

## Local Perturbations of Conditional Expectations\*

L. ACCARDI†

31, chemin Joseph Aiguier, F-13274 Marseille Cedex 2, France

Submitted by G.-C. Rota

*Contents.* 0. Statement of the problem. I. 1. Existence of states with preassigned conditional expectations. 2. Locally normal states. 3. Extremal  $(E_\alpha)$ -invariant states. 4. Integral decomposition. 5. Uniqueness. 6. Ergodic properties. 7. Invariant states.

II. 1. Locality and Markovianity. 2. Local conditional expectations. 3. Conditioned martingales. 4. Uniqueness: factorizable case.

III. 1. The local  $\sigma$ -algebras of the free Euclidean field. 2. The global Markov property.

### 0. STATEMENT OF THE PROBLEM

The problem of determining the existence or uniqueness of a measure on a function space with preassigned local characteristics is well known. If the local characteristics are the restrictions of the measure on an increasing net of  $\sigma$ -algebras, the Daniell–Kalmogorof–Prohorof theorem (cf. [4]) gives a general criterion.

In a series of memoirs, starting from 1968, Dobrushin [5] considered the problem of describing the probability measures with a preassigned family of conditional probabilities with respect to a given decreasing net of  $\sigma$ -algebras. The main feature of this problem, with respect to the previous one, is the lack of uniqueness of the solution even in the case of its existence. The attempt to extend Dobrushin's techniques from the case of discrete stochastic fields to that of continuous ones is motivated by Euclidean quantum field theory. In 2-dimensional Euclidean boson field theory it is given a standard Borel space  $(\Omega, O, \mu^0)$  and an increasing family  $(O_A)$  of sub- $\sigma$ -algebras of  $O$  indexed by the family  $\mathcal{F}$  of bounded open (regular) subsets of  $\mathbb{R}^2$ ; sub- $\sigma$ -algebras  $(O_{\mathbb{C}A})$  of  $O$  are indexed by the complements of elements  $A \in \mathcal{F}$  and, for each  $A \in \mathcal{F}$ , it is given a random variable  $U_A$  measurable for  $O_A$ , and satisfying

$$U_{A_1 \cup A_2} = U_{A_1} + U_{A_2}; \quad A_1 \cap A_2 = \emptyset. \quad (0.0)$$

\* Revised: November 1976.

† On leave of absence from Laboratorio di Cibernetica, Consiglio Nazionale Della Ricerche, 80072 Arco Felice, Napoli, Italy.

One studies the limits (taken in various ways) when  $\Lambda' \uparrow \mathbb{R}^2$ , absorbing eventually all bounded sets, of the perturbed measures (cf. [12], for example)

$$\mu^{(\Lambda')}(\cdot) = \mu^0(e^{-U_{\Lambda'} \cdot}) / \mu^0(e^{-U_{\Lambda'}}) \tag{0.1}$$

which satisfy the Osterwalder–Schrader axioms [18]. The properties of  $U_{\Lambda}$  imply that any weak limit (on the algebra  $\mathcal{A}$ , cf. Part III, Section 1)  $\mu$  of perturbed measures of the form (0.1) satisfies the following conditions:

$$\mu \cdot E_{\mathbb{G}_{\Lambda}} = \mu; \quad \mu \upharpoonright O_{\Lambda} < \mu^0 \upharpoonright O_{\Lambda}; \quad \Lambda \in \mathcal{F}, \tag{0.2}$$

where  $E_{\mathbb{G}_{\Lambda}}$  is defined by

$$E_{\mathbb{G}_{\Lambda}}(\cdot) = E_{\mathbb{G}_{\Lambda}}^0(k_{\Lambda} \cdot); \quad k_{\Lambda} = \frac{e^{-U_{\Lambda}}}{E_{\mathbb{G}_{\Lambda}}^0(e^{-U_{\Lambda}})} \tag{0.3}$$

and  $E_{\mathbb{G}_{\Lambda}}^0$  denotes the conditional expectation, with respect to  $O_{\mathbb{G}_{\Lambda}}$ , associated with  $\mu^0$ . In [13], Guerra, *et al.* (cf. also [7]) proposed to consider Eqs. (0.2) as an intrinsic approach to  $P(\phi)_2$  Euclidean quantum field theories; i.e., to solve Eqs. (0.2) and prove that (some of) the solutions satisfy the Osterwalder–Schrader axioms [18]. If  $\mu$  is locally absolutely continuous with respect to  $\mu^0$ , Eqs. (0.2) have a meaning on the norm closure of the union of the algebras  $L^\infty(\Omega, O_{\Lambda}, \mu_{\Lambda}^0)$ ,  $\Lambda \in \mathcal{F}$ . This leads to the following general situation: given a  $C^*$ -algebra  $\mathcal{A}$  which is the norm closure of the union of an increasing net  $(\mathcal{A}_{\alpha})_{\alpha \in \mathcal{F}}$  of sub-von Neumann algebras of  $\mathcal{A}$ ; a decreasing net  $(\mathcal{A}'_{\alpha})$  of sub- $C^*$ -algebras of  $\mathcal{A}$  ( $\mathcal{A}'_{\alpha} \simeq \mathcal{A} \cap L^\infty(\Omega, O_{\alpha}, \mu_{\alpha}^0)$ ;  $\alpha' = \mathbb{R}^2 - \alpha$ ); a projective (cf. (I.1.1)) family  $(E_{\alpha'})$  of locally normal conditional expectations  $E_{\alpha'}: \mathcal{A} \rightarrow \mathcal{A}'_{\alpha'}$ ; one looks at the locally normal solutions of the equations

$$\varphi \cdot E_{\alpha'} = \varphi; \quad \alpha \in \mathcal{F}. \tag{0.4}$$

One is interested in the structure of the solutions of (0.4), the Euclidean invariant ones, the integral decomposition by means of extremal states, and, most of all, the criteria of uniqueness. Such problems are studied in Part I. The conditional expectations (0.3) enjoy an important locality property, singled out by Nelson (cf., for example, [17])—the Markov property—which can be expressed by the relation

$$E_{\alpha'}(\mathcal{A}_{\alpha}) \subseteq \mathcal{A}_{\bar{\alpha}} \quad (\bar{\alpha} = \text{closure of } \alpha).$$

In Part II we discuss a more general concept of locality, namely,

$$E_{\alpha'}(\mathcal{A}_{\alpha}) \subseteq \mathcal{A}_{\beta}; \quad \text{for some } \beta \supseteq \bar{\alpha}; \quad \beta \in \mathcal{F}$$

and prove that, under some assumption on the local algebras  $\mathcal{A}_{\alpha}$ , this property is equivalent to (a generalization of) Dobrushin’s ( $d$ )-Markov property. Under

some locality (or regularity) assumption on the  $E_\alpha$ 's it is shown that any projective family  $(E_{\alpha'})$  is of the form

$$E_{\alpha'}(\cdot) = E_\alpha^0(k_{\alpha'} \cdot),$$

where  $E_\alpha^0$  is the conditional expectation associated with  $\mu^0$ . ( $\mathcal{A}_\alpha \simeq L^\infty(\Omega, O_\alpha, \mu_\alpha^0)$ );  $k_\alpha = dE_{\alpha'}/dE_\alpha^0$  is a local (or quasi-local) perturbation of  $E_\alpha^0$ , which can be considered as the Radon-Nikodym derivative of  $E_{\alpha'}$  with respect to  $E_\alpha^0$ ; and the  $(k_\alpha)$  satisfy

$$k_\beta = k_\alpha \cdot E_\alpha^0(k_\beta); \quad \alpha < \beta. \quad (0.5)$$

A family of random variables  $(k_\alpha)$  satisfying (0.5) is called a *conditional martingale*. The structure of Markovian conditional martingales  $(k_\alpha)$  with respect to a Markovian measure  $\mu^0$  is determined and it is shown that they are determined by a "potential function"  $(U_\alpha)$  satisfying (0.0) up to an additive "gauge transformation" which leaves the  $k_\alpha$  invariant.

In Part III local perturbations of the conditional expectations of the  $d$ -dimensional free Euclidean field are considered and it is proved that the solutions of Eqs. (0.2) satisfy the (hyperplane) Markov property.

## I

### 1. EXISTENCE OF STATES WITH PREASSIGNED CONDITIONAL EXPECTATIONS

In the following  $\mathcal{A}$  will denote a  $C^*$ -algebra (by this we shall always mean a  $C^*$ -algebra with unit);  $\mathcal{S}(\mathcal{A})$  the set of states (positive normalized linear functionals) on  $\mathcal{A}$ ;  $\mathcal{F}$  a set partially ordered by a filtering increasing relation  $<$  (i.e., if  $\alpha, \beta \in \mathcal{F}$  there is a  $\gamma \in \mathcal{F}$  such that  $\alpha < \gamma, \beta < \gamma$ );  $(\mathcal{A}_{\alpha'})_{\alpha \in \mathcal{F}}$  a family of sub- $C^*$ -algebras of  $\mathcal{A}$  indexed by the "opposite" of  $\mathcal{F}$  (i.e., the set of  $\alpha'$  such that  $\alpha \in \mathcal{F}$  with the order relation  $\beta' < \alpha' \Leftrightarrow \beta > \alpha \Leftrightarrow \alpha < \beta$ ) and such that  $\alpha < \beta \Rightarrow \mathcal{A}_{\alpha'} \supseteq \mathcal{A}_{\beta'}$ . By a conditional expectation from  $\mathcal{A}$  to a sub- $C^*$ -algebra  $\mathcal{A}_0 \subseteq \mathcal{A}$  we shall mean a norm one projector  $E: \mathcal{A} \rightarrow \mathcal{A}_0$ . Tomiama's theorem [23] asserts that such a projection enjoys all the properties (with the exception, at most, of normality) which, according to a result of Moy [15], characterize the usual probabilistic concept of conditional expectation and which are assumed by Umegaki [24] to be a definition of conditional expectation between two arbitrary  $C^*$ -algebras.

For each  $\alpha \in \mathcal{F}$  a conditional expectation  $E_{\alpha'}: \mathcal{A} \rightarrow \mathcal{A}_{\alpha'}$  is given so that the projective condition

$$\alpha < \beta \Rightarrow E_{\beta'} \cdot E_{\alpha'} = E_{\beta'} \quad (I.1.1)$$

is satisfied. A family of conditional expectations  $(E_{\alpha'})_{\alpha \in \mathcal{F}}$  satisfying (I.1.1) will be called projective.

A state  $\varphi \in \mathcal{S}(\mathcal{A})$  will be called compatible with the family  $(E_{\alpha'})$ , or simply  $(E_{\alpha'})$ -invariant, if

$$\varphi \cdot E_{\alpha'} = \varphi; \quad \forall \alpha \in \mathcal{F}. \tag{I.1.2}$$

**THEOREM 1.1.** *In the above notation the set  $(E_{\alpha'})$ -invariant states—which from now on will be denoted  $\mathcal{S}_I$ —is nonempty.*

*Proof.* For each  $\alpha \in \mathcal{F}$  the weakly compact convex set  $\mathcal{S}(\mathcal{A})$  is mapped into itself by the weakly continuous map  $\varphi \mapsto \varphi \cdot E_{\alpha'}$ ; denote  $\mathcal{S}_{\alpha'} = \mathcal{S}(\mathcal{A}) \circ E_{\alpha'}$ .  $\mathcal{S}_{\alpha'}$  as a nonempty weakly closed set. Moreover, if  $\alpha < \beta$  and  $\varphi \in \mathcal{S}_{\beta'}$  then

$$\varphi \cdot E_{\alpha'} = \varphi \cdot E_{\beta'} \cdot E_{\alpha'} = \varphi \cdot E_{\beta'} = \varphi.$$

Therefore  $\varphi \in \mathcal{S}_{\alpha'}(\mathcal{A})$  and  $\mathcal{S}_{\beta'} \subseteq \mathcal{S}_{\alpha'}$ . Thus the family  $(\mathcal{S}_{\alpha'})_{\alpha \in \mathcal{F}}$  of  $w$ -closed subsets of  $\mathcal{S}(\mathcal{A})$  has the finite intersection property, hence

$$\mathcal{S}_I = \bigcap_{\alpha \in \mathcal{F}} \mathcal{S}_{\alpha'} \neq \emptyset.$$

Clearly each  $\varphi \in \mathcal{S}_I$  is compatible with  $(E_{\alpha'})$  and this ends the proof.

*Remark.* Throughout the paper “weak topology” on  $\mathcal{A}^*$  (= the dual of  $\mathcal{A}$ ) will denote the weak topology induced on  $\mathcal{A}^*$  by the coupling  $\langle \mathcal{A}^*, \mathcal{A} \rangle$  ( $w^*$ -topology).

**LEMMA 1.2.** *Let  $\mathcal{F}_0$  be a subnet of  $\mathcal{F}$ ,  $(\psi_\gamma)_{\gamma \in \mathcal{F}_0}$  a family of states on  $\mathcal{A}$ ; and  $\psi$  a state on  $\mathcal{A}$  such that*

$$\psi = w\text{-}\lim_{\gamma \in \mathcal{F}_0} \psi_\gamma \cdot E_{\gamma'}.$$

*Then  $\psi$  is  $(E_{\alpha'})$ -invariant.*

*Proof.* For each  $\alpha \in \mathcal{F}$ , the projectivity condition (I.1.1) implies

$$\psi \cdot E_{\alpha'} = w\text{-}\lim_{\mathcal{F}_0} \psi_\gamma \cdot E_{\gamma'} \cdot E_{\alpha'} = w\text{-}\lim_{\mathcal{F}_0} \psi_\gamma \cdot E_{\gamma'} = \psi,$$

hence  $\psi$  is  $(E_{\alpha'})$ -invariant  $\forall \alpha \in \mathcal{F}$ .

## 2. LOCALLY NORMAL STATES

In the above notation let  $(\mathcal{A}_\alpha)_{\alpha \in \mathcal{F}}$  be an increasing net of  $C^*$ -subalgebras of  $\mathcal{A}$  ( $\alpha < \beta \Rightarrow \mathcal{A}_\alpha \subseteq \mathcal{A}_\beta$ ). Assume that

- (i) each  $\mathcal{A}_\alpha$  is a von Neumann algebra,
- (ii)  $\mathcal{A}$  = norm closure of  $\bigcup_{\alpha \in \mathcal{F}} \mathcal{A}_\alpha$ ,
- (iii)  $\mathcal{F}$  is countable generated (i.e., if  $\mathcal{F}_0$  is any subnet of  $\mathcal{F}$ , there exists a sequence  $(\alpha_n)$  in  $\mathcal{F}_0$  such that for each  $\alpha \in \mathcal{F}$ ,  $\alpha < \alpha_n$  for some  $n$ ).

Examples of countably generated nets  $\mathcal{F}$  are: (1) the net of finite subsets of  $\mathbb{Z}^d$ ; (2) the net of bounded open subsets of  $\mathbb{R}^d$ , the subnets being given by the

families  $\mathcal{F}_0$  such that for each bounded open set  $A$  in  $\mathcal{F}$ ,  $A \subseteq A_0$  for some  $A_0 \in \mathcal{F}$  (in both cases the order being given by inclusion). The conditional expectation  $E_{\alpha'}$  (resp. the state  $\varphi$ ) is called *locally normal* if for each  $\beta \in \mathcal{F}$ ,  $E_{\alpha'} \upharpoonright \mathcal{A}_\beta$  (resp.  $\varphi \upharpoonright \mathcal{A}_\beta$ ) is a normal map.

Under the above assumptions, if the topology of weak convergence on locally normal states is metrizable, then the set of limits  $w\text{-}\lim_{\mathcal{F}_0} \psi \cdot E_{\alpha'}$  where  $\mathcal{F}_0$  is a countable subnet of  $\mathcal{F}$  is nonempty for any  $\varphi \in \mathcal{S}(\mathcal{A})$  and, because of Lemma (1.2), such limits are contained in the set  $\mathcal{S}_I$  of  $(E_{\alpha'})$ -invariant states. By the sequential completeness of locally normal states (cf. [3]) we conclude that if the family  $(E_{\alpha'})$  is locally normal, the set of  $(E_{\alpha'})$ -invariant, locally normal states on  $\mathcal{A}$  is nonempty.

*Remark.* The metrizability of locally normal states is not realizable in many important cases. However, for the existence of locally normal states in  $\mathcal{S}_I$ , it is sufficient that on a closed,  $(E_{\alpha'})$ -invariant subset of locally normal states there is a metrizable topology finer than the weak topology. In some cases such a metrizable topology is provided by the ‘‘Wasserstein distance’’ (cf. [6]). The problem of existence and uniqueness of locally normal states will be discussed elsewhere.<sup>1</sup>

### 3. EXTREMAL $(E_{\alpha'})$ -INVARIANT STATES

In the notation of Theorem 1.1 let  $\mathcal{S}_I$  denote the set of  $(E_{\alpha'})$ -invariant states.  $\mathcal{S}_I$  is a nonempty  $w$ -compact convex subset of  $\mathcal{S}(\mathcal{A})$ . Extreme points of  $\mathcal{S}_I$  are called *ergodic*. Our analysis of the extremal points of  $\mathcal{S}_I$  is based on the analogy between the systems  $\{\mathcal{A}, (E_{\alpha'})_{\alpha \in \mathcal{F}}\}$  and  $\{\mathcal{A}, G\}$ —of a  $C^*$ -algebra acted upon by a group  $G$  of  $*$ -automorphisms—with the norm one projectors  $E_{\alpha'}$  playing the role of the  $*$ -automorphisms  $g \in G$ . The known results on  $G$ -Abelian systems, due to Ruelle [19] and Doplicher *et al.* [8], are extended without difficulty to  $(E_{\alpha'})$ -Abelian systems. For  $\varphi \in \mathcal{S}(\mathcal{A})$ , denote  $\{\mathcal{H}_\varphi, \pi_\varphi, 1_\varphi\}$  the Gelfand–Naimark–Segal representation of  $\mathcal{A}$  with respect to  $\varphi$  [20, p. 40]; denote

$$\mathcal{A}_\infty^\varphi = \bigcap_{\alpha \in \mathcal{F}} \pi_\varphi(\mathcal{A}_{\alpha'})''; \quad \mathcal{H}_\infty^\varphi = \bigcap_{\alpha \in \mathcal{F}} [\pi_\varphi(\mathcal{A}_{\alpha'})'' \cdot 1_\varphi]$$

$([\pi_\varphi(\mathcal{A}_{\alpha'}) \cdot 1_\varphi])$  = norm closure in  $\mathcal{H}_\varphi$  of  $\pi_\varphi(\mathcal{A}_{\alpha'}) \cdot 1_\varphi$ ; and  $e_\varphi: \mathcal{H}_\varphi \rightarrow \mathcal{H}_\infty^\varphi$ , the orthogonal projection. If  $\varphi \in \mathcal{S}_I$  Kadison’s inequality implies that the map  $\pi_\varphi(a) \cdot 1_\varphi \mapsto \pi_\varphi(E_{\alpha'}(a)) \cdot 1_\varphi$  that satisfies

$$\begin{aligned} \|\pi_\varphi(E_{\alpha'}(a)) \cdot 1_\varphi\|^2 &= \varphi(E_{\alpha'}(a)^* \cdot E_{\alpha'}(a)) \leq \varphi(E_{\alpha'}(a^*a)) \\ &= \|\pi_\varphi(a) \cdot 1_\varphi\|^2 \end{aligned}$$

<sup>1</sup> The author is grateful to Jean Bellissard for his correspondence concerning this argument.

is well defined, and its extension to  $\mathcal{H}_\varphi$  defines the orthogonal projection  $e_\alpha^\varphi: \mathcal{H}_\varphi \rightarrow [\pi_\alpha(\mathcal{A}_\alpha) \cdot 1_\varphi]$ . One has  $e_\varphi = \inf e_\alpha^\varphi = \text{strong-lim } e_\alpha^\varphi$ ; and  $\pi_\varphi(E_\alpha(a)) \cdot e_\alpha^\varphi = e_\alpha^\varphi \cdot \pi_\varphi(a) \cdot e_\alpha^\varphi$ ;  $a \in \mathcal{A}$ . Moreover

$$\mathcal{H}_\infty^\varphi = \{\xi \in \mathcal{H}_\varphi: e_\alpha^\varphi \xi = \xi; \forall \alpha \in \mathcal{F}\}.$$

From now on the net  $\mathcal{F}$  will be assumed to be countably generated, (cf. Part I, Section 2(iii)). The system  $\{\mathcal{A}, (\mathcal{A}_\alpha)\}$  is called *asymptotically Abelian* if, for any subnet  $\mathcal{F}_0 \subseteq \mathcal{F}$ ,

$$\lim_{\mathcal{F}_0} \|[a, b_{\beta'}]\| = 0; \quad b_{\beta'} \in \mathcal{A}_{\beta'}; \quad \|b_{\beta'}\| \leq 1; \quad a \in \mathcal{A}$$

( $[a, b] = ab - ba$ ). The system  $\{\mathcal{A}, (E_\alpha)\}$  is called  $(E_\alpha)$ -Abelian if  $\forall \varphi \in \mathcal{S}_1$ ,  $e_\varphi \cdot \pi_\varphi(\mathcal{A}) \cdot e_\varphi$  generates a commutative algebra. Asymptotic Abelianness implies  $(E_\alpha)$ -Abelianness.

LEMMA 3.1. *Let  $\{\mathcal{A}, (E_\alpha)\}$  be an asymptotically Abelian system and  $\varphi \in \mathcal{S}_1$ . Then*

(i) *There exists a conditional expectation strongly continuous on bounded sets  $E_\infty^\varphi: \pi_\varphi(\mathcal{A})'' \rightarrow \mathcal{A}_\infty^\varphi$  satisfying*

$$e_\varphi(\pi_\varphi(a) \cdot 1_\varphi) = E_\infty^\varphi(\pi_\varphi(a)) \cdot 1_\varphi. \tag{I.3.1}$$

(ii) *For every subnet  $\mathcal{F}_0$  of  $\mathcal{F}$ ,*

$$\mathcal{H}_\infty^\varphi = [\mathcal{A}_\infty^\varphi \cdot 1_\varphi] = \left[ \bigcap_{\alpha \in \mathcal{F}_0} \pi_\alpha(\mathcal{A}_\alpha)'' \cdot 1_\varphi \right].$$

*Proof.* (i) Because of asymptotic Abelianness  $\forall a, b \in \mathcal{A}$ ,

$$\begin{aligned} \pi_\varphi(b) \cdot e_\varphi(\pi_\varphi(a) \cdot 1_\varphi) &= \lim_\alpha \pi_\varphi(b) \cdot \pi_\alpha(E_\alpha(a)) \cdot 1_\varphi \\ &= \lim_\alpha \pi_\alpha(E_\alpha(a)) \cdot \pi_\varphi(b) \cdot 1_\varphi. \end{aligned}$$

Hence the map  $E_\infty^\varphi: \pi_\varphi(\mathcal{A})'' \rightarrow \mathcal{A}_\infty^\varphi$ , defined by  $E_\infty^\varphi(\pi_\varphi(a)) = s\text{-lim } \pi_\alpha(E_\alpha(a))$ , satisfies equality (I.3.1). If  $(\pi_\alpha(a_j)): a_j \in \mathcal{A}$  converges strongly to  $\bar{a} \in \pi_\varphi(\mathcal{A})''$ , then if  $b \in \mathcal{A}$ ,

$$E_\infty^\varphi(\pi_\varphi(a_j)) \cdot \pi_\varphi(b) \cdot 1_\varphi = \pi_\varphi(b) \cdot e_\varphi(\pi_\varphi(a_j) \cdot 1_\varphi) \rightarrow \pi_\varphi(b) e_\varphi(\bar{a} \cdot 1_\varphi).$$

By Kaplansky's density theorem [20, p. 22] one can assume  $\|\pi_\varphi(a_j)\| \leq \|\bar{a}\|$ , therefore  $E_\infty^\varphi(\pi_\varphi(a_j))$  converges strongly to a limit depending only on  $\bar{a}$ , and this allows us to extend  $E_\infty^\varphi$  to a map,  $\pi_\varphi(\mathcal{A})'' \rightarrow \mathcal{A}_\infty^\varphi$ , still denoted by  $E_\infty^\varphi$  which is strongly continuous on bounded sets. Clearly  $E_\infty^\varphi$  is a conditional expectation.

(ii) From (i) it follows that

$$e_\varphi(\pi_\varphi(\mathcal{A}) \cdot 1_\varphi) = E_\infty^\varphi(\pi_\varphi(\mathcal{A})) \cdot 1_\varphi \subseteq \mathcal{A}_\infty^\varphi \cdot 1_\varphi$$

which implies  $\mathcal{H}_\infty^\varphi \subseteq [\mathcal{A}_\infty^\varphi \cdot 1_\varphi] \subseteq [\bigcap_{\alpha \in \mathcal{F}_0} \pi_\varphi(\mathcal{A}_\alpha)' \cdot 1_\varphi]$ . To prove the converse inclusion let  $\bar{a} \in \bigcap_{\alpha \in \mathcal{F}_0} \pi_\varphi(\mathcal{A}_\alpha)''$  and for each  $\alpha \in \mathcal{H}_0$  let  $a_{\alpha'} \in \mathcal{A}_{\alpha'}$  such that  $\bar{a} \cdot 1_\varphi = s\text{-}\lim_{\alpha \in \mathcal{F}_0} \pi_\varphi(a_{\alpha'}) \cdot 1_\varphi$ . Then, for each  $\beta \in \mathcal{F}$  (since  $\mathcal{F}_0$  is a subset of  $\mathcal{F}$ )

$$\begin{aligned} e_\beta^\varphi(\bar{a} \cdot 1_\varphi) &= s\text{-}\lim_{\alpha \in \mathcal{F}_0} \pi_\varphi(E_{\beta'}(a_{\alpha'})) 1_\varphi \\ &= s\text{-}\lim_{\alpha \in \mathcal{F}_0} \pi_\varphi(a_\alpha) \cdot 1_\varphi = \bar{a} \cdot 1_\varphi, \end{aligned}$$

hence  $e_\varphi(\bar{a} \cdot 1_\varphi) = \bar{a} \cdot 1_\varphi$  and  $[\mathcal{A}_\infty^\varphi \cdot 1_\varphi] \subseteq \mathcal{H}_\infty^\varphi$ . And this ends the proof.

**THEOREM 3.2.** *Let  $\{\mathcal{A}, (E_\alpha)\}$  be an asymptotically Abelian system;  $\varphi \in \mathcal{S}_1$ . Then  $\mathcal{S}_1$  is a Choquet simplex and for any  $\varphi \in \mathcal{S}_1$  the following properties are equivalent:*

- (i1)  $\varphi$  is an extremal point in  $\mathcal{S}_1$ ,
- (i2)  $\{\pi_\varphi(\mathcal{A}), (I_\alpha^\varphi)\}' = \mathbb{C} \cdot 1_{\mathcal{H}_\varphi}$  ( $\{\cdots\}'$  denoting the commutant on  $\mathcal{H}_\varphi$ ),
- (i3)  $\mathcal{H}_\infty^\varphi = \mathbb{C} \cdot 1_\varphi$ ,
- (i4)  $\mathcal{A}_\infty^\varphi = \mathbb{C} \cdot 1_{\mathcal{H}_\varphi}$ ,
- (i5) for every  $\epsilon > 0$  and  $a \in \mathcal{A}$  there exists a  $\beta_0 = \beta_0(\epsilon, a)$  such that, for any  $\beta \succ \beta_0$ ,

$$|\varphi(a \cdot b_{\beta'}) - \varphi(a) \cdot \varphi(b_{\beta'})| \leq \epsilon \cdot \|\pi_\varphi(b_{\beta'})\|; \quad \forall b_{\beta'} \in \mathcal{A}_{\beta'}.$$

*Proof.* The fact that  $\mathcal{S}_1$  is a Choquet simplex and the equivalences (i1)  $\Leftrightarrow$  (i2)  $\Leftrightarrow$  (i3) can be proved by adapting to  $(E_\alpha)$ -Abelian systems the arguments which establish the corresponding result in the case of  $(E_\alpha)$ -Abelian systems (cf., for example, Sakai [20, Chapter 3]). The equivalence (i4)  $\Leftrightarrow$  (i5) is proved as in [14, Proposition (2.3)].

(i4)  $\Rightarrow$  (i3) follows from assertion (ii) in Lemma 3.1.

(i3)  $\Rightarrow$  (i4). Let  $p \in \mathcal{A}_\infty^\varphi$  be a projection. If (i3) holds, then

$$p \cdot 1_\varphi = \langle 1_\varphi, p \cdot 1_\varphi \rangle \cdot 1_\varphi = p^2 \cdot 1_\varphi = \langle 1_\varphi, p \cdot 1_\varphi \rangle^2 \cdot 1_\varphi.$$

Thus  $p \cdot 1_\varphi = 0$  or  $1$ . Asymptotic Abelianness implies that  $\mathcal{A}_\infty^\varphi \subseteq$  (center of  $\pi_\varphi(\mathcal{A})''$ ), hence by the cyclicity of  $1_\varphi$  for  $\pi_\varphi(\mathcal{A})$ ,  $p = 0$  or  $1$ .

#### 4. INTEGRAL DECOMPOSITION

If the system  $\{\mathcal{A}, (E_\alpha)\}$  is asymptotically Abelian and  $\mu \in \mathcal{S}_1$ , then  $\mathcal{A}_x^\mu$  is Abelian. If  $\mathcal{G}(K)$  is a functional realization of  $\mathcal{A}_x^\mu$ , the restriction on  $\mathcal{A}_x^\mu$  of the

state  $\bar{a} \in \pi_\mu(\mathcal{A})^n \mapsto \langle 1_\mu, \bar{a} \cdot 1_\mu \rangle$  defines a Radon measure  $\mu_\infty$  on  $K$ . For each  $a \in \mathcal{A}$  one has

$$\begin{aligned} \mu(a) &= \mu_\infty(E_\infty^\varphi(\pi_\mu(a))) \\ &= \int_k E_\infty^\mu(\pi_\mu(a))(\omega) \mu_\infty(d\omega). \end{aligned} \tag{I.4.1}$$

The definition of  $E_\infty^\mu$  (cf. Lemma 3.1) implies that  $E_\infty^\mu \pi_\mu(E_{\alpha'}(\cdot)) = E_\infty^\varphi \cdot \pi_\mu$  hence the states  $E_\omega^\mu$  of  $\mathcal{A}$  defined by

$$E_\omega^\mu(a) = E_\infty^\mu(\pi_\mu(a))(\omega); \quad \omega \in K;$$

belong to  $\mathcal{S}_I$ . One easily verifies that they satisfy condition (i5) of Theorem 3.2 for  $\mu_\infty$ —almost every  $\omega$ . Therefore equality (I.4.1) gives an integral decomposition of  $\mu$  by means of extremal states of  $\mathcal{S}_I$ . In the Abelian case (I.4.1) is the Dynkin–Föllmer decomposition (cf. [3, 10]).

### 5. UNIQUENESS

We keep the notation of the preceding sections. Throughout this section. The net  $\mathcal{F}$  is assumed to be countably generated.

**THEOREM 5.1.** *The following assertions are equivalent:*

- (i)  $\mathcal{S}_I$  consists of a single point.
- (ii)  $\bigcap_\alpha \mathcal{A}_{\alpha'} = \mathbb{C} \cdot 1$  and the net  $(E_{\alpha'})$  converges pointwise weakly in  $\mathcal{A}$ .
- (iii)  $\bigcap_\alpha \mathcal{A}_{\alpha'} = \mathbb{C} \cdot 1$  and the net  $(E_{\alpha'})$  converges pointwise in norm in  $\mathcal{A}$ .

*Proof.* Clearly (iii)  $\Rightarrow$  (ii). If (ii) holds, denoting  $E_\infty = w\text{-lim } E_{\alpha'}$ , one has  $E_\infty \cdot E_{\beta'} = E_{\beta'} \cdot E_\infty = E_\infty \forall \beta \in \mathcal{F}$ , therefore  $E_\infty$  is a conditional expectation on  $\bigcap_\alpha \mathcal{A}_\alpha = \mathbb{C} \cdot 1$ , hence  $E_\infty(a) = \varphi(a) \cdot 1; a \in \mathcal{A}$  with  $\varphi \in \mathcal{S}_I$ . If  $\psi \in \mathcal{S}_I$  then

$$\psi = w\text{-lim } \psi \cdot E_{\alpha'} = \psi(1) \cdot \varphi = \varphi.$$

Thus (ii)  $\Rightarrow$  (i).

To prove that (i)  $\Rightarrow$  (iii) assume first that there is an  $a \in \mathcal{A}^+$  such that the net  $(E_{\alpha'}(a))$  is not Cauchy in norm. Then there exists an  $\epsilon > 0$  such that for each  $\beta_0 \in \mathcal{F}$  there is a  $\gamma > \beta_0$  and a  $\delta(\gamma) > \beta_0$  for which

$$\|E_{\gamma'}(a) - E_{\delta(\gamma)'}(a)\| > \epsilon.$$

Since  $E_{\gamma'}(a) - E_{\delta(\gamma)'}(a)$  is self-adjoint this implies (cf. [20, p. 10]) that there exists a  $\psi_\gamma \in \mathcal{S}(\mathcal{A})$  such that

$$|\psi_\gamma(E_{\gamma'}(a)) - \psi_\gamma(E_{\delta(\gamma)'}(a))| > \epsilon/2.$$



Hence there exists a subnet  $\mathcal{F}_1$  of  $\mathcal{F}$  and  $\chi_1, \chi_2 \in \mathcal{S}(\mathcal{A})$  such that

$$\chi_1 = w\text{-}\lim_{\mathcal{F}_1} \psi_\gamma \cdot E_{\gamma'}; \quad \chi_2 = w\text{-}\lim_{\gamma \in \mathcal{F}_1} \psi_\gamma \cdot E_{\delta(\gamma)'}$$

By Lemma 1.2,  $\chi_1, \chi_2 \in \mathcal{S}_I$  and

$$|\chi_1(a) - \chi_2(a)| \geq \epsilon/2.$$

Therefore the pointwise norm convergence of  $(E_{\alpha'})$  is a necessary condition for  $\mathcal{S}_I$  to consist of a single state. If  $(E_{\alpha'})$  converges pointwise in norm its limit  $E_\infty$  is a conditional expectation on  $\bigcap_\alpha \mathcal{A}_{\alpha'}$  and for every state  $\psi_x$  on  $\bigcap_\alpha \mathcal{A}_{\alpha'}$ ,  $\psi_x \cdot E_\infty \in \mathcal{S}_I$ . Therefore, under our assumption,  $\mathcal{S}_I$  consists of a single state if and only if  $\mathcal{S}(\bigcap_\alpha \mathcal{A}_{\alpha'})$  contains a single state, i.e., if  $\bigcap_\alpha \mathcal{A}_{\alpha'} = \mathbb{C} \cdot 1$  and this ends the proof.

**PROPOSITION 5.2.** *If  $\mu \in \mathcal{S}_I$  is the only state on  $\mathcal{A}$  compatible with  $(E_{\alpha'})$  then*

$$\lim_{\beta} \frac{1}{\mu(b_{\beta'})} \mu((b_{\beta'})^{1/2} a \cdot (b_{\beta'})^{1/2}) = \mu(a); \quad a \in \mathcal{A};$$

*uniformly in  $b_{\beta'}$  such that  $b_{\beta'} \in \mathcal{A}_{\beta'}^+$ ,  $\mu(b_{\beta'}) > 0$ .*

*Proof.* Given a family  $(b_{\beta'})$  such that  $b_{\beta'} \in \mathcal{A}_{\beta'}^+$ ,  $\mu(b_{\beta'}) > 0$ , define the state  $\psi_{\beta'} \in \mathcal{S}(\mathcal{A})$  by

$$\psi_{\beta'}(a) = \mu((b_{\beta'})^{1/2} \cdot a \cdot (b_{\beta'})^{1/2}) / \mu(b_{\beta'}); \quad a \in \mathcal{A};$$

and assume that there are an  $\epsilon > 0$ , an  $a \in \mathcal{A}^+$ , and a subnet  $\mathcal{F}_0$  of  $\mathcal{F}$  such that

$$|\psi_{\beta'}(a) - \mu(a)| \geq \epsilon; \quad \beta \in \mathcal{F}_0.$$

Then there will exist a subnet  $\mathcal{F}_1$  of  $\mathcal{F}_0$  and a  $\psi \in \mathcal{S}(\mathcal{A})$  such that  $w\text{-}\lim_{\beta \in \mathcal{F}_1} \psi_{\beta'} = \psi$ . For every  $\beta \in \mathcal{F}_0$ ,  $\psi_{\beta'} \cdot E_{\beta'} = \psi_{\beta'}$ , hence by Lemma 1.2,  $\psi \in \mathcal{S}_I$  and

$$|\psi(a) - \mu(a)| \geq \epsilon$$

contrary to the assumption  $\mathcal{S}_I = \{\mu\}$ .

**COROLLARY 5.3.** *Let  $\mu \in \mathcal{S}_I$  and assume that the family  $(\mathcal{A}_{\alpha'})$  satisfies the following condition*

$$\lim_{\beta} \|\pi_\mu([a \cdot (b_{\beta'})^{1/2}] \cdot 1_\mu)\| = 0; \quad a \in \mathcal{A}; \quad (*)$$

*uniformly in  $b_{\beta'} \in \mathcal{A}_{\beta'}^+$  such that  $\|\pi_\mu(b_{\beta'})^{1/2} \cdot 1_\mu\| = \mu(b_{\beta'})^{1/2} = 1$ . Then if  $\mathcal{S}_I = \{\mu\}$ , for every  $a \in \mathcal{A}$ ,*

$$\lim_{\beta} \mu(\alpha \cdot b_{\beta'}) / \mu(b_{\beta'}) = \mu(a) \quad (\text{I.5.1})$$

*uniformly in  $b_{\beta'} \in \mathcal{A}_{\beta'}^+$  such that  $\mu(b_{\beta'}) > 0$ .*

*Proof.* For  $a$  and  $b_{\beta'}$  as above, one has

$$\begin{aligned} & \left| \frac{\mu(a \cdot b_{\beta'})}{\mu(b_{\beta'})} - \frac{\mu((b_{\beta'})^{1/2} \cdot a \cdot (b_{\beta'})^{1/2})}{\mu(b_{\beta'})} \right| \\ &= \frac{1}{\mu(b_{\beta'})} \cdot |\langle \pi_{\mu}([a, (b_{\beta'})^{1/2}]) \cdot 1_{\mu}, \pi_{\mu}(b_{\beta'}^{1/2}) \cdot 1_{\mu} \rangle| \\ &\leq \left\| \pi_{\mu} \left( \left[ a, \frac{(b_{\beta'})^{1/2}}{\| \pi_{\mu}((b_{\beta'})^{1/2}) \cdot 1_{\mu} \|} \right] \right) \cdot 1_{\mu} \right\|. \end{aligned}$$

Assumption (\*) implies that the right-hand side tends to 0 uniformly in  $b_{\beta'} \in \mathcal{A}_{\beta'}^+, \mu(b_{\beta'}) > 0$ . Hence the assertion follows from Proposition (5.2).

A state  $\mu$  on  $\mathcal{A}$  which satisfies condition (I.5.1) above for every  $a \in \bigcup_{\alpha} \mathcal{A}_{\alpha}$  will be called “uniformly regular from the outside” (cf. [5, (2.12)]).

*Remark.* The condition of asymptotic Abelianness used in Corollary (5.3) (i.e., condition (\*)) is not implied, in general, by the one defined before Lemma 3.1.

### 6. ERGODIC PROPERTIES

In the notations of Section 2 a family  $(E_{\alpha'})_{\alpha \in \mathcal{F}}$  of positive maps  $E_{\alpha'}: \mathcal{A} \rightarrow \mathcal{A}_{\alpha'}$ , will be called *ergodic* if  $\forall a \in \bigcup_{\alpha} \mathcal{A}_{\alpha}; a \geq 0; a \neq 0$ , and  $\psi \in \mathcal{S}(\mathcal{A})$  there exists an  $\alpha \in \mathcal{F}$  such that

$$\psi(E_{\alpha'}(a)) > 0. \tag{I.6.1}$$

If, given  $a \in \mathcal{A}_{\beta}^+ (\beta \in \mathcal{F})$  an  $\alpha \in \mathcal{F}$  can be found such that (I.6.1) holds for every  $\psi \in \mathcal{S}(\mathcal{A})$ , the family  $(E_{\alpha'})$  is called *positivity improving*. For a projective (i.e., satisfying (I.1.1)) family of positive maps  $(E_{\alpha'})$  such that  $E_{\alpha'}(1) = 1$  the two concepts are equivalent. In fact, if  $(E_{\alpha'})$  is ergodic and not positivity improving there will exist an  $a \in \mathcal{A}^+$ , a subnet  $\mathcal{F}_0$  of  $\mathcal{F}$ , and a family  $(\psi_{\gamma})_{\gamma \in \mathcal{F}_0}$  in  $\mathcal{S}(\mathcal{A})$  such that  $\psi_{\gamma} \cdot E_{\gamma}(a) = 0$ , for every  $\gamma \in \mathcal{F}_0$ . The net  $(\psi_{\gamma} \cdot E_{\gamma})$  can be assumed—considering, possibly, a subnet—convergent to a state  $\psi$  which because of Lemma 1.2 is  $(E_{\alpha'})$ -invariant. Therefore, for every  $\alpha \in \mathcal{F}$ ,

$$\psi(E_{\alpha'}(a)) = \psi(a) = \lim \psi_{\gamma} \cdot E_{\gamma}(a) = 0$$

contradicting the ergodicity of  $(E_{\alpha'})$ .

In particular, if the projective family of conditional expectations is ergodic, every  $(E_{\alpha'})$ -invariant state on  $\mathcal{A}$  is faithful on  $\bigcup_{\alpha} \mathcal{A}_{\alpha}$ .

Throughout the present section we shall use a stronger ergodic property; namely, we shall require that (I.6.1) holds for every  $a \in \mathcal{A}^+, a \neq 0$ . This implies that any  $(E_{\alpha'})$ -invariant state is faithful on  $\mathcal{A}$ .

Ergodic properties of the family  $(E_{\alpha'})$  arise naturally in the study of the uniqueness problem for the solutions of Eqs. (I.1.2). In fact if  $\mu$  is the unique  $(E_{\alpha'})$ -invariant state on  $\mathcal{A}$  and  $a \in \bigcup_{\alpha} \mathcal{A}_{\alpha}$ ,  $a \geq 0$ , is such that  $\mu(a) > 0$ , then Theorem 5.1 implies that, for every  $\psi \in \mathcal{S}(\mathcal{A})$ ,

$$\lim_{\alpha} \psi(E_{\alpha'}(a)) = \mu(a) > 0.$$

Thus, in order for the uniqueness problem to be well posed, it is necessary that condition (I.6.1) be satisfied for all  $a$  for which there is an  $(E_{\alpha'})$ -invariant state  $\nu$  such that  $\nu(a) > 0$ . The simplest way to describe this set of  $a$  in terms of the local data (i.e., the  $E_{\alpha'}$ 's) is to restrict one's attention to the families  $(E_{\alpha'})$  for which this set coincides with the positive part of  $\bigcup_{\alpha} \mathcal{A}_{\alpha}$ . This is done by introducing properties of ergodic type. Such properties are implicitly introduced in Dobrushin's papers (e.g., cf. [5, condition (2.18)]). Sometimes it is useful to restrict the definition of ergodicity by considering condition (I.6.1) only for locally normal states.

The asymptotic factorization properties of the measure  $\mu^0$  associated with the 2-dimensional free Euclidean field (deduced, for example, from [11, Sect. (2.2)]) imply that Markovian perturbations of the conditional expectations  $E_{\alpha'}^0$  ( $\alpha \subseteq \mathbb{R}^2$ , open bounded set) are ergodic in this restricted sense.

**THEOREM 6.2.** *If the family  $(E_{\alpha'})$  is ergodic then  $\mathcal{S}_1$  contains at most one state uniformly regular from the outside.*

*Proof.* We show, using an idea of Dobrushin, that if there exist two different states on  $\mathcal{A}$ , compatible with the ergodic family  $(E_{\alpha'})$ , then no  $(E_{\alpha'})$ -invariant state can enjoy the property of uniform regularity from the outside.

Let  $\varphi, \psi$  be two states on  $\mathcal{A}$ ,  $(E_{\alpha'})$ -invariant and such that there is an  $\alpha \in \mathcal{F}$ , an  $a_{\alpha} \in \mathcal{A}_{\alpha}^+$ , and a  $\rho > 0$  for which

$$\varphi(a_{\alpha}) - \psi(a_{\alpha}) = \rho > 0.$$

Let, for each  $\beta \in \mathcal{F}$ ,  $C_{\beta'}$  be the commutative  $C^*$ -algebra generated by  $E_{\beta'}(a)$  and 1 and  $\varphi_{\beta'}$  (resp.  $\psi_{\beta'}$ ) the restriction of  $\varphi$  (resp.  $\psi$ ) on  $\mathcal{A}_{\beta'}$ . Then

$$\begin{aligned} \rho &= \varphi_{\beta'} \otimes \psi_{\beta'}(E_{\beta'}(a) \otimes 1 - 1 \otimes E_{\beta'}(a)) \\ &\leq \|E_{\beta'}(a) \otimes 1 - 1 \otimes E_{\beta'}(a)\| \end{aligned} \tag{I.6.1}$$

(where  $\varphi_{\beta'} \otimes \psi_{\beta'}$  is defined on the tensor product of  $C^*$ -algebras  $\mathcal{A}_{\beta'} \otimes \mathcal{A}_{\beta'}$  with  $\alpha_0$ -norm, cf. [20, p. 60]).

Let  $\mathcal{C}(K_{\beta'})$  be a functional realization of  $C_{\beta'}$ . Then  $C_{\beta'} \otimes C_{\beta'} \cong \mathcal{C}(K_{\beta'} \times K_{\beta'})$  and if  $a_{\beta'} = a_{\beta'}(x)$ ,  $x \in K_{\beta'}$ , is the image of  $E_{\beta'}(a_{\alpha})$  under the isomorphism

$C_{\beta'} \rightarrow \mathcal{C}(K_{\beta'})$ , inequality (I.6.3) implies that there exist two points  $x_0, y_0 \in K_{\beta'}$  and disjoint neighborhoods  $U(x_0), U(y_0)$  of  $x_0, y_0$ , respectively, such that

$$a_{\beta'}(x) - a_{\beta'}(y) > \rho/2; \quad x \in U(x_0); \quad y \in U(y_0).$$

Let  $\hat{b}_1, \hat{b}_2 \in \mathcal{C}(K_{\beta'})$  be continuous functions  $\hat{b}_j: K_{\beta'} \rightarrow [0, 1]$ ,  $j = 1, 2$  such that

$$\begin{aligned} \hat{b}_1(x_0) &= 1; & \hat{b}_1(x) &= 0; & x &\in K_{\beta'} - U(x_0); \\ \hat{b}_2(y_0) &= 1; & \hat{b}_2(y) &= 0; & y &\in K_{\beta'} - U(y_0). \end{aligned}$$

Denoting  $b_{\beta'}^{(j)}$  ( $j = 1, 2$ ) the image of  $\hat{b}_j$  under the inverse isomorphism  $\mathcal{C}(K_{\beta'}) \rightarrow C_{\beta'}$  one has

$$E_{\beta'}(a_{\alpha}) \cdot b_{\beta'}^{(1)} \otimes b_{\beta'}^{(2)} - b_{\beta'}^{(1)} \otimes E_{\beta'}(a_{\alpha}) \cdot b_{\beta'}^{(2)} \geq \frac{\rho}{2} b_{\beta'}^{(1)} \otimes b_{\beta'}^{(2)}$$

with

$$b_{\beta'}^{(j)} \in C_{\beta'} \subset \mathcal{A}_{\beta'}; \quad b_{\beta'}^{(j)} \geq 0; \quad \|b_{\beta'}^{(j)}\| = 1; \quad j = 1, 2.$$

Therefore, for any  $\chi \in \mathcal{S}_I$

$$\chi(a_{\alpha} \cdot b_{\beta'}^{(1)}) \cdot \chi(b_{\beta'}^{(2)}) - \chi(b_{\beta'}^{(1)}) \cdot \chi(a_{\alpha} \cdot b_{\beta'}^{(2)}) \geq \frac{\rho}{2} \chi(b_{\beta'}^{(2)}) \cdot \chi(b_{\beta'}^{(2)}).$$

By the ergodicity of  $(E_{\alpha'})$  the right-hand side is strictly larger than 0 thus

$$\frac{\chi(a_{\alpha} \cdot b_{\beta'}^{(1)})}{\chi(b_{\beta'}^{(1)})} - \frac{\chi(a_{\alpha} \cdot b_{\beta'}^{(2)})}{\chi(b_{\beta'}^{(2)})} \geq \frac{\rho}{2}.$$

Since  $\beta$  is arbitrary, the above inequality implies that  $\chi$  is not uniformly regular from the outside. Since  $\chi \in \mathcal{S}_I$  is arbitrary, the theorem is proved.

**COROLLARY 6.3.** *Let  $\mu \in \mathcal{S}_I$ , and let  $\{\mathcal{A}, (E_{\alpha'})\}$  satisfy the (asymptotic Abelianness) condition (\*) of Lemma 5.3. A necessary condition for  $\mu$  to be the only  $(E_{\alpha'})$ -invariant state is that  $\forall \epsilon > 0; \forall a \in \mathcal{A}; \exists \beta_0 = \beta_0(\epsilon, a); \forall \beta > \beta_0;$*

$$|\mu(a \cdot b_{\beta'}) - \mu(a) \cdot \mu(b_{\beta'})| \leq \epsilon \cdot \mu(b_{\beta'})$$

*uniformly in  $b_{\beta'} \in \mathcal{A}_{\beta'}^+$ . If the family  $(E_{\alpha'})$  is ergodic the condition is also sufficient.*

*Proof.* The necessity has been proved in Lemma 5.3. The sufficiency follows from Theorem 6.2 where we have shown that the existence of different states in  $\mathcal{S}_I$  implies that no state in  $\mathcal{S}_I$  can be uniformly regular from the outside.

*Remark 1.* If the family  $(E_{\alpha'})$  is "local" in the sense that for each  $\beta \in \mathcal{F}$  there is a  $\gamma \in \mathcal{F}, \gamma \succ \beta$  such that  $E_{\beta'}(\mathcal{A}_{\beta}) \subseteq \mathcal{A}_{\gamma}$ , the term "ergodic" in the formulation of the theorem can be as meant in the sense of (I.6.1).

**THEOREM 6.4.** *Assume that the system  $\{\mathcal{A}, E_{\alpha'}\}$  satisfies (asymptotic Abelianness) condition (\*) of Lemma 5.3 and that the family  $(E_{\alpha'})$  is ergodic. Then  $\mathcal{S}_1$  consists of a single point if and only if the following condition is satisfied:  $\forall \alpha \in \mathcal{F}$ ;  $\forall a_{\alpha} \in \mathcal{A}_{\alpha}$ ;  $\forall \psi \in \mathcal{S}(\mathcal{A})$ ; there exists a constant  $M(a_{\alpha})$  such that for every  $\epsilon > 0$  there is a  $\beta = \beta(\epsilon, a_{\alpha}, \psi)$  and a  $\gamma_0 = \gamma_0(\epsilon, a_{\alpha}, \psi)$  such that*

$$|\psi(E_{\gamma'}(a_{\alpha} \cdot b_{\beta'})) - \psi(E_{\gamma'}(a_{\alpha}) \cdot E_{\gamma'}(b_{\beta'}))| \leq \epsilon \cdot M(a_{\alpha}) \cdot \psi(E_{\gamma'}(b_{\beta'}))$$

for every  $\gamma > \gamma_0$  and  $b_{\beta'} \in \mathcal{A}_{\beta'}^+$ .

*Proof.* Sufficiency: Assume that the above condition holds and let  $\psi \in \mathcal{S}_1$ . Then, for  $\beta, \gamma, b_{\beta'}$  as above

$$|\psi(a_{\alpha} \cdot b_{\beta'}) - \psi(a_{\alpha} \cdot E_{\gamma'}(b_{\beta'}))| \leq \epsilon \cdot M(a_{\alpha}) \cdot \psi(b_{\beta'}).$$

If  $\psi$  is an extremal point of  $\mathcal{S}_1$ , Theorem 3.2 and the above inequality imply

$$|\psi(a_{\alpha} \cdot b_{\beta'}) - \psi(a_{\alpha}) \cdot \psi(b_{\beta'})| \leq \epsilon \cdot M(a_{\alpha}) \cdot \psi(b_{\beta'}).$$

The ergodicity of  $(E_{\alpha'})$  and the independence of  $M(a_{\alpha})$  on  $\epsilon$  imply that  $\psi$  is uniformly regular from the outside. Therefore, by Corollary 6.3,  $\mathcal{S}_1 = \{\psi\}$ . (Note that it would have been sufficient to require  $\epsilon \cdot M_{\epsilon}(a_{\alpha}) \rightarrow 0$  as  $\epsilon \rightarrow 0$ .)

Necessity: Assume that the condition of the theorem is not satisfied. Then there exist  $\alpha \in \mathcal{F}$ ,  $a_{\alpha} \in \mathcal{A}_{\alpha}$ , and  $\epsilon > 0$  such that for every  $\gamma_0, \beta_0 \in \mathcal{F}$  there are  $\gamma > \gamma_0$ ,  $\beta > \beta_0$  and  $b_{\beta'} \in \mathcal{A}_{\beta'}$  for which

$$|\psi(E_{\gamma'}(a_{\alpha} \cdot b_{\beta'})) - \psi(E_{\gamma'}(a_{\alpha}) \cdot E_{\gamma'}(b_{\beta'}))| > \epsilon \psi(E_{\gamma'}(b_{\beta'})). \quad (\text{I.6.2})$$

The above inequality implies that  $\psi(E_{\gamma'}(b_{\beta'})) > 0$ . Moreover for any  $a \in \mathcal{A}$ ,  $b_{\beta'} \in \mathcal{A}_{\beta'}^+$ ,  $\psi(E_{\gamma'}(b_{\beta'})) > 0$ ,

$$\left| \frac{\psi_{\gamma}(a \cdot b_{\beta'})}{\psi_{\gamma}(b_{\beta'})} - \frac{\psi_{\gamma}((b_{\beta'})^{1/2} \cdot a \cdot (b_{\beta'})^{1/2})}{\psi_{\gamma}(b_{\beta'})} \right| \leq \left\| \pi_{\psi_{\gamma}} \left( \left[ a, \frac{(b_{\beta'})^{1/2}}{\psi_{\gamma}(b_{\beta'})} \right] \right) \cdot 1_{\psi_{\gamma}} \right\| = \epsilon_{\gamma, \beta}(a), \quad (\text{I.6.3})$$

where  $\psi_{\gamma} = \psi \cdot E_{\gamma'}$ , and  $\lim_{\beta} \epsilon_{\gamma, \beta}(a) = 0$ . Choose now a subnet  $\{\gamma\} \subseteq \mathcal{F}$  and, for each  $\gamma$  in this net, another subnet  $\{\beta'(\gamma)\} \subseteq \mathcal{F}$  and a family  $b_{\beta'(\gamma)} \in \mathcal{A}_{\beta'(\gamma)}^+$  such that (I.6.2) is satisfied for  $\gamma'$ ,  $\beta'(\gamma)$ ,  $b_{\beta'(\gamma)}$ , and  $\beta(\gamma) > \gamma$  for every  $\gamma$ .

For each  $\gamma$  in the net, let  $\chi_{\gamma}$  be a weak accumulation point of the family states

$$\psi_{\gamma}((b_{\beta'(\gamma)})^{1/2} \cdot (b_{\beta'(\gamma)})^{1/2}) / \psi_{\gamma}(b_{\beta'(\gamma)}) = \bar{\psi}_{\gamma(\beta(\gamma))}.$$

Clearly  $\chi_\gamma \in \mathcal{S}_I$  (i.e., it is  $(E_{\alpha'})$ -invariant). We can assume that the net  $\{\chi_\gamma\}$  converges to  $\chi \in \mathcal{S}_I$  (considering, eventually, a subnet). For such a  $\chi$  one has

$$\begin{aligned} & \left| \frac{\psi_\gamma(a \cdot b_{\beta'(\gamma)})}{\psi_\gamma(b_{\beta'(\gamma)})} - \chi(a) \right| \\ & \leq \left| \frac{\psi_\gamma(a \cdot b_{\beta'(\gamma)})}{\psi_\gamma(b_{\beta'(\gamma)})} - \tilde{\psi}_{\gamma, \beta'(\gamma)}(a) \right| + |\tilde{\psi}_{\gamma, \beta'(\gamma)}(a) - \chi_\gamma(a)| + |\chi_\gamma(a) - \chi(a)|. \end{aligned} \tag{I.6.4}$$

Therefore, for each  $\gamma$  one can choose a  $\beta(\gamma)$  and a  $b_{\beta'(\gamma)}$  such that the right-hand side of (I.6.4) tends to 0 with respect to the net  $\{\gamma\}$ . This proves that the family of linear functionals

$$a \in \mathcal{A} \mapsto \psi(E_{\gamma'}(ab_{\beta'})) / \psi(E_{\gamma'}(b_{\beta'}))$$

with  $\gamma, \beta', b_{\beta'}$  satisfying (I.6.2) has a weak accumulation point  $\chi$  in  $\mathcal{S}_I$ . Similar considerations also prove that the family

$$a \in \mathcal{A} \mapsto \psi(E_{\gamma'}(a) \cdot E_{\gamma'}(b_{\beta'})) / \psi(E_{\gamma'}(b_{\beta'}))$$

has an accumulation point  $\chi' \in \zeta_I$  which by construction satisfies

$$|\chi(a_\alpha) - \chi'(a_\alpha)| \geq \epsilon,$$

where  $a_\alpha \in \mathcal{A}_\alpha$  is the one appearing in (I.6.2). Hence  $\mathcal{S}_I$  contains at least two different points, and this proves the statement.

### 7. INVARIANT STATES

Let  $\mathcal{A}, (\mathcal{A}_\alpha), (\mathcal{A}_{\alpha'}), (E_{\alpha'})$  be as in Part I, Sections 1 and 2 and let  $G$  be a group. We assume that there is an action of  $G$  on  $\mathcal{F}$ ;  $g: \alpha \in \mathcal{F} \rightarrow g\alpha \in \mathcal{F}$  such that

$$\alpha < \beta \Leftrightarrow g\alpha < g\beta; \quad (g\alpha)' = g\alpha'; \quad \alpha, \beta \in \mathcal{F}; \quad g \in G;$$

and a (locally normal) covariant action of  $G$  on  $\mathcal{A}$  by \*-automorphisms. That is,  $g: a \in \mathcal{A} \rightarrow ga \in \mathcal{A}$  is a \*-automorphism of  $\mathcal{A}$  ( $g \in G$ ) and

$$g\mathcal{A}_\alpha = \mathcal{A}_{g\alpha}; \quad g\mathcal{A}_{\alpha'} = \mathcal{A}_{g\alpha'}; \quad \alpha \in \mathcal{F}.$$

A state  $\varphi \in \mathcal{S}(\mathcal{A})$  is called  $G$ -invariant if  $\varphi \cdot g = \varphi$ ;  $g \in G$  ( $\varphi \cdot g(a) = \varphi(ga)$ ). If  $\varphi \in \mathcal{S}_I$  the  $G$ -invariance of  $\varphi$  is equivalent to

$$\varphi_{g\alpha'}(E_{g\alpha'}(ga) \cdot gb_{\alpha'}) = \varphi_{g\alpha'}(g \cdot E_{\alpha'}(a) \cdot gb_{\alpha'})$$

for every  $a \in \mathcal{A}$ ,  $b_{\alpha'} \in \mathcal{A}_{\alpha'}$  which if  $\varphi$  is faithful on  $\mathcal{A}$  (in particular, if  $(E_{\alpha'})$  is ergodic) is equivalent to

$$g \cdot E_{\alpha'} = E_{g\alpha'} \cdot g. \quad (\text{I.7.1})$$

A family  $(E_{\alpha'})$  satisfying (I.7.1) is called  $(G)$ -covariant. Thus, for an ergodic family  $(E_{\alpha'})$ ,  $G$ -covariance is a necessary condition for  $\mathcal{S}_I$  to contain  $G$ -invariant states. Conversely, if  $(E_{\alpha'})$  is  $G$ -covariant, then the action of  $G$  leaves invariant the nonempty weakly compact convex set  $\mathcal{S}_I$  ( $\varphi \cdot g \cdot E_{\alpha'} = \varphi \cdot E_{g\alpha'} \cdot g = \varphi \cdot g$ ;  $\varphi \in \mathcal{S}_I$ ); therefore, by the Ryll–Nardzewski fixed-point theorem [16] the set of  $G$ -invariant points in  $\mathcal{S}_I$  is nonempty.

If  $G = E(d) = d$ -dimensional Euclidean group, following the procedure of Föllmer [10], one can use Wiener's ergodic theorem (i.e., take means on bounded sets and pass them to the limit) to construct a translation invariant state  $\varphi$ . Let  $\psi \in \mathcal{S}_I$  denote the mean of  $\varphi \cdot g$  over the  $d$ -dimensional group. Then, if  $t: \xi \in \mathbb{R}^d \rightarrow \xi - t$  is a translation, one has, by translation invariance of  $\varphi$

$$\psi \cdot t = \int \varphi \cdot g \cdot t \, dg = \int \varphi \cdot (gt) \cdot g \, dg = \int \varphi \cdot g \, dg = \psi.$$

Thus  $\psi$  is a locally normal  $E(d)$ -invariant state. We conclude

**THEOREM 7.1.** *In the notation above, if  $(E_{\alpha'})$  is a  $G$ -covariant family, the set of  $G$ -invariant states in  $\mathcal{S}_I$  is nonempty.*

Let  $\mu$  be an  $(E_{\alpha'})$ - and  $G$ -invariant state. Denoting  $\mathcal{A}_G^\mu$  the subalgebra of the elements of  $\mathcal{A}$  left fixed by the action of  $G$ , then—as remarked in [10]—if  $\mathcal{A}_G^\mu \subseteq \mathcal{A}_\infty^\mu$  and if the system  $\{\mathcal{A}, (E_{\alpha'})\}$  is  $(E_{\alpha'})$ -Abelian, then  $\mu$  admits an integral decomposition—associated with the conditional expectation  $E_G^\mu: \mathcal{A}_\infty^\mu \rightarrow \mathcal{A}_G^\mu$ .

## II

### 1. LOCALITY AND MARKOVIANITY

From now on we shall consider systems  $\{\mathcal{A}, (\mathcal{A}_\alpha)\}$  such that  $\mathcal{A}$  is an Abelian  $C^*$ -algebra and  $(\mathcal{A}_\alpha)$  a family of von Neumann local subalgebras of  $\mathcal{A}$  whose norm closure is equal to  $\mathcal{A}$ . In [2] it has been shown that a stochastic process is completely defined (up to stochastic equivalence, cf. [2, Df. (2.11)]) by a triple  $\{\mathcal{A}, (\mathcal{A}_\alpha), \mu^0\}$ , where  $\{\mathcal{A}, (\mathcal{A}_\alpha)\}$  is as above and  $\mu^0$  is a locally normal state (i.e., a measure) on  $\mathcal{A}$  and, conversely, each such triple determines an equivalence class of stochastic processes. Since local structures are intrinsically associated with stochastic processes and conditional expectations with measures, it is natural to analyze the structure of those stochastic processes whose local struc-

tures are preserved (or “almost” preserved) by the associated conditional expectations. Such processes will be called *local* (or *quasi-local*) (cf. Definition 2.3). It turns out that locality is the most general property of Markovian type. If the local algebras  $(\mathcal{A}_\alpha)$  satisfy some rather general conditions (cf. (II.2.1)–(II.2.4)), then locality is *equivalent* to (a generalization of) Dobrushin’s  $(d)$ -Markov property. (The implication: locality preserving  $\Rightarrow$   $(d)$ -Markovianness as well as some modified form of the perturbation theorems below, continue to take place in the noncommutative—at least uniformly hyperfinite—case. In this case the objects intrinsically associated to  $\mu^0$  are not the (quasi-)conditional expectations, but the modular automorphisms of the restrictions of  $\mu^0$  to the local algebras. Locality properties of these ones allow us to establish the Markovianity of  $\mu^0$ —in the sense of [1]—and to obtain an explicit form for the quasi-conditional expectations associated with  $\mu^0$  (cf. [25]).

## 2. LOCAL CONDITIONAL EXPECTATION

Let  $(\Omega, \mu^0)$  be a standard Borel probability space;  $T$  a locally compact metrizable space;  $\mathcal{F}$  a family of bounded open subsets of  $T$ . We assume that the empty set is not in  $\mathcal{F}$ ;  $\mathcal{F}$  contains all the bounded balls; if  $\alpha, \beta \in \mathcal{F}$  and  $\bar{\alpha} \subseteq \beta$  then  $\beta - \bar{\alpha} \in \mathcal{F}$  ( $\bar{\alpha}$ —the closure of  $\alpha$ ); finite unions and finite nonempty intersections of elements of  $\mathcal{F}$  are in  $\mathcal{F}$ .

Let for each  $\alpha \in \mathcal{F}$  be given a  $\mu^0$ -complete  $\sigma$ -algebra  $O_\alpha \subseteq O$ . We shall assume that the family  $(O_\alpha)_{\alpha \in \mathcal{F}}$  satisfies

$$\alpha \subseteq \beta; \quad \alpha, \beta \in \mathcal{F} \Rightarrow O_\alpha \subseteq O_\beta; \tag{II.2.1}$$

$$\bigvee_{\alpha \in \mathcal{F}} O_\alpha = O; \tag{II.2.2}$$

$$O_\alpha = \bigvee \{O_\beta: \beta \subseteq \alpha; \beta \in \mathcal{F}\}; \quad \alpha \in \mathcal{F}. \tag{II.2.3}$$

For any two open sets  $A, B$  ( $A \cap B \neq \emptyset$ )

$$O_A \cap O_B = O_{A \cap B}, \tag{II.2.4}$$

where  $V_\alpha O_\alpha$  denotes the complete  $\sigma$ -algebra generated by the  $O_\alpha$ ; for an open  $A \subseteq T$ ,  $O_A = \bigvee \{O_\alpha: \alpha \subseteq A; \alpha \in \mathcal{F}\}$  and for any  $C \subseteq T$   $O_C = \bigcap \{O_A: A \text{ open } \supseteq C\}$ . From relations (II.2.1)–(II.2.4) one deduces the following:

$$O_C \cap O_{C'} = O_{C \cap C'}; \tag{II.2.5}$$

$C, C'$  arbitrary sets (in the second case  $C \cap C' \neq \emptyset$ );

$$\bigvee_{(B)} O_B = O_A; \tag{II.2.6}$$



$A$  open;  $\{B\}$  any increasing net of open sets such that  $\bigcup_{\{B\}} B = A$ ;

$$\bigcap_{\{B\}} O_B = O_{A_0}; \quad (\text{II.2.7})$$

$A_0$  bounded closed ( $\neq \emptyset$ ) or the complement of a bounded open ( $\neq T$ ) set;  $\{B\}$  any decreasing net of open sets or of closed bounded sets such that  $\bigcap B = A_0$ . Moreover we shall assume that

$$O_B = O_{\bar{A}} \vee O_{B-\bar{A}} \quad (\text{II.2.8})$$

$\bar{A} \subseteq B$ ;  $A, B$  open or  $A, B$  closed; and  $B$  bounded or the complement of an open bounded set.

In the following we shall denote  $\alpha'$  the complement in  $T$  of a set  $\alpha \in \mathcal{F}$ . Denote, for bounded  $C$  and unbounded  $C'$ ,

$$\mathcal{A}_C = L^\infty(\Omega, O_C, \mu_C^0); \quad \bar{O}_{C'} = L^\infty(\Omega, O_{C'}, \mu_{C'}^0);$$

( $\bar{O} = \bar{O}_T$ ). The  $\mathcal{A}_C$  will be called local algebras. Relations (II.2.1)–(II.2.8) are equivalent to the corresponding relations for the algebras  $\mathcal{A}_C, \bar{O}_{C'}$ . A conditional expectation  $E_{\beta'}: \bar{O} \rightarrow \bar{O}_{\beta'}$  will be called *local* if it preserves the local structure; that is, if  $\forall \alpha \in \mathcal{F}, \exists \gamma \supseteq \bar{\beta}, \gamma \in \mathcal{F}$ , such that

$$E_{\beta'}(\mathcal{A}_\gamma) \subseteq \mathcal{A}_\gamma. \quad (\text{II.2.9})$$

Define  $(\beta)_a$  as the intersection of the closures of those  $\gamma$  for which (II.1.9) holds with  $\alpha = \beta$ . Clearly  $(\beta)_a \supseteq \bar{\beta}$ .

**PROPOSITION 2.1.** *Let  $E_{\beta'}: \bar{O} \rightarrow \bar{O}_{\beta'}$  be a locally normal conditional expectation.  $E_{\beta'}$  is local if and only if*

$$E_{\beta'}(\mathcal{A}_\beta) \subseteq \mathcal{A}_{(\beta)_a - \beta}. \quad (\text{II.2.10})$$

*Proof.* Necessity: By (II.2.7), if  $E_{\beta'}$  is local,  $E_{\beta'}(\mathcal{A}_\beta) \subseteq \mathcal{A}_{(\beta)_a}$ . Therefore, by (II.2.5)

$$E_{\beta'}(\mathcal{A}_\beta) \subseteq \mathcal{A}_{(\beta)_a} \cap \bar{O}_{\beta'} = \mathcal{A}_{(\beta)_a - \beta'}.$$

Sufficiency: Assume that (II.2.10) holds. Then, if  $\gamma \supseteq \bar{\beta}, \gamma \in \mathcal{F}$ , for any  $a_\beta \in \mathcal{A}_\beta$  and  $a_{\gamma - \bar{\beta}} \in \mathcal{A}_{\gamma - \bar{\beta}}$ ,

$$E_{\beta'}(a_\beta \cdot a_{\gamma - \bar{\beta}}) = E_{\beta'}(a_\beta) \cdot a_{\gamma - \bar{\beta}} \in \mathcal{A}_{(\beta)_a} \vee \mathcal{A}_\gamma \subseteq \mathcal{A}_\delta$$

for some  $\delta \in \mathcal{F}$ . Thus, by local normality (not used in the proof of the necessity) and (II.2.8),  $E_{\beta'}(\mathcal{A}_\gamma) \subseteq \mathcal{A}_\delta$ . Since  $\gamma$  is arbitrary ( $\supseteq \bar{\beta}$ ),  $E_{\beta'}$  is local.

*Remark 1.* Dobrushin's ( $d$ )-Markov property for  $E_{\beta'}$  corresponds to the case

$$(\beta)_d = \{x \in T: \text{dist}(x; \beta) \leq d\} \quad (d > 0);$$

the Markov property—to the case  $(\beta)_d = \bar{\beta}$ .

*Remark 2.* If  $E_{\beta'}$  is local then

$$\gamma \supseteq (\beta)_d \Rightarrow E_{\beta'}(\mathcal{A}_\gamma) \subseteq \mathcal{A}_\gamma.$$

In particular, denoting  $\mathcal{A}$  the norm closure of  $\bigcup_{\alpha \in \mathcal{F}} \mathcal{A}_\alpha$

$$E_{\beta'}(\mathcal{A}) \subseteq \mathcal{A}.$$

Any conditional expectation  $E_{\beta'}$  satisfying the latter relation will be called *quasi-local*. By “dualizing” the definition of locality for  $E_{\beta'}$  one obtains the condition;  $\forall \alpha \in \mathcal{F}, \exists \gamma \in \mathcal{F}, \bar{\gamma} \subseteq \beta$  such that

$$E_{\beta}(\bar{O}_\alpha) \subseteq \bar{O}_\gamma. \tag{II.2.11}$$

$E_{\beta}$  will be called *local* if the above condition holds for  $\alpha$  “big enough” (say  $\alpha$  contains a given open set  $\beta_0 \subseteq \beta$ ). Denote  $d\beta$  the union of the  $\gamma$ 's for which (II.2.11) is satisfied with  $\alpha = \beta$ . Then  $d\beta$  is open and contained in  $\bar{\beta}$ .

**PROPOSITION 2.2.** *A normal conditional expectation  $E_{\beta}: \bar{O} \rightarrow \mathcal{A}_{\beta}$  is local if and only if*

$$E_{\beta}(\bar{O}_{\beta'}) \subseteq \mathcal{A}_{\beta-d\beta}. \tag{II.2.12}$$

*Proof.* If  $E_{\beta}$  is local, then for each  $\gamma$  satisfying (II.2.11)

$$E_{\beta}(\bar{O}_{\beta'}) \subseteq \mathcal{A}_{\beta-\gamma} \subseteq \mathcal{A}_{\beta-\gamma}.$$

Therefore choosing an increasing sequence  $(\gamma_n)$  for which (II.2.11) is satisfied and such that  $\gamma_n \uparrow d\beta$ , (II.2.12) follows from (II.2.7). Conversely let (II.2.12) be satisfied and let  $\alpha \in \mathcal{F}$ . We have to prove that (II. .11) holds for some  $\gamma \in \mathcal{F}, \gamma \subseteq \beta$ . If  $\alpha \supseteq \beta$ , this is clear. If  $\alpha \subseteq \beta$ , then for any  $a_{\beta-\alpha} \in \mathcal{A}_{\beta-\alpha}$  and  $a_{\beta'} \in \bar{O}_{\beta'}$ :

$$\begin{aligned} E_{\beta}(\alpha_{\beta-\alpha} \cdot \alpha_{\beta'}) &= a_{\beta-\alpha} \cdot E_{\beta}(a_{\beta'}) \in \bar{O}_{\beta-\alpha} \vee \bar{O}_{\beta-d\beta} \\ &= O_{\beta(\alpha \cap d\beta)}. \end{aligned}$$

Thus, whenever  $\alpha \cap d\beta \neq \emptyset$ , the normality of  $E_{\beta}$  (not used in the first step) and (II.2.8) imply (II.2.11) for any  $\gamma \subseteq \alpha \cap d\beta; \gamma \in \mathcal{F}$ .

*Remark.* Dobrushin's  $d$ -Markov property for  $E_{\beta}$  corresponds to the case

$$d\beta = \{x \in \beta: \text{dist}(x, \beta') > d\} \quad (d > 0)$$

and the Markov property to the case  $d\beta = \beta$ .

In the following we shall use the notation  $f \hat{\in} O_C$  (or sometimes  $f \hat{\in} \mathcal{A}_C$ ) to denote the  $O_C$ -measurability of  $f$ .

**DEFINITION 2.3.** The measure  $\mu^0$  will be called *forward* (resp. *backward*) *local* (with respect to the family  $(O_\alpha)$ ) if the family  $(E_{\beta'}^0)$  (resp.  $(E_{\beta}^0)$ ) of its conditional expectations with respect to the  $\sigma$ -algebras  $O_{\beta'}$  (resp.  $O_{\beta}$ ) are local. If both conditions are verified,  $\mu^0$  is called *local*.

In the following, for backward local measures, we shall also require that  $\beta \uparrow T$  implies  $d\beta \uparrow T$ .

**PROPOSITION 2.4.** *Let  $\mu^0$  be forward local. Then a locally normal conditional expectation  $E_{\beta'}: \bar{O} \rightarrow \bar{O}_{\beta'}$  is local if and only if there exist a local  $\sigma$ -algebra  $O_{(\beta)_{d_1}}$  and a  $K_\beta \in L^1(\Omega, O_{(\beta)_{d_1}}, \mu_{(\beta)_{d_1}}^0)$  such that*

$$E_{\beta'}(a) = E_{\beta'}^0(K_\beta a); \quad a \in \mathcal{A}. \quad (\text{II.2.13})$$

*Proof.* Necessity: By the locality and local normality of  $E_{\beta'}$  and (II.2.8) there is  $\gamma \in \mathcal{F}$  (any  $\gamma \supseteq (\beta)_d$ ,  $(\beta)_d$  being defined as stated after (II.2.9)) such that  $E_{\beta'}(\mathcal{A}_\gamma) \subseteq \mathcal{A}_\gamma$ . Thus  $E_{\beta'} \upharpoonright \mathcal{A}_\gamma = E_{\beta'}^\gamma: \mathcal{A}_\gamma \rightarrow \mathcal{A}_\gamma \cap \bar{O}_{\beta'}$  is a normal conditional expectation. A theorem of Moy [15, Theorem (1.1)] implies then that there exists a  $K_\gamma \in L^1(\Omega, O_\gamma, \mu_\gamma^0)$  such that

$$E_{\beta'}^\gamma(a_\gamma) = E_{O_{\beta'} \cap O_\gamma}^0(K_\gamma a_\gamma); \quad a_\gamma \in \mathcal{A}_\gamma;$$

(where  $E_{O_{\beta'} \cap O_\gamma}^0$  is the  $\mu^0$ -conditional expectation on  $O_{\beta'} \cap O_\gamma$ ). Since  $\mu^0$  is forward local, if  $\gamma$  has been chosen so that  $\gamma \supseteq (\beta)_{d_0}$  ( $(\beta)_{d_0}$  being defined for  $E_{\beta'}^0$ , as after (II.2.9)), then one easily verifies that

$$E_{O_{\beta'} \cap O_\gamma}^0(K_\gamma a_\gamma) = E_{\beta'}^0(K_\gamma a_\gamma); \quad a_\gamma \in \mathcal{A}_\gamma.$$

Thus, by local normality and (II.2.8)

$$E_{\beta'}(a) = E_{\beta'}^0(K_\gamma a).$$

Since, by (II.2.2)  $\mathcal{A}$  is weakly dense in  $\bar{O}$ ,  $K_\gamma$  is the unique element of  $L^1(\Omega, O, \mu)$  satisfying the above inequality. Therefore  $K_\gamma = K_\delta$ , whenever  $\delta \in \mathcal{F}$  is such that  $(\beta)_d \cup (\beta)_{d_0} \subseteq \delta \subseteq \gamma$ , and  $K_\delta$  is defined by the above procedure. Thus

$$K_\gamma \hat{\in} \bigcap \{O_\delta: (\beta)_d \cup (\beta)_{d_0} \subseteq \delta \in \mathcal{F}\}.$$

Denoting

$$(\beta)_{d_1} = (\beta)_d \cup (\beta)_{d_0} \quad (\text{II.2.14})$$

the assertion follows. Since the sufficiency is clear, Proposition 2.4 is proved.

In the following if  $E_{\beta'}$  is a conditional expectation and  $K_{\beta}$  is any function satisfying (II.2.13), then  $K_{\beta}$ , which is necessarily the unique ( $\mu^0$ -a.e.) one with this property, will be called the *conditional density* of  $E_{\beta'}$  with respect to  $E_{\beta}$ , and denoted  $dE_{\beta'}/dE_{\beta}$ .

**PROPOSITION 2.5.** *Let  $\mu^0$  be local. The locally normal conditional expectation  $E_{\alpha'}: \bar{O} \rightarrow O_{\alpha'}$  is quasi-local if and only if*

$$E_{\alpha'}(a) = E_{\alpha'}^0(K_{\alpha}a); \quad a \in \mathcal{A};$$

where  $K_{\alpha} \in L^1(\Omega, O, \mu^0)$  and there is a sequence of conditioned densities  $(K_{\beta_n})$  such that  $\beta_n \in \mathcal{F}$ ;  $\beta_n \uparrow T$ ;

$$K_{\beta_n} \in L^1(\Omega, O_{\beta_n}, \mu_{\beta_n}^0) \quad (n \in \mathbb{N})$$

such that

$$\lim_n \| E_{\alpha'}^0[(K_{\beta_n} - K_{\alpha}) \cdot a] \| = 0; \quad a \in \mathcal{A}.$$

*Proof.* The sufficiency is clear. Let  $E_{\alpha'}$  be locally normal and quasi-local. The backward locality of  $\mu^0$  implies that, given  $\alpha$ , there is a  $\beta_0 \in \mathcal{F}$  and we can assume  $\beta_0 \supseteq \alpha$  such that, for  $\beta \supseteq \beta_0$ ,  $\beta \in \mathcal{F}$ ,  $E_{\beta}^0(\bar{O}_{\alpha'}) \subseteq \bar{O}_{\alpha'}$ . Therefore the map

$$E_{\beta}^0 \cdot E_{\alpha'} \upharpoonright \mathcal{A}_{\beta}: \mathcal{A}_{\beta} \rightarrow \mathcal{A}_{\beta} \cap \bar{O}_{\alpha'}$$

is a conditional expectation. The same reasoning as in Proposition 2.4 yields the existence of a  $K_{\beta} \in L^1(\Omega, \bar{O}_{\beta}, \mu_{\beta}^0)$  such that

$$E_{\beta}(E_{\alpha'}(a_{\beta})) = E_{\alpha'}^0(K_{\beta}a_{\beta}); \quad a_{\beta} \in \mathcal{A}_{\beta}$$

(if  $\beta$  is large enough). Choose, now,  $\beta_n \uparrow T$ ,  $\beta_n \in \mathcal{F}$ . If  $a \in \bigcup_{\gamma \in \mathcal{F}} \mathcal{A}_{\gamma}$ , then  $a \in \mathcal{A}_{\beta_n}$  for some  $n$  and, by the quasi-locality of  $E_{\alpha'}$ , given  $\epsilon > 0$ , there is a  $\beta_m \in \mathcal{F}$  and a  $b_{\beta_m} \in \mathcal{A}_{\beta_m}$  such that  $\| E_{\alpha'}(a) - b_{\beta_m} \| \leq \epsilon$ . Therefore, if  $\beta \supseteq \beta_m \supseteq \beta_n$

$$\| E_{\alpha'}^0(K_{\beta}a) - E_{\alpha'}(a) \| \leq \| E_{\beta}E_{\alpha'}(a) - b_{\beta_m} \| + \epsilon \leq 2\epsilon.$$

Thus  $\lim_n \| E_{\alpha'}(K_{\beta_n}a) - E_{\alpha'}(a) \| = 0$ ;  $a \in \mathcal{A}$ . Since  $(\Omega, O, \mu^0)$  is a standard Borel space and  $T$  is second commutable,  $E_{\alpha'}$  is normal on  $\bar{O}$ , hence it has a conditional density  $K$ . And this ends of proof.

### 3. CONDITIONED MARTINGALES

Let  $(E_{\alpha'})$  be a family of normal conditional expectations  $E_{\alpha'}: \bar{O} \rightarrow O_{\alpha'}$  and let  $K_{\alpha} = dE_{\alpha'}/dE_{\alpha}$ . The projectivity of the family  $(E_{\alpha'})$  (cf. (I.1.1)) is equivalent to

$$K_{\beta} = K_{\alpha} \cdot E_{\alpha'}^0(K_{\beta}); \quad \alpha \subseteq \beta; \quad \alpha, \beta \in \mathcal{F} \quad (\text{II.3.1})$$

Any family  $(K_\alpha)$  of positive random variables satisfying (III.1.1) and

$$E_\alpha^0(K_\alpha) = 1 \quad (\text{II.3.2})$$

will be called a *conditional martingale*. If  $(K_\alpha)$  is a conditioned martingale, then (III.1.1), (III.1.2), and the martingale theorem imply that

$$K_\beta / K_\alpha \hat{\in} O_{\alpha'}; \quad \alpha \subset \beta; \quad (\text{II.3.3})$$

$$w\text{-lim } K_{\alpha_n} = K_\beta; \quad \alpha_n \uparrow \beta \quad (\text{II.3.4})$$

(in (II.3.3), and always in the following, we have assumed that  $K_\alpha > 0$ ,  $\mu^0$ -a.e., which is equivalent to the fact that  $E_{\alpha'}$  is faithful). Remark, that (III.1.2) is equivalent to

$$K_\alpha = e^{-U_\alpha} / E_{\alpha'}^0(e^{-U_\alpha}) \quad (\text{II.3.5})$$

for some measurable function  $U_\alpha$ .

It is important to remark that  $U_\alpha$  is defined by  $K_\alpha$  up to a “gauge transformation”

$$U_\alpha \mapsto U_\alpha + g_{\alpha'}; \quad g_{\alpha'} \hat{\in} O_{\alpha'}.$$

Assume that the measure  $\mu^0$  is (locally) Markovian, i.e. (cf. (II.2.10), (II.2.12))

$$E_{\alpha'}^0(\mathcal{A}_\alpha) \subseteq \mathcal{A}_\alpha; \quad E_{\bar{\alpha}}^0(\bar{O}_{\alpha'}) \subseteq \mathcal{A}_{\partial\alpha}; \quad \alpha \in \mathcal{F}$$

and that the conditioned martingale  $(K_\alpha)$  satisfies the (strong locality) condition

$$K_\alpha \hat{\in} O_{\bar{\alpha}}; \quad \alpha \in \mathcal{F}. \quad (\text{II.3.6})$$

Proposition 2.4 and (II.2.14) show that (II.3.6) is equivalent to the Markovianity of  $E_{\alpha'} = E_\alpha^0(K_\alpha \cdot)$ . Clearly, if

$$U_\alpha \hat{\in} O_{\bar{\alpha}} \quad (\text{II.3.7})$$

then (II.3.6) holds, and one can verify that, conversely, if (II.3.6) holds then in the “gauge-equivalence” class of functions  $U_\alpha$  satisfying (II.3.5), there is an element satisfying (II.3.7).

Thus, under assumption (II.3.6)—i.e., Markovianity—one can limit oneself to the consideration of the  $U_\alpha$  satisfying (II.3.7). This restricts the class of admissible “gauge transformations” to those of the type

$$U_\alpha \mapsto U_\alpha + g_{\partial\alpha}; \quad g_{\partial\alpha} \hat{\in} O_{\partial\alpha}. \quad (\text{II.3.8})$$

**THEOREM 3.1.** *Let  $\mu^0$  be Markovian. A projective family  $(E_{\alpha'})$  of locally normal conditional expectations  $E_{\alpha'}: \bar{O} \rightarrow \bar{O}_{\alpha'}$  is Markovian if and only if there is a family of measurable functions  $(U_{\alpha})$  such that*

$$\frac{dE_{\alpha'}}{dE_{\alpha'}^0} = \frac{e^{-U_{\alpha}}}{E_{\alpha'}^0(e^{-U_{\alpha}})} \quad (\text{II.3.9})$$

and, modulo a gauge transformation

$$U_{\beta} = U_{\alpha} + U_{\beta-\bar{\alpha}}; \quad U_{\alpha} \hat{\in} O_{\bar{\alpha}}; \quad \bar{\alpha} \subseteq \beta. \quad (\text{II.3.10})$$

*Proof.* It is known, and immediately verified, that a family  $(U_{\alpha})$  satisfying (II.3.10) defines, by (II.3.9), a Markovian family  $(E_{\alpha'})$ . Conversely, let  $((E_{\alpha'})$  be Markovian.  $K_{\alpha} = dE_{\alpha'}/dE_{\alpha'}^0$ , and  $U_{\alpha}$  be defined by (II.3.0)  $U_{\alpha} \in O_{\bar{\alpha}}$ .

By (II.3.1), applied to  $\beta - \bar{\alpha} \subseteq \beta$

$$e^{-U_{\beta}} = e^{-U_{\beta-\bar{\alpha}}} \cdot E_{(\beta-\bar{\alpha})'}^0(e^{-U_{\beta}}) / E_{(\beta-\bar{\alpha})'}^0(e^{-U_{\beta-\bar{\alpha}}})$$

and since  $U_{\alpha} \hat{\in} O_{\bar{\alpha}}$

$$e^{-(U_{\beta}-U_{\alpha})} = e^{-U_{\beta}} \cdot E_{(\beta-\bar{\alpha})'}^0(e^{-(U_{\beta}-U_{\alpha})}) / E_{(\beta-\bar{\alpha})'}^0(e^{-U_{\beta}}).$$

Therefore

$$e^{-(U_{\beta}-U_{\alpha})} = e^{-(U_{\beta-\bar{\alpha}}+g_{(\beta-\bar{\alpha})'})},$$

where

$$e^{-g_{(\beta-\bar{\alpha})'}} = E_{(\beta+\bar{\alpha})'}^0(e^{-(U_{\beta}-U_{\alpha})}) / E_{(\beta+\bar{\alpha})'}^0(e^{-U_{\beta-\bar{\alpha}}}).$$

Equations (II.3.3) and (II.3.7) imply that

$$e^{-(U_{\beta}-U_{\alpha})} \hat{\in} O_{\bar{\beta}} \cap O_{\alpha'} = O_{\bar{\beta}-\alpha},$$

therefore, by Markovianity

$$g_{(\beta-\bar{\alpha})'} \hat{\in} O_{\partial(\beta-\bar{\alpha})}.$$

Hence  $g_{(\beta-\bar{\alpha})'}$  is a gauge transformation and this establishes (II.3.10).

*Remark.* By imposing, besides (II.3.7), the invariance of the equalities (II.3.10) under “gauge transformations” one determines a global constraint on the action of such transformations on  $(U_{\alpha})$ , namely,

$$g_{\partial(\beta-\bar{\alpha})} = g_{\partial\beta} - g_{\partial\alpha}; \quad \bar{\alpha} \subseteq \beta. \quad (\text{II.3.11})$$

The relations (II.3.10) and (II.3.11) agree with the usual interpretation of  $U_\alpha$  and  $g_{\partial\alpha}$  as random variable-valued integrals, respectively, over the volume  $\alpha$  and the surface  $\partial\alpha$ .

The connection between conditioned martingales and martingales is described, in the local case, by the Dobrushin–Lanford–Ruelle equations (cf. [13, Proposition (VII, 2)] for the Markovian case).

**PROPOSITION 3.2.** *Let  $\mu^0$  be (forward) local, and let  $(E_{\alpha'})$  be a projective family of locally normal, local conditional expectations  $E_{\alpha'}: \mathcal{A} \rightarrow \bar{O}_{\alpha'}$ . A locally normal state  $\mu$  is  $(E_{\alpha'})$ -invariant if and only if  $\forall \alpha \in \mathcal{F}$*

$$\frac{d\mu_{\bar{\alpha}}}{d\mu_{\bar{\alpha}}^0} = E_{\bar{\alpha}}^0 \left( \frac{dE_{\alpha'}}{dE_{\alpha'}^0} \cdot \frac{d\mu_{(\alpha)_{d-\alpha}}}{d\mu_{(\alpha)_{d-\alpha}}^0} \right) \quad (\text{II.3.12})$$

( $(\alpha)_{\bar{a}}$  being defined by the remarks after (III.1.9)).

*Proof.* Necessity: Let  $\alpha \in \mathcal{F}$ . By the locality of  $E_{\alpha'}$ ,

$$\begin{aligned} \mu(a_{\bar{\alpha}}) &= \mu_{(\alpha)_{d-\alpha}}(E_{\alpha'}(a_{\bar{\alpha}})) \\ &= \mu \left( \frac{d\mu_{(\alpha)_{d-\alpha}}}{d\mu_{(\alpha)_{d-\alpha}}^0} \cdot E_{\alpha'}^0 \left( \frac{dE_{\alpha'}}{dE_{\alpha'}^0} a_{\bar{\alpha}} \right) \right) \\ &= \mu^0 \left( E_{\bar{\alpha}}^0 \left( \frac{dE_{\alpha'}}{dE_{\alpha'}^0} \cdot \frac{d\mu_{(\alpha)_{d-\alpha}}}{d\mu_{(\alpha)_{d-\alpha}}^0} \right) \cdot a_{\bar{\alpha}} \right) \end{aligned}$$

which is equivalent to (II.3.12).

Sufficiency: Let  $\alpha \in \mathcal{F}$  and  $\beta \in \mathcal{F}$  be such that  $\beta \supseteq (\alpha)_d$ . Then,  $\forall a_\beta \in \mathcal{A}_\beta$ ,

$$\mu(a_\beta) = \mu^0 \left( \frac{dE_{\alpha'}}{dE_{\alpha'}^0} \cdot \frac{d\mu_{(\beta)_{d-\beta}}}{d\mu_{(\beta)_{d-\beta}}^0} \cdot a_\beta \right).$$

Since  $(E_{\beta'})$  is projective, one has

$$\begin{aligned} \frac{dE_{\beta'}}{dE_{\beta'}^0} &= \frac{dE_{\alpha'}}{dE_{\alpha'}^0} \cdot E_{\alpha'}^0 \left( \frac{dE_{\beta'}}{dE_{\beta'}^1} \right); \\ \mu(a_\beta) &= \mu^0 \left( E_{\alpha'}^0 \left( \frac{dE_{\alpha'}}{dE_{\alpha'}^0} \cdot a_\beta \right) \cdot E_{\alpha'}^0 \left( \frac{dE_{\beta'}}{dE_{\beta'}^0} \cdot \frac{d\mu_{(\beta)_{d-\beta}}}{d\mu_{(\beta)_{d-\beta}}^0} \right) \right). \end{aligned}$$

Since  $E_{\alpha'}$  is local and  $\beta \supseteq (\alpha)_d$

$$\begin{aligned} \mu(a_\beta) &= \mu^0 \left( E_{\beta'} \left( \frac{dE_{\beta'}}{dE_{\beta'}^0} \cdot \frac{d\mu_{(\beta)_{d-\beta}}}{d\mu_{(\beta)_{d-\beta}}^0} \right) \cdot E_{\alpha'}(a_\beta) \right) \\ &= \mu(E_{\alpha'}(a_\beta)) \end{aligned}$$

by (II.3.12). Since  $\beta \supseteq (\alpha)_d$  is arbitrary, this implies  $\mu = \mu \cdot E_{\alpha'}$ , therefore the proof is completed.

4. UNIQUENESS: FACTORIZABLE CASE

Let  $\mathcal{F}$  be as in the preceding sections. Let  $\mathcal{A}$  be a commutative  $C^*$ -algebra,  $(\mathcal{A}_\alpha)_{\alpha \in \mathcal{F}}$ ,  $(\mathcal{A}'_\alpha)_{\alpha \in \mathcal{F}}$  be families of  $C^*$ -sub-algebras of  $\mathcal{A}$  such that

$$\mathcal{A} \cong \mathcal{A}_\alpha \otimes \mathcal{A}'_\alpha; \quad \alpha \in \mathcal{F}; \quad (\text{II.4.1})$$

$$\mathcal{A}_\alpha \cong \mathcal{A}_{\alpha_1} \otimes \mathcal{A}_{\alpha-\alpha_1}; \quad \alpha \in \mathcal{F}; \quad \alpha_1 \subseteq \alpha. \quad (\text{II.4.2})$$

Let  $(E'_\alpha)$  be a projective family of conditional expectations  $E'_\alpha: \mathcal{A} \rightarrow \mathcal{A}'_\alpha$  and  $\lambda^0$  a faithful state on  $\mathcal{A}$  which is compatible with  $(E'_\alpha)$  and such that, for every  $\alpha \in \mathcal{F}$ ,  $\lambda^0 \cong \lambda_{\alpha^0} \otimes \lambda'_{\alpha^0}$ . If  $\Omega$  (resp.  $\Omega_\alpha$ ,  $\Omega'_\alpha$ ) denotes the spectrum of  $\mathcal{A}$  (resp.  $\mathcal{A}_\alpha$ ,  $\mathcal{A}'_\alpha$ ), then  $\Omega \cong \Omega_{\alpha_1} \times \Omega_{\alpha-\alpha_1} \times \Omega'_\alpha$ , whenever  $\alpha_1 \subseteq \alpha$ . Let, for  $\alpha \in \mathcal{F}$ , a function be given  $K_\alpha: \Omega \rightarrow \mathbb{R}$  such that the operator

$$E_{\alpha'} = E'_\alpha(K_\alpha \cdot) \quad (\text{II.4.3})$$

defines a conditional expectation  $E_{\alpha'}: \mathcal{A} \rightarrow \mathcal{A}'_{\alpha'}$ . Thus  $K_\alpha \geq 0$  and  $E'_\alpha(K_\alpha) = 1$ ,  $\lambda^0$ -a.e. In the following it will be assumed that the family  $(K_\alpha)$  is a conditional martingale (cf. (II.3.1)), that is the family  $(E_{\alpha'})$  is projective in the sense of (I.1.1) and that the following conditions are satisfied

$$K_\alpha: \Omega \rightarrow \mathbb{R} \quad \text{is continuous} \quad (\alpha \in \mathcal{F}); \quad (\text{II.4.4})$$

$$K_\alpha(x_\alpha, x_{\alpha'}) > 0; \quad \forall x_{\alpha'} \in \Omega_{\alpha'}; \quad \lambda_\alpha^0 - \forall x_\alpha \in \Omega_\alpha \quad (\text{II.4.5})$$

(we use the notation:  $\omega = (x_\alpha, x_{\alpha'}) \in \Omega \cong \Omega_\alpha \times \Omega_{\alpha'}$ ). If the  $K_\alpha$  are local (i.e., there is some  $\beta = \beta(\alpha)$  such that  $K_\alpha = K_\alpha \circ \pi_\beta$ ,  $-\pi_\beta: \Omega \rightarrow \Omega_\beta$  denoting the canonical projection), instead of (II.4.4) it will be required simply that  $K_\alpha$  is strictly positive  $\lambda_\beta^0$ -a.e. These conditions guarantee the ergodicity of  $(E_{\alpha'})$ .

LEMMA 4.1. *In the notation above, the uniqueness of the state compatible with the projective family  $(E_{\alpha'})$  is equivalent to the following condition:*

$$\begin{aligned} \forall \alpha \in \mathcal{F}; \quad \forall a_\alpha \in \mathcal{A}_\alpha; \quad \exists M(a_\alpha) > 0; \quad \forall \epsilon > 0; \\ \exists \beta = \beta(\epsilon, a_\alpha); \quad \exists \gamma_0 = \gamma_0(\epsilon, a_\alpha), \end{aligned}$$

such that

$$|E_{\gamma'}(a_\alpha \cdot b_{\beta'}) (\omega) - E_{\gamma'}(a_\alpha) (\omega) \cdot E_{\gamma'}(b_{\beta'}) (\omega)| \leq \epsilon \cdot M(a_\alpha) \cdot E_{\gamma'}(b_{\beta'}) (\omega)$$

for every  $\gamma \supseteq \gamma_0$ ,  $b_{\beta'} \in \mathcal{A}_{\beta'}^+$ , and  $\omega \in \Omega$ .

*Proof.* Sufficiency: Assume that the above condition is satisfied and let  $\psi$



be a state on  $\mathcal{A} \cong \mathcal{C}(\Omega)$ . Then  $\psi$  defines a Radon measure on  $\Omega$ , still denoted  $\psi$ , and one has, for  $\alpha, a_\alpha, M(a_\alpha), \epsilon, \beta, \gamma, b_{\beta'}$  as in the condition above,

$$\begin{aligned} & |\psi(E_{\gamma'}(a_\alpha \cdot b_{\beta'})) - \psi(E_{\gamma'}(a_\alpha) \cdot E_{\gamma'}(b_{\beta'}))| \\ & \leq \int_{\Omega} |E_{\gamma'}(a_\alpha b_{\beta'}) (\omega) - E_{\gamma'}(a_\alpha) (\omega) \cdot E_{\gamma'}(b_{\beta'}) (\omega)| \psi(d\omega) \\ & \leq \epsilon \cdot M(a_\alpha) \cdot \psi(E_{\gamma'}(b_{\beta'})). \end{aligned}$$

Therefore the condition of Theorem 6.4 is fulfilled, hence uniqueness takes place.

Necessity: Assume that the condition of the lemma does not hold. This means that:  $\exists \alpha \in \mathcal{F}; \exists a_\alpha \in \mathcal{A}_\alpha; \forall M(a_\alpha) > 0; \exists \epsilon > 0$  ( $\epsilon = \epsilon(\alpha, a_\alpha, M(a_\alpha))$ ) such that for every  $\beta \supseteq \alpha$  there are a  $\gamma \supseteq \beta$ , an  $\omega \in \Omega$ , and a  $b_{\beta'} \in \mathcal{A}_{\beta'}^+$ , for which

$$|E_{\gamma'}(a_\alpha \cdot b_{\beta'}) (\omega) - E_{\gamma'}(a_\alpha) (\omega) \cdot E_{\gamma'}(b_{\beta'}) (\omega)| > \epsilon \cdot M(a_\alpha) \cdot E_{\gamma'}(b_{\beta'}) (\omega).$$

In the following we shall take  $M(a_\alpha) = 1$ . The strict inequality above implies that  $E_{\gamma'}(b_{\beta'}) (\omega) > 0$ . Dividing the above inequality by  $E_{\gamma'}(b_{\beta'}) (\omega)$ , and defining the states

$$\begin{aligned} \chi_\beta^1: a \in \mathcal{A} & \mapsto E_{\gamma'}(a \cdot b_{\beta'}) (\omega) / E_{\gamma'}(b_{\beta'}) (\omega); \\ \chi_\beta^2: a \in \mathcal{A} & \mapsto E_{\gamma'}(a_\alpha) (\omega); \end{aligned}$$

one has:  $\chi_\beta^j \cdot E_{\beta'} = \chi_\beta^j; j = 1, 2$ , and

$$|\chi_\beta^1(a_\alpha) - \chi_\beta^2(a_\alpha)| > \epsilon. \quad (\text{II.4.6})$$

One can assume that  $w\text{-}\lim_{\beta \in \mathcal{F}_0} \chi_\beta^j = \chi^j$  ( $j = 1, 2$ ) for some subnet  $\mathcal{F}_0 \subseteq \mathcal{F}$ . Because of (II.4.6)  $\chi^1 \neq \chi^2$ . By Lemma 1.2 in I,  $\chi^1, \chi^2$  are  $(E_{\alpha'})$ -invariant. Thus if the condition of the lemma does not hold, there are at least two different invariant states. And this ends the proof.

In the following, on account of the identification

$$\Omega \cong \Omega_\alpha \times \Omega_{\beta-\alpha} \times \Omega_{\gamma-\beta} \times \Omega_\gamma, \quad (\text{II.4.7})$$

( $\alpha \subset \beta \subset \gamma$ ) which follows from (II.4.2), any point  $\omega \in \Omega$  will be written in the form

$$\omega = (x_\alpha, x_{\beta-\alpha}, x_{\gamma-\beta}, x_{\gamma'}); \quad \alpha \subset \beta \subset \gamma.$$

For any function  $f: \Omega \rightarrow \mathbb{R}$  we shall use the notation

$$f(x_\alpha, \Omega_{\beta-\alpha}, x_{\gamma-\beta}, x_{\gamma'}) = \int_{\Omega_{\beta-\alpha}} f(x_\alpha, y_{\beta-\alpha}, x_{\gamma-\beta}, x_{\gamma'}) \cdot \lambda_{\beta-\alpha}^0(dy_{\beta-\alpha}).$$

LEMMA 4.2. *Let  $(K_\alpha)$  be a conditioned martingale (cf. (II.3.1)) satisfying (II.4.4) and (II.4.5); and let  $E_\alpha = E_\alpha^0(K_\alpha \cdot)$  be the associated projective family of conditional expectations. Assume that the  $K_\alpha$ 's satisfy the following condition:*

*For every  $\alpha \in \mathcal{F}$  and  $\epsilon > 0$ , there exists  $\beta_0 = \beta_0(\epsilon, \alpha)$  such that for every  $\gamma \supset \beta \supset \beta_0(\supset \alpha)$*

$$\left| \frac{K_\gamma(\tilde{x}_\alpha, \Omega_{\beta-\alpha}, \hat{x}_{\gamma-\beta}, x_{\gamma'})}{K_\gamma(\tilde{x}_\alpha, \Omega_{\beta-\alpha}, \tilde{x}_{\gamma-\beta}, x_{\gamma'})} \cdot \frac{K_\gamma(\hat{x}_\alpha, \Omega_{\beta-\alpha}, \tilde{x}_{\gamma-\beta}, x_{\gamma'})}{K_\gamma(\hat{x}_\alpha, \Omega_{\beta-\alpha}, \hat{x}_{\gamma-\beta}, x_{\gamma'})} - 1 \right| < \epsilon. \quad (\text{II.2.8})$$

*Uniformly in  $\tilde{x}_\alpha, \hat{x}_\alpha \in \Omega_\alpha; \tilde{x}_{\gamma-\beta}, \hat{x}_{\gamma-\beta} \in \Omega_{\gamma-\beta}, x_{\gamma'} \in \Omega_{\gamma'}$ . Then there exists only one  $(E_\alpha)$ -invariant state.*

*Proof.* Assume that the condition of the theorem is fulfilled. Then

$$\left| \frac{K_\gamma^{\wedge\wedge} \cdot K_\gamma^{\wedge\wedge}}{K_\gamma^{\sim\sim} \cdot K_\gamma^{\sim\sim}}(\hat{y}_\alpha, \hat{y}_\alpha; \tilde{x}_{\gamma-\beta}; \hat{x}_{\gamma-\beta}; x_{\gamma'}) - 1 \right| < \epsilon \quad (\text{II.4.9})$$

uniformly in  $\tilde{y}_\alpha, \hat{y}_\alpha \in \Omega_\alpha; \tilde{x}_{\gamma-\beta}, \hat{x}_{\gamma-\beta} \in \Omega_{\gamma-\beta}, x_{\gamma'} \in \Omega_{\gamma'}$  for every  $\gamma \supset \beta \supset \beta_0(\epsilon, \alpha)$ , where we have introduced the notation

$$\begin{aligned} & \frac{K_\gamma^{\wedge\wedge} \cdot K_\gamma^{\wedge\wedge}}{K_\gamma^{\sim\sim} \cdot K_\gamma^{\sim\sim}}(\tilde{y}_\alpha, \hat{y}_\alpha; \tilde{x}_{\gamma-\beta}, \hat{x}_{\gamma-\beta}, x_{\gamma'}) \\ &= \frac{K_\gamma(\tilde{y}_\alpha, \Omega_{\beta-\alpha}, \hat{x}_{\gamma-\beta}, x_{\gamma'})}{K_\gamma(\tilde{y}_\alpha, \Omega_{\beta-\alpha}, \tilde{x}_{\gamma-\beta}, x_{\gamma'})} \cdot \frac{K_\gamma(\hat{y}_\alpha, \Omega_{\beta-\alpha}, \tilde{x}_{\gamma-\beta}, x_{\gamma'})}{K_\gamma(\hat{y}_\alpha, \Omega_{\beta-\alpha}, \hat{x}_{\gamma-\beta}, x_{\gamma'})}. \end{aligned} \quad (\text{II.4.10})$$

Therefore, if  $a_\alpha \in \mathcal{A}_\alpha, b_{\beta'} \in \mathcal{A}_{\beta'}^+$

$$\begin{aligned} & |E_{\gamma'}(a_\alpha b_{\beta'}) (x_{\gamma'}) - E_{\gamma'}(a_\alpha) (x_{\gamma'}) \cdot E_{\gamma'}(b_{\beta'}) (x_{\gamma'})| \\ &= \left| \int_{\Omega_\alpha} \int_{\Omega_\alpha} \int_{\Omega_{\lambda-\beta}} \int_{\Omega_{\lambda-\beta}} a_\alpha(\tilde{x}_\alpha) \cdot b_{\beta'}(\hat{x}_{\gamma-\beta}, x_{\gamma'}) \cdot \{K_\gamma(\tilde{x}_\alpha, \Omega_{\beta-\alpha}, \hat{x}_{\gamma-\beta}, x_{\gamma'}) \right. \\ & \quad \times K_\gamma(\hat{x}_\alpha, \Omega_{\beta-\alpha}, \tilde{x}_{\gamma-\beta}, x_{\gamma'}) - K_\gamma(\tilde{x}_\alpha, \Omega_{\beta-\alpha}, \tilde{x}_{\gamma-\beta}, x_{\gamma'}) \\ & \quad \times K_\gamma(\hat{x}_\alpha, \Omega_{\beta-\alpha}, \hat{x}_{\gamma-\beta}, x_{\gamma'}) \cdot \lambda_\alpha^0(d\tilde{x}_\alpha) \cdot \lambda_\alpha^0(d\hat{x}_\alpha) \cdot \lambda_{\gamma-\beta}^0(d\tilde{x}_{\gamma-\beta}) \cdot \lambda_{\gamma-\beta}^0(d\hat{x}_{\gamma-\beta}) \\ & \leq \epsilon \cdot \int_{\Omega_\alpha} \int_{\Omega_\alpha} \int_{\Omega_{\beta-\alpha}} \int_{\Omega_{\beta-\alpha}} |a_\alpha(\tilde{x}_\alpha)| \cdot b_{\beta'}(\hat{x}_{\gamma-\beta}, x_{\gamma'}) \cdot K_\gamma(\tilde{x}_\alpha, \Omega_{\beta-\alpha}, \tilde{x}_{\gamma-\beta}, x_{\gamma'}) \\ & \quad \times K_\gamma(\hat{x}_\alpha, \Omega_{\beta-\alpha}, \hat{x}_{\gamma-\beta}, x_{\gamma'}) \cdot \lambda_\alpha^0(d\tilde{x}_\alpha) \cdot \lambda_\alpha^0(d\hat{x}_\alpha) \cdot \lambda_{\gamma-\beta}^0(d\tilde{x}_{\gamma-\beta}) \cdot \lambda_{\gamma-\beta}^0(d\hat{x}_{\gamma-\beta}) \\ & = \epsilon \cdot E_{\gamma'}(|a_\alpha|) (x_{\gamma'}) \cdot E_{\gamma'}(b_{\beta'}) (x_{\gamma'}) \\ & \leq \epsilon \cdot \|a_\alpha\| \cdot E_{\gamma'}(b_{\beta'}) (x_{\gamma'}). \end{aligned}$$

Since  $E_{\gamma'}(a) (x_\gamma, x_{\gamma'}) = E_{\gamma'}(a) (x_{\gamma'})$ , and the above estimate is uniform in  $x_{\gamma'}$ , the condition of Lemma 4.1 is satisfied, hence uniqueness holds.

LEMMA 4.3. *Assume that for some  $\alpha \in \mathcal{F}$  and  $\tilde{x}_\alpha, \hat{x}_\alpha \in \Omega_\alpha$  there exist neighborhoods  $U(\tilde{x}_\alpha), U(\hat{x}_\alpha)$  of  $\tilde{x}_\alpha, \hat{x}_\alpha$ , respectively, and  $\epsilon > 0$  such that for every  $\beta_0$  there exist  $\gamma \supset \beta \supset \beta_0$  and  $\tilde{x}_{\gamma-\beta}, \hat{x}_{\gamma-\beta} \in \Omega_{\gamma-\beta}, x_{\gamma'} \in \Omega_{\gamma'},$  such that  $\forall (\hat{y}_\alpha, \hat{y}'_\alpha) \in U(\tilde{x}_\alpha) \times U(\hat{x}_\alpha)$*

$$\left| \frac{K_{\tilde{y}_\alpha}^{\sim} \cdot K_{\hat{y}'_\alpha}^{\sim}}{K_{\tilde{y}_\alpha}^{\sim\sim} \cdot K_{\hat{y}'_\alpha}^{\sim\sim}} (\hat{y}_\alpha, \hat{y}'_\alpha; \tilde{x}_{\gamma-\beta}, \hat{x}_{\gamma-\beta}, x_{\gamma'}) - 1 \right| > \epsilon$$

(cf. (II.4.10) for the notation). Then there exist two different states compatible with the projective family  $E_{\alpha'} = E_{\alpha'}(K_\alpha \cdot), \alpha \in \mathcal{F}$ .

*Proof.* By continuity one can assume, possibly considering subneighborhoods and exchanging the roles of  $\tilde{x}_{\gamma-\beta}$  and  $\hat{x}_{\gamma-\beta}$ , that

$$\frac{K_{\tilde{y}_\alpha}^{\sim} \cdot K_{\hat{y}'_\alpha}^{\sim}}{K_{\tilde{y}_\alpha}^{\sim\sim} \cdot K_{\hat{y}'_\alpha}^{\sim\sim}} (\hat{y}_\alpha, \hat{y}'_\alpha; \tilde{x}_{\gamma-\beta}, \hat{x}_{\gamma-\beta}, x_{\gamma'}) > 1 + \epsilon.$$

There exist two functions  $\tilde{a}_\alpha, \hat{a}_\alpha \in \mathcal{A}_\alpha (\cong \mathcal{C}(\Omega_\alpha))$  such that

$$\begin{aligned} \tilde{a}_\alpha(\tilde{x}_\alpha) &= \hat{a}_\alpha(\hat{x}_\alpha) = 1; \\ \tilde{a}_\alpha \upharpoonright \Omega_\alpha - U(\tilde{x}_\alpha) &= \hat{a}_\alpha \upharpoonright \Omega_\alpha - U(\hat{x}_\alpha) = 0; \\ \tilde{a}_\alpha, \hat{a}_\alpha: \Omega &\rightarrow [0, 1]. \end{aligned}$$

For such  $\tilde{a}_\alpha$  and  $\hat{a}_\alpha$  one will have

$$\begin{aligned} (K_{\tilde{y}_\alpha}^{\sim} \cdot K_{\hat{y}'_\alpha}^{\sim} \cdot \tilde{a}_\alpha \cdot \hat{a}_\alpha) (\hat{y}_\alpha, \hat{y}'_\alpha; \tilde{x}_{\gamma-\beta}, \hat{x}_{\gamma-\beta}, x_{\gamma'}) \\ \geq (1 + \epsilon) (K_{\tilde{y}_\alpha}^{\sim\sim} \cdot K_{\hat{y}'_\alpha}^{\sim\sim} \cdot \tilde{a}_\alpha, \hat{a}_\alpha) (\hat{y}_\alpha, \hat{y}'_\alpha; \tilde{x}_{\gamma-\beta}, \hat{x}_{\gamma-\beta}, x_{\gamma'}) \end{aligned}$$

hence, integrating with respect to  $\lambda_\alpha^0(d\tilde{x}_\alpha), \lambda_\alpha^0(d\hat{x}_\alpha)$  one finds

$$\begin{aligned} (K_{\tilde{y}_\alpha} \tilde{a}_\alpha) (\hat{y}_{\gamma-\beta}, x_{\gamma'}) \cdot (K_{\hat{y}'_\alpha} \hat{a}_\alpha) (\tilde{x}_{\gamma-\beta}, x_{\gamma'}) \\ \geq (1 + \epsilon) (K_{\tilde{y}_\alpha} \tilde{a}_\alpha) (\hat{y}_{\gamma-\beta}, x_{\gamma'}) \cdot (K_{\hat{y}'_\alpha} \hat{a}_\alpha) (\hat{y}_{\gamma-\beta}, x_{\gamma'}), \end{aligned}$$

where

$$(K_{\tilde{y}_\alpha} a_\alpha) (y_{\gamma-\beta}, x_{\gamma'}) = \int_{\Omega_\alpha} K(z_\alpha, \Omega_{\beta-\alpha}, y_{\gamma-\beta}, x_{\gamma'}) \cdot a_\alpha(z_\alpha) \cdot \lambda_\alpha^0(dz_\alpha).$$

By continuity there are neighborhoods  $U(\tilde{x}_{\gamma-\beta})$  and  $U(\hat{x}_{\gamma-\beta})$  of  $\tilde{x}_{\gamma-\beta}, \hat{x}_{\gamma-\beta}$ , respectively, such that

$$\begin{aligned} (K_{\tilde{y}_\alpha} a_\alpha) (\hat{y}_{\gamma-\beta}, x_{\gamma'}) \cdot (K_{\hat{y}'_\alpha} \hat{a}_\alpha) (y_{\gamma-\beta}, x_{\gamma'}) \\ > (1 + \epsilon/2) \cdot (K_{\tilde{y}_\alpha} \tilde{a}_\alpha) (y_{\gamma-\beta}, x_{\gamma'}) \cdot (K_{\hat{y}'_\alpha} \hat{a}_\alpha) (\hat{y}_{\gamma-\beta}, x_{\gamma'}) \end{aligned} \quad (\text{II.4.11})$$

for every  $\tilde{y}_{\gamma-\beta} \in U(\tilde{x}_{\gamma-\beta})$ ,  $y_{\gamma-\beta} \in U(\hat{x}_{\gamma-\beta})$ . Therefore there will exist functions  $\tilde{b}_{\beta'}, \hat{b}_{\beta'} \in \mathcal{A}_{\gamma-\beta} \subseteq \mathcal{A}_{\beta'}$  such that

$$\begin{aligned} \tilde{b}_{\beta'}(\tilde{x}_{\gamma-\beta}) &= \hat{b}_{\beta'}(\hat{x}_{\gamma-\beta}) = 1; \\ \tilde{b}_{\beta'} \upharpoonright \Omega_{\gamma-\beta} - U(\tilde{x}_{\gamma-\beta}) &= \hat{b}_{\beta'} \upharpoonright \Omega_{\gamma-\beta} - U(\hat{x}_{\gamma-\beta}) = 0; \\ \tilde{b}_{\beta'}, \hat{b}_{\beta'}: \Omega_{\gamma-\beta} &\rightarrow [0, 1]. \end{aligned}$$

Multiplying (II.4.11) by  $\tilde{b}_{\beta'}(\tilde{y}_{\gamma-\beta}, x_{\gamma'}) \cdot \hat{b}_{\beta'}(\hat{y}_{\gamma-\beta}, x_{\gamma'})$  and integrating with respect to  $\lambda_{\gamma-\beta}^0(d\tilde{y}_{\gamma-\beta}) \cdot \lambda_{\gamma-\beta}(d\hat{y}_{\gamma-\beta}^0)$  one finds

$$\begin{aligned} E_{\gamma'}(\tilde{a}_\alpha \cdot \hat{b}_{\beta'})(x_{\gamma'}) \cdot E_{\gamma'}(\hat{a}_\alpha \cdot \tilde{b}_{\beta'})(x_{\gamma'}) \\ > (1 + \epsilon/2) \cdot E_{\gamma'}(\tilde{a}_\alpha \cdot \tilde{b}_{\beta'})(x_{\gamma'}) \cdot E_{\gamma'}(\hat{a}_\alpha \cdot \hat{b}_{\beta'})(x_{\gamma'}). \end{aligned}$$

By (II.4.5)  $E_{\gamma'}(\tilde{b}_{\beta'})(x_{\gamma'}) \cdot E_{\gamma'}(\hat{b}_{\beta'})(x_{\gamma'}) > 0$ , therefore

$$\begin{aligned} \frac{E_{\gamma'}(\tilde{a}_\alpha \hat{b}_{\beta'})(x_{\gamma'})}{E_{\gamma'}(\tilde{b}_{\beta'})(x_{\gamma'})} \cdot \frac{E_{\gamma'}(\hat{a}_\alpha \cdot \tilde{b}_{\beta'})(x_{\gamma'})}{E_{\gamma'}(\hat{b}_{\beta'})(x_{\gamma'})} \\ > (1 + \epsilon/2) \frac{E_{\gamma'}(\tilde{a}_\alpha \cdot \tilde{b}_{\beta'})(x_{\gamma'})}{E_{\gamma'}(\tilde{b}_{\beta'})(x_{\gamma'})} \cdot \frac{E_{\gamma'}(\hat{a}_\alpha \hat{b}_{\beta'})(x_{\gamma'})}{E_{\gamma'}(\hat{b}_{\beta'})(x_{\gamma'})}. \end{aligned} \tag{II.4.12}$$

Assume, by contradiction, that there is a unique state  $\mu$  on  $\mathcal{A}$  compatible with the family  $(E_{\alpha'})$ . Then, taking the limit of (II.4.12) with respect to  $\beta$  one finds, reasoning as in Theorem 6.4 in *I*, that

$$\mu(\tilde{a}_\alpha) \cdot \mu(\hat{a}_\alpha) \geq (1 + \epsilon/2) \mu(\tilde{a}_\alpha) \cdot \mu(\hat{a}_\alpha).$$

Since  $(E_{\alpha'})$  is ergodic,  $\mu(\tilde{a}_\alpha) \cdot \mu(\hat{a}_\alpha) > 0$ , therefore the above inequality is impossible. Thus, if the condition of the lemma is not fulfilled, then there are two different compatible states.

**COROLLARY 4.4.** *Assume that  $\mathcal{F}$  is the set of finite parts of  $\mathbb{Z}^d$  ( $d \in \mathbb{N}$ ) and that  $\Omega_\alpha \simeq \prod_{\beta \neq \alpha} \mathcal{S}$ ;  $\mathcal{S}$ —a finite set;  $\alpha \in \mathcal{F}$ . Then the condition of Lemma 4.2 is equivalent to the uniqueness of the  $(E_{\alpha'})$ -invariant state.*

*Proof.* The sufficiency follows from Lemma 4.2. If this condition is not satisfied, then there are  $\alpha \in \mathcal{F}$ ,  $\epsilon > 0$ ,  $\tilde{x}_\alpha, \hat{x}_\alpha \in \Omega_\alpha$ , such that for every  $\beta_0 \supseteq \alpha$ , there are  $\gamma \supset \beta \supset \beta_0$  and  $\tilde{x}_{\gamma-\beta}, \hat{x}_{\gamma-\beta} \in \Omega_{\gamma-\beta}$ ,  $x_{\gamma'} \in \Omega_{\gamma'}$  such that

$$\left| \frac{K_{\gamma'}^{\sim\sim} \cdot K_{\gamma'}^{\wedge\wedge}}{K_{\gamma'}^{\sim\sim} \cdot K_{\gamma'}^{\wedge\wedge}}(\tilde{x}_\alpha, \hat{x}_\alpha; \tilde{x}_{\gamma-\beta}, \hat{x}_{\gamma-\beta}, x_{\gamma'}) - 1 \right| > \epsilon.$$

Therefore, since  $\Omega_\alpha$  is discrete, the condition of Lemma 4.3 is fulfilled, hence uniqueness fails.

## III

I. THE LOCAL  $\sigma$ -ALGEBRAS OF THE FREE EUCLIDEAN FIELD

Let  $N$  be the completion of the real Schwartz space  $\mathcal{S}(\mathbb{R}^d)$  with respect to the norm defined by the scalar product  $\langle f, (-\Delta + m^2)^{-1} h \rangle$ , where  $\Delta$  is the Laplacian and  $m > 0$  ( $m \geq 0$  if  $d \geq 3$ ). The free Euclidean field is defined (cf. [17, p. 106]) as the unit Gaussian process on  $N$ , extended by linearity to the complexification of  $N$ . Let  $(\Omega, O, \mu^0)$  be a probability space underlying the free Euclidean field.

Denote, for an open or closed set  $A \subseteq \mathbb{R}^d$ ,  $N_A$  the closure of the set of distributions in  $N$  with support in  $A$ ;  $e_A: N \rightarrow N_A$  the orthogonal projection (cf. [22, p. 94]). If  $A$  is an open set  $\subseteq \mathbb{R}^d$ , define  $O_A$  as the  $\mu^0$ -complete sub- $\sigma$ -algebra of  $O$  generated by  $\{\varphi(f): f \in N_A\}$ ; if  $C \subseteq \mathbb{R}^d$  is any set define

$$O_C = \cap \{O_A: A \text{ open } \supseteq C\}.$$

LEMMA 1.1. *The family of  $\sigma$ -algebras, defined above, satisfies the following relations:*

*If  $\{A_i\}$  is an arbitrary family of open subsets of  $\mathbb{R}^d$  then*

$$O_{\cup_i A_i} = \bigvee_i O_{A_i}. \quad (\text{III.1.1})$$

$\bigvee_i O_{A_i}$  (resp.  $O_{A_1} \vee O_{A_2}$ ) denotes, here and in the following, the  $\mu^0$ -complete  $\sigma$ -algebra generated by the family  $\{A_i\}$  (resp. by  $A_1$  and  $A_2$ .) We shall use the same symbols referred to von Neumann algebras.

*If  $C, D$  are open or closed subsets of  $\mathbb{R}^d$ , then*

$$O_C \cap O_D = O_{C \cap D}. \quad (\text{III.1.2})$$

*Proof.* (III.1.1) follows from the fact that  $S(\mathbb{R}^d)$  admits partitions of the identity.

For  $A$  open or closed, denoting  $E_{O_A}^0$  the conditional expectation on  $O_A$  associated with  $\mu^0$ , one has

$$\Gamma(l_A) = E_{O_A}^0 \quad \text{restricted on } L^2(\Omega, O, \mu^0)$$

(cf. [22] for the notation). Therefore if  $A, B$  are open or closed

$$\begin{aligned} E_{O_{A \cap B}}^0 &= \Gamma(e_A) \wedge \Gamma(e_B) = \lim_n (\Gamma(e_A) \cdot \Gamma(e_B))^n \\ &= \Gamma(e_A \wedge e_B) = \Gamma(e_{A \cap B}) \\ &= E_{O_{A \cap B}}^0 \end{aligned}$$

which implies  $O_A \cap O_B = O_{A \cap B}$ . Taking intersections one establishes (III.1.2) for any two sets  $C, D$ .

In particular, if  $\alpha, \beta$  are bounded open sets with piecewise smooth boundary, and  $\alpha \subseteq \beta$ , then

$$O_\beta = O_\alpha \vee O_{\beta - \bar{\alpha}}. \tag{III.1.3}$$

The free Euclidean field is a Markov field, in the sense of Nelson (cf., for example, [17, p. 107]), that is, it satisfies

$$E_{\mathbb{C}A}^0 \cdot E_A^0 = E_{\partial A}^0 \cdot E_A^0; \quad A \subseteq \mathbb{R}^d, \text{ open.}$$

We shall need a stronger property of the free field, namely,

$$E_A^0 \cdot E_B^0 = E_{\partial A \cup (A \cap B)}^0 \cdot E_B^0 \tag{III.1.4}$$

for any two closed sets  $A, B \subseteq \mathbb{R}^d$ .

Relation (III.1.4), isolated by Guerra *et al.* [13, Proposition II.3], is stronger than the Markov property since it gives information on the statistical correlation among observables localized on two *arbitrary* (closed) sets, while the latter is limited to one set and its complement (for sufficiently regular sets the two properties are equivalent due to (III.1.3)).

Denote  $\mathcal{F}$  the family of bounded open subsets of  $\mathbb{R}^d$  with piecewise smooth boundary. For each *bounded* set  $A_0$ , define

$$\mathcal{A}_{A_0} = L^\infty(\Omega, O_{A_0}, \mu_{A_0}^0).$$

If  $C \subseteq \mathbb{R}^d$  is any set, define

$$\mathcal{A}_C = \text{norm closure of } \cup \{ \mathcal{A}_D; D \text{ bounded } \subseteq C \}.$$

Because of (3.1.4),  $\mathcal{A}_C$  is a weakly dense subalgebra of  $L^\infty(\Omega, O_C, \mu_C^0)$ ,  $\mathcal{A}_{\mathbb{R}^d}$  will be denoted  $\mathcal{A}$ .

LEMMA 1.2. *The Abelian  $c^*$ -algebras defined above enjoy the following properties*

$$C \subseteq D \Rightarrow \mathcal{A}_C \subseteq \mathcal{A}_D; \quad C, D \text{ arbitrary sets.} \tag{III.1.5}$$

$$C \subseteq D \text{ bounded regular } \Rightarrow \mathcal{A}_D = \mathcal{A}_C \vee \mathcal{A}_{D-C}. \tag{III.1.6}$$

If  $C$  is a closed set with bounded boundary, then

$$\mathcal{A}_C = \mathcal{A} \cap L^\infty(\Omega, O_C, \mu_C^0). \tag{III.1.7}$$

If  $\Lambda \in \mathcal{F}$  and  $E_{\mathbb{C}\Lambda}^0$  denotes the conditional expectation on  $O_{\mathbb{C}\Lambda}$  associated with  $\mu^0$ , then

$$E_{\mathbb{C}\Lambda}^0(\mathcal{A}) \subseteq \mathcal{A}_{\mathbb{C}\Lambda}. \tag{III.1.8}$$

*Proof.* (III.1.6), (III.1.7) are evident.

(III.1.8) Clearly  $\mathcal{A}_C \subseteq \mathcal{A} \cap L^\infty(\Omega, O_C, \mu_C^0)$ .

Conversely, let  $a \in \mathcal{A} \cap L^\infty(\Omega, O_C, \mu_C^0)$ . Then there exist a sequence of open bounded sets  $(A_n)$  such that  $A_n \uparrow \mathbb{R}^d$  and a sequence  $(a_n)$  such that  $a_n \in \mathcal{A}_{A_n}$  and  $\|a_n - a\| \rightarrow 0$ . Because of (III.1.4)

$$E_C^0(a_n) = E_C^0 E_{A_n}^0(a_n) = E_{\partial C \cup (\bar{A}_n \cap C)}^0(a_n).$$

Thus, since  $\partial C$  is bounded,  $E_C^0(a_n) \in \mathcal{A}_C$ . But then  $a \in \mathcal{A}_C$ , because

$$\|E_C^0(a_n) - a\| = \|E_C^0(a_n - a)\| \leq \|a_n - a\| \rightarrow 0$$

and this establishes (III.1.7).

(III.1.9) Because of (III.1.7) one needs only to prove that  $E_{\mathbb{C}A}^0(\mathcal{A}) \subseteq \mathcal{A}$ . For this it is sufficient that, for any  $\beta \in \mathcal{F}$ ,  $E_{\mathbb{C}A}^0(\mathcal{A}_\beta) \subseteq \mathcal{A}$ . If  $\beta \supseteq \bar{A}$  then, by (III.1.7),  $\mathcal{A}_\beta = \mathcal{A}_{\bar{A}} \vee \mathcal{A}_{\beta - \bar{A}}$ . By the Markov property  $E_{\mathbb{C}A}^0(\mathcal{A}_{\bar{A}}) \subseteq \mathcal{A}_{\partial A} \subseteq \mathcal{A}_\beta$ . This implies  $E_{\mathbb{C}A}^0(\mathcal{A}_\beta) \subseteq \mathcal{A}_\beta \subseteq \mathcal{A}$ , concluding the proof.

*Remark.* Since  $N_A = N_{\bar{A}}$  for sufficiently regular sets (cf. [22, Remark 2, p. 94]) the definition of the algebras  $\mathcal{A}_C$  imply that, for sufficiently regular sets  $C$  (e.g., finite unions or intersections of open half spaces), one has  $\mathcal{A}_C = \mathcal{A}_{\bar{C}}$ .

## 2. THE GLOBAL MARKOV PROPERTY

Let  $\mathcal{F}$  be the family of bounded open sets in  $\mathbb{R}^d$  with piecewise smooth boundary. In the following  $(E_{\mathbb{C}A_0})_{A_0 \in \mathcal{F}}$  will denote a locally normal  $E(d)$ -covariant family of Markovian conditional expectations  $E_{\mathbb{C}A_0}: \mathcal{A} \rightarrow \mathcal{A}_{\mathbb{C}A_0}$ ; the family will be supposed projective and the associated conditioned martingale (cf. Part II, Proposition (2.5) and (II.3.1)) will be denoted  $(K_{A_0})_{A_0 \in \mathcal{F}}$ . Thus

$$E_{\mathbb{C}A_0}(\cdot) = E_{\mathbb{C}A_0}^0(K_{A_0} \cdot)$$

and

$$K_{A_0} \hat{\in} O_{\bar{A}_0} = O_{A_0}; \quad (A_0 \in \mathcal{F}).$$

(III.2.1)

The localization property (III.1.5) of the conditional expectations of the free field is partially transferred to the family  $(E_{\mathbb{C}A_0})$ . In fact, if  $A_1, A_2 \in \mathcal{F}$ , and  $A_1 \cap A_2 = \emptyset$  then

$$\begin{aligned} E_{\mathbb{C}A_1} \cdot E_{\mathbb{C}A_2}(a) &= E_{\mathbb{C}A_1}^0(K_{A_1} \cdot E_{\mathbb{C}A_2}^0(K_{A_2} a)) \\ &= E_{\mathbb{C}A_1}^0 \cdot E_{\mathbb{C}A_2}^0(K_{A_1} \cdot K_{A_2} \cdot a) \\ &= E_{\mathbb{C}(A_1 \cup A_2)}^0(K_{A_1} \cdot K_{A_2} \cdot a). \end{aligned}$$

Therefore the projectivity of the family  $(E_{\mathfrak{C}A_0})$  implies

$$E_{\mathfrak{C}A_1} \cdot E_{\mathfrak{C}A_2} = E_{\mathfrak{C}(A_1 \cup A_2)}; \quad A_1 \cap A_2 = \emptyset; \quad A_i \in \mathcal{F}. \quad (\text{III.2.2})$$

(Relation (III.2.2) can also be verified directly using Theorem (3.1) in Part II.)

Let  $\mu$  be any state on  $\mathcal{A}$  compatible with the family  $(E_{\mathfrak{C}A_0})_{A_0 \in \mathcal{F}}$ . Denote  $\{\mathcal{H}_\mu, \pi_\mu, 1_\mu\}$  the Gelfand–Naimark–Segal representation of  $\mathcal{A}$  associated with  $\mu$ , and define for  $B \subseteq \mathbb{R}^d$  the *bounded* or the *complement of a bounded* set

$$\mathcal{O}_B = \pi_\mu(\mathcal{A}_B)''.$$

If  $C$  is any other set, define

$$\mathcal{O}_C = \bigcap \{ \pi_\mu(\mathcal{A}_{\mathfrak{C}B})'' : B \text{ bounded closed } \subseteq \mathfrak{C}C \}.$$

Thus, in any case,  $\mathcal{O}_C \supseteq \pi_\mu(\mathcal{A}_C)''$ . This definition of the algebras  $\mathcal{O}_{\mathfrak{C}C}$ , localized on sets with unbounded boundary, is motivated by the necessity of controlling the “effects at infinity” arising when the  $(E_{\mathfrak{C}A})$ -invariant state is not unique (i.e., in presence of phase transitions). When the algebra at infinity is trivial (i.e., for extremal states) our definition coincides with the one proposed by Nelson in [17].

Let, for  $C \subseteq \mathbb{R}^d$ ,  $\mathcal{H}_\mu(C) = [\mathcal{O}_C \cdot 1_\mu]$  (= the closure in  $\mathcal{H}_\mu$  of  $\mathcal{O}_C \cdot 1_\mu$ ) and denote  $e_C^\mu$  the corresponding orthogonal projection.

If  $A_0 \in \mathcal{F}_0$  one has (cf. Part I, Section 3)

$$e_{\mathfrak{C}A_0}^\mu(\pi_\mu(a) \cdot 1_\mu) = \pi_\mu(E_{\mathfrak{C}A_0}(a)) \cdot 1_\mu.$$

Therefore, if  $A_1, A_2 \in \mathcal{F}$ ,  $A_1 \cap A_2 = \emptyset$ , (III.2.2) implies that

$$e_{\mathfrak{C}A_1}^\mu \cdot e_{\mathfrak{C}A_2}^\mu = e_{\mathfrak{C}(A_1 \cup A_2)}^\mu. \quad (\text{III.2.3})$$

LEMMA 2.1. *Denote  $\Lambda$  the open half-space*

$$\Lambda = \{(x, t) \in \mathbb{R}^{d-1} \times \mathbb{R} : t < 0\}$$

*and  $\rho$  the reflection with respect to  $\partial\Lambda$  (= the boundary of  $\Lambda$ ). If  $(\Lambda_n)$  is any increasing sequence of sets in  $\mathcal{F}$  such that*

$$\bar{\Lambda}_n \subseteq \Lambda; \quad \bigsqcup_n \Lambda_n = \Lambda$$

*then*

$$\mathcal{O}_{\mathfrak{C}\Lambda} = \bigcap_n \pi_\mu(\mathcal{A}_{\mathfrak{C}\Lambda_n})''; \quad (\text{III.2.4})$$

$$\mathcal{O}_{\partial\Lambda} = \bigcap_n \pi_\mu(\mathcal{A}_{\mathfrak{C}(\Lambda_n \cup \rho\Lambda_n)})''. \quad (\text{III.2.5})$$



*Proof.* If  $B$  is a bounded closed set  $\subseteq A$ , then  $B \subseteq A_n$  for some  $n$ , since  $B$  is compact and  $\{A_n\}$  is an open cover of  $A$ , hence of  $B$ . Thus

$$\pi_\mu(\mathcal{A}_{\mathbf{C}B})'' \supseteq \pi_\mu(\mathcal{A}_{\mathbf{C}A_n})'' \supseteq \bigcap_k \pi_\mu(\mathcal{A}_{\mathbf{C}A_k})''.$$

Since  $B$  is an arbitrary bounded closed set  $\subseteq A$ , this implies

$$\mathcal{O}_C = \bigcap \{ \pi_\mu(\mathcal{A}_{\mathbf{C}B})'' : B \text{ bounded closed } \subseteq A \} \supseteq \bigcap_n \pi_\mu(\mathcal{A}_{\mathbf{C}A_n})''.$$

The converse inclusion is clear since  $\bar{A}_n \subseteq A$  and  $\pi_\mu(\mathcal{A}_{\mathbf{C}A_n})'' = \pi_\mu(\mathcal{A}_{\mathbf{C}\bar{A}_n})''$ , because  $A_n \in \mathcal{F}$ . Thus (III.2.4) is proved. (III.2.5) is proved with analogous reasoning.

**PROPOSITION 2.2.** *In the notation introduced above, one has*

$$e_{\mathbf{C}\rho A}^\mu \cdot e_{\mathbf{C}A}^\mu = e_{\partial A}^\mu. \quad (\text{III.2.6})$$

*Proof.* Let  $(A_n)$  be a sequence as in Lemma 2.1. Then

$$\mathcal{O}_{\mathbf{C}A} = \bigcap_n \pi_\mu(\mathcal{A}_{\mathbf{C}A_n})''; \quad \mathcal{O}_{\mathbf{C}\rho A} = \bigcap_n \pi_\mu(\mathcal{A}_{\mathbf{C}\rho A_n})''.$$

Therefore, by the martingale theorem (or, directly, using a modification of Lemma 3.1 in Part I),

$$\mathcal{H}_\mu(\mathbf{C}A) = \bigcap_n \mathcal{H}_\mu(\mathbf{C}A_n); \quad \mathcal{H}_\mu(\mathbf{C}\rho A) = \bigcap_n \mathcal{H}_\mu(\mathbf{C}\rho A_n).$$

Thus, since multiplication is strongly continuous on bounded sets

$$e_{\mathbf{C}\rho A}^\mu \cdot e_{\mathbf{C}A}^\mu = s\text{-lim } e_{\mathbf{C}\rho A_n}^\mu \cdot e_{\mathbf{C}A_n}^\mu$$

therefore, using (III.2.3) and (III.2.5), one finds

$$e_{\mathbf{C}\rho A}^\mu \cdot e_{\mathbf{C}A}^\mu = s\text{-lim } e_{\mathbf{C}(\rho A_n \cup A_n)}^\mu = e_{\partial A}^\mu$$

and this ends the proof.

Let now  $\mu$  be locally normal; denote for any subset  $C \subseteq \mathbb{R}^d$ ,  $\mathcal{B}_C = \mathcal{B}_C^\mu$  the  $\mu$ -complete  $\sigma$ -algebra defined by the projections of  $\mathcal{O}_C$  ( $\mathcal{B} = \mathcal{B}_{\mathbb{R}^d}$ ), and  $E_C^\mu$  the conditional expectation defined by  $\mu$  on  $\mathcal{B}_C$ . One has

$$e_C^\mu(\bar{a} \cdot 1_\mu) = E_C^\mu(\bar{a}) \cdot 1_\mu; \quad \bar{a} \in \pi_\mu(\mathcal{A})'';$$

therefore (III.2.6) is equivalent to

$$E_{\mathbb{C}^{\rho A}}^{\mu} \cdot E_{\mathbb{C}^A}^{\mu} = E_{\partial A}^{\mu}$$

which is the Markov property with respect to the hyperplane  $\partial A$ .

*Remark.* The proof of Proposition 2.2 can be adapted to the case when  $A$  is any region with smooth boundary.

The action of the  $d$ -dimensional Euclidean group  $E(d)$  on  $(\Omega, O, \mu^0)$  (cf. [17, p. 107]) induces an action  $g: a \in \mathcal{A} \rightarrow ga \in \mathcal{A}$ , of  $E(d)$  on  $\mathcal{A}$ , by \*-automorphisms, which is covariant, i.e.,

$$g \cdot \mathcal{A}_C = \mathcal{A}_{gC}; \quad g \in E(d); \quad C \subseteq \mathbb{R}^d.$$

If  $\mu$  is  $E(d)$ -invariant, this action induces a unitary representation  $T: g \in E(d) \rightarrow T_g$  of  $E(d)$  on  $\mathcal{H}_{\mu}$ , defined by

$$T_g \cdot \pi_{\mu}(a) \cdot 1_{\mu} = \pi_{\mu}(ga) \cdot 1_{\mu}; \quad g \in E(d); \quad a \in \mathcal{A}.$$

Covariance implies that  $T_g e_C^{\mu} = e_{gC}^{\mu} \cdot T_g$ .

LEMMA 2.3. *Let  $\mu$  be an  $(E_{\mathbb{C}^{\rho A}})$ -compatible,  $E(d)$ -invariant state on  $\mathcal{A}$ . The following conditions are equivalent:*

$$e_{\mathbb{C}^A}^{\mu} \cdot T_{\rho} \cdot e_{\mathbb{C}^A}^{\mu} \geq 0; \tag{III.2.7}$$

$$T_{\rho} \cdot e_{\partial A}^{\mu} = e_{\partial A}^{\mu}. \tag{III.2.8}$$

*Proof.* By covariance and (III.2.6), it is clear that (III.2.8)  $\Rightarrow$  (III.2.7). Conversely, since  $T_{\rho} \cdot e_{\partial A}^{\mu} = e_{\partial A}^{\mu} \cdot T_{\rho}$  one has

$$(e_{\mathbb{C}^A}^{\mu} \cdot T_{\rho} \cdot e_{\mathbb{C}^A}^{\mu})^2 = (T_{\rho} \cdot e_{\partial A}^{\mu})^2 = e_{\partial A}^{\mu}.$$

Hence if  $e_{\mathbb{C}^A}^{\mu} \cdot T_{\rho} \cdot e_{\mathbb{C}^A}^{\mu}$  is positive it is equal to  $e_{\partial A}^{\mu}$ , thus (III.2.8) holds.

Note that (III.2.5) implies that

$$\mathcal{O}_{\partial A} \supseteq \mathcal{A}_{\infty}^{\mu},$$

hence  $\mathcal{H}_{\infty}^{\mu} \subseteq \mathcal{H}_{\mu}(\partial A)$ . Therefore a necessary condition for property (III.2.8) to take place is that

$$T_{\rho} \cdot e_{\infty}^{\mu} = e_{\infty}^{\mu} \tag{III.2.9}$$

( $e_{\infty}^{\mu}$  denoting the orthogonal projector  $\mathcal{H}_{\mu} \rightarrow \mathcal{H}_{\infty}^{\mu}$ ). Relation (III.2.9) is called the reflection property at infinity. It is possible that (III.2.9) is equivalent to (III.2.8). In such a case all  $T_{\rho}$ -invariant solutions of the equations

$$\mu = \mu \cdot E_{\mathbb{C}^{\rho A_0}}; \quad A_0 \in \mathcal{F}; \tag{III.2.10}$$

with the cluster property defined by Theorem 3.2 (Part I) would satisfy (III.2.7) (i.e.,  $T$ -positivity).<sup>2</sup>

If  $f \in N$  has compact support then for any  $\mu$  satisfying (III.2.10), and  $n \in N$ ,  $\mu(|\varphi(f)|^n)$  is well defined. If the map

$$(f_1, \dots, f_n) \in \mathcal{D}(\mathbb{R}^d)^n \rightarrow \mu(\varphi(f_1) \cdots \varphi(f_n))$$

extends to a continuous map  $S_n: \mathcal{S}(\mathbb{R}^d)^n \rightarrow \mathbb{R}$  we say, following [17], that  $\mu$  satisfies assumption B. We sum up our conclusions in the following.

**THEOREM 2.4.** *In the above notation, let  $(E_{\mathfrak{C}_{A_0}})_{A_0 \in \mathcal{F}}$  be a projective family of locally normal,  $E(d)$ -covariant, Markovian conditional expectations  $E_{\mathfrak{C}_{A_0}}: \mathcal{A} \rightarrow \mathcal{A}_{\mathfrak{C}_{A_0}}$ . Then for any solution of Eqs. (III.2.10) satisfying assumption B and (III.2.8), the sequence of distributions defined by the maps  $S_n$  satisfies the axioms E0, E1, E2, and E3 of Osterwalder and Schrader [18].*

*Proof.* The assertion follows from Theorem (2) in [17] and Lemma 1.3.

#### ACKNOWLEDGMENTS

The author wishes to thank Professor R. Stora for the kind hospitality extended to him at the Centre de Physique Théorique of the CNRS during the year 1976.

#### REFERENCES

1. L. ACCARDI, Non-commutative Markof chains, "Proceedings of the School in Math. Phys., Camerino, October 1974."
2. L. ACCARDI, Nonrelativistic quantum mechanics as a noncommutative Markof Process, *Advances in Math.* **20** (1976), 329–366.
3. C. A. ACKEMANN, The dual space of an operator algebra, *Trans. Amer. Math. Soc.* **126** (1967), 286–302.
4. N. BOURBAKI, "Integration," Chap. IX, Hermann, Paris, 1969.
5. R. L. DOBRUSHIN, Description of a random field by means of conditional probabilities and the condition of its regularity, *Theor. Probability Appl.* **13**, No. 2 (1968), 201–229 (in Russian).
6. R. L. DOBRUSHIN, Assignment of a system of random variables by means of conditional distributions, *Theor. Probability Appl.* **15**, No. 3 (1970), 469–497 (in Russian).
7. R. L. DOBRUSHIN AND R. A. MINLOS, *Functional Anal. Appl.* **7**, No. 4 (1973), 81–82 (in Russian).
8. S. DOPLICHER, D. KASTLER, AND D. W. ROBINSON, Covariance algebras in field theory and statistical mechanics, *Comm. Math. Phys.* **3** (1966), 1–8.

<sup>2</sup> In a recent preprint S. Albeverio and R. Hoegh-Kröhn have established the equivalence of (III.2.8) and (III.2.9) when  $\mathcal{A}_\infty$  is trivial. It is quite likely that this equivalence holds in the general case.

9. E. B. DYNKIN, "Markov Processes," Die Grundlehren der Math. Wiss., Band 121, Springer-Verlag, Berlin/New York, 1965.
10. H. FOLLMER, Phase transition and Martin boundary, "Sém. Probabilités, Strasbourg IX, 1975."
11. J. GLIMM, A. JAFFE, AND T. SPENCER, The particle structure of the weakly coupled Model and other applications of high temperature expansions, I, II, in Lecture Notes in Physics No. 25, pp. 132-242, Springer-Verlag, Berlin/New York, 1973.
12. J. GLIMM, The mathematics of quantum fields, *Advances in Math.* 16 (1975), 221-232.
13. F. GUERRA, L. ROSEN, AND B. SIMON, The  $P(\phi)_2$  Euclidean quantum field theory as classical statistical mechanics, I, *Ann. of Math.* 101, No. 1 (1975), 111-189.
14. O. E. LANFORD III AND D. RUELLE, Observables at infinity and states with short range correlations in statistical mechanics, *Comm. Math. Phys.* 13 (1969), 194-215.
15. S. C. MOY, Characterization of conditional expectation as a transformation on function spaces, *Pacific J. Math.* 4 (1954), 47-64.
16. I. NAMIOKA AND E. ASPLUND, A geometric proof of the Ryll-Nardzewski's fixed point theorem, *Bull. Amer. Math. Soc.* (1967), 443-445.
17. E. NELSON, Probability theory on Euclidean field theory, in Lecture Notes in Physics, No. 25, pp. 94-124, Springer-Verlag, Berlin/New York, 1973.
18. K. OSTERWALDER, Euclidean Green's functions and Wightman distributions, in Lecture Notes in Physics, No. 25, pp. 71-93, Springer-Verlag, Berlin/New York, 1973.
19. D. RUELLE, States of physical systems, *Comm. Math. Phys.* 3 (1966), 133-150.
20. S. SAKAI, "C\*-Algebras and W\*-Algebras," *Ergebnisse der Math.*, Band 60, Springer-Verlag, Berlin/New York, 1971.
21. B. SIMON, Ergodic semi-groups of positivity preserving self-adjoint operators, *J. Functional Anal.* 10 (1972), 251-258.
22. B. SIMON, "The  $P(\phi)_2$  Euclidean (Quantum) Field Theory," Princeton Univ. Press, Princeton, N. J., 1974.
23. J. TOMIYAMA, On the projection of norm one in W\*-algebras, *Proc. Japan Acad.* 33 (1957), 608-612.
24. H. UMEGAKI, Conditional expectations in an operator algebra, I, *Tôhoku Math. J.* 6 (1954), 177-181.
25. L. ACCARDI, Markovian cocycles, *Proc. Roy. Irish Acad.*, to appear.