for each $n=1,\ldots,q$, the product

$$(2.6)\quad \chi^{\varepsilon(2n-1)}_{[s_{j_n-1},s_{j_n-1}+d_T)}(s_{j_n})\times\chi^{\varepsilon(2n)}_{[s_{j_n},s_{j_n}+d_T)}(s_{j_n+1})$$

surely depends on s_{j_n} but not necessarily on s_{j_n+1} or s_{j_n-1}. So if we denote (2.7)

$$F(s_1,\dots,s_p\ ;\ \{j_n\}_{n=1}^q,\varepsilon) := \prod_{n=1}^q \chi_{[s_{j_n-1},s_{j_n-1}+d_T)}^{\varepsilon(2n-1)}(s_{j_n}) \times \chi_{[s_{j_n},s_{j_n}+d_T)}^{\varepsilon(2n)}(s_{j_n+1})$$

then F depends on $s_{j_1},\dots,s_{j_q}$, but not necessarily on the other variables, i.e. F is a function which depends on at least the q variables $s_{j_1},\dots,s_{j_q}$. In the multiple integral (2.4), if the variable s_j does not appear in any characteristic function, then we majorize the corresponding integral with $\nu([0,T))$. If it appears we would like to majorize the correspoding integral with the factor $\nu([0,d_T))$. However in doing so we should keep in mind that some of the characteristic functions can coincide. This can happen only if, for some $r=1,\dots,q$,

$$j_{r-1} = j_r - 1. \tag{2.8}$$

In this case we have the factor

(2.9)

$$\chi_{[s_{j_{r-1}-1},s_{j_{r-1}-1}+d_T]}(s_{j_{r-1}})^{\varepsilon(2(r-1)-1)} \cdot \chi_{[s_{j_{r-1}},s_{j_{r-1}}+d_T]}(s_{j_{r-1}+1})^{\varepsilon(2(r-1))}$$
$$\cdot\chi_{[s_{j_{r-1}},s_{j_{r-1}}+d_T]}(s_{j_r})^{\varepsilon(2r-1)} \cdot \chi_{[s_{j_r},s_{j_r}+d_T]}(s_{j_r+1})^{\varepsilon(2r)}.$$

So if

$$\varepsilon(2r) = \varepsilon(2(r-1)-1) = 0 \tag{2.10}$$

then the product (2.9) becomes

$$\chi_{[s_{j_{r-1}},s_{j_{r-1}}+d_T]}(s_{j_{r-1}+1}) \cdot \chi_{[s_{j_{r-1}},s_{j_{r-1}}+d_T]}(s_{j_r})$$

and in the view of (2.5a) and (2.5b), this is equal to $\chi_{[s_{j_{r-1}},s_{j_{r-1}}+d_T]}(s_{j_r})$. Thus, if both conditions (2.8) and (2.10) are satisfied, then from two singletons we get only one characteristic function.

Since there are q singletons, the worst case is when we get only $\left[\frac{q+1}{2}\right]$ characteristic functions. This is clear if q is even and, if $q=2m+1$ is odd, then after having formed m pairs, the remaining term will surely produce a characteristic function, because if condition (2.8) is satisfied by three indices, say j_{r-2},j_{r-1},j_r, then condition (2.10) cannot be simultaneously verified for the two pairs (j_{r-2},j_{r-1}), (j_{r-1},j_r). In conclusion, if q is odd we have at least $m+1=\frac{q+1}{2}$ different characteristic functions.

As a consequence of this we obtain the estimate

$$\nu^p(A(T,d_T,p,\{j_h\}_{h=1}^q)) \leqq \nu([0,T))^{p-[\frac{q+1}{2}]} \cdot \nu([0,d_T))^{[\frac{q+1}{2}]}. \tag{2.11}$$

Therefore if we denote
(2.12)
$$A(t,d_T,p,q) := \sum_{1\leqq j_1<\ldots<j_q\leqq p;\ \{S_{j_h}\}_{h=1}^q \text{ are singletons}} A\left(T,d_T,p,\{j_h\}_{h=1}^q\right)$$

then

(2.12a) $$\nu^p(A(t,d_T,p,q)) \leqq \binom{p}{q} \cdot \nu([0,T))^{p-[\frac{q+1}{2}]} \cdot \nu([0,d_T))^{[\frac{q+1}{2}]}.$$

Moreover by (2.12a) we obtain

(2.13) $$\frac{1}{\nu([0,T))^{ak}} \int\limits_{A(T,d_T,p,q)} |E(j_{s_1}(b_{S_1})\ldots j_{s_p}(b_{S_p}))|\nu(ds_1)\ldots\nu(ds_p) \leqq$$
$$\leqq \frac{1}{\nu([0,T))^{ak}}\nu([0,T))^{p-[\frac{q+1}{2}]} \cdot \nu([0,d_T))^{[\frac{q+1}{2}]} \cdot C(b_1,\ldots,b_k) \cdot \binom{p}{q}.$$

But if $(S_1,\ldots,S_p)$ contains exactly q singletons, then there are $p-q$ non-sigletons, therefore

(2.14) $$k = \sum_{j=1}^{p} |S_j| \geqq q+2(p-q) = 2p-q$$

and this implies that

(2.14a) $$p \leqq \frac{1}{2}(k+q) \leqq \frac{k}{2} + \left[\frac{q+1}{2}\right].$$

Therefore, if $a > \frac{1}{2}$, we have

(2.15) $$\frac{1}{\nu([0,T))^{ak}}\nu([0,T))^{p-[\frac{q+1}{2}]} \cdot \nu([0,d_T))^{[\frac{q+1}{2}]} \leqq$$
$$\leqq \frac{1}{\nu([0,T))^{k(a-\frac{1}{2})}}\nu([0,d_T))^{[\frac{q+1}{2}]} \to 0$$

as $T\to\infty$. Moreover, $p,k\in\mathbf{N}$ implies that in (2.14a) it is possible to have equality only if k is even, therefore if $a=\frac{1}{2}$ and k is odd, then

$$p < \frac{k}{2} + \left[\frac{q+1}{2}\right] \quad \text{i.e.} \quad p - [\frac{q+1}{2}] < \frac{k}{2}.$$

Thus
(2.16)

$$\frac{1}{\nu([0,T))^{ak}}\nu([0,T))^{p-[\frac{q+1}{2}]}\cdot\nu([0,d_T))^{[\frac{q+1}{2}]}=\frac{\nu([0,d_T))^{[\frac{q+1}{2}]}}{\nu([0,T))^{\frac{k}{2}-(p-[\frac{q+1}{2}])}}\to 0$$

as $T\to\infty$. Define now the set

$$A_p=\{(s_1,\dots,s_p)\in\mathbf{R}_+^p : s_1<\dots<s_p\leqq T\}\backslash A(T,d_T,p)$$

by (2.3) and (2.12). One knows that for each $(s_1,\dots,s_p)\in A_p$ there exists a $q\leqq p$, $1\leqq j_1<\dots<j_q\leqq p$ such that

$$|s_{j_r}-s_{j_r-1}|>d_T \quad\text{and}\quad |s_{j_r+1}-s_{j_r}|>d_T,\ \ \forall\ r=1,\dots,q. \tag{2.17a}$$

Therefore the mean zero condition ($b_j\in B$, $j=1,\dots,k$) and (1.6) imply that

$$\frac{1}{\nu([0,T))^{ak}}\int\limits_{A_p}|E(j_{s_1}(b_{S_1})\dots j_{s_p}(b_{S_p}))|\nu(ds_1)\dots\nu(ds_p)\leqq \tag{2.17}$$

$$\leqq\frac{1}{\nu([0,T))^{ak}}\nu([0,T))^p\cdot O(\delta_T)\cdot C(b_1,\dots,b_k)\longrightarrow 0,\quad\text{as}\quad T\to\infty.$$

Putting together ((2.16) and (2.17) we obtain that (2.1) is equal to

$$\lim_{T\to\infty}\frac{1}{\nu([0,T))^{ak}}\Bigg(\int\limits_{A_p}+\int\limits_{A(T,d_T,p,q)}\Bigg)\big|E(j_{s_1}(b_{S_1})\dots j_{s_p}(b_{S_p}))\big|\cdot \tag{2.18}$$
$$\cdot\nu(ds_1)\dots\nu(ds_p)=0.$$

COROLLARY 2.2. *Let E be as specified in Section* 1 *and let $a>\frac{1}{2}$, $p\leqq$ $\leqq k\in\mathbf{N}$, $b_1,\dots,b_k\in B$, $(S_1,\dots,S_p)\in\mathcal{P}_{k,p}$. Then*

$$\lim_{T\to\infty}\frac{1}{\nu([0,T))^{ak}}\int\limits_{0\leqq s_1<\dots<s_p\leqq T}\big|E(j_{s_1}(b_{S_1})\dots j_{s_p}(b_{S_p}))\big|\cdot \tag{2.19}$$
$$\cdot\nu(ds_1)\dots\nu(ds_p)=0.$$

PROOF. We distinguish two cases:

i) if $p\leqq k/2$ then $ak>p$;

ii) if $p>k/2$ then there exist singletons among $S_1,\dots,S_p$.

The proof follows that of Lemma 2.1.

§3. Normal order in abstract algebras

In this section we generalize some techniques widely used in quantum field theory and known under the name of *normal order* or *Wick order.* In an abstract setting, the problem giving rise to these techniques can be formulated as follows: one starts from elements a_x $(x \in \mathbf{N})$ of an algebra $\mathcal{Q}$, satisfying some commutation relations of the form (3.3); one considers products of the a_x, of the form (3.1) and, by repeated application of the commutation relations (3.3), one wants to write the product (3.1) in such a way that the indices $x_1, \ldots, x_n$ appear in a preassigned order. In the present paper, the preassigned order will be the increasing one (3.2). The *normal order,* usually considered by the physicists is different: the indices x_j take only the values 0 (corresponding to creation operators) and 1 (corresponding to annihilation operators), and one wants to write the product (3.1) as a sum of products in which all the zeros are to the left of all the ones and the original order among the zeros and among the ones is preserved. In that case the factor ε corresponds to a scalar product and the factor σ corresponds to 1 (Boson case) or -1 (Fermi case). The situation considered by us corresponds to a *time ordering.* The basic techniques are the same in both cases. The techniques developed below are a natural generalization of those, introduced by the authors, to deal with the weak coupling and the low density problem (cf. [3], [4]).

DEFINITION 3.1. For any algebra $\mathcal{Q}$, $n, N \in \mathbf{N}$, $\{a_h\}_{h=1}^N \subset \mathcal{Q}$, $1 \leqq x_1, \ldots, x_n \leqq N$, we say that the product

$$a_{x_1} \cdots a_{x_n} \tag{3.1}$$

is *ordered* if the indices $\{x_h\}_{h=1}^n$ are ordered, i.e.

$$x_1 \leqq x_2 \leqq \ldots \leqq x_n. \tag{3.2}$$

In the following we shall investigate the ordered form of products of the form (3.1), where the a_j satisfy the commutation relations

$$a_x \cdot a_y = \sigma(x,y) \cdot a_y \cdot a_x + \varepsilon(x,y), \quad \forall \ x, y \in \mathbf{N}, \ x \neq y \tag{3.3}$$

with σ, ε in the center of $\mathcal{Q}$.

For each $n, N \in \mathbf{N}$, $n \leqq N$ and $1 \leqq x_1, \ldots, x_n \leqq N$, there exists a unique n-permutation $\pi \in \mathcal{S}_n$ (the permutation group on $\{1, \ldots, n\}$) such that π is a composition of k consecutive exchanges

$$x_{\pi(1)} \leqq x_{\pi(2)} \leqq \cdots \leqq x_{\pi(n)} \tag{3.4}$$

and for any other n-permutation π', if π' is a composition of k' consecutive exchanges with $k' < k$, then π' does *not* satisfy (3.4). An exchange is called *consecutive* if it exchanges two consecutive indices and leaves the remaining ones fixed.

In the following for any given $x = \{x_1, \ldots, x_n\}$, we shall denote this permutation by π^x.

LEMMA 3.2. *In the notations* (1.2a),

$$S_T(b_1)\cdots S_T(b_n) = \tag{3.5}$$
$$= \sum_{p=1}^{n} \sum_{(S_1,\ldots,S_p)\in\mathcal{P}_{n,p}} \int_{[S_1,\ldots,S_p]_T} j_{t_1}(b_1)\cdots j_{t_n}(b_n)\nu(t_1)\cdots\nu(t_n).$$

PROOF. (3.5) is an immediate consequence of

$$[0,T)^n = \bigcup_{p=1}^{n} \bigcup_{(S_1,\ldots,S_p)\in\mathcal{P}_{n,p}} \bigcup_{t\in[S_1,\ldots,S_p]_T} \{t\}. \tag{3.6}$$

LEMMA 3.3. *For each $n, N \in \mathbf{N}$, $n \leqq N$ and $1 \leqq x_1,\ldots,x_n \leqq N$, the ordered form of the product* (3.1) *is equal to*

(3.7)
$$\sum_{m=0}^{[n/2]} \sum_{1\leqq p_1<\ldots<p_m\leqq n} \sum_{(q_1,\ldots,q_m)}^{*} \prod_{h=1}^{m} \varepsilon(x_{p_h}, x_{q_h})\sigma(x_1,\ldots,x_n)\cdot a_{x_{\pi^x(r_1)}}\cdots a_{x_{\pi^x(r_{n-2m})}}$$

where and in the following

i) *for each fixed m and $1 \leqq p_1 < \ldots < p_m \leqq n$, $\sum^{*}_{(q_1,\ldots,q_m)}$ means the sum over all $1 \leqq q_1,\ldots,q_m \leqq n$ satisfying*

$$\operatorname{card}(\{q_h\}_{h=1}^m) := |\{q_h\}_{h=1}^m| = m, \tag{3.8a}$$
$$\{q_h\}_{h=1}^m \subset \{1,\ldots,n\} \setminus \{p_h\}_{h=1}^m, \tag{3.8b}$$
$$p_h < q_h, \quad \forall\, h = 1,\ldots,m \tag{3.8c}$$

and

$$x_{p_h} > x_{q_h}, \quad \forall\, h = 1,\ldots,m; \tag{3.8d}$$

ii) *for each fixed m and $\{p_h, q_h\}_{h=1}^m$,*

$$\{r_h\}_{h=1}^{n-2m} := \{1,\ldots,n\} \setminus \{p_h, q_h\}_{h=1}^m, \quad r_1 < \ldots < r_{n-2m} \tag{3.9a}$$

and $x_{\pi^x(r_1)} \leqq \ldots \leqq x_{\pi^x(r_{n-2m})}$;

iii) *$\sigma(x_1,\ldots,x_n)$ is a product factor of the form $\sigma(x_i,x_j)$.*

PROOF. From the commutation relation (3.3) we know that in the ordered form of the product (3.1), some elements of $\{a_{x_h}\}_{h=1}^n$ will be used to produce an ε-factor and in order to get an ε-factor we use two elements of $\{a_{x_h}\}_{h=1}^n$, therefore the number m of ε-factors can be equal to $0,1,\ldots,[n/2]$.

For each fixed m, let $\{p_h\}_{h=1}^m$ denote indices $\{x_h\}_{h=1}^n$ such that a_{p_h} is used to produce an ε-factor with some element a_{q_h} with $q_h > p_h$. By relabeling the order, one may suppose that $p_1 < p_2 < \ldots < p_m$. If $p_1 < p_2 < \ldots < p_m$ and $q_1, q_2, \ldots, q_m \subset \{1, \ldots, n\}$ are chosen as above then obviously:

— q_h cannot be in $\{p_h\}_{h=1}^m$ for any $h = 1, \ldots, m$, i.e. $\{q_h\}_{h=1}^m \subset \subset \{1, \ldots, n\} \setminus \{p_h\}_{h=1}^m$;

— q_h cannot be equal to another $q_{h'}$, i.e. $|\{q_h\}_{h=1}^m| = m$.

— $q_h > p_h$ for each $h = 1, \ldots, m$;

— if for some $i < j$, $x_i < x_j$ then we do not exchange the order of the two elements a_{x_i} and a_{x_j}, so there is no factor $\varepsilon(x_i, x_j)$, i.e. $x_{p_h} > x_{q_h}$ for each $h = 1, \ldots, m$.

For each fixed m and $\{p_h, q_h\}_{h=1}^m$, denoting $\{r_h\}_{h=1}^{n-2m} := \{1, \ldots, n\} \setminus \setminus \{p_h, q_h\}_{h=1}^m$, the $\{a_{r_h}\}_{h=1}^{n-2m}$ are not used to produce ε-factors, therefore in order to bring their product to the ordered form one can apply the restriction of the permutation π^x to the set $\{r_h\}_{h=1}^{n-2m}$. Thus one obtains the product $a_{x_{\pi^x(r_1)}} \cdots a_{x_{\pi^x(r_{n-2m})}}$, where, by the definition of π^x (cf. (3.4)),

$$x_{\pi^x(r_1)} < \ldots < x_{\pi^x(r_{n-2m})}.$$

Since each exchange gives rise to one σ-factor, eventually we obtain a factor $\sigma(x_1, \ldots, x_n)$ which is a product of some factors $\sigma(x_i, x_j)$.

As a special case of Lemma 3.3, for each $T > 0$, $n \in \mathbf{N}$, $\{t_h\}_{h=1}^n \subset [0, T]$ and $b_1, \ldots, b_n$, we can obtain the ordered form of the product

$$j_{t_1}(b_1) \cdots j_{t_n}(b_n). \tag{3.9}$$

COROLLARY 3.4. *In the notations of Lemma 3.3, for each $T > 0$, $n \in \in \mathbf{N}$, $t := \{t_h\}_{h=1}^n \subset [0, T]$ and $b_1, \ldots, b_n$, the ordered form of the product* (3.9) *is equal to*

$$\sum_{m=0}^{[n/2]} \sum_{1 \leqq p_1 < \ldots < p_m \leqq n} \sum_{(q_1, \ldots, q_m)}^{*} \prod_{h=1}^{m} \varepsilon(t_{p_h}, t_{q_h}, b_{p_h}, b_{q_h}) \tag{3.10}$$
$$\sigma(t_1, \ldots, t_n, b_1, \ldots, b_n) \cdot j_{t_{\pi^t(r_1)}}(b_{\pi^t(r_1)}) \cdots j_{t_{\pi^t(r_{n-2m})}}(b_{\pi^t(r_{n-2m})}).$$

Moreover we have the following

COROLLARY 3.5. *In the notations of Lemma 3.3, for each $T > 0$, $n \in \in \mathbf{N}$, and $b_1, \ldots, b_n$, the product*

$$S_T(b_1) \cdots S_T(b_n) \tag{3.11a}$$

is equal to

$$\sum_{p=1}^{n} \sum_{(S_1, \ldots, S_p) \in \mathcal{P}_{2n,p}} \int_{t \in [S_1, \ldots, S_p]_T} \sum_{m=0}^{[n/2]} \sum_{1 \leqq p_1 < \ldots < p_m \leqq n} \sum_{(q_1, \ldots, q_m)}^{*} \tag{3.11b}$$

$$\prod_{h=1}^{m} \varepsilon(t_{p_h}, t_{q_h}, b_{p_h}, b_{q_h}) \sigma(t_1, \dots, t_n, b_1, \dots, b_n) \cdot j_{t_{\pi^t(r_1)}}(b_{\pi^t(r_1)}) \cdots$$
$$\cdots j_{t_{\pi^t(r_{n-2m})}}(b_{\pi^t(r_{n-2m})}) \nu(t_1) \cdots \nu(t_n).$$

PROOF. Corollary 3.5 follows immediately from Lemma 3.1 and Corollary 3.4.

REMARK. Notice that since $(S_1, \dots, S_p)$ is an ordered partition, one has that *inside* each S_h, $h = 1, \dots, p$, π^t will keep the order, i.e. if $l_h < k_h \in S_h$, $h \in \{1, \dots, p\}$ and $l_h < k_h$,then $\pi^t(l_h) < \pi^t(k_h)$ and $t_{\pi^t(l_h)} = t_{\pi^t(k_h)}$. In other terms, π^t acts on the blocks S_j, keeping the order inside each block.

Now let us consider Corollary 3.5 from another point of view: for each $p = 1, \dots, n$ and any n-permutation π, let $[S_1, \dots, S_p]_T^\pi$ be the subset of $t \in [S_1, \dots, S_p]_T$ such that $\pi^t = \pi$ with π^t defined by (3.4), i.e. $t_{\pi(1)} \leqq \leqq \dots \leqq t_{\pi(n)}$. Clearly we have

(3.12)
$$S_T(b_1) \dots S_T(b_n) = \sum_{p=1}^{n} \sum_{(S_1,\dots,S_p) \in \mathcal{P}_{n,p}} \sum_{\pi \in \mathcal{S}_n} \sum_{t \in [S_1,\dots,S_p]_T^\pi} j_{t_1}(b_1) \dots j_{t_n}(b_n).$$

For each $t \in [S_1, \dots, S_p]_T^\pi$, the product $j_{t_1}(b_1) \dots j_{t_n}(b_n)$ is not ordered (unless π is the identity) and the permutation which makes the product $j_{t_1}(b_1) \cdots j_{t_n}(b_n)$ ordered is π. Therefore we can write the product as

$$j_{t_{\pi^{-1}(1)}}(b_1) \cdots j_{t_{\pi^{-1}(n)}}(b_n) \tag{3.13}$$

with $t \in I_T(S_1, \dots, S_p)$.

In the following for each fixed partition $(S_1, \dots, S_p) \in \mathcal{P}_{n,p}$, we shall use the notation

$$\mathcal{S}_n^{(S,p)} := \{\pi \in \mathcal{S}_n : [S_1, \dots, S_p]_T^\pi \text{ non-empty}\}. \tag{3.14}$$

That is, $\mathcal{S}_n^{(S,p)}$ consists of all permutations on $\{1, \dots, n\}$ which permute among themselves the blocks S_j, considered as individual objects. Thus $\mathcal{S}_n^{(S,p)}$ is isomorphic to $\mathcal{S}_p$.

Applying Corollary 3.4 to the product (3.13) we find the following result.

LEMMA 3.6. *In the notations of Lemma* 3.3, *for each* $T > 0$, $n \in \mathrm{N}$ *and* $b_1, \dots, b_n$, *the product* (3.11a) *is equal to*

$$\sum_{p=1}^{n} \sum_{(S_1,\dots,S_p) \in \mathcal{P}_{n,p}} \sum_{\pi \in \mathcal{S}_n^{(S,p)}} \int_{t \in I_T(S_1,\dots,S_p)} \sum_{m=0}^{[n/2]} \sum_{1 \leqq p_1 < \dots < p_m \leqq n} \sum_{(q_1,\dots,q_m)}^{*} \tag{3.15}$$
$$\prod_{h=1}^{m} \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h})$$
$$\sigma(t_1, \dots, t_n, b_1, \dots, b_n) \cdot j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{n-2m}}}(b_{\pi(r_{n-2m})}) \nu(t_1) \cdots \nu(t_n).$$

REMARK. Notice that (3.11b) and (3.15) are two different ways to write the product (3.11a).

PROOF. From the definition of $[S_1,\dots,S_p]_T^\pi$ and the identity

$$\int_{t\in[S_1,\dots,S_p]_T^\pi} j_{t_1}(b_1)\cdots j_{t_n}(b_n)\nu(t_1)\cdots\nu(t_n) = \tag{3.16}$$
$$= \sum_{t\in I_T(S_1,\dots,S_p)} j_{t_{\pi^{-1}(1)}}(b_1)\cdots j_{t_{\pi^{-1}(n)}}(b_n)\nu(t_1)\cdots\nu(t_n)$$

immediately follows.

For each fixed partition $(S_1,\dots,S_p)\in\mathcal{P}_{n,p}$, the sum $\sum^*_{(q_1,\dots,q_m)}$ is the same as in Lemma (3.2), the only difference being that (3.8d) is replaced by

$$t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)},\quad h=1,\dots,m. \tag{3.17}$$

It will be useful, in the following, to perform the summation first in the m, p_h, q_h indices and then in the π, t indices. This goal is achieved in the following lemma

LEMMA 3.7. *For each $T>0$, $n\in\mathbf{N}$ and $b_1,\dots,b_n$, the product* (3.11a) *is equal to*

(3.18)
$$\sum_{p=1}^{n}\sum_{m=0}^{[n/2]}\sum_{1\leqq p_1<\dots<p_m\leqq n}\sum'_{(q_1,\dots,q_m)}\sum_{(S_1,\dots,S_p)\in\mathcal{P}_{n,p}}\sum_{\pi\in\mathcal{S}_n^{(S,p)}}\int_{t\in I_T(S_1,\dots,S_p,\{p_h,q_h\}_{h=1}^m,\pi)}$$
$$\prod_{h=1}^{m}\varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})\cdot\sigma(t_1,\dots,t_n,b_1,\dots,b_n)$$
$$j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{n-2m}}}(b_{\pi(r_{n-2m})})\nu(t_1)\cdots\nu(t_n)$$

where and in the following, $\sum'_{(q_1,\dots,q_m)}$ means summation over all $1\leqq$ $\leqq q_1,\dots,q_m\leqq n$ satisfying the conditions (3.8a), (3.8b) *and* (3.8c) *(but without the condition* (3.8d)*), and*

$$I_T(S_1,\dots,S_p,\{p_h,q_h\}_{h=1}^m,\pi) := \tag{3.19}$$
$$:= \left\{t\in I_T(S_1,\dots,S_p):\ \ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)},\ \ h=1,\dots,m\right\}.$$

PROOF. The Lemma is proved with the following procedure: First we choose m and $\{p_h,q_h\}_{h=1}^m$ as in Lemma (3.6) but without the condition (3.17). Second, for each fixed m and $\{p_h,q_h\}_{h=1}^m$ and $\pi\in\mathcal{S}_n^{(S,p)}$, we define $I_T(S_1,\dots,S_p,\{p_h,q_h\}_{h=1}^m,\pi)$ as (3.19).

§4. The negligible terms

In analogy with weak coupling and low density limit, the following lemma corresponds to proving that, in these limits, the so called "type II terms" tend to zero. Recall that, in this analogy, the index k (resp. t) is interpreted as time, the ε-factor as scalar product, and the type II terms are those products of ε-factors which contain at least one factor of the form $\varepsilon(h,k)$ with $k-h \geqq 2$.

LEMMA 4.2. *Suppose that* $\varepsilon_1, \varepsilon_2 : \mathbf{N}^2 \to C$ *are in* $s - L^1(C, dn)$, *then*

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{1 \leqq k_1 < k_2 < k_3 < k_4 \leqq N} |\varepsilon_1(k_4,k_2)| \cdot |\varepsilon_2(k_3,k_1)| = 0 \tag{4.1a}$$

and

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{1 \leqq k_1 < k_2 < k_3 < k_4 \leqq N} |\varepsilon_1(k_4,k_1)\varepsilon_2(k_3,k_2)| = 0. \tag{4.1b}$$

PROOF. Since $|\cdot|$ is a semi-norm on C, we may suppose that the $\varepsilon_j(h,k)$ are positive numbers. Thus ε_i are $s - L^1(\mathbf{R}_+, dn)$, $i = 1,2$ and, because of (1.11a), there exists a finite constant M_i such that

$$\max\left\{ \sum_{k=h+1}^{\infty} \varepsilon_i(k,h),\ \sum_{k=0}^{h} \varepsilon_i(h,k),\ \ h \in \mathbf{N} \right\} \leqq M_i < +\infty. \tag{4.2}$$

Then, because of (4.2), for each $\eta > 0$, there exists a $K \in \mathbf{N}$ such that for any $h_1, h_2 \in \mathbf{N}$

$$\sum_{k=K+h_1}^{\infty} \varepsilon_1(k,h_1) + \sum_{k=K+h_2}^{\infty} \varepsilon_2(k,h_2) < \eta. \tag{4.3}$$

We rewrite

$$\frac{1}{N^2} \sum_{1 \leqq k_1 < k_2 < k_3 < k_4 \leqq N} \varepsilon_1(k_4,k_2)\varepsilon_2(k_3,k_1)$$

as

(4.4)

$$\frac{1}{N^2} \sum_{k_1=1}^{N-3} \sum_{k_2=k_1+1}^{N-2} \left(\sum_{k_3=k_2+1}^{k_1+K} + \sum_{k_3=k_1+K+1}^{N-1} \right) \sum_{k_4=k_3+1}^{N} \varepsilon_1(k_4,k_2)\varepsilon_2(k_3,k_1).$$

Then the first term of (4.4) becomes

$$\text{(4.4a)}\quad \frac{1}{N^2}\sum_{k_1=1}^{N-3}\sum_{k_2=k_1+1}^{N-2}\sum_{k_3=k_2+1}^{k_1+K}\varepsilon_2(k_3,k_1)\sum_{k_4=k_3+1}^{N}\varepsilon_1(k_4,k_2)\leqq$$
$$\leqq\frac{1}{N^2}\sum_{k_1=1}^{N-3}\sum_{k_2=k_1+1}^{N-2}\sum_{k_3=k_2+1}^{k_1+K}\varepsilon_2(k_3,k_1)\sum_{k_4=k_2+1}^{\infty}\varepsilon_1(k_4,k_2)\leqq$$
$$\leqq\frac{1}{N^2}\sum_{k_1=1}^{N-3}\sum_{k_2=k_1+1}^{N-2}\sum_{k_3=k_2+1}^{k_1+K}\varepsilon_2(k_3,k_1)\cdot M_1.$$

Notice that on the right hand side of (4.4a), $k_1 < k_3 \leqq k_1 + K$, hence one has $k_1 < k_2 < k_1 + K$. This implies that the right hand side of (4.4a) is less than or equal to

$$\text{(4.4b)}\quad \frac{1}{N^2}\sum_{k_1=1}^{N-3}\sum_{k_2=k_1+1}^{k_1+K-1}\sum_{k_3=k_2+1}^{k_1+K}\varepsilon_2(k_3,k_1)\cdot M_1\leqq$$
$$\leqq\frac{1}{N^2}\sum_{k_1=1}^{N-3}\sum_{k_2=k_1+1}^{k_1+K-1}M_2\cdot M_1=\frac{1}{N^2}\sum_{k_1=1}^{N-3}K\cdot M_2\cdot M_1$$

and this tends to zero as $N\to\infty$. By (4.3), the second term of (4.4) is majorized by

$$\text{(4.4c)}\quad \eta\cdot M_1\cdot\frac{(N-3)^2}{N^2}$$

therefore

$$\text{(4.5)}\quad \limsup_{N\to\infty}\frac{1}{N^2}\sum_{1\leqq k_1<k_2<k_3<k_4\leqq N}\varepsilon_1(k_4,k_2)\varepsilon_2(k_3,k_1)\leqq\eta\cdot M_1$$

and since $\eta > 0$ is arbitrary, this proves (4.1a).

In order to prove (4.1b) we rewrite

$$\frac{1}{N^2}\sum_{1\leqq k_1<k_2<k_3<k_4\leqq N}\varepsilon_1(k_4,k_1)\varepsilon_2(k_3,k_2)$$

as

$$\text{(4.6)}\quad \frac{1}{N^2}\sum_{k_1=1}^{N-3}\sum_{k_2=k_1+1}^{N-2}\sum_{k_3=k_2+1}^{N-1}\left(\sum_{k_4=k_3+1}^{k_1+K}+\sum_{k_4=k_1+K+1}^{N}\right)\varepsilon_1(k_4,k_1)\varepsilon_2(k_3,k_2).$$

By the same arguments as above, (4.6) is dominated by

$$\eta \cdot M_2 + M_1 \cdot M_2 \frac{K}{N^2}(N-3). \tag{4.6a}$$

Hence,

$$\limsup_{N\to\infty} \frac{1}{N^2} \sum_{1\leqq k_1<k_2<k_3<k_4\leqq N} \varepsilon_1(k_4,k_1)\varepsilon_2(k_3,k_2) \leqq \eta \cdot M_2 \tag{4.7}$$

and this implies (4.2a) by the arbitrariness of $\eta > 0$.

LEMMA 4.3. *Suppose that* $\varepsilon_1, \varepsilon_2 : \mathrm{N}^2 \to \mathcal{C}$ *are in* $s - L^1(\mathcal{C}, dn)$, *then*

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{\substack{1\leqq h_1<k_1, h_2<k_2\leqq N \\ |\{h_j,k_j\}_{j=1}^2|<4}} \left|\varepsilon_1(k_1,h_1)\right| \cdot \left|\varepsilon_2(k_2,h_2)\right| = 0. \tag{4.8}$$

REMARK. The condition $\left|\{h_j,k_j\}_{j=1}^2\right| < 4$ means that in the sum $\sum_{1\leqq h_1<k_1,h_2<k_2\leqq N}$ some of the indices h_j, k_j are equal.

PROOF. Let us denote

$$\sum_2 := \frac{1}{N^2} \sum_{\substack{1\leqq h_1<k_1, h_2<k_2\leqq N \\ |\{h_j,k_j\}_{j=1}^2|<4}} \left|\varepsilon_1(k_1,h_1)\right| \cdot \left|\varepsilon_2(k_2,h_2)\right|$$

and discuss separately all the possibilities according to which indices are equal.

i) If $h_1 = h_2$ then

$$\sum_2 = \frac{1}{N^2} \sum_{h_1=1}^{N} \sum_{k_1=h_1+1}^{N} \left|\varepsilon_1(k_1,h_1)\right| \sum_{k_2=h_1+1}^{N} \left|\varepsilon_2(k_2,h_1)\right|. \tag{4.9}$$

Since $\varepsilon_1, \varepsilon_2$ are in $s - L^1(\mathcal{C}, dn)$ one has, in the notation (4.3b),

$$\sum_2 \leqq M_1 \cdot M_2 \cdot \frac{1}{N^2} \sum_{h_1=1}^{N} 1 = \frac{M_1 \cdot M_2}{N} \longrightarrow 0. \tag{4.10}$$

ii) If $h_1 = k_2$ then

$$\sum_2 = \frac{1}{N^2} \sum_{h_1=1}^{N} \sum_{k_1=h_1+1}^{N} \left|\varepsilon_1(k_1,h_1)\right| \sum_{h_2=1}^{h_1-1} \left|\varepsilon_2(h_1,h_2)\right|. \tag{4.11}$$

Changing the order of summation on the right hand side of (4.11), we find that

$$(4.12)\quad \sum_2 = \frac{1}{N^2}\sum_{h_2=1}^{N}\sum_{h_1=h_2+1}^{N}\sum_{k_1=h_1+1}^{N}\left|\varepsilon_1(k_1,h_1)\right|\cdot\left|\varepsilon_2(h_1,h_2)\right| \leqq$$

$$\leqq M_1\cdot\frac{1}{N^2}\sum_{h_2=1}^{N}\sum_{h_1=h_2+1}^{N}\left|\varepsilon_2(h_1,h_2)\right| \leqq M_1M_2\cdot\frac{1}{N}\to 0.$$

The cases $k_1 = h_2$ and $k_1 = k_2$ follow from the same arguments.

Now we prove the generalization of Lemma 4.2 to the case of a product of n ε-factors. These products are analogues of the type II terms of [3].

LEMMA 4.3. *For each $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n : \mathrm{N}^2 \to \mathcal{C}$ which are in $s-L^1(\mathcal{C}, dn)$, we have*

$$(4.13)\quad \lim_{N\to\infty}\frac{1}{N^n}\sum_{\substack{1\leqq h_1<k_1,\dots,h_n<k_n\leqq N\\ |\{h_j,k_j\}_{j=1}^n|<2n}}\prod_{j=1}^{n}\left|\varepsilon_j(k_j,h_j)\right| = 0.$$

REMARK. *The condition $\left|\{h_j,k_j\}_{j=1}^n\right| < 2n$ means that in the sum $\sum_{1\leqq h_1<k_1,\dots,h_n<k_n\leqq N}$ some of the indices h_j, k_j are equal.*

PROOF. Let us first consider the case in which h_1, k_1 are free indices, i.e.

$$(4.14)\quad \{h_1,k_1\}\cap\{h_j,k_j\}_{j=2}^n = \emptyset.$$

Then, since $\varepsilon_1(k_1, h_1)$ is in $s - L^1(\mathcal{C}, dn)$, it follows that

$$(4.15)\quad \sum_n := \frac{1}{N^n}\sum_{\substack{1\leqq h_1<k_1,\dots,h_n<k_n\leqq N\\ |\{h_j,k_j\}_{j=1}^n|<2n}}\prod_{j=1}^{n}\left|\varepsilon_j(h_j,k_j)\right| =$$

$$= \frac{1}{N}\sum_{h_1=1}^{N}\sum_{k_1=h_1+1}^{N}\varepsilon(k_1,h_1)\sum_{\substack{1\leqq h_2<k_2,\dots,h_n<k_n\leqq N\\ |\{h_j,k_j\}_{j=2}^n|<2n-2}}\prod_{j=2}^{n}\left|\varepsilon_j(h_j,k_j)\right| \leqq$$

$$\leqq M_1\cdot\frac{1}{N^{n-1}}\sum_{\substack{1\leqq h_2<k_2,\dots,h_n<k_n\leqq N\\ |\{h_j,k_j\}_{j=2}^n|<2n-2}}\prod_{j=2}^{n}\left|\varepsilon_j(h_j,k_j)\right|.$$

Thus if (4.14) is true then (4.15) and the induction gives our proof. Therefore we may assume that (4.14) is *not true*, i.e. that there exists a $j = 2, \dots, n$ such that

$$\{h_1, k_1\} \cap \{h_j, k_j\} \neq \emptyset. \tag{4.16}$$

In any case, because of (4.1),

$$\frac{1}{N} \sum_{1 \leqq h_m < k_m \leqq N} |\varepsilon_m(k_m, h_m)| \leqq M_m \tag{4.17}$$

therefore

$$\sum_n \leqq C_1 \frac{1}{N^2} \sum_{\substack{1 \leqq h_1 < k_1, h_j < k_j \leqq N \\ |\{h_1, k_1, h_j, k_j\}| < 4}} \left|\varepsilon_1(h_1, k_1)\right| \left|\varepsilon_j(h_j, k_j)\right| \tag{4.18}$$

for some constant C_1 and the statement follows from Lemma 4.2.

§5. The non-negligible terms

LEMMA 5.1. *Suppose that* $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n : \mathbf{N}^2 \to C$ *are* $S - L^1(C, dn)$ *and that* $F_1, \dots, F_m$: $\mathbf{N} \to \mathbf{C}$ *are such that the limits*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} F_j(k) := F_j \quad , \quad j = 1, 2, \dots, m, \tag{5.1}$$

exist. Then, for each $\{i_1, \dots, i_m\} \subset \mathbf{N}$, *we have*

(5.2)

$$\lim_{N \to \infty} \frac{1}{N^{n+m}} \sum_{\substack{1 \leqq k_1 < k_1' < \dots < k_{i_1} < k_{i_1}' < r_1 < k_{i_1+1} < k_{i_1+1}' < \dots < \\ < k_{i_2} < k_{i_2}' < r_2 < \dots < r_m < \dots < k_n < k_n' \leqq N}} \prod_{i=1}^{n} \varepsilon_i(k_i', k_i) \cdot \prod_{j=1}^{m} F_j(r_j) =$$

$$= \frac{1}{(n+m)!} \prod_{j=1}^{m} F_j \cdot \prod_{i=1}^{n} f_i$$

where

$$f_i := \sum_{k=h+1}^{\infty} \varepsilon_i(k, h), \qquad i = 1, 2, \dots, n. \tag{5.3}$$

PROOF. For $n+m=1$, (5.2) is clearly true. Suppose that (5.2) is true for $m+n \leqq q$. We have, for $m+n=q+1$

(5.4)

$$\frac{1}{N^{q+1}} \sum_{1 \leqq k_1 < k_1' < \ldots < k_{i_1} < k_{i_1}' < r_1 < \ldots < r_m < \ldots < k_n < k_n' \leqq N} \prod_{i=1}^{n} \varepsilon_i(k_i', k_i) \cdot \prod_{j=1}^{m} F_j(r_j) =$$

$$= \frac{1}{N^{m+n}} \sum_{k_n'=2n+m}^{N} \sum_{k_n=2n+m-1}^{k_n'-1}$$

$$\sum_{\substack{1 \leqq k_1 < k_1' < \ldots < k_{i_1} < k_{i_1}' < r_1 < \ldots < \\ < r_m < \ldots < k_{n-1} < k_{n-1}' \leqq k_n - 1}} \prod_{i=1}^{n} \varepsilon_i(k_i', k_i) \cdot \prod_{j=1}^{m} F_j(r_j) =$$

$$= \frac{1}{N^{n+m}} \sum_{k_n'=2n+m}^{N} \sum_{k_n=2n+m-1}^{k_n'-1} \varepsilon_n(k_n', k_n)(k_n - 1)^{n+m-1} \cdot$$

$$\cdot \frac{1}{(k_n-1)^{n+m-1}} \sum_{\substack{1 \leqq k_1 < k_1' < \ldots < k_{i_1} < k_{i_1}' < r_1 < \ldots < \\ < r_m < \ldots < k_{n-1} < k_{n-1}' \leqq k_n - 1}} \prod_{i=1}^{n-1} \varepsilon_i(k_i', k_i) \cdot \prod_{j=1}^{m} F_j(r_j) =$$

$$= \frac{1}{N^{n+m}} \sum_{k_n'=2n+m}^{N} \left(\sum_{k_n=2n+m-1}^{K+1} + \sum_{k_n=K+2}^{k_n'-1} \right) \varepsilon_n(k_n', k_n)(k_n - 1)^{n+m-1} \cdot$$

$$\cdot \frac{1}{(k_n-1)^{n+m-1}} \sum_{\substack{1 \leqq k_1 < k_1' < \ldots < k_{i_1} < k_{i_1}' < r_1 < \ldots < \\ < r_m < \ldots < k_{n-1} < k_{n-1}' \leqq k_n - 1}} \prod_{i=1}^{n-1} \varepsilon_i(k_i', k_i) \cdot \prod_{j=1}^{m} F_j(r_j).$$

For each $\eta > 0$, we take K such that

$$\left| \frac{1}{K^{n+m-1}} \sum_{1 \leqq k_1 < k_1' < \ldots < k_{n-1} < k_{n-1}' \leqq K} \prod_{i=1}^{n-1} \varepsilon_i(k_i', k_i) \cdot \prod_{j=1}^{m} F_j(r_j) \right. \tag{5.5}$$

$$\left. - \frac{1}{(n+m-1)!} \prod_{j=1}^{m} F_j \cdot \prod_{i=1}^{n-1} f_i \right| \leqq \eta$$

and

$$\left| \sum_{k=K+h+1}^{\infty} \varepsilon_i(k, h) \right| \leqq \eta. \tag{5.5a}$$

First of all let us see the absolute value of the first term on the right hand side of (5.4), i.e.

$$(5.6)\qquad \left|\frac{1}{N^{n+m}}\sum_{k'_n=2n+m}^{N}\ \sum_{k_n=2n+m-1}^{(K+1)\wedge(k'_n-1)}\varepsilon_n(k'_n,k_n)\cdot(k_n-1)^{n+m-1}\right.$$
$$\left.\frac{1}{(k_n-1)^{n+m-1}}\cdot\sum_{\substack{1\leqq k_1<k'_1<\ldots<k_{i_1}<k'_{i_1}<r_1<\ldots<\\ <r_m<\ldots<k_{n-1}<k'_{n-1}\leqq k_n-1}}\ \prod_{i=1}^{n-1}\varepsilon_i(k'_i,k_i)\cdot\prod_{j=1}^{m}F_j(r_j)\right|.$$

By the assumption of the induction, (5.6) is dominated by, with a constant M,

$$(5.7)\qquad M^{m+n-1}\frac{1}{N}\sum_{k'_n=2n+m}^{N}\left|\sum_{k_n=2n+m-1}^{(K+1)\wedge(k'_n-1)}\varepsilon_n(k'_n,k_n)\right|=$$
$$=M^{m+n-1}\frac{1}{N}\left(\sum_{k'_n=2n+m}^{K+2}+\sum_{k'_n=K+2}^{N}\right)\left|\sum_{k_n=2n+m-1}^{(K+1)\wedge(k'_n-1)}\varepsilon_n(k'_n,k_n)\right|.$$

The first term on the right hand side of (5.7) is equal to

$$(5.8)\qquad O(1)\frac{K+2}{N}\longrightarrow 0,\quad \text{as}\quad N\to\infty.$$

The second term on the right hand side of (5.7) is equal to

$$(5.9)\qquad M^{m+n-1}\frac{1}{N}\sum_{k'_n=K+2}^{N}\left|\sum_{k_n=2n+m-1}^{(K+1)}\varepsilon_n(k'_n,k_n)\right|\leqq$$
$$\leqq M^{m+n-1}\frac{1}{N}\sum_{k_n=2n+m-1}^{(K+1)}\ \sum_{k'_n=K+2}^{N}|\varepsilon_n(k'_n,k_n)|=$$
$$=O(1)\frac{K+1}{N}\longrightarrow 0,\quad \text{as}\quad N\to\infty.$$

Let us now consider the second term of (5.4) and rewrite it as

$$(5.10)\qquad \frac{1}{N^{n+m}}\sum_{k'_n=2n+m}^{N}\ \sum_{k_n=K+2}^{k'_n-1}\varepsilon_n(k'_n,k_n)(k_n-1)^{n+m-1}.$$

$$\cdot \frac{1}{(n+m-1)!} \prod_{j=1}^{m} F_j \cdot \prod_{i=1}^{n-1} f_i +$$

$$+ \frac{1}{N^{n+m}} \sum_{k'_n=2n+m}^{N} \sum_{k_n=K+2}^{k'_n-1} \varepsilon_n(k'_n, k_n)(k_n-1)^{n+m-1}$$

$$\left(\frac{1}{(n+m-1)!} \prod_{j=1}^{m} F_j \cdot \prod_{i=1}^{n-1} f_i - \right.$$

$$\left. - \frac{1}{(k_n-1)^{n+m-1}} \sum_{\substack{1 \leqq k_1 < k'_1 < \ldots < k_{i_1} < k'_{i_1} < r_1 < \ldots < \\ < r_m < \ldots < k_{n-1} < k'_{n-1} \leqq k_n - 1}} \prod_{i=1}^{n-1} \varepsilon_i(k'_i, k_i) \cdot \prod_{j=1}^{m} F_j(r_j) \right).$$

By (5.4) one knows that the absolute value of the second term of (5.10) is less than or equal to

$$(5.11) \quad \eta \cdot \frac{1}{N^{n+m}} \sum_{k'_n=2n+m}^{N} \sum_{k_n=K+2}^{k'_n-1} |\varepsilon_n(k'_n, k_n)|(k_n-1)^{n+m-1} = \eta \cdot O(1).$$

Moreover, since n, m, K are fixed so that the limit, as $N \to \infty$, of the first term of (5.10) is equal to the limit of the following quantity:
(5.12)

$$\frac{1}{(n+m-1)!} \prod_{j=1}^{m} F_j \cdot \prod_{i=1}^{n-1} f_i \cdot \frac{1}{N^{n+m}} \sum_{k'_n=2}^{N} \sum_{k_n=1}^{k'_n-1} \varepsilon_n(k'_n, k_n)(k_n-1)^{n+m-1}.$$

Exchanging the order of summations in (5.12), it becomes
(5.12a)

$$\frac{1}{(n+m-1)!} \prod_{j=1}^{m} F_j \cdot \prod_{i=1}^{n-1} f_i \cdot \frac{1}{N^{n+m}} \sum_{k_n=1}^{N-1} (k_n-1)^{n+m-1} \sum_{k'_n=k_n+1}^{N} \varepsilon_n(k'_n, k_n).$$

Letting N tend to infinity, we obtain the limit of (5.12a):

$$(5.13) \qquad \frac{1}{(n+m)!} \prod_{j=1}^{m} F_j \cdot \prod_{i=1}^{n} f_i$$

and this ends the proof.

More generally we have the following

LEMMA 5.2. *With the same notations and assumptions as in Lemma 5.1,*

$$\lim_{N\to\infty} \frac{1}{N^{n+m}} \sum_{\substack{1\leqq k_1<k_1'<\ldots<k_{i_1}<k_{i_1}'<r_1<k_{i_1+1}<k_{i_1+1}'<\ldots< \\ <k_{i_2}<k_{i_2}'<r_2<\ldots<r_m<\ldots<k_n<k_n'\leqq N \\ k_{j_1}'\leqq k_{j_1}+d_N,\ldots,k_{j_p}'\leqq k_{j_p}+d_N}} \varepsilon_i(k_i',k_i)\cdot\prod_{j=1}^m F_j(r_j) = \frac{1}{(n+m)!}\prod_{j=1}^m F_j\cdot\prod_{i=1}^n f_i \tag{5.14}$$

where $d_N\to\infty$ and $N-d_N\to\infty$ as $N\to\infty$.

PROOF. Notice that the only difference is that on the left hand side of (5.14), $k_{j_h}'-k_{j_h}\leqq d_N$ for each $h=1,\ldots,p$ but on the left hand side of (5.2), $k_{j_h}'-k_{j_h}$ can be greater than d_N ($\leqq N-1$) for some $h\in\{1,\ldots,p\}$. Since the series $\sum_{k=h}^{\infty}\varepsilon(h,k)$ converges and $d_N\to\infty$, we know that

$$\lim_{N\to\infty}\sum_{k=h}^{N-1}\varepsilon(h,k)=\lim_{N\to\infty}\sum_{k=h}^{d_N}\varepsilon(h,k). \tag{5.15}$$

This ends the proof.

In the continuous case, the analogue of Lemmata 5.1 and 5.2 are the following

LEMMA 5.3. *Let $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n$: $\mathbf{N}^2\to\mathcal{C}$ be in $S-L^1(\mathcal{C},dt)$ and $F_1,\ldots,F_m$: $\mathbf{R}_+\to\mathbf{C}$ such that the limits*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T F_j(t)dt=F_j \quad , \quad j=1,2,\ldots,m \tag{5.16}$$

exist, then for each $\{i_1,\ldots,i_m\}\subset\mathbf{N}$ we have

$$\lim_{T\to\infty}\frac{1}{T^{n+m}} \int_{\substack{0\leqq t_1<t_1'<\ldots<t_{i_1}<t_{i_1}'<s_1<t_{i_1+1}<t_{i_1+1}'<\ldots< \\ t_{i_2}<t_{i_2}'<s_2<\ldots<s_m<\ldots<t_n<t_n'\leqq T}} \prod_{i=1}^n\varepsilon_i(t_i',t_i)\cdot\prod_{j=1}^m F_j(s_j)dt_1\ldots dt_n\ \ dt_1'\ldots dt_n'\ \ ds_1\ldots ds_p = \frac{1}{(n+m)!}\cdot\prod_{j=1}^m F_j\cdot\prod_{i=1}^n f_i \tag{5.17}$$

where

$$f_i := \int\limits_{[s,\infty)} \varepsilon_i(t,s)\nu(dt), \quad i = 1,\dots,n. \tag{5.18}$$

LEMMA 5.4. *Let* $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n : \mathbf{N}^2 \to C$ *be in* $S - L^1(C, dt)$ *and let* $F_1, \dots, F_m \colon \mathbf{R}_+ \to \mathbf{C}$ *be such that the limits*

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T F_j(t)dt = F_j \quad , \quad j = 1,2,\dots,m \tag{5.19}$$

exist. Then for each $\{i_1,\dots,i_m\} \subset \mathbf{N}$ *and* $\{j_1,\dots,j_p\} \subset \{0,1,\dots,n\}$, *we have*

$$\lim_{T\to\infty} \frac{1}{T^{n+m}} \int\limits_{\substack{0 \leqq t_1 < t'_1 < \dots < t_{i_1} < t'_{i_1} < s_1 < t_{i_1+1} < t'_{i_1+1} < \dots < \\ t_{i_2} < t'_{i_2} < s_2 < \dots < s_m < \dots < t_n < t'_n \leqq T \\ t'_{j_h} \leqq t_{j_h} + d_T,\ h=1,\dots,p}} \tag{5.20}$$

$$\prod_{i=1}^{n} \varepsilon_i(t'_i, t_i) \cdot \prod_{j=1}^{m} F_j(s_j) dt_1 \dots dt_n \quad dt'_1 \dots dt'_n \quad ds_1 \dots ds_p =$$

$$= \frac{1}{(n+m)!} \cdot \prod_{j=1}^{m} F_j \cdot \prod_{i=1}^{n} f_i$$

where $d_T \to \infty$, *and* $T - d_T \to \infty$ *as* $T \to \infty$.

The proofs of these two lemmata are the same as those of Lemmata 5.1 and 5.2.

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