THE WEAK COUPLING LIMIT FOR NONLINEAR INTERACTIONS

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ABSTRACT

We study the weak coupling limit of a quantum "Resevoir+System" model coupled to a Boson field with a polynomial interaction. We show that some new quantum central limit phenomena arise which have no counterpart for linear interactions. In the case of a quadratic interaction we prove a uniform estimate which allows to establish the term by term convergence of the iterated series. The limiting evolution is also derived and it is shown that it is a non unitary deterministic evolution. We relate this fact to the choice of the collective coherent vectors and we interpret it as an indication that in the weak coupling limit for nonlinear interactions the collective vectors, with respect to which one considers the matrix elements of the wave operator, should be choosen in a interaction dependent way. This indication has been confirmed by subsequent work.

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§1. Introduction.

This paper extends the results of [1], [2], ..., [7], where, we have studied the weak coupling limit for the interaction

$$V_1 := -\frac{1}{i}(D \otimes A^+(g) - D^+ \otimes A(g)) \tag{1.1}$$

In the present paper we shall investigate the same problem for interacions of the more general form

$$V_d := -\frac{1}{i}(D \otimes ((A^+(g))^d - D^+ \otimes (A(g))^d) \tag{1.2}$$

where d is an integer ≥ 2.

In the present paper, we shall follow the notations and the basic assumptions of [1] ,[2], ... , [7], which we sum up briefly in the following for commodity of the the reader.

We consider a quantum "System +Resevoir" model. Let H_0 be the system Hilbert space; H_1 the one particle resevoir Hilbert space; H_S the system Hamiltonian; $H_R = d\Gamma(-\Delta)$, where Δ is a self-adjoint operator, the resevoir Hamiltonian and $S_t^0 := e^{-it\Delta}$. The total space is

$$H_0 \otimes \Gamma(H_1)$$

where $\Gamma(H_1)$ is the Fock space over H_1 and the Hamiltonian of the composite system is

$$H^{(\lambda)} = H_S \otimes 1 + 1 \otimes H_R + \lambda V_d \tag{1.3}$$

Now let $H^{(o)} = H^{(\lambda)} - \lambda V_d = H_S \otimes 1 + 1 \otimes H_R$, then the wave operator at time t

$$U^{(\lambda)}(t) := e^{-itH^{(o)}} e^{itH_\lambda} \tag{1.4}$$

satisfies the differential equation

$$\frac{d}{dt}U^{(\lambda)}(t) = \frac{1}{i}V_d(t)U^{(\lambda)}(t) \tag{1.5}$$

where

$$V_d(t) := -\frac{1}{i}(D(t) \otimes (A^+(S_t^0 g))^d - D^+(t) \otimes (A(S_t^0 g))^d) \tag{1.6}$$

We shall assume the rotating wave approximation, i.e.

$$D(t) = \exp(-i\omega t)D \tag{1.7}$$

so that the interaction can be rewritten in the following form

$$V_d(t) := -\frac{1}{i}(D \otimes (A^+(S_t g))^d - D^+ \otimes (A(S_t g))^d) \tag{1.6a}$$

with

$$S_t := \exp(-i\frac{\omega}{d}t)S_t^0 \tag{1.8}$$

We shall deal only with the Fock case. The general quasi-free case can be reduced to the Fock's one with a standard technique (cf. [3]).

The formal solution of equation (1.5) is

$$U^{(\lambda)}(t) = \sum_{n=0}^{\infty}(-i)^n\lambda^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n V_d(t_1)V_d(t_2)\cdots V_d(t_n) \tag{1.9}$$

and for each $n \in \mathbf{N}$, we define

$$\Delta_n^{(\lambda)}(t) := (-i)^n\lambda^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n V_d(t_1)V_d(t_2)\cdots V_d(t_n) \tag{1.10}$$

so that, formally

$$U^{(\lambda)}(t/\lambda^2) = \sum_{n=0}^{\infty} \Delta_n^{(\lambda)}(t/\lambda^2) \tag{1.11}$$

Our goal is to compute the formal limit

$$\lim_{\lambda \to 0} < u \otimes W(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f du)\Phi, U^{(\lambda)}(t/\lambda^2)v \otimes W(\lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f' du)\Phi > \tag{1.12}$$

for $u, v \in H_0$, the initial (system) Hilbert space, $f, f' \in K$, $S, T, S', T' \in \mathbf{R}, t \geq 0$, where, $W(.)$ is Weyl operator and Φ is vacuum. By " formal limit" (1.12), we mean the limit (1.12) with $U^{(\lambda)}(t/\lambda^2)$ replaced by $\Delta_n^{(\lambda)}(t/\lambda^2)$ for each natural integer n, i.e. the limit of the matrix elements of each term of the formal expansion (1.11).

Or main results are:

- Theorem (2.5), where we derive the explicit form of the limit of the matrix elements, with respect to arbitrary collective coherent vectors, of the n-th term of the iterated series (1.11) for an arbitrary degree d of the interaction (1.2).

- Theorem (3.5), where in case of a quadratic interaction and for short time intervals, we prove a uniform estimate for these matrix elements.

- Theorem (4.1), where we prove that, under the conditions of Theorem (3.5), the van Hove rescaled time t wave operator $U^{(\lambda)}(t/\lambda^2)$ converges, in the sense of the matrix elements in the collective coherent vectors, to the semi-group

$$\exp\{-tD^+D \cdot \| g \|_{2,-}^2\}$$

where, by definition,

$$\| g \|_{2,-}^2 := \int_{\infty}^{0} 2 < g, S_u g >^2 du$$

The reason why we get in the limit neither a unitary evolution nor a stochastic differential equation should be looked for in the fact that we have considered matrix elements of (1.11) in the collective coherent vectors

$$W(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f du)\Phi$$

i.e. the same conidered in the case of linear interactions. Motivated by the apparently strange results of the present paper, we have now proved that, by considering **interaction dependent collective vectors** , we obtain, in the limit, a bona fide unitary operator which satisfies a quantum stochastic differential equation (cf. [12] for a detailed discussion).

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§2. The Limit of $\Delta_n^{(\lambda)}(t/\lambda^2)$ as $\lambda \to 0$

In this section we compute, for each $n \in \mathrm{N}$, the limit, as $\lambda \to 0$, of the matrix element

$$< u \otimes W(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f du), \Delta_n^{(\lambda)}(t/\lambda^2) v \otimes W(\lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f' du) >$$

with $\Delta_n^{(\lambda)}(t)$ given by (1.10).

From (1.2) it follows that

$$(-i)^n V_d(t_1)\cdots V_d(t_n) = \sum_{\varepsilon\in\{0,1\}^n} D_{\varepsilon(1)}\cdots D_{\varepsilon(n)} \otimes (A^{\varepsilon(1)}(S_{t_1}g))^d \cdots (A^{\varepsilon(n)}(S_{t_n}g))^d$$

The normally ordered form of this expression will be

$$\sum_{k=0}^{n} \sum_{1\le j_1<\cdots<j_k\le n} D_n(j_1,\cdots,j_k)\otimes$$

$$\otimes \sum_{(n,m,k;\{p_h\}_{h=1}^m,\{q_h\}_{h=1}^m,\{j_h\}_{h=1}^k)} (A^+\cdots)_{r_1}(A\cdots)_{r_2} \prod_{h=1}^{m} < S_{t_{p_h}} g, S_{t_{q_h}} g > \tag{2.0}$$

where, for each $k = 0, 1, \cdots, n$, the symbol

$$\sum_{(n,m,k;\{p_h\}_{h=1}^m,\{q_h\}_{h=1}^m,\{j_h\}_{h=1}^k)} (A^+\cdots)_{r_1}(A\cdots)_{r_2} \prod_{h=1}^{m} < S_{t_{p_h}} g, S_{t_{q_h}} g > \tag{2.0a}$$

denotes the Wick ordered form of

$$(A^{\varepsilon(1)}(S_{t_1}g))^d\cdots(A^{\varepsilon(n)}(S_{t_n}g))^d = (A(S_{t_1}g))^d\cdots(A^+(S_{t_{j_1}}g))^d\cdots(A^+(S_{t_{j_k}}g))^d\cdots(A(S_{t_n}g))^d \tag{2.0b}$$

and $j_1, \cdots, j_k$ is the ordered (increasing) k-tuple of time indices corresponding to the creation operators in the product (2.0b).

We keep the notations of our previous, i.e., the indices p_h (resp. q_h) in (2.0) correspond to those annihilation (resp. creation) operators which have given rise to some scalar products by application of the CCR. Thus, in the linear case one must have $|\{p_h\}_{h=1}^m| = m$, $|\{q_h\}_{h=1}^m| = m$, where $|\cdot|$ denotes the cardinality, i.e. all the p_h (resp. q_h) are different among themself (this is becomes at each time there is either a creator or an annihilator). In the non linear case each creator (resp. annihilator) appears at the power d and therefore several p_h (resp. q_h) could be equal to some other p_k (resp. some other q_k), so that one only has $|\{p_h\}_{h=1}^m| \le m$, $|\{q_h\}_{h=1}^m| \le m$. Of course, since in (2.0b) there are nd operators, one has always

$$r_1 + r_2 + 2m = dn \tag{2.1}$$

where r_1 (resp. r_2) is the number of creation (resp. annihilation) operators and m, the number of scalar products in the term. This fact creates some technical (both combinatorial and analytical) difficulties. Notice however that because of the form (1.6) of the interaction, no index p_h can be equal to some index q_h. In the Wick ordered product, the creation and annihilation operators not used to produce scalar products, will be denoted with the simplifying notations $(A^+\cdots)_{r_1}(A\cdots)_{r_2}$.

In the expression (2.0), let us first of all the product

$$\prod_{h=1}^{m} < S_{t_{p_h}} g, S_{t_{q_h}} g > \tag{2.2}$$

Since some of the indices $\{p_h\}_{h=1}^m$ (resp. $\{q_h\}_{h=1}^m$) might coincide, the expressions (2.2), considered as a function of the variables t_j, is a function of at most $2m$-variables. Denote m' the cardinality of the set $\{q_h\}_{h=1}^m$, i.e. the maximum number of mutually different indices q_j. Then we can choose a subset $\{q'_h\}_{h=1}^{m'} \subset \{q_h\}_{h=1}^m$ such that $q'_1 < \cdots < q'_{m'}$. After that, we choose a subset $\{p'_h\}_{h=1}^{m'} \subset \{p_h\}_{h=1}^m$ with the following rule:

$$\text{if } \ q'_h = q_{h_0}, \ \text{ then } \ p'_h = p_{h_0}, \quad h = 1,\cdots,m', \ h_0 \in \{1,\cdots,m\} \tag{2.2a}$$

This choice implies that, if $p_h < q_h - 1$ for some $h \in \{1,\cdots,m\}$, then $p'_{h'} < q'_{h'} - 1$ for some $h' \in \{1,\cdots,m'\}$. In fact, if $h \in \{1,\cdots,m\}$ is such that $p_h < q_h - 1$ and $h' = 1,\cdots,m'$ is such that $q_h = q'_{h'}$ then, because of (2.2a), $p'_{h'} = p_h$ and therefore, $p'_{h'} = p_h < q_h - 1 = q'_{h'} - 1$. Thus the module of the product (2.2) is majorized by

$$\|g\|^{2(m-m')} \cdot \prod_{h=1}^{m'} | < S_{t_{p'_h}} g, S_{t_{q'_h}} g > | \tag{2.3}$$

Notice that the cardinality of the subset $\{p'_h\}_{h=1}^{m'}$ might be less than m'. Moreover the cardinality of the subset $\{p'_h\}_{h=1}^{m'}$ is strictly less than m' only if $p_h < q_h - 1$ for some $h = 1,\cdots,m$ (otherwise because of (2.2a) $p'_h = q'_h - 1$ for each $h = 1,\cdots,m'$ and therefore $p'_h = q'_h - 1 \neq q'_{h'} - 1 = p'_{h'}$ for each $h,h' = 1,\cdots,m'$). This gives rise to the necessity of generalizing Lemma (4.1) of [2] (which asserts that $\Delta_{m,n}^{(\lambda)}$ is bounded uniformly in λ) and Lemma (5.2) (the uniform eatimate) of [2]. The uniform estimate can be obtained form the generalized Pulé inequality proved in [11] and the generalization of Lemma (4.1) of [2] will be the follwing:

<u>*LEMMA* (2.1)</u> Let $m,n \in \mathbf{N}$, $m \le n$, $\{p_h,q_h\}_{h=1}^m$ be a subset of $\{1,\cdots,n\}$ such that
(i). $q_1 < \cdots < q_m$,
(ii). $p_h < q_h$ for each $h = 1,\cdots,m$
then for each $f \in L^1(\mathbf{R})$ and uniformly in λ, one has

$$\Big|\lambda^{-2m}\int_0^{t} dt_1 \cdots \int_0^{t_{n-1}} dt_n \prod_{h=1}^{m} f(\frac{t_{q_h}-t_{p_h}}{\lambda^2})\Big| \le t^{n-m}\cdot\Big(\int_{-\infty}^{\infty}|f(t)|dt\Big)^m \tag{2.4}$$

<u>*REMARK*</u> Notice that the difference between Lemma (2.1) above and Lemma (4.1) of [2] is not only that the cardinality of $\{p_h\}_{h=1}^m$ can be less than m but also that we don't distinguish the type *II* terms ($p_h < q_h - 1$ for some $h = 1,\cdots,m$) from the type *I* terms (resp. $p_h = q_h - 1$ for each $h = 1,\cdots,m$).

<u>*PROOF*</u> Without loss of generality we can assume that $f \ge 0$, otherwise one may consider its module, let us make the change of variables in (2.4)

$$t_{q_j} - t_{p_j} = s_{q_j} \quad ; \quad j = 1,\cdots,m \tag{2.5}$$

$$t_\alpha = s_\alpha \ , \alpha \neq q_j \ , j = 1,\cdots,m \tag{2.6}$$

then (2.4) becomes equal to:

$$\lambda^{-2m}\int_0^t ds_1 \cdots \int_0^{s_{q_1-2}} dt_{s_1-1}\int_{-s_{p_1}}^{s'_{q_1-1}-s_{p_1}} ds_{q_1}\int_0^{s_{q_1}+s_{p_1}} ds_{q_1+1}\cdots$$
$$\cdots\int_0^{s_{q_m-2}} ds_{q_m-1}\int_{-s_{p_m}}^{s'_{q_m-1}-s_{p_m}} ds_{q_m}\int_0^{s_{q_m}+s_{p_m}} ds_{q_m+1}$$
$$\int_0^{s_{q_m+1}} ds_{q_m+2}\cdots\int_0^{s_{n-1}} ds_n \prod_{h=1}^m f(s_{q_h}/\lambda^2) \tag{2.7}$$

where

$$s'_{q_j-1} = \begin{cases} s_{q_j-1}, & \text{if } q_j - 1 \neq q_{j-1} \\ s_{q_{j-1}} + s_{p_{j-1}}, & \text{if } q_j - 1 = q_{j-1} \end{cases} \tag{2.8}$$

The further change of variable

$$s_{q_j}/\lambda^2 = t_{q_j}, \quad j = 1,\cdots,m \tag{2.9}$$

brings the expression (2.7) to the form:

$$\int_0^t ds_1\cdots\int_0^{s_{q_1-2}} ds_{q_1-1}\int_{-s_{p_1}/\lambda^2}^{(s'_{q_1-1}-s_{p_1})/\lambda^2} dt_{q_1}\int_0^{\lambda^2 t_{q_1}+s_{p_1}} ds_{q_1+1}\cdots \tag{2.10}$$
$$\cdots\int_0^{s_{q_m-2}} ds_{q_m-1}\int_{-s_{p_m}/\lambda^2}^{(s'_{q_m-1}-s_{p_m})/\lambda^2} dt_{q_m}$$
$$\int_0^{\lambda^2 t_{q_m}+s_{p_m}} ds_{q_m+1}\int_0^{s_{q_m}+1} ds_{q_m+2}\cdots\int_0^{s_{n-1}} ds_n\prod_{j=1}^m f(t_{q_h})$$

Now, for each $h \in \{1,\cdots,m\}$ one has $t_{q_h} \in [-s_{p_h}/\lambda^2, (s'_{q_h-1}-s_{p_h})/\lambda^2]$ and therefore $\lambda^2 t_{q_h} + s_{p_h} \in [0, s'_{q_h-1}]$ for each $h = 1,\cdots,m$. From (2.8) one knows that, if $q_h \neq q_h - 1$ then $s'_{q_j-1} = s_{q_j-1}$ and, because of (2.6), $s_{q_j-1} = t_{q_j-1} \le t$. On the other hand, if

$q_h = q_h - 1$ then $s'_{q_h-1} = s_{q_h-1} + s_{p_{h-1}} = (t_{q_{h-1}} - t_{p_{h-1}}) + s_{p_{h-1}} \le s_{p_{h-1}} = t_{p_{h-1}} \le t$ since $t_{q_h} - t_{p_h} \le 0$. Therefore repalcing the integrals

$$\int_{-s_{p_h}/\lambda^2}^{(s'_{q_h-1}-s_{p_h})/\lambda^2} dt_{q_h} f(t_{q_h}) \quad , \quad h = 1, \cdots, m \tag{2.11}$$

by

$$\int_{-\infty}^{\infty} dt_{q_h} f(t_{q_h}) \quad , \quad h = 1, \cdots, m \tag{2.11a}$$

and $\lambda^2 t_{q_h} + s_{p_h}$ by t, the expression (2.10) is majorized by

$$t^{n-m} \Big(\int_{-\infty}^{\infty} dt f(t) \Big)^m \tag{2.12}$$

This ends the proof.

REMARK The following Lemma shows that if we take the matrix elements of (2.0) in some collective coherent vectors, then in the limit $\lambda \to 0$ the only term which does not vanish is the one corresponding to

$$r_1 = r_2 = 0$$

This phenomenon is very similar to what happens in the proof of the quantum central limit theorem with the method of momenta (cf. [8], [9], [10]). Also the idea of the proof is similar, i.e. one shows that if $2m < dn$ then there are "too many" λ and nothing survives in the limit $\lambda \to 0$.

LEMMA(2.2). In the notation (2.0a), for $n, m \in \mathbf{N}, q_1, \cdots, q_m, p_1, \cdots, p_m \in \{1, \cdots, n\}$ and $f, f' \in K$, $S, T, S', T' \in \mathbf{R}$, $t \ge 0$, if $2m < dn$,

$$\lim_{\lambda \to 0} < W(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f du)\Phi, \lambda^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$

$$\prod_{h=1}^{m} < S_{t_{p_h}} g, S_{t_{q_h}} > (A^+ \cdots)_{r_1} (A \cdots)_{r_2} W(\lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f' du)\Phi >= 0 \tag{2.13}$$

PROOF. Consider the expression

$$| < W(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f du)\Phi, \lambda^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$

$$\prod_{h=1}^{m} < S_{t_{p_h}} g, S_{t_{q_h}} > (A^+ \cdots)_{r_1} (A \cdots)_{r_2} W(\lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f' du)\Phi > | \tag{2.14}$$

letting $(A^+ \cdots)_{r_1} (A \cdots)_{r_2}$ act on the coherent vectors and labeling the product $(A^+ \cdots)_{r_1}$ by $\prod_{h=1}^{r_1} A^+(S_{t_{\alpha_h}} g)$ and $(A \cdots)_{r_2}$ by $\prod_{h=1}^{r_2} A(S_{t_{\beta_h}} g)$ with $\alpha_1 \le \cdots \le \alpha_{r_1}$ and $\beta_1 \le \cdots \le \beta_{r_2}$, we know that the expression (2.14) is less than or equal to

$$\lambda^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \lambda^{r_1+r_2} \prod_{h=1}^{m} | < S_{t_{p_h}} g, S_{t_{q_h}} g > |$$

$$\prod_{h=1}^{r_1} | < S_{t_{\alpha_h}} g, \int_{S/\lambda^2}^{T/\lambda^2} S_u f \, du > | \prod_{h=1}^{r_2} | < S_{t_{\beta_h}} g, \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f' \, du > | \tag{2.15}$$

since the coherent vectors have norm 1.

Now (2.1) implies that $r_1 + r_2 = dn - 2m$, so with the change of variables $\lambda^2 t_j = t'_j$, $j = 1, \cdots, n$ and using in (2.15), the majorizations

$$| < S_{t_{\alpha_h}} g, \int_{S/\lambda^2}^{T/\lambda^2} S_u f \, du > | \le \int_{-\infty}^{\infty} | < g, S_t f > | dt \tag{2.16}$$

$$| < S_{t_{\beta_h}} g, \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f' \, du > | \le \int_{-\infty}^{\infty} | < g, S_t f' > | dt \tag{2.17}$$

we see that the expression (2.15) becomes less than or equal to

$$\lambda^{(d-1)n-2m} C \int_0^t dt'_1 \int_0^{t'_1} dt'_2 \cdots \int_0^{t'_{n-1}} dt'_n \prod_{h=1}^{m} | < S_{t'_{p_h}/\lambda^2} g, S_{t'_{q_h}/\lambda^2} g > | \tag{2.18}$$

where,

$$C = \Big(\max_{F \in \{f, f'\}} \int_{-\infty}^{\infty} | < g, S_u F > | du \Big)^{r_1+r_2} \tag{2.19}$$

We shall prove that, under the condition $2m < dn$, the expression (2.18) tends to zero as $\lambda \to 0$. To this goal first notice that, as already remarked before, some p_h (resp. q_h) might be equal to $p_{h'}$ (resp. $q_{h'}$) for some $h \ne h'$. But by the arguments before Lemma (2.1), the product $\prod_{h=1}^{m} | < S_{t_{p_h}} g, S_{t_{q_h}} g > |$ is majorized by

$$\|g\|^{2(m-m')} \prod_{h=1}^{m'} | < S_{t_{p'_h}} g, S_{t_{q'_h}} g > |$$

where $m' \le m$ is the cardinality of the set $\{q_h\}_{h=1}^{m}$ and $q'_1 < q'_2 < \cdots < q'_{m'}$. Moreover we know that, if $h \in \{1, \cdots, m\}$ is such that $p_h < q_h - 1$, then one has $p'_{h'} < q'_{h'} - 1$ for some $h' = 1, \cdots, m'$. So the expression (2.18) is majorized by

$$\|g\|^{2(m-m')} C \lambda^{(d-1)n-2m+2m'} \frac{1}{\lambda^{2m'}} \int_0^t dt'_1 \cdots \int_0^{t'_{n-1}} dt'_n \prod_{j=1}^{m'} | < S_{t'_{p'_j}/\lambda^2} g, S_{t'_{q'_j}/\lambda^2} g > | \tag{2.20}$$

By Lemma (2.1), we know that the quantity

$$\frac{1}{\lambda^{2m'}}\int_0^t dt'_1 \cdots \int_0^{t'_{n-1}} dt'_n \prod_{j=1}^{m'} |< S_{t'_{p'_j}/\lambda^2} g, S_{t'_{q'_j}/\lambda^2} g >|$$

is uniformly bounded in λ. Therefore to prove the lemma it is enough to show that, for $0 \le 2m < dn$, one has

$$(d-1)n - 2m + 2m' > 0 \tag{2.21}$$

But, since at each time there are exactly d creators, it follows that for each $h = 1, \cdots, m'$, the cardinality of set

$$\{q_j : j = 1, \cdots, m, \; q_j = q'_h\}$$

is less than or equal to d, hence

$$m = \sum_{h=1}^{m'} |\{q_j : j = 1, \cdots, m, \; q_j = q'_h\}| \le m'd \tag{2.22}$$

therefore,

$$(d-1)n - 2m + 2m' \ge (d-1)n - 2m + 2\frac{m}{d} = (d-1)n - 2m(1 - \frac{1}{d}) \tag{2.23}$$

But, since by assumption $2m < dn$, the right hand side of (2.23) is strictly greater than

$$(d-1)n - dn(1 - \frac{1}{d}) = 0$$

hence (2.21) holds. This finishes the proof.

As a direct consequence of Lemma (2.2), one has the following

<u>COROLLARY (2.3)</u> For each $n \in \mathbb{N}, \varepsilon \in \{0,1\}^n$ and $f, f' \in K$, let k be the number of ones among the $\varepsilon(j)$ (i.e. $k = \sum_{j=1}^n \varepsilon(j)$). Then, for n odd or $n = 2m$ is even but $k \ne m$, the limit as $\lambda \to 0$ of

$$\lambda^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n < W(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f du)\Phi,$$

$$(A^{\varepsilon(1)}(S_{t_1}g))^d \cdots (A^{\varepsilon(n)}(S_{t_n}g))^d W(\lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f' du)\Phi > \tag{2.24}$$

is equal to zero.

<u>PROOF</u> For n odd or $n = 2m$ and $k \ne m$, the numbers of creation and annihilation operators are different (recall that k is the number of creators), so in the normally ordered form of the product (2.0b) we can't use all the operators to produce scalar products. Therefore, by putting (2.0b) in the normal ordered form, the expression (2.24) becomes equal to a sum of expressions of the form (2.14) all with $r_1 + r_2 > 0$, So the limit of (2.24) is zero by Lemma (2.2).

Corollary (2.3) shows that in the normally ordered form of (2.0b), we need only to consider the purely scalar terms, i.e. those of the form

$$\sum_{1 \le q_1 \le \cdots \le q_{kd} \le 2k} \; \sum_{\{p_h\}_{h=1}^{kd} = \{1,\cdots,2k\} \setminus \{q_h\}_{h=1}^{k};\; p_h < q_h,\; h=1,\cdots,kd} \; \prod_{h=1}^{kd} < S_{t_{p_h}} g, S_{t_{q_h}} g > \tag{2.25}$$

because the other terms vanish as $\lambda \to 0$.

Now we investigate the behaviour, as $\lambda \to 0$, of the expression (2.25). To this goal, we have to analyse firstly the sets of indices $\{q'_h\}_{h=1}^{m'}$ and $\{p'_h\}_{h=1}^{m'}$ in this expression.

<u>LEMMA (2.4)</u> In (2.25) the cardinality of the set $\{q_h\}_{h=1}^{kd}$ is exactly k (i.e. $m' = k$) and moreover we can choose the set $\{p'_h\}_{h=1}^{m'}$ so that the following three properties are satisfied:

(i). $p'_h < q'_h$, for each $h = 1, \cdots, k$,

(ii). if there exists a $h \in \{1, \cdots, kd\}$ such that $p_h < q_h - 1$, then there exists a $h' \in \{1, \cdots, k\}$ such that $p'_{h'} < q'_{h'} - 1$,

(iii). the cardinality of the set $\{p'_h\}_{h=1}^{k}$ is equal to k.

<u>PROOF</u> The first assertion ($m' = k$) is clear because the creator (time) indices are exactly k and for each time index there are exactly d creators, and we use them all to produce scalar products.

Choosing in (2.25) the subset $\{q'_h\}_{h=1}^{k}$ of the index set $\{q_h\}_{h=1}^{kd}$ as described before (2.2a), then for each $h = 1, \cdots, k$, there exist exactly d-elements in $\{q_h\}_{h=1}^{kd}$ are equal to q'_h. Since the set $\{q_h\}_{h=1}^{kd}$ is ordered, we must have

$$q'_1 = q_1 = \cdots = q_d, \; q'_2 = q_{d+1} = q_{d+2} = \cdots = q_{2d}, \cdots,$$

$$q'_k = q_{(k-1)d+1} = q_{(k-1)d+2} = \cdots = q_{kd} \tag{2.26}$$

Moreover in (2.25), for each $h = 1, \cdots, kd$, the set

$$M(p_h) := \{p_j : j = 1, \cdots, kd, \; p_j = p_h\}$$

has exactly d-elements. Now having fixed $\{q'_h\}_{h=1}^{k}$, we want to constructe a subset $\{p'_h\}_{h=1}^{k}$ of $\{p_h\}_{h=1}^{kd}$ with the properties (i), (ii) and (iii) above.

We choose $p'_1 \in \{p_1, \cdots, p_d\}$ to be the index with the property:

$$q'_1 - p'_1 = \max\{q'_1 - p_h : h = 1, \cdots, d\} = \max\{q_h - p_h : h = 1, \cdots, d\} \tag{2.27}$$

Notice that if $h \in \{1,\cdots,d\}$ is such that $p_h < q_h - 1$ then, by construction, $p'_1 < q'_1 - 1$. Having fixed p'_1, certainly there exists a $h = d+1,\cdots,2d$ such that $p_h \neq p'_1$ (otherwise $M(p'_1)$ would have at least $d+1$-elements). So we can choose $p'_2 \in \{p_{d+1},\cdots,p_{2d}\}$ such that

$$q'_2 - p'_2 = \max\{q'_2 - p_h \, : \, h = d+1,\cdots,2d \text{ and } p_h \neq p'_1\} \tag{2.28}$$

Obviously, $p'_2 \neq p'_1$. Iterating the procedure for each $h = 3,\cdots,k$, we choose $p'_h \in \{p_{(h-1)d+1},\cdots,p_{hd}\}$ so that

$$q'_h - p'_h = \max\{q'_h - p_j \, : \, j = (h-1)d+1,\cdots,dh \text{ and } p_j \notin \{p'_1,\cdots,p'_{h-1}\}\} \tag{2.29}$$

The set $\{p'_h\}_{h=1}^k$ clearly satifies the conditions (i) and (iii) above. Now we show that it satisfies also the condition (ii), i.e. if there exists a $h \in \{1,\cdots,kd\}$ such that $p_h < q_h - 1$, then there exists a $h' \in \{1,\cdots,k\}$ such that $p'_{h'} < q'_{h'} - 1$. We have already proved this for $h = 1$. In general, let us suppose that the set

$$\{h \in \{1,\cdots,kd\} \, : \, p_h < q_h - 1\} \tag{2.30}$$

is nonempty and denote h_0 its minimum. Then if α is defined by

$$(\alpha-1)d + 1 \le h_0 \le \alpha d \tag{2.31}$$

we claim that $p_\alpha < q_\alpha - 1$. In fact, (2.31) imply that $p_h = q_h - 1$ for each $h = 1,\cdots,(\alpha-1)d$ and therefore, due to (2.29), $p'_h = q'_h - 1$ for each $h = 1,\cdots,\alpha-1$. But then because of (2.26) one has also

$$p'_1 = p_1 = \cdots = p_d, \;\cdots,\; p'_{\alpha-1} = p_{(\alpha-2)d+1} = \cdots = p_{(\alpha-1)d} \tag{2.31a}$$

Therefore, since for each $h = 1,\cdots,kd$, $M(p_h)$ has exactly d elements, it follows that the set $\{p_h\}_{h=(\alpha-1)d+1}^{\alpha d}$ is disjoint with the set $\{p_h\}_{h=1}^{(\alpha-1)d}$ and since, by (2.31a), the later set contains $p'_1,\cdots,p'_{\alpha-1}$, it follows that

$$\{p_j \, : \, j = (\alpha-1)d+1,\cdots,\alpha d \text{ and } p_j \notin \{p'_1,\cdots,p'_{\alpha-1}\}\}$$
$$= \{p_j \, : \, j = (\alpha-1)d+1,\cdots,\alpha d\} \tag{2.32}$$

Therefore, (2.29), (2.31) and (2.32) imply that

$$q'_\alpha - p'_\alpha = \max\{q'_\alpha - p_j \, : \, j = (\alpha-1)d+1,\cdots,\alpha d\} \ge q'_\alpha - p_{h_0} = q_{h_0} - p_{h_0} > 1 \tag{2.33}$$

Thus the proof is finished.

Now we can prove the following Theorem which improves Corollary (2.3).

<u>*THEOREM*(2.5)</u> For each $n \in \mathbf{N}, \varepsilon \in \{0,1\}^n$ and $f, f' \in K$, the limit

$$\lim_{\lambda\to 0} \lambda^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n < W(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f du)\Phi,$$

$$(A^{\varepsilon(1)}(S_{t_1}g))^d \cdots (A^{\varepsilon(n)}(S_{t_n}g))^d W(\lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f' du)\Phi > \tag{2.34}$$

exists. Moreover,

(i) For each n, if either n is odd or if it is even but ε is not the form

$$0,1,0,1,\cdots,0,1 \tag{2.35}$$

then, the limit (2.34) is equal to 0;

(ii) For each n even, if ε has the form (2.35), then the limit (2.34) is equal to

$$t^{n/2}\frac{1}{(n/2)!}\left(\int_{-\infty}^0 d! < g, S_u g >^d du\right)^{n/2} < W(\chi_{[S,T]} \otimes f)\Psi, W(\chi_{[S',T']} \otimes f')\Psi > \tag{2.36}$$

where, Ψ and $W(.)$ are the Fock-vacuum and Weyl operator on Fock space $\Gamma(L^2(\mathbf{R} \otimes (K,(.|.)))$.

<u>*PROOF*</u>. (i) From Corollary (2.3) we know that the only nonzero contribution to the limit (2.34) can come from the pure scalar term in the Wick ordered form of the product (2.0b), i.e. the term in which all the creators and annihilators have been used to produce scalar products. If n is even and ε is not of the form (2.35), then certainly in each summand of the pure scalar product term of the Wick ordered form of the product (2.0b), some of the annihilators must have produced scalar products with some non time-consecutive creators, i.e., in the development (2.0a) there will exist a $h = 1,\cdots,m$, such that $q_h - p_h > 1$. This implies, because of Lemma (2.4), that there exists $h = 1,\cdots,m'$ such that $q'_h - p'_h > 1$. Moreover for the pure scalar product term, because of Lemma (2.4), we can choose the index set $\{p'_h, q'_h\}_{h=1}^{m'}$ so that the cardinalities of both $\{q'_h\}_{h=1}^{m'}$ and $\{p'_h\}_{h=1}^{m'}$ are exactly m'. Hence, the scalar term, corresponding to this ε, will give rise to an expression which is of type II in the sense of Lemma (4.1) of [3] and therefore it goes to 0 as $\lambda \to 0$.

(ii) If $n = 2k$ is even and ε is the form of (2.35), then the product (2.0a) is equal to

$$(A(S_{t_1}g))^d (A^+(S_{t_2}g))^d \cdots (A(S_{t_{2k-1}}g))^d (A^+(S_{t_{2k}}g))^d$$

and the scalar term in its Wick ordered form is equal to the product (for $j = 1,\cdots,k$) of the scalar terms of the Wick ordered forms of

$$(A(S_{t_{2j-1}}g))^d (A^+(S_{t_{2j}}g))^d \quad ; \quad j = 1,\cdots,k$$

i.e. to

$$\prod_{j=1}^k d! < S_{t_{2j-1}}g, S_{t_{2j}}g >^d$$

It follows that the limit (2.34) is equal to

$$\lim_{\lambda\to 0} \lambda^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \prod_{j=1}^k d! < S_{t_{2j-1}}g, S_{t_{2j}}g >^d$$
$$< W(\chi_{[S,T]} \otimes f)\Psi, W(\chi_{[S',T']} \otimes f')\Psi >$$

$$= t^{n/2}\frac{1}{(n/2)!}\left(\int_{-\infty}^0 d! < g, S_u g >^d du\right)^{n/2} < W(\chi_{[S,T]} \otimes f)\Psi, W(\chi_{[S',T']} \otimes f')\Psi >$$

So the lemma is proved.

§3. The Uniform Estimate: the case $d = 2$

In the Section 2.) we have derived the explicit form of the limit, as $\lambda \to 0$, of the matrix elements of each term of the formal iterated series. In this Section we show that, if $d = 2$, then the formal iterated series is in fact uniformly convergent in λ, so that the limiting series gives the limit of the matrix element (1.12).

To get the limit (1.12), it is necessary to give a uniform estimate of each term so that we can exchange the limit $\lambda \to 0$ and the sum of $n = 0, 1, \cdots$. In this Section, we shall assume $d = 2$ so that in the notation (1.2) with $d = 2$, for each $n \in \mathbb{N}$,

$$V_2(t_1)V_2(t_2)\cdots V_2(t_n)$$

$$= \sum_{\varepsilon\in\{0,1\}^n} i^n D_{\varepsilon(1)}\cdots D_{\varepsilon(n)} \otimes (A^{\varepsilon(1)}(S_{t_1}g))^2 \cdots (A^{\varepsilon(n)}(S_{t_n}g))^2 \tag{3.0a}$$

In order to apply Theorem (6.1) of [11], we shall write the normally ordered form of the product

$$(A^{\varepsilon(1)}(S_{t_1}g))^2 \cdots (A^{\varepsilon(n)}(S_{t_n}g))^2 \tag{3.0}$$

in a way which is particularly well suited for the application of the Pulé inequality.

<u>*LEMMA* (3.1)</u> For each $k, n \in \mathbb{N}$, $1 \le j_1 < \cdots < j_k \le n$ and $f_1, \cdots, f_n \in K$, the <u>normally ordered form</u> of the product

$$A(f_1)\cdots A^+(f_{j_1})\cdots A^+(f_{j_k})\cdots A(f_n) \tag{3.1}$$

(i.e., $j_1, \cdots, j_k$ are the creation indices) is given by

$$\sum_{m=0}^{k\wedge(n-k)} \sum_{\substack{1\le\alpha_1<\cdots<\alpha_m\le n \\ \{\alpha_h\}_{h=1}^m \subset (\{1,\cdots,n\}\setminus\{j_h\}_{h=1}^k)}} \sum_{\substack{1\le\beta_1<\cdots<\beta_m\le n;\ \alpha_h<\beta_h,\ h=1,\cdots,m \\ \{\beta_h\}_{h=1}^m\subset\{j_h\}_{h=1}^k}}$$

$$\sum_{\sigma\in S_m^1}\prod_{h=1}^m < f_{\alpha_h}, f_{\beta_h} > \prod_{\alpha\in\{j_h\}_{h=1}^k\setminus\{\beta_h\}_{h=1}^m} A^+(f_\alpha) \prod_{\alpha\in\{1,\cdots,n\}\setminus(\{j_h\}_{h=1}^k\cup\{\beta_h\}_{h=1}^m)} A(f_\alpha) \tag{3.2}$$

where,

$$S_m^1 := \{\sigma \in S_m :\ \alpha_h < \beta_{\sigma(h)} \ \text{ for all } \ h = 1, \cdots, m\} \tag{3.3}$$

and S_m is the group of m-permutations.

<u>*PROOF*.</u> The normally ordered form of the product (3.1) has the form

$$\sum_{m=0}^{k} C_{n,k,m}\cdot(A^+\cdots)_{k-m}\cdot(A\cdots)_{n-k-m} \tag{3.4}$$

where m is the number of pairs of creators and annihilators used to produce a scalar product, $(A^+\cdots)_{k-m}$ is the product of a subset of $\{A^+(f_{j_1}), \cdots, A^+(f_{j_k})\}$ of cardinality $k-m$ and $(A\cdots)_{n-k-m}$ is the product of a subset of annihilators $\{A(f_\alpha) :\ \alpha \in \{1, \cdots, n\} \setminus \{j_1, \cdots, j_k\}\}$ of cardinality $n-k-m$. All the $A(f_j)$ and $A^+(f_k)$ not appearing in the product $(A^+\cdots)_{k-m}\cdot(A\cdots)_{n-k-m}$ have produced a scalar product $< f_j, f_k >$ according to the rules that we are going to make explicit. For each $m = 0, \cdots, k$, there are $\binom{n-k}{m}$ ways in which we can choose a subset of $\{A(f_\alpha) :\ \alpha \in \{1, \cdots, n\} \setminus \{j_1, \cdots, j_k\}\}$ such that all elements in this subset produce scalar products. Each choice defines a subset $\{\alpha_1, \cdots, \alpha_m\}$ of $\{1, \cdots, n\} \setminus \{j_1, \cdots, j_k\}$ with $\alpha_1 < \cdots < \alpha_m$. For each scalar product, a pair A^+, A disappears, so $k - m \ge 0$ and $n - k - m \ge 0$, that is $m \le k \wedge (n-k)$.

Now fix $m = 0, 1, \cdots, k\wedge(n-k)$, $1 \le \alpha_1 < \cdots < \alpha_m \le n$ and a subset of annihilation opeartors $\{A(f_{\alpha_1}), \cdots, A(f_{\alpha_m})\}$ in $\{A(f_\alpha) :\ \alpha \in \{1, \cdots, n\} \setminus \{j_1, \cdots, j_k\}\}$ with which we are going to produce scalar products with some of the creators $A^+(f_{r_1}), \cdots, A^+(f_{r_m})$. Notice that clearly one must have $\{r_h\}_{h=1}^m \subset \{j_h\}_{h=1}^k$ and $r_1 > \alpha_1, r_2 > \alpha_2, \cdots, r_m > \alpha_m$. Conversely, each choice of $r_1, \cdots, r_m$ with these properties will give rise to a term in which the only scalar products arising are the $< f_{\alpha_h}, f_{r_h} >$ $(h = 1, \cdots, m)$. Therefore, in the notation (3.4), one has

$$A(f_1)\cdot\ldots\cdot A^+(f_{j_1})\cdot\ldots\cdot A^+(f_{j_k})\cdot\ldots\cdot A(f_n)$$

$$= \sum_{m=0}^{k\wedge(n-k)} \sum_{\substack{1\le\alpha_1<\cdots\alpha_m\le n \\ \{\alpha_h\}_{h=1}^m\subset(\{1,\cdots,n\}\setminus\{j_h\}_{h=1}^k)}} \sum_{\substack{1\le r_1,\cdots,r_m\le n;\ r_h>\alpha_h,\ h=1,\cdots,m \\ \{r_h\}_{h=1}^m\subset\{j_h\}_{h=1}^k}}$$

$$\prod_{h=1}^m < f_{\alpha_h}, f_{r_h} > \prod_{\alpha\in\{j_h\}_{h=1}^k\setminus\{r_h\}_{h=1}^m} A^+(f_\alpha) \prod_{\alpha\in\{1,\cdots,n\}\setminus(\{j_h\}_{h=1}^k\cup\{\beta_h\}_{h=1}^m)} A(f_\alpha) \tag{3.5}$$

Notice that the index set $\{r_1, \cdots, r_m\}$ is unordered. Let $\beta_1 < \cdots < \beta_m$ be its ordered rearrangement, then for each fixed $\alpha_1 < \cdots < \alpha_m$ and for each $\{r_1, \cdots, r_m\} \subset \{j_h\}_{h=1}^k$ satisfying $r_h > \alpha_h$, $h = 1, \cdots, m$ there exists a unique $\sigma \in S_m^1$ such that $r_h = \beta_{\sigma(h)}$, $h = 1, \cdots, m$. On the other hand, for each $\sigma \in S_m^1$, we have $r_h = \beta_{\sigma(h)} > \alpha_h$ for any $h = 1, \cdots, m$, so that

$$\bigcup\{r_1, \cdots, r_m :\ r_h > \alpha_h,\ h = 1, \cdots, m,\ \{r_h\}_{h=1}^m \subset \{j_h\}_{h=1}^k\} =$$

$$= \bigcup_{\substack{\beta_1<\cdots<\beta_m;\ \beta_h>\alpha_h,\ h=1,\cdots,m \\ \{\beta_h\}_{h=1}^m\subset\{j_h\}_{h=1}^k}} \bigcup_{\sigma\in S_m^1} \{\beta_{\sigma(1)}, \cdots, \beta_{\sigma(m)}\} \tag{3.6}$$

Applying (3.6) to (3.5), (3.2) follows.

Now we apply Lemma (3.1) to obtain the explicit form of the normally ordered form of the product (3.0) and the uniform estimate. For each $\varepsilon \in \{0,1\}^n$ and $0 \le k \le n$, $1 \le j_1 < \cdots < j_k \le n$ satisfying

$$\varepsilon(j) = \begin{cases} 1, & \text{if } j \in \{j_h\}_{h=1}^k, \\ 0, & \text{otherwise} \end{cases} \tag{3.7}$$

the product (3.0) has the form:

$$(A(S_{t_1}g))^2\cdots(A^+(S_{t_{j_1}}g))^2\cdots(A^+(S_{t_{j_k}}g))^2\cdots(A(S_{t_n}g))^2 \tag{3.8}$$

Introducing the notation

$$f_{1,t}=f_{2,t}=S_{t_1}g,\quad f_{3,t}=f_{4,t}=S_{t_2}g,\cdots,\quad f_{2n-1,t}=f_{2n,t}=S_{t_n}g \tag{3.9}$$

(3.8) becomes

$$A(f_{1,t})A(f_{2,t})\cdots A^+(f_{2j_1-1,t})A^+(f_{2j_1,t})\cdots A^+(f_{2j_k-1,t})A^+(f_{2j_k,t})\cdots A(f_{2n-1,t})A(f_{2n,t}) \tag{3.10}$$

Applying Lemma (3.1) to (3.10) and noticing that now instead of $\{j_h\}_{h=1}^k$ in Lemma (3.1) we have $\{j_{2h-1},j_{2h}\}_{h=1}^k$, the following conclusion is easily obtained

LEMMA (3.2) The normally ordered form of the product (3.10) is given by

$$\sum_{m=0}^{2(k\wedge(n-k))}\ \sum_{\substack{1\le\alpha_1<\cdots<\alpha_m\le 2n\\ \{\alpha_h\}_{h=1}^m\subset(\{1,\cdots,2n\}\setminus\{j_{2h-1},j_{2h}\}_{h=1}^k)}}\ \sum_{\substack{1\le\beta_1<\cdots<\beta_m\le 2n;\ \alpha_h<\beta_h,\ h=1,\cdots,m\\ \{\beta_h\}_{h=1}^m\subset\{j_{2h-1},j_{2h}\}_{h=1}^k}}$$

$$\sum_{\sigma\in S_m^1}\prod_{h=1}^m<f_{\alpha_h,t},f_{\beta_h,t}>\prod_{\alpha\in\{j_{2h-1},j_{2h}\}_{h=1}^k\setminus\{\beta_h\}_{h=1}^m}A^+(f_{\alpha,t})$$

$$\prod_{\alpha\in\{1,\cdots,n\}\setminus(\{j_{2h-1},j_{2h}\}_{h=1}^k\cup\{\beta_h\}_{h=1}^m)}A(f_{\alpha,t}) \tag{3.11}$$

PROOF Apply (3.2) eith the notation (3.10).

Now we can apply the normally ordered form (3.11) to the coherent vectors and get the following estimate

LEMMA (3.3) The module of

$$<W(\lambda\int_{S/\lambda^2}^{T/\lambda^2}S_uf\,du)\Phi,\lambda^n\int_0^{t/\lambda^2}dt_1\cdots\int_0^{t_{n-1}}dt_n(A(S_{t_1}g))^2\cdots(A^+(S_{t_{j_1}}g))^2\cdots$$

$$(A^+(S_{t_{j_k}}g))^2\cdots(A(S_{t_n}g))^2W(\lambda\int_{S'/\lambda^2}^{T'/\lambda^2}S_uf'du)\Phi> \tag{3.12}$$

is majorized by

$$\sum_{m=0}^{2(k\wedge(n-k))}\ \sum_{\substack{1\le\alpha_1<\cdots<\alpha_m\le 2n\\ \{\alpha_h\}_{h=1}^m\subset(\{1,\cdots,2n\}\setminus\{j_{2h-1},j_{2h}\}_{h=1}^k)}}\ \sum_{\substack{1\le\beta_1<\cdots<\beta_m\le 2n;\ \alpha_h<\beta_h,\ h=1,\cdots,m\\ \{\beta_h\}_{h=1}^m\subset\{j_{2h-1},j_{2h}\}_{h=1}^k}}$$

$$\sum_{\sigma\in S_m^1}\lambda^{3n-2m}\int_0^{t/\lambda^2}dt_1\cdots\int_0^{t_{n-1}}dt_n\prod_{h=1}^m|<f_{\alpha_h,t},f_{\beta_{\sigma(h)},t}>|\cdot C^{2(n-m)} \tag{3.13}$$

where, the constant C is defined by (2.19)

PROOF Applying the normally ordered form (3.11) to (3.12), and letting the product of creators

$$\prod_{\alpha\in\{j_{2h-1},j_{2h}\}_{h=1}^k\setminus\{\beta_h\}_{h=1}^m}A^+(f_{\alpha,t}) \tag{3.14}$$

act on the coherent vector

$$W(\lambda\int_{S/\lambda^2}^{T/\lambda^2}S_uf\,du)\Phi$$

we get the following quantity

$$\lambda^{2k-m}\prod_{\alpha\in\{j_{2h-1},j_{2h}\}_{h=1}^k\setminus\{\beta_h\}_{h=1}^m}\int_{S/\lambda^2}^{T/\lambda^2}<S_uf,f_{\alpha,t}>du \tag{3.15}$$

Since each $f_{\alpha,t}$ is equal to some $S_{t_h}g$, it follows that the module of (3.15) is controlled by

$$\lambda^{2k-m}(\int_{\infty}^{\infty}|<S_tf,g>|dt)^{2k-m} \tag{3.16}$$

By the same argument, from the action of the product of annihilation operators

$$\prod_{\alpha\in\{1,\cdots,n\}\setminus(\{j_{2h-1},j_{2h}\}_{h=1}^k\cup\{\beta_h\}_{h=1}^m)}A(f_{\alpha,t}) \tag{3.17}$$

on the coherent vector

$$W(\lambda\int_{S'/\lambda^2}^{T'/\lambda^2}S_uf'du)\Phi$$

one gets the bound

$$\lambda^{2(n-k)-m}(\int_{\infty}^{\infty}|<g,S_tf'>|dt)^{2(n-k)-m} \tag{3.18}$$

From this (3.13) follows.

In [2],$\cdots$, [7], we have used Pulé type inequalities to estimate expressions like (3.13). However in the present case there are some differences which prevent a direct application of this inequality. First of all the index m, in the product in (3.13) can vary in $\{1,\cdots,2n\}$, while in the Pulé inequality it must be vary in $\{1,\cdots,n\}$. Moreover, in the Pulé Lemma all the $f_{\beta_h,t}$ should be different, but in our case from (3.8) and (3.9), we see that some of them could be equal. The basic goal of the following combinatorial arguments is to majorize (3.13) with another expression which satisfies the conditions of the Pulé Lemma.

The first step of our strategy is to collect a subset of $\{f_{\beta_h,t}\}_{h=1}^{m}$ (or $\{f_{\alpha_h,t}\}_{h=1}^{m}$) whose elements are different among themself.

For each m and $\{\alpha_h,\beta_h\}_{h=1}^{m}$, since $f_{2h-1,t} = f_{2h,t} = S_{t_h}g$, there exist unique $\bar{m} \le m$ and $1 \le r_1 < \cdots < r_{\bar{m}} \le m$ (determined by m and $\{\alpha_h,\beta_h\}_{h=1}^{m}$) such that

$$f_{\beta_{r_1},t} = f_{\beta_{r_1+1},t}, \cdots, f_{\beta_{r_{\bar{m}}},t} = f_{\beta_{r_{\bar{m}}+1},t} \tag{3.19}$$

On the other hand, by the definition of $\{f_{h,t}\}_{h=1}^{2n}$, $f_{\beta,t} = f_{\beta',t}$ for some $1 \le \beta < \beta' \le 2n$, if and only if $\beta = 2h-1$, $\beta' = 2h$ for some $h = 1,\cdots,n$. Therefore (3.19) is equavelent to

$$\beta_{r_1} + 1 = \beta_{r_1+1}, \cdots, \ \beta_{r_{\bar{m}}} + 1 = \beta_{r_{\bar{m}}+1} \tag{3.20}$$

and

$$\{\beta_{r_h}\}_{h=1}^{\bar{m}} \subset \{j_{2h-1}\}_{h=1}^{k} \ \text{ and } \ \{\beta_{r_h+1}\}_{h=1}^{\bar{m}} \subset \{j_{2h}\}_{h=1}^{k} \tag{3.21}$$

Moreover

$$\{\beta_{r_h}, \beta_{r_h+1}\}_{h=1}^{\bar{m}} \subset \{\beta_h\}_{h=1}^{m} \tag{3.22}$$

and because of (3.21) one has $\{\beta_{r_h}\}_{h=1}^{\bar{m}} \cap \{\beta_{r_h+1}\}_{h=1}^{\bar{m}} = \emptyset$, so that

$$2\bar{m} \le m \tag{3.23}$$

This together with $m \le 2(k \wedge (n-k)) \le n$ implies that

$$2\bar{m} \le m \le n \tag{3.24}$$

On the other hand, since $\{\beta_{r_h+1}\}_{h=1}^{\bar{m}}$ is uniquely determined by $\{\beta_{r_h}\}_{h=1}^{\bar{m}}$ via (3.20), it follows that for each fixed chioce $\alpha_1,\cdots,\alpha_m$, the sum over all $1 \le \beta_1 < \cdots < \beta_m \le 2n$ in the expression (3.13) can be replaced by a sum over the $m - \bar{m}$ indices

$$1 \le \beta_1 < \cdots < \beta_{r_1} < \beta_{r_1+2} < \cdots < \beta_{r_{\bar{m}}} < \beta_{r_{\bar{m}}+2} < \cdots < \beta_m \le 2n \tag{3.27}$$

whose precise expression is :

$$\sum_{\substack{1\le\beta_1<\cdots<\beta_{r_1}<\beta_{r_1+2}<\cdots<\beta_{r_{\bar{m}}}<\beta_{r_{\bar{m}}+2}<\cdots<\beta_m\le 2n;\ \alpha_h<\beta_h,\ h=1,\cdots,m;\ \beta_{r_h+2}-\beta_{r_h}>2,\ h=1,\cdots,\bar{m} \\ \{\beta_h\}_{1\le h\le m,h\notin\{r_h+1\}_{h=1}^{\bar{m}}}\subset\{j_{2h-1},j_{2h}\}_{h=1}^{k};\ \{\beta_{r_h+1}\}_{h=1}^{\bar{m}}\subset\{j_{2h}\}_{h=1}^{k}}} \tag{3.28}$$

In fact, in (3.28), the first condition eliminates the indices $\{\beta_{r_h+1}\}_{h=1}^{\bar{m}}$ using (3.20); the second is as in (3.13); the third comes again from (3.20); the fourth corresponds to the third condition in (3.13); and the fifth comes from (3.21). In the following the sum (3.28) will be simply denoted $\sum_{(\beta)}$.

The next step in our strategy is reduce the sum over $\mathcal{S}_m^1$ in (3.13) to a sum over a set of permutations which act only on the $m - \bar{m}$ indices of (3.28). To this goal, in the following we shall relabel the indices set $\{\beta_h\}_{1\le h\le m, h\notin\{r_h+1\}_{h=1}^{\bar{m}}}$ by $\{q_h\}_{h=1}^{m-\bar{m}}$ and

$\{\alpha_h\}_{1\le h\le m, h\notin\{r_h+1\}_{h=1}^{\bar{m}}}$ by $\{p_h\}_{h=1}^{m-\bar{m}}$. For each fixed $\{\alpha_h,\beta_h\}_{h=1}^{m}$ (so fixed $\bar{m}$ and $\{r_h\}_{h=1}^{\bar{m}}$) and for each $\sigma \in \mathcal{S}_m^1$ there exist a unique subset $\{y_h\}_{h=1}^{\bar{m}}$ of $\{1,\cdots,m\}$ such that $y_1 < \cdots < y_{\bar{m}}$ and

$$\{\sigma(y_h)\}_{h=1}^{\bar{m}} = \{r_h + 1\}_{h=1}^{\bar{m}} \tag{3.29}$$

(notice that it is not necessary that $\sigma(y_h) = r_h + 1$ for each $h = 1,\cdots,\bar{m}$). Now we decompose each $\sigma \in \mathcal{S}_m^1$ in the two parts: one (σ_1) which acts on the relevant indices and the other (σ_2) which acts on the irrelevant ones. More precisely, denote

$$\{y_h^c\}_{h=1}^{m-\bar{m}} := \{1,\cdots,m\} \setminus \{y_h\}_{h=1}^{\bar{m}} \tag{3.30}$$

with $y_1^c < \cdots < y_{m-\bar{m}}^c$ and define

$$\sigma_1 : \{1,\cdots,m-\bar{m}\} \longrightarrow \{x_1,\cdots,x_{m-\bar{m}}\} := \{1,\cdots,m\} \setminus \{r_h+1\}_{h=1}^{\bar{m}} \tag{3.31a}$$

by

$$\sigma_1(h) = \sigma(y_h^c), \quad h = 1,\cdots,m-\bar{m}, \tag{3.31}$$

and

$$\sigma_2 : \{1,\cdots,\bar{m}\} \longrightarrow \{r_1+1,\cdots,r_{\bar{m}}+1\} \tag{3.32a}$$

by

$$\sigma_2(h) = \sigma(y_h), \quad h = 1,\cdots,\bar{m} \tag{3.32}$$

notice that, both σ_1 and σ_2 are one-to-one maps which uniquely define σ. With these notations, for each fixed $\{\alpha_h\}_{h=1}^{m}$ one has that

$$\sum_{\substack{1\le\beta_1<\cdots<\beta_m\le 2n;\ \alpha_h<\beta_h,\ h=1,\cdots,m \\ \{\beta_h\}_{h=1}^{m}\subset\{j_{2h-1},j_{2h}\}_{h=1}^{k}}}$$

$$\sum_{\sigma\in\mathcal{S}_m^0} \lambda^{3n-2m} \int_0^{t/\lambda^2} dt_1 \cdots \int_0^{t_{n-1}} dt_n \prod_{h=1}^{m} |< f_{\alpha_h,t}, f_{\beta_{\sigma(h)},t} >|$$

$$= \sum_{(\beta)} \sum_{\sigma\in\mathcal{S}_m^0} \lambda^{3n-2m} \int_0^{t/\lambda^2} dt_1 \cdots \int_0^{t_{n-1}} dt_n \prod_{h=1}^{m} |< f_{\alpha_h,t}, f_{\beta_{\sigma(h)},t} >|$$

$$= \sum_{(\beta)} \sum_{\sigma\in\mathcal{S}_m^0} \lambda^{3n-2m} \int_0^{t/\lambda^2} dt_1 \cdots \int_0^{t_{n-1}} dt_n$$

$$\prod_{h=1}^{m-\bar{m}} |< f_{\alpha_{y_h^c},t}, f_{\beta_{\sigma_1(h)},t} >| \prod_{h=1}^{\bar{m}} |< f_{\alpha_{y_h},t}, f_{\beta_{\sigma_2(h)},t} >| \tag{3.33}$$

Notice that the last product in the right hand side of (3.33) has the form

$$\prod_{h=1}^{\bar{m}} |< S_{t_{l_h}} g, S_{t_{l_h'}} g >| \tag{3.34}$$

with $1 \le l_h < l'_h \le n,\ h = 1, \cdots, \bar{m}$ and it is bounded by $\|g\|^{2\bar{m}}$. So the right hand side of (3.33) is majorized by

$$\sum_{(\beta)} \sum_{\sigma \in \mathcal{S}^0_m} \lambda^{3n-2m} \int_0^{t/\lambda^2} dt_1 \cdots \int_0^{t_{n-1}} dt_n \prod_{h=1}^{m-\bar{m}} | < f_{\alpha_{y_h^c}, t}, f_{\beta_{\sigma_1(h)}, t} > | \cdot \|g\|^{2\bar{m}} \tag{3.35}$$

From the definition of σ_1, we deduce that in (3.35)
- the cardinality of set $\{f_{\beta_{\sigma_1(h)}, t}\}_{h=1}^{m-\bar{m}}$ is exactly $m - \bar{m}$ for each σ_1;
- $\alpha_{y_h^c} < \beta_{\sigma_1(h)}$ for each $h = 1 \cdots, m - \bar{m}$.

Now, from the definition of $\{f_{h,t}\}_{h=1}^m$ (cf. formula (3.9)), we know that each fixed $\{\alpha_h\}_{h=1}^m$ and $\{\beta_h\}_{h=1}^m$ satisfying:
(i). $\alpha_h < \beta_h,\ h = 1, \cdots, m$;
(ii). $\beta_{r_h+2} - \beta_{r_h} > 2,\ h = 1, \cdots, \bar{m}$;
(iii). $\{\beta_h\}_{1 \le h \le m, h \notin \{r_h+1\}_{h=1}^{\bar{m}}} \subset \{j_{2h-1}, j_{2h}\}_{h=1}^k$;
(iv). $\{\beta_{r_h}+1\}_{h=1}^{\bar{m}} \subset \{j_{2h}\}_{h=1}^k$
define a unique set $\{q_h\}_{h=1}^{m-\bar{m}}$ satisfying

$$f_{\beta_{\sigma_1(h)}, t} = S_{t_{q_{\sigma_1(h)}}} g, \quad h = 1, \cdots, m - \bar{m} \tag{3.36}$$

and

$$1 \le q_1 < \cdots < q_{m-\bar{m}} \le n \tag{3.37}$$

By the same argument one can rewrite $\{f_{\alpha_{y_h^c}, t}\}_{h=1}^{m-\bar{m}}$ as $\{S_{t_{p_h}} g\}_{h=1}^{m-\bar{m}}$ with

$$1 \le p_1 \le \cdots \le p_{m-\bar{m}} \le n \tag{3.37a}$$

$$p_h < q_{\sigma_1(h)}, \quad h = 1, \cdots, m - \bar{m} \tag{3.37b}$$

Therefore, if of all these conditions we only keep (3.37), (3.37a), (3.37b), the expression (3.35) can be majorized by

$$\sum_{1 \le q_1 < \cdots < q_{m-\bar{m}} \le n,\ q_h > p_h, h=1,\cdots,m-\bar{m}} \ \sum_{\sigma \in \mathcal{S}^0_m}$$

$$\lambda^{3n-2m} \int_0^{t/\lambda^2} dt_1 \cdots \int_0^{t_{n-1}} dt_n \prod_{h=1}^{m-\bar{m}} | < S_{t_{p_h}} g, S_{t_{q_{\sigma_1(h)}}} g > | \cdot \|g\|^{2\bar{m}} \tag{3.38}$$

The expression (3.38) is almost of the type which allows the use of Pulé type inequality, the only difficulty being that σ_1 is not a permutation but only a one-to-one map form $\{1, \cdots, m - \bar{m}\}$ to a set of $m - \bar{m}$ elements. Identifying this set with $\{1, \cdots, m - \bar{m}\}$ we get a permutation σ', with the same basic property as σ_1, namely: $q_{\sigma'(h)} > p_h,\ h = 1, \cdots, m - \bar{m}$. So, one can rewrite (3.38) as

$$\sum_{1 \le q_1 < \cdots < q_{m-\bar{m}} \le n,\ q_h > p_h, h=1,\cdots,m-\bar{m}} \ \sum_{\sigma \in \mathcal{S}^0_m}$$

$$\lambda^{3n-2m} \int_0^{t/\lambda^2} dt_1 \cdots \int_0^{t_{n-1}} dt_n \prod_{h=1}^{m-\bar{m}} | < S_{t_{p_h}} g, S_{t_{q_{\sigma'(h)}}} g > | \cdot \|g\|^{2\bar{m}} \tag{3.39}$$

The last step will be to reduce the $\mathcal{S}^1_m$-sum to a $\mathcal{S}^1_{m-\bar{m}}$-sum. To this goal notice that because of (3.31), (3.32) and the above remark, each $\sigma \in \mathcal{S}^1_m$ determines a unique $\sigma' \in \mathcal{S}^1_{m-\bar{m}}$ and a unique permutation $\sigma'_2 : \{1, \cdots, m\} \longrightarrow \{1, \cdots, m\}$, where,

$$\mathcal{S}^1_{m-\bar{m}} = \mathcal{S}^1_{m-\bar{m}}(\{p_h, q_h\}_{h=1}^{m-\bar{m}}) := \{\epsilon \in \mathcal{S}_{m-\bar{m}} :\ q_{\epsilon(h)} > p_h,\ h = 1, \cdots, m - \bar{m}\} \tag{3.40}$$

Since the product in (3.39) depends only on σ' and not on σ'_2, it follows that (3.39) is majorized by

$$\sum_{1 \le q_1 < \cdots < q_{m-\bar{m}} \le n,\ q_h > p_h, h=1,\cdots,m-\bar{m}} \ \sum_{\sigma' \in \mathcal{S}^1_{m-\bar{m}}}$$

$$\bar{m}! \lambda^{3n-2m} \int_0^{t/\lambda^2} dt_1 \cdots \int_0^{t_{n-1}} dt_n \prod_{h=1}^{m-\bar{m}} | < S_{t_{p_h}} g, S_{t_{q_{\sigma'(h)}}} g > | \cdot \|g\|^{2\bar{m}} \tag{3.41}$$

where, the factor $\bar{m}!$ comes from that σ'_2 runs over the group of $\bar{m}$-permutations. With the change of variables

$$\lambda^2 t_h = s_h, \quad h = 1, \cdots, n \tag{3.42}$$

(3.41) becomes

$$\sum_{1 \le q_1 < \cdots < q_{m-\bar{m}} \le n,\ q_h > p_h, h=1,\cdots,m-\bar{m}} \ \sum_{\sigma' \in \mathcal{S}^1_{m-\bar{m}}} \bar{m}! \lambda^{n-2m+2(m-\bar{m})}$$

$$\lambda^{-2(m-\bar{m})} \int_0^t ds_1 \cdots \int_0^{s_{n-1}} ds_n \prod_{h=1}^{m-\bar{m}} | < S_{s_{p_h}/\lambda^2} g, S_{s_{q_{\sigma'(h)}}/\lambda^2} g > | \cdot \|g\|^{2\bar{m}} \tag{3.43}$$

Applying the generalized Pulé inequality (Theorem (4.1) of [11]) we majorize (3.43) by

$$\sum_{1 \le q_1 < \cdots < q_{m-\bar{m}} \le n,\ q_h > p_h, h=1,\cdots,m-\bar{m}} \bar{m}! \lambda^{n-2\bar{m}} \frac{t^{n-(m-\bar{m})}}{(n-(m-\bar{m}))!}$$

$$\left(\int_{-\infty}^0 | < g, S_t g > | dt \right)^{m-\bar{m}} \|g\|^{2\bar{m}} \tag{3.44}$$

Since

$$\frac{\bar{m}!}{(n-(m-\bar{m}))!} \le \frac{1}{(n-m)!} \tag{3.45}$$

$$\sum_{1 \le q_1 < \cdots < q_{m-\bar{m}} \le n,\ q_h > p_h, h=1,\cdots,m-\bar{m}} 1 \le \binom{n}{m-\bar{m}} \tag{3.46}$$

$$\sum_{\substack{1\le\alpha_1<\cdots<\alpha_m\le 2n \\ \{\alpha_h\}_{h=1}^m \subset (\{1,\cdots,2n\}\setminus\{j_{2h-1},j_{2h}\}_{h=1}^k)}} 1 \le \binom{2n}{m} \tag{3.47}$$

In conclusion we obtain that (3.13) is dominated by

$$\sum_{m=0}^{2(k\wedge(n-k))} \binom{n}{m-\bar m}\cdot\binom{2n}{m}\cdot\frac{1}{(n-m)!}\cdot\lambda^{n-2\bar m}\cdot t^{n-(m-\bar m)}.$$

$$\left(\int_{-\infty}^0 |<g,S_tg>|dt\right)^{m-\bar m}\cdot\|g\|^{2\bar m}\cdot C^{2(n-m)} \tag{3.47a}$$

with $\bar m \le [n/2]$ (see formula (3.24)). Moreover since the $q_1,\cdots,q_{m-\bar m}$ label the times t_j in which different creators have been used to produce scalar products in the expression (3.13) and since at each time there are 2 creators, we have at least $2(m-\bar m)$ creators. On the other hand the scalar products are m, hence there are at least m annihilation operators. This gives the relation:

$$m + 2(m-\bar m) \le 2n$$

i.e.

$$\bar m \ge \frac{3}{2}m - n \tag{3.48}$$

Summing up:

<u>*THEOREM*(3.4)</u> For each $k,n \in \mathbf{N}$ $k\le n$, $1\le j_1<\cdots<j_k\le n$, the module of (3.12) is majorized by

$$\sum_{m=0}^{2(k\wedge(n-k))} \max_{0\vee(3m/2-n)\le p\le[n/2]} \binom{n}{m-p}\cdot\binom{2n}{m}\cdot\frac{1}{(n-m)!}\cdot t^{n-(m-p)}\cdot\lambda^{n-2p}$$

$$\|g\|_-^{2(m-p)}\cdot\|g\|^{2p}\cdot C^{2(n-m)} \tag{3.49}$$

where,

$$\|g\|_-^2 := \int_{-\infty}^0 |<g,S_tg>|dt \tag{3.50}$$

<u>*PROOF*</u> From the considerations above.

This Theorem gives the uniform estimate, namely:

<u>*THEOREM* (3.5)</u>
(i). For each $g\in K$, $D\in B(H_0)$, there exist a $t_0\in(0,1)$ and $\delta\in(0,1)$ such that for any $t\in(0,t_0)$, the module of

$$< u\otimes W(\lambda\int_{S/\lambda^2}^{T/\lambda^2} S_ufdu)\Phi, \lambda^n\int_0^{t/\lambda^2}dt_1\cdots\int_0^{t_{n-1}}dt_n$$

$$V_2(t_1)\cdots V_2(t_n), v\otimes W(\lambda\int_{S'/\lambda^2}^{T'/\lambda^2} S_uf'du)\Phi > \tag{3.51}$$

has bound δ^n;

(ii). If, for $g\in K$ and $D\in B(H_0)$ given by (1.1), there exists $\epsilon>0$ such that

$$\|D\|\cdot\max\{\|g\|_-^2,\|g\|^2\} < \frac{1}{16+\epsilon} \tag{3.52}$$

then there exists a $\delta'\in(0,1)$ such that for any $t\in(0,1]$, the module of (3.51) has bound δ'^n.

<u>*PROOF*</u> First of all notice that, since $2p\le n$ (cf. (3.24)), we can take away the factor λ^{n-2p} in the formula (3.49). Denote

$$G := \max\{\|g\|_-^2,\|g\|^2\} \tag{3.53}$$

and without loss of generality we suppose that

$$G \ne 0$$

Expanding the product $V_2(t_1)\cdots V_2(t_n)$ according the formula (3.0a), we find that the module of (3.51) is bounded by

$$2^n\cdot\|D\|^n\cdot\sum_{m=0}^{2(k\wedge(n-k))}\max_{0\vee(3m/2-n)\le p\le[n/2]}\binom{n}{m-p}\cdot\binom{2n}{m}\cdot\frac{1}{(n-m)!}$$

$$\|u\|\cdot\|v\|\cdot t^{n-m+p}\cdot G^m\cdot C^{2(n-m)} \tag{3.54}$$

Since

$$\binom{n}{m-p}\le 2^n,\quad \binom{2n}{m}\le 4^n \tag{3.55}$$

(3.54) is less than or equal to

$$\|u\|\cdot\|v\|\cdot 16^n\cdot\|D\|^n\cdot\sum_{m=0}^n\max_{0\vee(3m/2-n)\le p\le[n/2]}\cdot\frac{(Ct)^{n-m}}{(n-m)!}\cdot G^m\cdot t^p$$

$$= \|u\|\cdot\|v\|\cdot(16\|D\|\cdot G)^n\sum_{m=0}^n\frac{(G^{-1}Ct)^{n-m}}{(n-m)!}\cdot\max_{0\vee(3m/2-n)\le p\le[n/2]}t^p \tag{3.55a}$$

Therefore for $t\le 1$, the right hand side of (3.55a) is dominated by

$$\|u\|\cdot\|v\|\cdot(16\|D\|\cdot G)^n\exp(G^{-1}Ct) \tag{3.56}$$

so we finish the proof of (ii).

To prove (i) we see that for each $\eta \in (0,1)$, if $m \le \eta n$ then for $t \in \mathbf{R}_+$, (3.55) implies that

$$\max_{0\vee(3m/2-n)\le p\le[n/2]} \binom{n}{m-p}\cdot\binom{2n}{m}\cdot\frac{1}{(n-m)!}\cdot t^{n-m+p}\cdot G^m\cdot C^{2(n-m)}$$

$$\le \frac{(t\vee 1)^n}{[(1-\eta)n]!}\cdot 8^n\cdot G^m\cdot C^{2(n-m)} \le \frac{(t\vee 1)^n}{[(1-\eta)n]!}\cdot 8^n\cdot (G\bigvee C^2)^n \tag{3.57}$$

On the other hand, if $m \ge \eta n$, then for $t \le 1$, the left hand side of (3.57) is controlled by

$$t^{n-m+3m/2-n}\cdot 8^n\cdot (G\bigvee 1)^n\cdot\frac{1}{(n-m)!}\cdot C^{2(n-m)}$$

$$\le t^{\eta n/2}\cdot 8^n\cdot (G\bigvee 1)^n\cdot\frac{1}{(n-m)!}\cdot C^{2(n-m)} \tag{3.58}$$

Moreover, we know that

$$C_0 := \sup_{0\le m\le n<\infty}\frac{1}{(n-m)!}\cdot C^{2(n-m)}$$

is less than ∞. Therefore (3.54) is less than or equal to

$$n\cdot(16\cdot\|D\|)^n\cdot\|u\|\cdot\|v\|\cdot$$

$$\max\Big\{\frac{1}{[(1-\eta)n]!}\cdot(G\bigvee C^2)^n,\ t^{\eta n/2}\cdot(G\bigvee 1)^n\cdot C_0\Big\} \tag{3.59}$$

Summing up, if we fix $\eta \in (1/2,1)$, we have that for

$$t < \frac{1}{(16\cdot\|D\|\cdot(G\vee 1))^4} < \frac{1}{(16\cdot\|D\|\cdot(G\vee 1))^{2/\eta}} \tag{3.60}$$

one can easily find a $\delta < 1$ such that the quantity (3.59) is majorized by δ^n. Putting

$$t_0 = \frac{1}{(16\cdot\|D\|\cdot(G\vee 1))^4} \tag{3.61}$$

we obtain (i).

In the following we shall call the conditions in Theorem (3.5) as uniformity conditions, then Theorem (3.5) can be stated simply as:

With the uniformity conditions we have the uniform estimate.

§4. The Weak Coupling Limit

Putting togehter the conclusions in §2 and §3, now, we can get the main result of this paper, namely

<u>*THEOREM*(4.1)</u>. For each $g \in K, D \in B(H_0), u, v \in H_0, f, f' \in K, T, S, T', S' \in \mathbf{R}$ with the uniformity conditions, the limit

$$\lim_{\lambda\to 0} < u\otimes W(\lambda\int_{S/\lambda^2}^{T/\lambda^2} S_u f du)\Phi, U^{(\lambda)}(t/\lambda^2)v\otimes W(\lambda\int_{S'/\lambda^2}^{T'/\lambda^2} S_u f'\ du)\Phi > \tag{4.1}$$

exists and is equal to

$$\sum_{n=0}^{\infty} < u, (-D^+D)^n v > \frac{t^n}{n!}\Big(\int_{-\infty}^0 2 < g, S_u g >^2 du\Big)^n$$

$$< W(\chi_{[S,T]}\otimes f)\Psi, W(\chi_{[S',T']}\otimes f')\Psi > \tag{4.2}$$

$$=< u, \exp\Big(-tD^+D\cdot\int_{-\infty}^0 2 < g, S_u g >^2 du\Big)v > \cdot < W(\chi_{[S,T]}\otimes f)\Psi, W(\chi_{[S',T']}\otimes f')\Psi >$$

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