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Nonrelativistic Quantum Mechanics as a Noncommutative Markof Process

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INTRODUCTION

It is well known that quantum mechanics presents many analogies with the theory of Markof processes: In both cases one is concerned with a statistical theory in which the states of a system undergo a deterministic evolution; the analogy between the Green function of the Schrödinger equation and the transition probabilities of a Markof process, together with the fact that a quantum mechanical system is determined by the assignment of a functional on a space of trajectories, are guiding ideas to Feynman's approach to quantum mechanics [9]; the formal analogy between the diffusion equation and the Schrödinger equation has now become, through the systematic use of techniques of analytic continuation, a powerful tool in the treatment of the latter [18]; a one-to-one correspondence between wave functions of a large class of quantum systems and a class of Markof processes has been constructed in such a way that the corresponding statistical theories, at fixed times, coincide [19]; and, more recently, ideas and techniques of the theory of Markof processes have been used with success also in boson quantum field theory [20, 11].

The connection between the two theories lies at a deep level: The fact that the evolution of quantum systems is described by a differential equation of first order in time expresses the locality of the correlation between observables at different times; and the most general way of expressing, in a statistical theory, a property of local correlation is given by the Markof (or, more generally, (d)—Markof [6]) property.

The present work is concerned with the analysis of the property of "local statistical correlation" in the particular context of nonrelativistic quantum mechanics—as described by the axiomatics of von Neumann–

Segal–Mackey's type—and of some consequences of it. It is proven that the mathematical edifice of quantum mechanics, as characterized by the axiomatics of the above mentioned type, can be naturally embedded in the framework of a theory of noncommutative Markof processes. Noncommutative Markof processes are constructed by including the theory of stochastic processes (without assumptions of linearity) in von Neumann [32, 33] Segal's [23, 24, 26] algebraic formulation of probability theory (cf. N. 2) and by using a noncommutative variant of the multidimensional Markof property as formulated by Dobruscin [6], and Nelson [20] (cf. (3.4.1)). The class of Markof processes thus defined is strictly larger than that of quantum systems. However, for the processes in this class a rather rich theory can be developed which, apart from a nontrivial difference (cf. N. 5) is quite similar to the classical theory of Markof processes. In particular a system of evolution equations, naturally associated with the systems of this class, is derived and these turn out to be the noncommutative formulation of the "backward" and "forward" Kolmogorof equations well known in probability theory. It turns out that the Schrödinger equation is the simplest example of a noncommutative forward Kolmogorof equation, and that quantum systems can be characterized as those noncommutative Markof processes whose forward equation is the Schrödinger equation (cf. (6.1)). Another characterization of the quantum systems is that they are the only noncommutative Markof processes to which a reversible time evolution is associated (i.e. whose "transition operators" are invertible and depend regularly enough on the time parameter). From the latter characterization and a theorem of R. Kadison [14] it follows that quantum systems are exactly those noncommutative Markof processes whose transition operators map in a one-to-one way pure states onto pure states (and depend regularly enough on time); this justifies a posteriori the "mechanical character" of quantum systems among Markof processes.

Any attempt to embed Quantum Mechanics in a theory of stochastic processes faces the problem of the joint probabilities which, at the present time, have no natural interpretation in the framework of quantum theory. In the approach discussed in the present work this problem does not arise because it is proven (cf. (6.1.2)) that, for the noncommutative Markof processes corresponding to quantum systems the joint expectations are trivial: the expectation of a product of observables at different times is the product of the expectations of the single observables. This circumstance has a *partial* analogue in the classical theory of Markof processes: in fact quantum systems have been characterized as those noncommutative Markof processes with invertible transition operators, and the joint expectations of a classical Markof process with invertible transition operators factorize *if the initial state is pure.* The analogy between the noncommutative and the classical case is thus broken only when the initial distribution is a mixture and the transition operators are invertible. This is a direct consequence of the fact that the algebra of the observables at any fixed time in the quantum case is a factor.

Examples of irreversible Markof processes (i.e. not corresponding to quantum systems) are constructed with a procedure which corresponds to forming "mixtures of pure dynamics" (cf. (6.5)); other examples have been discussed (in the case of discrete parameter) in [2].

1. The Axioms

A theory is specified by its objects and the type of the assertions which can be formulated on them. A mathematical model of a theory is determined by a correspondence which to every object associates a mathematical entity and to every statement on objects a statement on the corresponding mathematical entities. Objects of a physical theory are dynamical systems, observable physical quantities associated with them, their states[1]. Since the same set of observables or states can correspond to many dynamical systems it will be appropriate, in the specification of the mathematical model, to distinguish the statements which characterize the single dynamical systems, among all those to which the same classes of observables and states are associated, from the statements which describe the mathematical entities corresponding to such observables and states.

In the present work the above-mentioned distinction will be carried out in the case of nonrelativistic quantum mechanics, as follows: the mathematical model of the theory will be determined, as usual, by means of axioms and the system of axioms will be subdivided into two groups: the first one (Static Axioms) comprises the characteristics of the theory common to all the dynamical systems considered; the second one (Dynamical Axioms) gives the characterization of the single dynamical systems and of their evolution law. For the construction of the mathematical model we shall assume, as a postulate, the following:

[1] We shall assume here these notions as primary without probing the questions arising from the attempt at giving a precise definition of these entities independently from the mathematical models used to describe them (cf. [21]).

Fundamental Phenomenological Principle

Every physical system is defined in all its physically observable aspects, by the set of all its bounded observables.

This principle has been formulated by I. E. Segal [27, Chap. 1], to whom is due the proposal of formulating the axiomatization of Quantum Mechanics in the context of abstract C^*-algebras rather than in that of Hilbert space.

The "Fundamental Phenomenological Principle" (for a discussion of which we refer to the above cited monograph) allows formulating the first group of axioms as follows:

STATIC AXIOMS

(I.) At each instant of time the bounded observables are in a one-to-one correspondence with the hermitean elements of a C^*-algebra A.

(II.) The physical states are in a one-to-one correspondence with a subset S_0 of the set of all the states of the C^*-algebra A.

(III.) If to the bounded observable $\mathfrak{A}$ corresponds the hermitean operator a in A and to the physical state Φ the state φ of the C^*-algebra A, then the mean value (or expectation value) of $\mathfrak{A}$ in the sate Φ is $\varphi(a)$.

The Axioms I, II, III, as formulated above do not determine uniquely the model. On the contrary, as follows from the analysis of J. von Neumann [32 or 34, p. 297] and I. E. Segal [23, 24] one can assert that they describe the most general mathematical model for a statistical theory of physical systems for which the validity of the "Fundamental Phenomenological Principle is assumed. The specific character of nonrelativistic quantum mechanics is determined by the following specifications of the Axioms I and II, respectively:

(I′.) The algebra A is the algebra $\mathfrak{B}(\mathscr{H})$ of all the bounded linear operators on a complex separable Hilbert space $\mathscr{H}$.

(II′.) The set S_0 is the set of all the normal states on $\mathfrak{B}(\mathscr{H})$.

The choices I′, II′ for A and S_0, respectively, completely specify the model in the sense that, as shown by J. von Neumann (cf. [32]) Axiom III is the only plausible way, compatible with these choices, to define a "mean (or expectation) value" of an observable in a state. Axiom III specifies the statistical character of quantum mechanics:

the assertions of the theory concern mean (or expectation) values of observable quantities. There are, in the literature, many ways of expressing the statistical assertions of quantum theory; the formulation given by Axiom III is von Neumann's original one (34).

In the model specified by I′, II′ Gleason's theorem and the spectral theorem allow establishing the equivalence between von Neumann's formulation and Mackey's (cf. [16, Chapter 2]), however von Neumann's formulation has the advantage of leaving a complete freedom in the choice of the C^*-algebra A, and in the following we shall make essential use of this.

The axioms listed above characterize the quantum (static) description of an arbitrary system. The fact that dynamical systems in "physical space" are considered is expressed by postulating the existence of a "group of symmetries" for the system and of a representation of such a group into the automorphisms of the C^*-algebra A. Being concerned, in the present work, only with the analysis of the statistical aspect of quantum mechanics, we shall not formulate the corresponding postulate and refer, for this, to Mackey's monograph [16] (cf. also [17]).

(2.1) As far as the dynamical postulate is concerned, many authors (cf. for example [34]) directly postulate the Schrödinger equation as the (time) evolution law of quantum systems. One of the first, mathematically rigorous, attempts of giving a theoretical foundation to the evolution law of a quantum system is due to G. W. Mackey who, in analogy with the classical case (cf. [16, pg. 81]) formulates the dynamical postulate of Quantum mechanics in the following way:

Dynamical Postulate (*Mackey*)

The temporal evolution of a quantum dynamical system is described by a one-parameter group $(V_t)_{t\in\mathbb{R}}$ of one to one maps of S_0 onto itself such that for each $t \in \mathbb{R}$:

$$V_t\Big(\sum_\iota \lambda_\iota\varphi_\iota\Big) = \sum_\iota \lambda_\iota V_t(\varphi_\iota)$$

$$\lambda_\iota \geqslant 0; \qquad \sum_\iota \lambda_\iota = 1; \qquad \varphi_\iota \in S_0\,.$$

Using a Theorem due to R. Kadison it is possible to prove (cf. [16, pg. 82]) that each such group is induced by a one-parameter group U_t of unitary operators in $\mathfrak{B}(\mathscr{H})$. Thus, from the dynamical postulate one deduces the existence of a Hamiltonian (the infinitesimal generator of

(U_t)) and, by differentiation of the function $t \mapsto U_t\zeta$, where $\zeta \in \mathscr{H}$ is a vector in the domain of the Hamiltonian, the Schrödinger equation.

At this point, however, there are some remarks to make: first of all there is the methodological question, pointed out by Mackey himself (cf. [16, page 81]) that in a statistical theory, as quantum mechanics turns out to be from the Static Axioms, the dynamical postulate is introduced as a strictly deterministic statement, namely that the assignment of a system at a given instant of time determines, through the transformation group whose existence is postulated, the state of the system at any future instant. Moreover, the analogy with the classical situation, although highly desirable, is not a satisfying requirement, from the physical point of view, as a theoretical foundation of an Axiom.

In the formulation of the axiomatic of Quantum Mechanics discussed in the present work the system of the static axioms will be kept unaltered, while the dynamical postulate will be radically changed and based not on an analogy with classical deterministic systems, but on an analogy with classical *stochastic* systems which will be translated into a requirement of *purely physical* character.

Now, without any doubt, as already von Neumann repeatedly points out, the deterministic character of the evolution of the states is a fundamental feature of quantum systems. However, if the term "state of a classical system" is meant in the wide sense of probability measure on its phase (or configuration) space, this feature is not peculiar to the deterministic systems of classical mechanics. There are classical stochastic processes whose state (i.e. probability distribution) at each fixed time uniquely determines the state at any future time. These processes are the so-called Markof processes.

Therefore, quantum mechanics being a statistical theory whose states at any time evolve deterministically, it is natural to attempt to describe its mathematical structure in analogy with Markof processes, rather than with classical deterministic processes. Among stochastic processes, Markof processes are characterized by the following property of their random variables (observables):

(P.) For any fixed instant of time t_0, the observables at any time $t > t_0$ are statistically correlated with the observables at time t_0 and are not statistically correlated with the observables at any time $s < t_0$.

Property (P.) is a qualitative formulation of the "Markof Property" for stochastic processes indexed by the parameter $t \in \mathbb{R}$ (time). Clearly in classical probability theory terms like "observables at time t," "statistical correlation," are given a precise, quantitative form. However, the above formulation expresses a purely physical requirement which makes sense for any mathematical model of statistical theory, as specified by the Static Axioms; in particular it makes sense for the model of nonrelativistic quantum mechanics.

In the following we shall refer to property (P.) as to the "Principle of local correlation" of nonrelativistic quantum observables at different times. The subsequent analysis will show then that there is essentially a unique way of formulating in mathematical terms the "Principle of local correlation" stated above, namely:

(IV.) A quantum system is a noncommutative Markof process.

This assertion, which is the corresponding one, in the mathematical model, to property (P.) will be taken as the Dynamical Postulate of quantum mechanics.

Thus, as the Static Axioms define the mathematical entities corresponding to physical *objects* which are not defined independently of this correspondence, so the Dynamical Axiom is the mathematical formulation of a physical *property* which only through this correspondence assumes a precise meaning.

Axiom (IV) defines a class of processes strictly larger than the class of usual quantum systems. It will be proven (cf. N. 6) that this enlargement essentially amounts to the inclusion, among quantum systems, of systems with an irreversible time evolution.

2. Noncommutative Stochastic Processes

In classical probability theory a random variable on the probability space $(\Omega, \mathscr{B}, \mu)$ with values on the measurable space $(S, \mathfrak{B})$ is defined as the μ-equivalence class of a function $x: \Omega \to S$ measurable for the respective structures.

In the following, the space $(\Omega, \mathscr{B}, \mu)$ will always be assumed *complete* (i.e., if $B \in \mathscr{B}$; $\mu(B) = 0$; and $B_0 \subseteq B$, then $B_0 \in \mathscr{B}$) and the space $(S, \mathfrak{B})$ a standard Borel space in the sense of [17].

Each random variable X defines a homomorphism of σ-algebras $\bar{X}: \mathfrak{B} \to \mathscr{B}/\mu$ which preserves the boolean units (if x is a representative of X then, $\forall B \in \mathfrak{B}$, $x^{-1}(B)$ is a representative of $\bar{X}(B)$), where $\mathscr{B}/\mu$ denotes the quotient σ-algebra of $\mathscr{B}$ by the σ-ideal of the μ-null sets. Conversely, from a theorem due to J. von Neumann (35) one can deduce that each such homomorphism defines a unique random variable. Therefore, according to I. Segal [23], a (generalized) random variable can be defined as a homomorphism of σ-algebras which preserves the boolean units. In the following we shall denote X any random variable and $\bar{X}$ the corresponding homomorphism of σ-algebras.

A stochastic process on $(\Omega, \mathscr{B}, \mu)$ with values in $(S, \mathfrak{B})$, indexed by the set T is determined by the assignment of a family $(X_t)_{t\in T}$ of random variables. The finite-dimensional joint-distributions of the process $(X_t)_{t\in T}$ are the probability measures defined, for each finite subset $F \subseteq T$ and $B_j \in \mathfrak{B}$, $t \in F$ by

$$(B_t)_{t\in F} \in \Pi_F \mathfrak{B} \mapsto \mu\left(\bigcap_{t\in F} \bar{X}_t(B_t)\right).$$

Stochastic processes are usually classified according to their finite-dimensional joint distributions; i.e., two stochastic processes are called equivalent if their finite-dimensional joint-distributions coincide (cf. for example [7, pg. 47]). In order to formulate this concept in a more precise and slightly more general way, let us introduce the following notations: for any subset $I \subseteq T$,

$$\bar{X}_I(\mathfrak{B}) = \bigvee_{t\in I} \bar{X}_t(\mathfrak{B})$$

denotes the sub-σ-algebra of $\mathscr{B}/\mu$ spanned by the family $(\bar{X}_t(\mathfrak{B}))_{t\in I}$ and $\bar{\mu}_I$ the restriction of μ on $\bar{X}_I(\mathfrak{B})$.

Definition (2.1.) Two stochastic processes $(X_t^\iota)_{t\in T}$ indexed by the set T, defined on $(\Omega^\iota, \mathscr{B}^\iota, \mu^\iota)$, with values in $(S^\iota, \mathfrak{B}^\iota)$; $\iota = 1, 2$; respectively will be called equivalent if there exists an isomorphism of σ-algebras ϕ: $\bar{X}_T^1(\mathfrak{B}^1) \to \bar{X}_T^2(\mathfrak{B}^2)$ such that:

$$\bar{\mu}_T^2 \cdot \phi = \bar{\mu}_T^1$$

and, for every finite subset $F \subseteq T$:

$$\phi(\bar{X}_F^1(\mathfrak{B}^1)) = \bar{X}_F^2(\mathfrak{B}^2).$$

Definition (2.1) classifies stochastic processes according to their "local algebras" and the corresponding classes of measures. A further weakening of the equivalence relation above could be obtained by allowing the two stochastic processes to be indexed by different sets T_1 and T_2 such that there exists an isomorphism α: $T_1 \to T_2$ compatible (in the obvious sense) with the isomorphism of definition (2.1). The latter classification is also meaningful for "continuous" stochastic processes, i.e., such that the set of indices T_1, T_2 are, in their turn, endowed with a structure (e.g., topological; linear topological; ...); the subsets $F \subseteq T$ are defined in terms of this structure (e.g., open subsets; closed subspaces; ...); and the isomorphism α: $T_1 \to T_2$ is compatible with it (i.e., continuous; linear continuous; ...).

However, in the case of discrete stochastic processes, i.e., processes classified according to their finite-dimensional joint distributions, this classification is less meaningful. Therefore, in the present work, where only discrete stochastic processes will be considered, the index set will be given once and for all and the equivalence of stochastic processes will be understood in the sense of Definition (2.1).

LEMMA (2.2). *There exists a one-to-one correspondence among random variables on $(\Omega, \mathscr{B}, \mu)$ with values in $(S, \mathfrak{B})$ and homomorphisms of C^*-algebras*

$$\bar{X}: L^\infty(S, \mathfrak{B}) \to L^\infty(\Omega, \mathscr{B}, \mu)$$

such that:

(i) $X(1_S) = 1_\Omega$ *1_S(resp. 1_Ω) is the function (resp. μ-class of functions) identically equal to* 1 *on S (resp. Ω).*

(ii) *If (f_α) is a filtering increasing family in $L_+^\infty(S, \mathfrak{B})$ such that $f = \operatorname{Sup} f_\alpha \in L^\infty(S, \mathfrak{B})$ then $\bar{X}(f) = \operatorname{Sup} \bar{X}(f_\alpha)$.*

Proof. Let x: $\Omega \to S$, be a function in the class defined by the random variable X. Then the mapping

$$f \in L^\infty(S, \mathfrak{B}) \mapsto X(f) = f \circ x \in L^\infty(\Omega, \mathscr{B}, \mu)$$

does not depend on the choice of $x \in X$, and clearly satisfies (i), (ii).

Conversely, any homomorphism of C^*-algebras $\bar{X}: L^\infty(S, \mathfrak{B}) \to L^\infty(\Omega, \mathscr{B}, \mu)$ induces by restriction on the characteristic functions a homomorphism of boolean algebras X: $\mathfrak{B} \to \mathscr{B}/\mu$. If (i) holds, X preserves the boolean units; if (ii) holds X is a homomorphism of σ-algebras and therefore a random variable.

Thus the assignment of a stochastic process indexed by T on $(\Omega, \mathscr{B}, \mu)$

with values in $(S, \mathfrak{B})$ is equivalent to the assignment of a family $(\bar{X}_t)_{t\in T}$ of C^*-algebra homomorphisms

$$\bar{X}_t : L^\infty(S, \mathfrak{B}) \to L^\infty(\Omega, \mathscr{B}, \bar{\mu})$$

each of which satisfies the conditions (i), (ii) of Lemma (2.2).

LEMMA (2.3). *Let $(\bar{X}_t{}^\iota)_{t\in T}$ be stochastic processes (as specified above) defined on $(\Omega^\iota, \mathscr{B}^\iota, \mu^\iota)$ and with values in $(S^\iota, \mathfrak{B}^\iota)$, respectively ($\iota = 1, 2$). The two stochastic processes are equivalent if and only if there exists a von Neumann algebra isomorphism:*

$$u: L^\infty(\Omega^1, \bar{X}_T{}^1(\mathfrak{B}^1), \bar{\mu}_T{}^1) \to L^\infty(\Omega^2, \bar{X}_T{}^2(\mathfrak{B}^2), \bar{\mu}_T{}^2)$$

with the following properties:

(j1) $\bar{\bar{\mu}}_T{}^2 \cdot u = \bar{\bar{\mu}}_T{}^1$. *$\bar{\bar{\mu}}_T{}^\iota$ is the state on $L^\infty(\Omega^\iota, \bar{X}_T{}^\iota(\mathfrak{B}^\iota), \bar{\mu}_T{}^\iota)$ induced by $\bar{\mu}_T{}^\iota$ ($\iota = 1, 2$).*

(j2) *For any finite subset $F \subseteq T$, if $A^\iota(F)$ denote the von Neumann sub-algebra of L^∞ $(\Omega^\iota, \bar{X}_T{}^\iota(\mathfrak{B}^\iota), \bar{\mu}_T{}^\iota)$ of the $\bar{X}_T{}^\iota(\mathfrak{B}^\iota)$-measurable classes of functions, one has: $u(A^1(F)) = A^2(F)$.*

Proof. From the above-mentioned von Neumann's theorem [35, pg. 302] one deduces the existence of a one-to-one correspondence between isomorphisms of boolean σ-algebras $\phi : \bar{X}_T{}^1(\mathfrak{B}^1) \to \bar{X}_T{}^2(\mathfrak{B}^2)$ such that $\bar{\mu}_T{}^2 \cdot \phi = \bar{\mu}_T{}^1$ and isomorphisms of von Neumann algebras

$$u: L^\infty(\Omega^1, \bar{X}_T{}^1(\mathfrak{B}^1), \bar{\mu}_T{}^1) \to L^\infty(\Omega^2, \bar{X}_T{}^2(\mathfrak{B}^2), \bar{\mu}_T{}^2)$$

satisfying (j1).

Since the algebras $A^\iota(F)$ are spanned by their projection operators, it is clear that isomorphisms of σ-algebras such that $\phi(\bar{X}_T{}^1(\mathfrak{B}^1)) = \bar{X}_T{}^2(\mathfrak{B}^2)$ will correspond in a one-to-one way to w^*-algebra isomorphisms satisfying (j2).

Let now $(\bar{X}_t{}^\iota)_{t\in T}$ ($\iota = 1, 2$) be two stochastic processes as in Lemma (2.3). Assume that

$$\bar{X}_T{}^\iota(\mathfrak{B}^\iota) = \mathscr{B}^\iota/\mu^\iota \tag{2.3.1}$$

i.e. that the process $(\bar{X}_t{}^\iota)_{t\in T}$ is "determining," in the sense of Segal [23], for $(\Omega^\iota, \mathscr{B}^\iota, \mu^\iota)$. In this case,

$$L^\infty(\Omega^\iota, \bar{x}_T{}^\iota(\mathfrak{B}^\iota), \bar{\mu}_T{}^\iota) = L^\infty(\Omega^\iota, \mathscr{B}^\iota, \mu^\iota) = A_{\mu^\iota},$$

where A_{μ^ι} denotes the von Neumann algebra obtained by $L^\infty(\Omega^\iota, \mathscr{B}^\iota)$ and μ^ι by means of the Gelfand–Neumark–Segal (GNS) construction. In this case the algebras $A^\iota(F)$ defined in (j2) of Lemma (2.3) are naturally identified with von Neumann sub-algebras of A_{μ^ι}.

COROLLARY (2.4). *If equality* (2.3.1) *holds, the conditions* (j1), (j2) *of Lemma* (2.3) *are equivalent to the following: if* $\{A_{\mu^\iota}, \mathscr{H}_{\mu^\iota}, \zeta_{\mu^\iota}\}$ *denotes the* GNS *triple associated to* L^∞ $(\Omega^\iota, \mathscr{B}^\iota)$ *and* μ^ι, *there exists an unitary transformation* $U: \mathscr{H}_{\mu^1} \to \mathscr{H}_{\mu^2}$ *such that*:

$$U \cdot A_{\mu^1} \cdot U^* = A_{\mu^2}$$

$$U \cdot \zeta_{\mu^1} = \zeta_{\mu^2}$$

$$U \cdot A^1(F) \cdot U^* = A^2(F)$$

for every finite $F \subseteq T$.

Proof. Follows immediately from Lemma (2.3) and equality (2.3.1).

COROLLARY (2.5). *In the notations of Lemma* (2.3) *property* (j2) *is satisfied if and only if for every* $t \in T$:

$$u(A^1(\{t\})) = A^2(\{t\}). \tag{2.5.1}$$

Proof. Clearly (j2) of Lemma (2.3) implies (2.5.1). Conversely, if (2.5.1) holds then, for every finite subset $F \subseteq T$, $\forall t \in F$

$$u(A^1(\{t\})) \subseteq A^2(F); \qquad u(A^1(F)) \supseteq A^2(\{t\})$$

and the above inclusions imply $u(A^1(F)) = A^2(F)$, since the family $(A^\iota(\{t\}))_{t\in F}$ is generating for $A^\iota(F)$; $\iota = 1, 2$.

Thus to every stochastic process indexed by the set T a triple $\{A, A(F), \mu\}$ is canonically associated where A is a von Neumann algebra, μ a faithful normal state on A, $(A(F))$ a family of von Neumann sub-algebras of A indexed by the sub-sets $F \subseteq T$ with the following properties:

(i1). If F and G are finite subsets of T then A $(F \cup G)$ is the von Neumann sub-algebra of A spanned by $A(F)$ and $A(G)$.

(i2). A is the von Neumann algebra spanned by the family $(A(F))$.

Two stochastic processes are equivalent if and only if, denoting by

$\{A^{\iota}, (A^{\iota}(F)), \mu^{\iota}\}$ the triples associated to them, there exists a von Neumann algebras isomorphism u: $A^1 \rightarrow A^2$ such that:

$$\mu^2 \cdot u = \mu^1 \tag{2.5.2}$$

$$u(A^1(F)) = A^2(F) \quad \text{for each finite } F \subseteq T. \tag{2.5.3}$$

DEFINITION (2.6). Two triples $\{A^{\iota}, (A^{\iota}(F)), \mu^{\iota}\}$, $\iota = 1, 2$ with the properties (i1), (i2) above, are called equivalent if there exists an isomorphism of von Neumann algebras u: $A^1 \rightarrow A^2$ satisfying (2.5.1) and (2.5.2).

Corollary (2.5) shows that for the equivalence of stochastic processes it is sufficient to limit the consideration to the subsets of T containing only one element.

This fact is connected with the limitation to discrete stochastic processes. For stochastic processes of more general type the classification given by Lemma (2.3) still holds, with the difference that the subsets $F \subseteq T$ no longer belong to the class of finite subsets, but to more general classes. For example the "infinitely decomposable" processes or the stochastic processes considered in euclidean field theory are of this kind. The relations (i1), (i2) among the "local algebras" $A(F)$ are universal in the sense that they take place for any stochastic process, independently of the eventual specific relations among the random variables of the process. In general, relations of this last type will be translated in terms of algebraic relations among the "local algebras" $A(F)$.

In what follows it will be shown that, in the case of discrete stochastic processes, it is possible, remaining inside the same equivalence class, to deduce some universal relations among the "local algebras" which are more precise (and useful) than (i1), (i2).

It is well known (cf. [7, pg. 621]) that every stochastic process indexed by the set T with values in the standard Borel space $(S, \mathfrak{B})$ is equivalent (in the sense of Definition (2.1) to the stochastic process determined, on the product space $\Pi_T(S, \mathfrak{B})$ with a given probability measure, by the assignment of the family of the canonical projections.

Every such stochastic process will be called of "product type"; if a stochastic process is equivalent to one of product type, the latter will be called a "product representation" of the former. Thus any discrete stochastic process has a product representation and any two such representations are equivalent. By duality the canonical projections induce the C^*-algebra immersions:

$$\Pi_t' : L^\infty(S, \mathfrak{B}) \rightarrow L^\infty(\Pi_T S, \Pi_T \mathfrak{B}).$$

Denote by A_t the image of $L^\infty(S, \mathfrak{B})$ under the immersion Π_t' $(t \in T)$, by A_0 the norm closure of the algebra spanned by the family $(A_t)_{t\in T}$, and by μ_0 the restriction of μ onto A_0. Let A_μ, $\mathscr{H}_\mu$, μ be respectively the von Neumann algebra, the Hilbert space, the faithful normal state on A_μ obtained from A_0 and μ_0 by the GNS construction; and let, for each finite subset $F \subseteq T$, $A_\mu(F)$ be the von Neumann sub-algebra of A_μ spanned by the images of the A_t $(t \in F)$ by the GNS representation.

Definition (2.7). *Two triples* $\{A_0{}^\iota, (A_t{}^\iota)_{t\in T}, \mu_0{}^\iota\}$ where $A_0{}^\iota$ is a C^*-algebra, $\mu_0{}^\iota$ a state on $A_0{}^\iota$ and $(A_t{}^\iota)_{t\in T}$ any family of C^*-sub-algebras of $A_0{}^\iota$, will be called equivalent if the triples $\{A_{\mu^\iota}, (A_{\mu^\iota}(F)), \mu^\iota\}$ obtained by them as described above are equivalent in the sense of Definition (2.6).

Lemma (2.8). *Two stochastic processes of product type are equivalent if and only if the triples* $\{A_0{}^\iota, (A_t{}^\iota)_{t\in T}, \mu_0{}^\iota\}$ *associated to them in the way described above are equivalent.*

Proof. There is a natural identification of the image of A_0 by the GNS representation with a determining sub-algebra of $L^\infty(\Pi_T S, \Pi_T \mathfrak{B}, \mu)$ which sends the images of the A_t onto the $\bar{X}_{\{t\}}(\mathfrak{B})$-measurable functions. Hence the assertion follows from Lemma (2.3).

Lemma (2.9). *Let A_0 and $(A_t)_{t\in T}$ be as above. Then A_0 is naturally identified with the infinite tensor product of the family $(A_t)_{t\in T}$.*

Proof. For each $t \in T$, $A_t = \Pi_t'(L^\infty(S, \mathfrak{B}))$ is a commutative C^*-algebra with identity hence on the algebraic tensor product of any two of them there is a unique C^*-cross-norm (cf. [22, pg. 62]). Therefore the infinite tensor product of the family $(A_t)_{t\in T}$ is uniquely determined. It will be therefore sufficient to prove that, for any finite subset $F \subseteq T$, the sub-algebra of A_0 algebraically spanned by the family $(A_t)_{t\in T}$ is isomorphic to the algebraic tensor product of the A_t $(t \in F)$. First let $F = \{s, t\}$; $s \neq t$. Assume that, for a finite set G, $f_s{}^\iota \in A_s$, $f_t{}^\iota \in A_t$, $\iota \in G$ one has:

$$\sum_{\iota\in G} f_s{}^\iota \cdot f_t{}^\iota = 0. \tag{2.9.1}$$

There is always a finite set $(g_t{}^J)_{J\in H}$ of linearly independent elements of A_t and a set of complex numbers $(a_{\iota J})$ $(\iota \in G,\ J \in H)$ such that

$$f_t{}^\iota = \sum_{J\in H} a_{\iota J} \cdot g_t{}^J; \qquad \iota \in G.$$

Consequently

$$0 = \sum_{\iota \in J} f_s^{\iota} \cdot a_{\iota J} \cdot g_t^{J} = h \in A_s \vee A_t .$$

By our assumptions the elements of $A_s \vee A_t$ (= the algebra spanned by A_s and A_t), are identified with functions $S \times S \to \mathbb{C}$. Since $h = 0$, the function $x_t \in S \mapsto h(x_s, x_t)$ is identically zero for every $x_s \in S$, and this, because of the linear independence of the g implies

$$\sum_{\iota \in G} f_s^{\iota} \cdot a_{\iota J} = 0. \tag{2.9.2}$$

Thus (2.9.1) takes place if and only if there are complex numbers $(a_{\iota J})$ satisfying (2.9.2); this is equivalent to the isomorphism of $A_s \vee A_t$ with the algebraic tensor product of A_s and A_t. The case of an arbitrary finite $F \subseteq T$ is reduced to the preceding one by induction, and this ends the proof.

We sum up our analysis in the following:

THEOREM (2.10). *To every stochastic process indexed by the set T a triple $\{A, (A_t)_{t \in T}, \mu\}$ is naturally associated, where is a C^*-algebra, μ a state on A and $(A_t)_{t \in T}$ is a family of sub-C^*-algebras of A such that*:

(i1) *A is the C^*-algebra spanned by $(A_t)_{t \in T}$*

(i2) *For any finite $F \subseteq T$, $V_{t \in F} A_t \approx \bigotimes_{t \in F} A_t$*

(i3) *The A_t $(t \in T)$ are mutually isomorphic.*

The C^*-algebras A_t are commutative, and two stochastic processes are equivalent if and only if the triples associated to them are equivalent. Conversely, given any triple as above there exists a stochastic process such that the triple naturally associated to it, according to the first part of the theorem, is equivalent to the initial one.

Proof. The first two assertions follow from Lemmas (2.8), (2.9). Let now $\{A, (A_t)_{t \in T}, \mu\}$ be a triple as specified above. Because of (i3) the spectrum S of A_t can be chosen independent of $t \in T$ and, by a theorem of Takeda [29], the spectrum of A can be identified with $\Pi_T S$. Denote by $\mathfrak{B}$ the Baire σ-algebra on S and by μ_0 the measure induced on $\Pi_T(S, \mathfrak{B})$ by μ. Then, to the stochastic process determined on $\Pi_T(S, \mathfrak{B})$ by μ_0 and the canonical projections (Π_t), the triple $\{A_0,$ $(A_t^0), \mu_0\}$, is naturally associated, where $A_t^0 = \Pi_t'(L^\infty(S, \mathfrak{B}))$, and A_0 is the norm closure in $L^\infty(\Pi_T S, \Pi_T \mathfrak{B})$ of the sub-algebra spanned by the

$(A_t^0)_{t\in T}$. The equivalence of the two triples $\{A, (A_t), \mu\}$, and $\{A_0, (A_t^0), \mu_0\}$ is clear, and therefore the theorem is proved.

Remark that, interpreting the parameter $t \in T$ as "time," from the above discussion it follows that the algebra A_t has a natural interpretation as the algebra of all the bounded observables of the system described by the stochastic process at time t. For example, if S is the space of the "positions" of the system, then a point in $\Pi_T S$ is a trajectory; an element of A_t is a bounded Baire function of the position of the system at time t; an element of A is a functional on the path space of the process.

The passage from the classical to the noncommutative theory of discrete stochastic processes will be accomplished by postulating that the universal relations (i1), (i2), (i3), derived from such processes in the commutative case, are preserved; and by allowing that the algebras A_t (of the "observables" at a fixed time) are arbitrary C^*-algebras. More precisely:

DEFINITION (2.11). A discrete symmetric stochastic process indexed by a set T is a triple $\{A, (A_t)_{t\in T}, \mu\}$ where A is a C^*-algebra, μ a state on A, and $(A_t)_{t\in T}$ a family of sub-algebras of A such that:

(i1) $A = V_{t\in T} A_t$

(i2) For each finite $F \subseteq T$; $V_{t\in F} A_t \approx \bigotimes_{t\in F} A_t$

(i3) The C^*-algebras A_t $(t \in T)$ are mutually isomorphic.

Two discrete symmetric stochastic processes will be called equivalent if the triples defining them are equivalent in the sense of Definition (2.7).

Remark 1. The tensor products appearing in (i2) of the above Definition are not uniquely determined in the noncommutative case; thus a symmetric stochastic process is also defined by the choice of the C^*-cross-norms. However in the most interesting cases there is a "natural" choice for the C^*-cross-norms arising, for example, from the fact that the algebras A_t are realized as algebras of operators on some Hilbert space (cf. also the following N. 3). For this reason the dependence of the process on the C^*-cross-norms has been left implicit in the above definition.

Remark 2. Property (i2) of Definition (2.11) is a kind of "compensation" of the noncentrality of the state μ which, according to I.E. Segal [23] seems to be a serious hindrance to the development of a sufficiently rich theory. The fact that, in the commutative case, property (i2) is universal up to equivalence, is typical of the class of discrete stochastic processes. For continuous ones, a property like $A\,(F \cup G) \approx A(F) \otimes A(G)$ will be the expression of specific relations among the random variables and the regions where they are localized.

3. Noncommutative Markof Processes

In the following we shall consider symmetric stochastic processes $\{A, (A_t)_{t\in\mathbb{R}^+}, \varphi\}$ indexed by $\mathbb{R}^+$ and with the following properties:

(3.1) There is a complex separable Hilbert space $\mathscr{H}$ such that, for each $t \in T$, there is a normal isomorphism $J_t\,;\ \mathfrak{B}(\mathscr{H}) \to A_t$.

(3.2) For any Finite $F \subseteq \mathbb{R}^+$ the C^*-cross-norm on $\bigotimes_{t\in F} A_t$ is the one induced by the identification of the algebraic tensor product of the family $(A_t)_{t\in F}$ with an algebra of operators on $\bigotimes_F \mathscr{H}$.

(3.3) For each finite $T \subseteq \mathbb{R}^+$ the restriction of φ on the C^*-algebra spanned by $(A_t)_{t\in F}$ has a normal extension on the weak closure of this algebra (identified with an algebra of operators on $\bigotimes_F \mathscr{H}$).

A state with property (3.3) will be called "locally normal." The fact that the "local algebras" are von Neumann algebras and that the state is locally normal corresponds, in a commutative context, to the fact that the stochastic processes considered are determined by measures (on the path space) locally absolutely continuous with respect to a given (privileged) measure. In particular (3.3) implies that the restriction of φ on A_t $(t \in \mathbb{R}^+)$ induces a normal state on $\mathfrak{B}(\mathscr{H})$. Thus a symmetric stochastic process satisfying (3.1), (3.2), (3.3) is such that the statistical theory arising when restricting the process at *any fixed time* is compatible with the static axioms of quantum mechanics.

By property (i1) of Definition (2.11) the state φ is completely determined by the family $\{\varphi_{t_0,\dots,t_n}\}_{0\leqslant t_0<\cdots<t_n}$ of its restrictions on the local algebras $A_{\{t_0,\dots,t_n\}}$; and, by (i2), each of the states $\varphi_{t_0,\dots,t_n}$ is completely determined by its values on the products $a_{t_0}\cdot\cdots\cdot a_{t_n}$; $a_{t_\iota} \in A_{t_\iota}$; $\iota = 0,\dots, n \in \mathbb{N}$.

Among the symmetric stochastic processes we shall single out those which satisfy the "Principle of local correlation" i.e. (cf. N. 1), those which determine, for each instant t, a "measure of statistical correlation" for which the observables $a_t \in A_t$ relative to a (future) time $t > t_0$ are statistically correlated with the observables at t_0 but not with those relative to a (past) time $s < t$.

In the commutative case it is well known that the stochastic processes with this property are the (strictly) Markovian ones (cf. [7, pg. 81]); and that the appropriate "measure of statistical correlation" is given by the conditional expectation on the σ-algebra spanned by the random variables relative to the past history ($s < t$) of the process.

From a probabilistic standpoint, a satisfactory analogue in a noncommutative context for the concept of conditional expectation is given by the concept of quasi-conditional expectation in terms of which noncommutative Markof processes will be now defined (cf. [1, 2] for a discussion of the inadequacy, for purely probabilistic purposes, of the usual concept of conditional expectation on arbitrary C^*-algebras as well as for a definition of quasi-conditional expectation more general than the one given below, which is limited to the particular type of processes discussed in the present paper).

Definition (3.4). A quasi-conditional expectation with respect to the triple of C^*-algebras:

$$A_{[0,s[} \subseteq A_{[0,s]} \subseteq A_{[0,t]}\,; \qquad s < t$$

is a linear map $E_{t,s} : A_{[0,t]} \to A_{[0,s]}$ such that

(i1) $E_{t,s}(a) \geqslant 0$; if $a \in A_{[0,t]}$; $a \geqslant 0$

(i2) $E_{t,s}(a_r \cdot b) = a_r \cdot E_{t,s}(b)$; $r < s < t$ for any $a_r \in A_r$; $b \in A_{[0,t]}$

(i3) $\| E_{t,s}(b)\| \leqslant \| b \|$; $b \in A_{[0,t]}$.

The quasi-conditional expectation $E_{t,s}$ will be called "normalized" if

(i4) $E_{t,s}(1) = 1$

Property (i1) implies that $E_{t,s}$ commutes with the involution. Thus, for any $a_r \in A_r$; $b \in A_{[0,t]}$; $r < s < t$:

$$E_{t,s}(b \cdot a_r) = E_{t,s}(a_r^* \cdot b^*)^* = \{a_r^* \cdot E_{t,s}(b)^*\}^* = E_{t,s}(b) \cdot a_r\,.$$

Consequently, denoting by $A'_{[0,s]}$ the relative commutant with respect to A of $A_{[0,s[}$ (i.e. $\{a \in A : ab = ba;\ \forall b \in A_{[0,s[}\}$) one has:

$$E_{t,s}(A'_{[0,s[} \cap A_{[0,t]}) \subseteq A'_{[0,s[} \cap A_{[0,s]}\,.$$

But our hypothesis on the local algebras $A_{[0,t]}$ imply that, for $s < t$

$$A'_{[0,s[} \cap A_{[0,t]} = A_{[s,t]} .$$

Thus a quasi-conditional expectation $E_{t,s}$ with respect to the triple

$$A_{[0,s[} \subseteq A_{[0,s]} \subseteq A_{[0,t]}$$

enjoys the property:

$$E_{t,s}(A_{[s,t]}) \subseteq A_s . \tag{3.4.1}$$

In the commutative case the relation (3.4.1) is an equivalent formulation of the (strict) Markof property (cf. [7, pg. 81]).

Remark. The use of *three* C^*-algebras in Definition (3.4) is not a mathematical device, but a conceptual necessity each time that one wants to introduce a property of Markof type (i.e. a condition of locality on the statistical dependence). In the case considered above, for example, the use of three C^*-algebras reflects the necessity of distinguishing between observables relative to the past ($A_{[0,s[}$), the present (A_s), and the future ($A_{]s,t]}$). The situation is perfectly analogous to the classical case, where three σ-algebras are needed for the formulation of the Markof property (cf. [15, pg. 562; and 20] for the multidimensional case).

Definition (3.4) shows that a (normalized) quasi-conditional expectation from $A_{[0,t]}$ to $A_{[0,s]}$ differs from an usual conditional expectation on the same algebras, only for its behaviour on the "local algebra" corresponding to the boundary point s. However this modification is essential in order to guarantee the existence of a class of nontrivial noncommutative Markof processes (cf. Lemma (3.6)).

DEFINITION (3.5.) Let A, $(A_{[0,t]})$ be as above; a state φ on A will be called a Markof state with respect to the family of local algebras $(A_{[0,t]})$ if for every $0 \leqslant s < t$, there exists a quasi-conditional expectation $E_{t,s}$ with respect to the triple

$$A_{[0,s[} \subseteq A_{[0,s]} \subseteq A_{[0,t]}$$

such that

$$\varphi_{[0,t]} = \varphi_{[0,s]} \cdot E_{t,s} ; \qquad 0 \leqslant s < t . \tag{3.5.1}$$

Remark 1. If the local algebras $A_t (t \in \mathbb{R}^+)$ satisfy (3.1), (3.2), (3.3) the $E_{t,s}$ will be required to be "locally normal" in the sense that for each n-tuple $0 \leqslant t_0 < t_1 < \cdots < t_{n-1} < t$, the restriction of $E_{t,s}$ on $A_{\{t_0,\ldots,t_{n-1},t\}}$ has a normal extension in the same sense as specified in (3.3).

Remark 2. In the commutative case any state φ determines a family of conditional expectations satisfying (3.5.1). The Markof states are those for which (3.4.1) is satisfied. In the noncommutative case, due to the lack of a sufficiently general Radon–Nikodyn Theorem on C^*-algebras, the existence of a family of quasi-conditional expectations satisfying (3.5.1) is not guaranteed for general local algebras $A_{[0,t]}$ and states φ. Even in case of existence the $E_{t,s}$ will be quasi-conditional expectations with respect to triples different from the

$$A_{[0,s[} \subseteq A_{[0,s]} \subseteq A_{[0,t]}$$

(i.e. they will not enjoy the Markof property). Moreover in the noncommutative case the uniqueness of the family $(E_{t,s})$ is assured in the following (weak) sense; if $(E_{t,s})$ and $(E'_{t,s})$ are families of quasi-conditional expectations with respect to the triples $A_{[0,s[} \subseteq A_{[0,s]} \subseteq A_{[0,t]}$ both satisfying (3.5.1) then:

$$\varphi_{[0,s]}(b \cdot E_{t,s}(a)) = \varphi_{[0,s]}(bE'_{t,s}(a)); \qquad b \in A_{[0,s[}\,; \qquad a \in A_{[0,t]}$$

In this sense we shall speak, in the following, of "the" family of quasi-conditional expectations associated to the Markof state φ.

LEMMA (3.6). *Let φ be a Markof state with respect to the family of local algebras $(A_{[0,t]})$ and $(E_{t,s})$ the family of quasi-conditional expectations associated to it. Then if each $E_{t,s}$ is a conditional expectation, φ is a product state, i.e.:*

$$\varphi = \bigotimes_{t \in \mathbb{R}^+} \varphi_t\,; \qquad \varphi_t = \varphi\,|_{A_t}\,. \tag{3.6.1}$$

Conversely each product state is a Markof state and the $E_{t,s}$ to it associated can always be chosen to be conditional expectations.

Proof. If $\varphi \equiv (\varphi_{t_0,\ldots,t_n})$ is any Markof state, the properties of the quasi-conditional expectations imply that for each $0 \leqslant t_0 < t_1 < \cdots < t_n$, and $a_{t_\iota} \in A_{t_\iota}$ $(\iota = 0,\ldots, n)$, one has:

$$\begin{aligned}
&\varphi_{t_0,\ldots,t_n}(a_{t_0} \cdot \cdots \cdot a_{t_n}) \\
&\qquad = \varphi_{t_0}(E_{t_1,t_0}(a_{t_0} \cdot E_{t_2,t_1}(a_{t_1} \cdot \cdots \cdot E_{t,t_n}(a_{t_n})) \cdots)
\end{aligned} \tag{3.6.1}$$

where the right-hand side of the equality does not depend on t. If $E_{t,s}$ is a conditional expectation, then for each $a_s \in A_s$ and $a_t \in A_t$: $a_s \cdot E_{t,s}(a_t) = E_{t,s}(a_s \cdot a_t) = E_{t,s}(a_t \cdot a_s) = E_{t,s}(a_t) \cdot a_s$ thus $E_{t,s}(a_t) \in A_s'$. But, because of the Markof property, $E_{t,s}(a_t) \in A_s$ hence $E_{t,s}(a_t)$ must be a scalar (because $A_s \approx \mathfrak{B}(\mathscr{H})$) and (3.5.1) implies that $E_{t,s}(a_t) = \varphi_t(a_t) \cdot 1$. Thus if for each $s < t$, $E_{t,s}$ is a conditional expectation, (3.6.1) implies that:

$$\varphi t_0, \dots, t_n(a_{t_0} \cdots a_{t_n}) = \varphi_{t_0}(a_{t_0} \cdot E_{t_1,t_0}(a_{t_1} \cdot \cdots \cdot E_{t_n,t_{n-1}}(a_{t_n})\dots)$$

$$= \varphi_{t_0}(a_{t_0}) \cdot \cdots \cdot \varphi_{t_n}(a_{t_n})$$

Conversely, let φ be a product state; then (3.6.2) holds. Since for $s < t$, $A_{[0,t]} \approx A_{[0,s]} \otimes A_{]s,t]}$ there exists a unique conditional expectation $E_{t,s} : A_{[0,t]} \to A_{[0,s]}$, which extends the map $ab \mapsto a \cdot \varphi(b)$; $a \in A_{[0,s]}$; $b \in A_{]s,t]}$. It is clear that $E_{t,s}$ satisfies (3.5.1) and, moreover, if φ is locally normal each $E_{t,s}$ is such. Therefore the Lemma is proven.

Since any state φ on A is completely determined by the projective family $(\varphi_{0,t_1,\dots,t_n})$, in equality (3.6.1) one can always choose $t_0 = 0$. Thus any Markof state is determined, through the equalities

$$\varphi_{0,t_1,\dots,t_n}(a_0 \cdot a_{t_1} \cdot \cdots \cdot a_{t_n})$$

$$= \varphi_0(\bar{E}_{t_1,0}(a_0 \cdot \bar{E}_{t_2,t_1}(a_{t_1} \cdot \cdots \cdot \bar{E}_{t,t_n}(a_{t_n})\dots). \tag{3.6.3}$$

by the couple $\{\varphi_0 ; (\bar{E}_{t,s})\}$ where $(E_{t,s})$ is the family of quasi-conditional expectations associated to φ and $\bar{E}_{t,s}$ denotes the restriction of $E_{t,s}$ on the C^*-algebra spanned by A_s and A_t. For any such a state the agreement conditions for the family $(\varphi_{t_0,\dots,t_n})$ can be expressed uniquely in terms of the couple $\{\varphi_0, (\bar{E}_{t,s})\}$ in fact they are quivalent to the validity of the equalities:

$$\varphi_0(\bar{E}_{t_1,0}(a_0 \cdot \cdots \cdot \bar{E}_{s,r}(a_r \cdot \bar{E}_{t,s}(1_s \cdot \bar{E}_{n,t}(a_t \cdot \cdots \cdot \bar{E}_{v,t_n}(a_{t_n}) \cdots)$$

$$= \varphi_0(\bar{E}_{t_1,0}(a_0 \cdot \cdots \cdot \bar{E}_{t,s}(a_r \cdot \bar{E}_{n,t}(a_t \cdot \cdots \cdot \bar{E}_{v,t_n}(a_{t_n}) \cdots) \tag{3.6.4}$$

$$\varphi_0(\bar{E}_{t_1,0}(a_0 \cdot \cdots \cdot \bar{E}_{s,r}(a_r \cdot \bar{E}_{t,s}(1)) \cdots) = \varphi_0(\bar{E}_{t_1,0}(a_0 \cdot \cdots \cdot \bar{E}_{s,r}(a_r)\dots) \tag{3.6.5}$$

for any $0 < t_1 \leqslant r < s < t < u < \cdots < t_{n-1} < t_n < v$, and $a_J \in A_J$ $(J = 0, t_1, \dots, v)$.

In equalities (3.6.4), (3.6.5) we have used the notation 1_s to mean that the corresponding expressions are obtained by the right-hand side of

(3.6.3) by putting $a_s = 1$. In the following we shall use the shortened notations:

$$\bar{E}_{s,r}(a_r \cdot \bar{E}_{t,s}(1_s \cdot b_t)) = \bar{E}_{t,r}(a_r \cdot b_t); \quad (\mathrm{mod}\{\varphi_0 ; (\bar{E}_{t,s})\}) \tag{3.6.6}$$

$$\bar{E}_{t,s}(1) = 1; \quad (\mathrm{mod}\{\varphi_0 ; (\bar{E}_{t,s})\}) \tag{3.6.7}$$

$0 \leqslant r < s < t$; $a_r \in A_r$; $b_t \in A_t$; to denote the validity of (3.6.4), (3.6.5) respectively, for any choice of $n \in \mathbb{N}$, $t_J \in \mathbb{R}^+$, and $a_{t_J} \in A_{t_J}$. From the properties of the quasi-conditional expectations and the Markof property, it follows that the $\bar{E}_{t,s} : A_s \vee A_t \to A_s$ are completely positive linear maps.[2]

PROPOSITION (3.7). *In the notations above, let φ_0 be a state on A_0 and, for $s < t$, let $\bar{E}_{t,s} : A_s \vee A_t \to A_s$, be a linear map which* (i) *is completely positive*; (ii) *is "locally normal"* (*cf. Remark* 1. *after Definition* (3.5)); (iii) *satisfies* (3.6.6), (3.6.7). *Then the couple $\{\varphi_0 ; (\bar{E}_{t,s})\}$ determines a unique state $\varphi \equiv (\varphi_{0,t_1,\ldots,t_n})$ on A by means of the equalities*:

$$\begin{gathered}\varphi_{0,t_1,\ldots,t}(a_{t_0} \cdot a_{t_1} \cdots a_{t_n}) = \varphi_0(\bar{E}_{t_1,0}(a_0 \cdot \bar{E}_{t_2,t_1}(a_{t_1} \cdot \bar{E}_{t,t_n}(a_{t_n}) \cdots) \\ 0 < t_1 < \cdots < t_n < t; \qquad a_{t_J} \in A_{t_J}\,(t_J = 0; \ldots, t_n).\end{gathered} \tag{3.7.1}$$

If φ_0 is normal (*cf.* (3.1)) *then φ is locally normal* (*cf.* (3.3)).

Proof. From (3.6.6) and (3.6.7) it follows

$$\bar{E}_{t,t_n}(a_{t_n}) = \bar{E}_{t,t_n}(a_{t_n} \cdot \bar{E}_{n,t}(1)) = \bar{E}_{n,t_n}(a_{t_n}); \; (\mathrm{mod}\{\varphi_0 ; (\bar{E}_{t,s})\})$$

thus the right-hand side of (3.7.1) is independent on $t > t_n$.

From (i) and (ii) it follows that each $\varphi_{0,t_1,\ldots,t_n}$ is positive; therefore (iii) implies that $(\varphi_{0,t_1,\ldots,t_n})$ is an agreeing family of states, hence it defines a unique state φ on A. Finally if φ_0 is normal, then $\varphi_{0,t_1,\ldots,t_n}$ is obtained by composition of $(n + 2)$ normal maps, hence is normal; and this concludes the proof.

Let $\{\varphi_0, (\bar{E}_{t,s})\}$ be as in Proposition (3.7).

The complete positivity of $\bar{E}_{t,s}$ implies that it can be extended to a

[2] For a C^*-algebra A, denote $M_n(A)$ the C^*-algebra of $(n \times n)$-matrices (a_{IJ}) with coefficients in A and the natural operations. A linear map α: $A \to B$, (A, B — C^*-algebras) is called completely positive if, for each $n \in \mathbb{N}$, the map α_n: $M_n(A) \to M_n(B)$, defined by: $\alpha_n(a_{IJ}) = (\alpha(a_{IJ}))$, is positive (cf. [28]).

positive linear map $A_{[0,s]} \vee A_t \rightarrow A_{[0,s]}$ (still denoted $\bar{E}_{t,s}$) by means of the equality:

$$\bar{E}_{t,s}(b \cdot a) = b \cdot \bar{E}_{t,s}(a); \qquad b \in A_{[0,s[}\,; \qquad a \in A_s \vee A_t\,.$$

If $\bar{E}_{t,s}$ (1) $= 1$ (without the restriction mod. $\{\varphi_0\,;\,(\bar{E}_{t,s})\}$), Kadison's inequality for completely positive linear maps implies that $\bar{E}_{t,s}$ is a quasi-conditional expectation with respect to the triple

$$A_{[0,s[} \subseteq A_{[0,s]} \subseteq A_{[0,s]} \vee A_t\,.$$

In these notations the state φ, defined by Proposition (3.7), can be expressed by the formula, particularly useful in explicit computations:

$$\varphi = \lim_{\substack{0<t_1<\cdots<t_n \\ \operatorname{Max}|t_{j+1}-t_j|\to 0 \\ t_n\to\infty}} \varphi_0 \cdot \bar{E}_{t_1,0} \cdot \bar{E}_{t_2,t_1} \cdot \cdots \cdot \bar{E}_{t_n,t_{n-1}} \tag{3.7.2}$$

The $\bar{E}_{t,s}$ defined above enjoy the Markof property ($\bar{E}_{t,s}(A_s \vee A_t) \subseteq A$), however the state φ defined by (3.7.2) will not be, in general, a Markof state in the sense of Definition (3.5). One can prove that it will be such if and only if:

$$\bar{E}_{t,s}(a_s) = a_s\,; (\text{mod}\{\varphi_0\,;\,(\bar{E}_{t,s})\}; s < t,\, a \in A_s\,. \tag{3.7.3}$$

Definition (3.8). A "Markof chain" is a state determined by a couple $\{\varphi_0\,,\,(\bar{E}_{t,s})\}$ as described in Proposition (3.7).

The distinction between Markof *states* and Markof *chains* is typical of the noncommutative context. These two classes of states have an extremely similar structure and most of their main properties in common. It is relatively easy (cf. (6.5)) to give explicit examples of families $(\bar{E}_{t,s})$ which satisfy (3.6.6), (3.6.7) *without the restriction* (mod. $\{\varphi_0\,;\,(\bar{E}_{t,s})\}$). In such cases from Proposition (3.7) follows that the couple $\{\varphi_0\,;\,(\bar{E}_{t,s})\}$ defines a state on A for *any* state φ_0 on A_0. This fact allows to give simple non-trivial examples of Markof chains.

Definition (3.9). A "noncommutative Markof process" is a symmetric stochastic process $\{A,\,(A_t)_{t\in T}\,,\,\varphi)$ such that the state φ is a Markof chain.

Summing up: the analysis of a classical discrete stochastic process in canonical (i.e. product) form naturally leads to the definition of symmetric stochastic processes; the class of non-commutative Markof processes is the most general class of symmetric stochastic processes to which is naturally associated, for each choice of the "present instant," a "measure of statistical correlation" (i.e. a quasi-conditional expectation) enjoying the Markof property which is the quantitative formulation of what we have called the "Principle of local correlation."

It is in this sense that we assert that Axiom (IV) is the mathematical formulation of the above mentioned principle.

4. Stationary Markof Processes

In the notations of the preceding number, let $J_t : \mathfrak{B}(\mathscr{H}) \to A_t$ be the (normal) isomorphism defined in (3.1). Denote $s \in \mathbb{R}^+ \mapsto T_s \in \mathrm{End}(A)$ the action of $\mathbb{R}^+$ on A uniquely determined by:

$$T_s \cdot J_t = J_{t+s}\,; \qquad s, t \in \mathbb{R}^+.$$

Definition (4.1). A state φ on A is stationary if it is T-invariant: i.e.: $\varphi \cdot T_s = \varphi$; $\forall s \in \mathbb{R}^+$.

Correspondingly, a symmetric stochastic process $\{A, (A_t)_{t\in\mathbb{R}^+}, \varphi\}$ will be called stationary if the state φ is stationary. Let now φ be a Markof state on A; $(E_{t,s})$ the family of quasi-conditional expectations associated to it; $\bar{E}_{t,s}$ the restriction of $E_{t,s}$ on $A_s \vee A_j$ ($s, t \in \mathbb{R}^+$; $s < t$). Each $\bar{E}_{t,s}$ induces a completely positive linear map $\mathfrak{E}_{t,s} : \mathfrak{B}(\mathscr{H}) \otimes \mathfrak{B}(\mathscr{H}) \to \mathfrak{B}(\mathscr{H})$ defined by the relation:

$$\bar{E}_{t,s} \cdot (J_s \otimes J_t) = J_s \cdot \mathfrak{E}_{t,s}\,. \tag{4.1.1}$$

One easily sees that the Markof state φ is stationary if and only if for every $r, s, t \in \mathbb{R}^+$; $a \in A_{[0,t]}$; $b \in A_{[r,s+r]}$; one has:

$$\varphi_{[r,s+r]}(b \cdot T_r \cdot E_{t,s}(a)) = \varphi_{[r,s+r]}(b \cdot E_{t+r,s+r}(T_r a)). \tag{4.1.2}$$

In the following, in agreement with the notations employed up to now, we shall write simply:

$$T_r \cdot E_{t,s} = E_{t+r,s+r} \cdot T_r\,; \text{(mod } \varphi) \tag{4.1.3}$$

to denote the (weak) covariance property expressed by the stationarity condition (4.1.2).

The following lemma proves that the stationarity of a Markof state is a purely local property, i.e. expressible only in terms of φ_0 and of the $\mathfrak{E}_{t,s}$.

LEMMA (4.2). *In the notations above, the Markof state $\varphi \equiv \{\varphi_0, (\mathfrak{E}_{t,s})\}$ is stationary if and only if $\forall s, t \in \mathbb{R}^+$, $s < t$, one has:*

$$\bar{\varphi}_s = \bar{\varphi}_0 ; \qquad s \in \mathbb{R}^+; \qquad \bar{\varphi}_t = \varphi_t \cdot J_t \tag{4.2.1}$$

$$\mathfrak{E}_{t,s} = \mathfrak{E}_{t+r,s+s} ; (\mathrm{mod}\ \varphi) \tag{4.2.2}$$

where (4.2.2) *means*:

$$\varphi_{[0,s]}(b \cdot J_s \cdot \mathfrak{E}_{t,s}(x)) = \varphi_{[0,s]}(b \cdot J_s \cdot \mathfrak{E}_{t+r,s+r}(x)) \tag{4.2.3}$$

for every $b \in A_{[0,s[}$; $x \in \mathfrak{B}(\mathscr{H}) \otimes \mathfrak{B}(\mathscr{H})$.

Proof. If φ is stationary, then, for every $b \in A_{[0,s[}$ and $x \in \mathfrak{B}(\mathscr{H}) \otimes \mathfrak{B}(\mathscr{H})$:

$$\varphi(b \cdot (J_s \otimes J_t)(x)) = \varphi_{[0,s]}(b \cdot \bar{E}_{t,s}(J_s \otimes J_t)(x)) = \varphi_{[0,s]}(b \cdot J_s \cdot \mathfrak{E}_{t,s}(x))$$

and for each $r \in \mathbb{R}^+$:

$$\begin{aligned} \varphi(b \cdot (J_s \otimes J_t)(x)) &= \varphi(T_r(b) \cdot (J_{s+r} \otimes J_{t+r}(x)) \\ &= \varphi_{[0,s+r]}(T_r(b) \cdot \bar{E}_{t+r,s+r}((J_{s+r} \otimes J_{t+r})(x)) \\ &= \varphi_{[0,s+r]}(T_r(b) \cdot J_{s+r} \cdot \mathfrak{E}_{t+r,s+r}(x)) \\ &= \varphi_{[0,s]}(b \cdot J_s \cdot \mathfrak{E}_{t+r,s+r}(x)) \end{aligned}$$

and this proves (4.2.2); (4.2.1) is obvious. Conversely, assume that (4.2.1), (4.2.2) hold. Then, for each $0 < t_0 < \cdots < t_n < t$, and $a_{t_J} \in \mathfrak{B}(\mathscr{H})$ $(J = 0,..., n)$, one has:

$$\begin{aligned} &\varphi(J_{t_0}(a_{t_0}) \cdot \cdots \cdot J_{t_n}(a_{t_n})) \\ &\quad = \varphi_{[0,t_n]}(J_{t_0}(a_{t_0}) \cdot \cdots \cdot J_{t_{n-1}}(a_{t_{n-1}}) \cdot \bar{E}_{t,t_n}(J_{t_n}(a_{t_n}))) \\ &\quad = \varphi_{[0,t_n]}(J_{t_0}(a_{t_0}) \cdot \cdots \cdot J_{t_{n-1}}(a_{t_{n-1}}) \cdot J_{t_n} \cdot \mathfrak{E}_{t,t_n}(a_{t_n} \otimes 1)) \\ &\quad = \varphi_{[0,t_n]}(J_{t_0}(a_{t_0}) \cdot \cdots \cdot J_{t_{n-1}}(a_{t_{n-1}}) \cdot J_{t_n} \cdot \mathfrak{E}_{t+r,t_n+r}(a_{t_n} \otimes 1)). \end{aligned}$$

Thus, iterating the procedure and applying (4.2.1) one finds:

$$\begin{aligned} &\varphi(J_{t_0}(a_{t_0}) \cdot \cdots \cdot J_{t_n}(a_{t_n})) \\ &\quad = \bar{\varphi}_{t_0+r}(\mathfrak{E}_{t_1+r,t_0+r}(a_{t_0} \otimes \mathfrak{E}_{t_2+r,t_1+r}(a_{t_1} \otimes \cdots \otimes \mathfrak{E}_{t+r,t_n+r}(a_{t_n} \otimes 1) \cdots) \end{aligned}$$

and the right-hand side of this equality is nothing but:

$$\varphi(J_{t_0+r}(a_{t_0}) \cdot \cdots \cdot J_{t_n+r}(a_{t_n}))$$

from the arbitrarity of the t_J and of the a_{t_J}, it follows

$$\varphi = \varphi \cdot T_r ; \qquad \forall r \in \mathbb{R}^+,$$

i.e. is stationary, and therefore the lemma is proven.

From the proof of Lemma (4.2) it follows that conditions (4.2.1), (4.2.2) are sufficient conditions for stationarity also for a Markof chain. In the following we shall use the notations $\mathfrak{E}_{t-s}$ for the operators $\mathfrak{E}_{t,s}$ associated, by (4.1.1) to a stationary Markof chain.

Remark. The restriction of the maps $\mathfrak{E}_{t,s}$ on sub-algebras of the type $\mathfrak{B}_0 \otimes \mathfrak{B}(\mathscr{H})$ where $\mathfrak{B}_0$ is an abelian von Neumann sub-algebra of $\mathfrak{B}(\mathscr{H})$ enjoy, in particular, all the properties which define an "expectation" in the sense of E. B. Davies [3].

5. Evolution Equations

In the present No. it is shown that, like the commutative case, to the non-commutative Markof processes, some evolution equations are naturally associated.

Let $\{A, (A_t)_{t\in\mathbb{R}^+}, \varphi\}$ be a non-commutative Markof process and let the Markof chain φ be determined by the couple $\{\varphi_0, (\bar{E}_{t,s})\}$. Denote $Z(t, s)$ the restriction of $\bar{E}_{t,s}$ on A_t; then the properties of the $\bar{E}_{t,s}$ imply that $Z(t, s)$: $A_t \to A_s$ is a completely positive linear map satisfying:

$$Z(t, s)[1] = 1 \tag{5.0.1}$$

$$Z(s, r) \cdot Z(t, s) = Z(t, r); \quad r < s < t. \tag{5.0.2}$$

In the above equalities, as well as in the following, the restriction (mod$\{\varphi_0, (\bar{E}_{t,s})\}$) is understood. In all the processes considered in the present paper, the equalities (3.6.6), (3.6.7) thus, in particular (5.0.1), (5.0.2) will be satisfied without any restriction and, therefore the couple $\{\varphi_0, (\bar{E}_{t,s})\}$ defines a Markof chain for any choice of the "initial state" φ_0. Moreover, denoting φ_t $(t > 0)$, the restriction of φ on A_t, one has:

$$\varphi_t = \varphi_s Z(t, s) \colon 0 < s < t \tag{5.0.3}$$

where we have used the notation $\varphi_s \mapsto \varphi_s \cdot Z(t, s)$ to denote the adjoint of $Z(t, s)$.

In the classical case (i.e. all the algebras A_s are commutative), φ_t is the distribution of the process at time t; (5.0.3) is the evolution equation of φ_t; (5.0.1) is the Chapman-Kolmogorof equation $Z(t, s)$ is the transition operator, from time s to time t, associated to the Markof process, and any (completely) positive linear operator $A_t \to A_s$, which preserves the identity is called a transition operator.

Equations (5.0.2), (5.0.3) can be considered as evolution equations in "integral form"; in order to write them in the more convenient differential form let us introduce some regularity conditions. First of all remark that from the local normality of the $\bar{E}_{t,s}$ it follows that the operator induced by $Z(t, s)$ on $\mathfrak{B}(\mathcal{H})$ is normal. In the following, unless explicitly stated the contrary, we shall still denote $Z(t, s)$ this operator, and we shall identify states on (resp. operators in) A_t with states on (resp. operators in) $\mathfrak{B}(\mathcal{H})$.

LEMMA (5.1). *Assume that the transition operators satisfy the following conditions*:

(i1) *For every* $t \in \mathbb{R}^+$ *and every* $a \in \mathfrak{B}(\mathcal{H})$, *the map* $s \in [0, t[\mapsto Z(t, s)[a]$ *extends to a weakly continuous map of* $[0, t]$ *in* $\mathfrak{B}(\mathcal{H})$.

(i2) $\lim_{\epsilon \to 0} Z(t - \epsilon, s) = Z(t, s)$; (*pointwise weakly*).

(*In the following the term "weakly continuous" will be meant in the sense of the duality* $\langle \mathfrak{B}(\mathcal{H})_*, \mathfrak{B}(\mathcal{H})\rangle$.) *Then the* (*pointwise weak*) *limit*:

$$\lim_{s \to t^-} Z(t, s) = P(t)$$

exists and is a projector satisfying the relations:

$$Z(t, s) \cdot P(t) = Z(t, s); \qquad \varphi_t = \varphi_t \cdot P(t); \qquad 0 \leqslant s < t.$$

Proof. For every $s < t$; $0 < \epsilon < t - s$; $a \in \mathfrak{B}(\mathcal{H})$; $\psi \in \mathfrak{B}(\mathcal{H})_*$; denote $P(t)[a]$ the weak limit, for $s \to t^-$ of $Z(t, s)[a]$, existing by (i1). Then,

$$\begin{aligned} &| \psi(Z(t, s) \cdot P(t)[a]) - \psi(Z(t, s)[a])| \\ &\quad \leqslant | \psi(Z(t, s) \cdot [P(t) - Z(t, t - \epsilon)](a))| \\ &\qquad + | \psi([Z(t, s) - Z(t - \epsilon, s)] \cdot Z(t, t - \epsilon)[a])|, \end{aligned} \tag{5.1.1}$$

From (i1) it follows that the set $\{Z(t, t - \epsilon)[a] : \epsilon \in [0, \delta]\}$ is weakly compact, hence totally bounded. Because of (i2) and the Banach-Steinhaus theorem the set $\{[Z(t, s)^* - Z(t - \epsilon, s)^*]\psi : \epsilon \in [0, \delta]\}$ is equicontinuous, hence on it the topology of pointwise convergence coincides with that of uniform convergence on totally bounded sets. This implies that the right hand side of (5.1.1) is of order ϵ. The arbitrariety of ϵ, ψ, a, imply $Z(t, s) \cdot P(t) = Z(t, s)$; hence, by (i1), $P(t)$ is a projector. The second equality of the Lemma follows from the first and (5.0.3). And this ends the proof.

In a similar way one proves:

LEMMA (5.2). *If the transition operators* $(Z(t, s))$ *satisfy:*

(j1) *For any* $s < t$, *and* $a \in \mathfrak{B}(\mathscr{H})$ *the map* $u \in]s, t] \mapsto Z(u, s)[a]$ *extends to a weakly continuous map of* $[s, t]$ *on* $\mathfrak{B}(\mathscr{H})$.

(j2) $\lim_{\epsilon\to 0} Z(t, s + \epsilon) = Z(t, s)$ *(pointwise weakly) then the (pointwise weak) limit*

$$\lim_{t\to s^+} Z(t, s) = Q(s)$$

exists and satisfies the relation

$$Q(s) \cdot Z(t, s) = Z(t, s).$$

If, in the hypothesis of Lemma (5.1), for each $t \in \mathbb{R}^+$ there are dense sets $\mathscr{D}(t) \subseteq \mathfrak{B}(\mathscr{H})$, $\mathscr{D}_*(t) \subseteq \mathfrak{B}(\mathscr{H})_*$ such that for every $\psi \in \mathscr{D}_*(t)$ and $a \in \mathscr{D}(t)$ the limit

$$\lim_{\epsilon\to 0} \psi\left(\left[\frac{Z(t + \epsilon) - P(t)}{\epsilon}\right](a)\right) = \psi(S(t)[a])$$

exists, then the evolution equation (5.0.3) can be written in differential form:

$$\frac{d}{dt}\varphi_t = \varphi_t \cdot S(t); \qquad \varphi_t \in D_*(t); \qquad t \in \mathbb{R}^+$$

where the (right) derivative is meant in the weak sense for the duality $\langle \mathscr{D}_*(t), \mathscr{D}(t)\rangle$.

Analogously, in the hypothesis of Lemma (5.2) and under the same conditions as above, the limit

$$\lim_{\epsilon\to 0} \psi\left(\left[\frac{Q(s) - Z(s, s - \epsilon)}{\epsilon}\right](a)\right) = \psi(R(t)[a])$$

exists, then for fixed a_t, the function $s \in [0, t[\mapsto a_s = Z(t, s)[a_t]$ is derivable in $]0, t[$, in the topology specified above, and $d/ds\, a_s = R(s)[a_s]$. If, moreover $P(t) = Q(t)$, i.e. if for each t:

$$\lim_{\epsilon \to 0} Z(t + \epsilon, t) = \lim_{\epsilon \to 0} Z(t, t - \epsilon). \tag{5.2.1}$$

The above equations take the form:

$$\frac{d}{dt} \varphi_t = \varphi_t \cdot S(t) \tag{5.2.2}$$

$$\frac{d}{ds} a_s = -S(s)[a_s]. \qquad [5.2.3]$$

These are the noncommutative analogue of the well known Kolmogorof equations of probability theory; consequently (5.2.2) (resp. (5.2.3)) will be called the *noncommutative forward* (resp. *backward*) *Kolmogorof* equation associated to the Markof process $\{A, (A_t), \varphi\}$.

Remark (1). Both equations (5.2.2) and (5.2.3) follow respectively from:

$$\frac{\partial}{\partial t} Z(t, s) = Z(t, s) \cdot S(t) \tag{5.2.4}$$

$$\frac{\partial}{\partial s} Z(t, s) = -S(s) \cdot Z(t, s) \tag{5.2.5}$$

and it is in this form, i.e. as equations on the transition operators, that they are often introduced in the probabilistic literature (cf. for example, [7, pg. 254]).

Remark (2). The regularity conditions for the validity of the noncommutative Kolmogorof equations have not been completely specified. Also in the commutative case, for nonstationary processes, there is no set of regularity conditions which is both natural and general enough.

In the stationary case however, the situation is simpler since, in this case, $Z(t, s) = Z(t - s)$; and, by the Chapman Kolmogorof equation $(Z(t))$ is a semi-group. In this case the appropriate regularity conditions come from semi-group theory, and the operators $S(t)$ in the Kolmogorof equations do not depend on t.

Thus the family of transition operators of a non-commutative Markof process is, under regularity conditions, the Green function of equation (5.2.2) or (5.2.3).

DEFINITION (5.3). A family of densely defined linear operators $(S(t))$ of $\mathfrak{B}(\mathscr{H})$ into itself, will be called a family of (noncommutative) Kolmogorof operators if the Green function of equation (5.2.2) or (5.2.3) is univoquely determined and is a family of transition operators (i.e. completely positive and preserving the identity).

A characterization of non-commutative Kolmogorof operators in the case when $\mathscr{H}$ is finite-dimensional has been given in [10].

One can prove that if $(S(t))$ is a family of Kolmogorof operators then there is a Markof process such that (5.2.2) (or (5.2.3)) is the non-commutative forward (backward) Kolmogorof equation of the process.

However the process above will not be, in general, unique, independently on the regularity conditions on the $S(t)$. More specifically: *in general the family of the transition operators of a noncommutative Markof process does not determine univoquely the process.*

This is a nontrivial difference between non-commutative and classical Markof processes which stems from the circumstance that, in the first case the quasiconditional expectations $E_{t,s}$ in general are not projection operators and therefore $\bar{E}_{t,s}$ is not determined by its restrictions on A_t, i.e. $Z(t, s)$ (cf. (6.5) for an example).

However the following assertion holds:

PROPOSITION (5.4). *Let φ be a non-commutative Markof chain and $Z(t, s)$ the family of its transition operators. If for each $s < t$, $Z(t, s)$ is invertible, one has:*

$$\varphi = \bigotimes_{t \in \mathbb{R}^+} \varphi_t\,; \qquad \varphi_0 = \varphi \mid A_0\,; \qquad \varphi_t = \varphi_s \cdot Z(t, s).$$

Proof. Let $\{\varphi_0\,; (\bar{E}_{t,s})\}$ be the couple determining the Markof chain (cf. Proposition (3.7)); and $\mathfrak{E}_{t,s} : \mathfrak{B}(\mathscr{H}) \otimes \mathfrak{B}(\mathscr{H}) \to \mathfrak{B}(\mathscr{H})$ the map induced by $\bar{E}_{t,s}$ (cf. No. 4). Define:

$$\bar{\mathfrak{E}}_{t,s}(x) = 1 \otimes \mathfrak{E}_{t,s}(x); \qquad x \in \mathfrak{B}(\mathscr{H}) \otimes \mathfrak{B}(\mathscr{H})$$

$$\bar{Z}(t, s)[1 \otimes b] = \bar{\mathfrak{E}}_{t,s}(1 \otimes b) = 1 \otimes Z(t, s)[b]; \qquad b \in \mathfrak{B}(\mathscr{H}).$$

By hypothesis $Z(t, s)$ is invertible, hence

$$F_{t,s} = \bar{Z}(t, s)^{-1} \cdot \bar{\mathfrak{E}}_{t,s} : \mathfrak{B}(\mathscr{H}) \otimes \mathfrak{B}(\mathscr{H}) \to 1 \otimes \mathfrak{B}(\mathscr{H})$$

is a conditional expectation. Therefore $F_{t,s}(\mathfrak{B}(\mathscr{H}) \otimes 1) = \mathbb{C} \cdot (1 \otimes 1)$.

Hence there is a state $\chi_{t,s}$ on $\mathfrak{B}(\mathscr{H})$ such that

$$F_{t,s}(a \otimes b) = \chi_{t,s}(a) \cdot 1 \otimes b$$

and this is equivalent to:

$$\bar{E}_{t,s}(\bar{a}_s \cdot \bar{b}_t) = \chi_{t,s}(a_s) \cdot Z(t, s)[b_t]; \; \bar{a}_t = J_t(\bar{a}_t).$$

By definition of Markof chain one has:

$$\varphi_{0,t_1,\ldots,t_n}(a_0 \cdot a_{t_1} \cdot \cdots \cdot a_{t_n}) = \varphi_0(\bar{E}_{t_1,0}(a_0 \cdot \bar{E}_{t_2,t_1}(a_{t_1} \cdot \cdots \cdot \bar{E}_{t,t_n}(a_{t_n})\ldots)$$

for $0 < t_1 < \cdots < t_n < t$; $a_{t_J} \in A_{t_J}$. Taking $t_n = s$; $a_{t_J} = 1$, for $t_J = 0, t_1, \ldots, t_{n-1}$, one finds, for any $s < t$:

$$\begin{aligned}\varphi_s(J_s(a_s)) &= \chi_{t,s}(a_s) \cdot \varphi_0(\bar{E}_{t_1,0}(1 \cdot \cdots \cdot \bar{E}_{t,t_n}(1) \cdots) \\ &= \chi_{t,s}(a_s).\end{aligned}$$

Thus one concludes:

$$\varphi_{0,t_1,\ldots,t_n}(a_0 \cdot a_{t_1} \cdot \cdots \cdot a_{t_n}) = \varphi_0(a_0) \cdot \varphi_{t_1}(a_{t_1}) \cdot \cdots \cdot \varphi_{t_n}(a_{t_n})$$

for any choice of n, t_J, a_{t_J}; and this establishes the first assertion. The second one is true for every Markof chain; therefore the proposition is proved.

6. Quantum Systems

Identifying the predual of $\mathfrak{B}(\mathscr{H})$ with the space of trace-class operators $T(\mathscr{H})$; denoting $V \in T(\mathscr{H}) \mapsto V \cdot Z(t, s)$, the action induced on $T(\mathscr{H})$ by the adjoint of $Z(t, s)$; and using the same notations for the action induced by the adjoint of the Kolmogorof operator $S(t)$, one can write the noncommutative forward Kolmogorof equation (5.2.2) in terms of the density matrix of the state φ_t:

$$\frac{d}{dt} W_t = W_t \cdot S(t). \tag{6.0.1}$$

The simplest example of a noncommutative Kolmogorof operator is obtained by taking $S(t) = S$ (independent of t) and

$$W \cdot S = i[W, H] = i(WH - HW),$$

where H is a self-adjoint operator. In fact, for this choice of S equation (6.0.1) becomes

$$\frac{d}{dt} W_t = i[W_t, H] \tag{6.0.2}$$

and the Green function of Eq. (6.0.2) is a (uniquely determined) one-parameter group of inner automorphisms of $\mathfrak{B}(\mathscr{H})$ which, clearly is completely positive and preserves the identity; i.e. *the Schrödinger equation* (6.0.2) *can be considered as the forward Kolmogorof equation of a noncommutative Markof process.* More precisely one has the following:

THEOREM (6.1). *Given an arbitrary quantum system (as univoquely specified by a (time-dependent) Hamiltonian $H(t)$ and an arbitrary initial state W_0) there exists exactly one noncommutative Markof process $\{A, (A_t)_{t\in\mathbb{R}^+}, \varphi\}$ with the property that the forward Kolmogorof equation associated to it coincides with the Schrödinger equation (in Heisenberg's form) of the quantum system.*

Proof. Consider the Schrödinger equation in Heisenberg's form:

$$\frac{d}{dt} W_t = i[W_t, H(t)]. \tag{6.1.1}$$

The hypothesis that the family $(H(t))$ univoquely determines the quantum process means that the Green function $(G(t,s))$ of Eq. (6.1.1) is univoquely determined (this always happens, for example, if the family $(H(t))$ satisfied the conditions: Domain $(H(t)) = \mathscr{D}$ (independent of t);

$$\|(i - H(t)) \cdot (i - H(s))^{-1} - 1\| \leqslant K \cdot |t - s|; K > 0)$$

and, for each $s < t$, $G(t, s)$ in an inner automorphism of $\mathfrak{B}(\mathscr{H})$.[3] Therefore the operators $V \in T(\mathscr{H}) \mapsto i[V, H(t)]$ constitute a family of Kolmogorof operators and the $(G(t, s))$ are the transition operators of a noncommutative Markof process. Since each $G(t, s)$ is invertible, the result of Proposition (5.4) is applicable, and implies that the Markof state φ whose transition operators are the $G(t, s)$ and whose initial (i.e. at time $t = 0$) state φ_0 has density matrix W_0 is univoquely determined by:

$$\varphi = \bigotimes_{t\in\mathbb{R}^+} \varphi_t ; \qquad \varphi_t = \varphi_s \cdot Z(t, s). \tag{6.1.2}$$

[3] The author is grateful to Giuseppe Da Prato for having shown him a proof of this assertion.

By construction Eq. (6.1.1) is the non-commutative forward Kolmogorof equation associated to this state. And this ends the proof.

COROLLARY (6.2). *If a noncommutative Markof process is such that each transition operator of it maps in a one-to-one way pure states, in* $\mathfrak{B}(\mathscr{H})_*$ *into pure states, then the family of the transition operators univoquely determines the process through* (6.1.2).

Proof. A theorem of R. Kadison [14] implies that each such transition operator is induced by an inner automorphism of $\mathfrak{B}(\mathscr{H})$. Hence the assertion follows from Proposition (5.4).

Thus if, under the hypothesis of Corollary (6.2), the transition operators $(Z(t, s))$ satisfy differentiability conditions (in t and s), the process is the non-commutative Markof process associated to a quantum system.

COROLLARY (6.3). *A non-commutative Markof process with stationary transition operators is the process associated (as described in Theorem* (6.1)) *to a Quantum system if and only if each transition operator of the process maps in a one-to-one way pure states into pure states. In such a case the corresponding Quantum System is conservative.*

Proof. The stationarity of the transition operators and Corollary (6.2) imply that the family of transition operators of such a process is a pointwise weakly continuous one-parameter group of inner automorphisms of $\mathfrak{B}(\mathscr{H})$. Thus Mackey's analysis (cf. [16, pg. 82]) is applicable and yields that to such a process it is associated the Kolmogorof operator $V \mapsto i[V, H]$, where H is a self-adjoint operator on $\mathscr{H}$. Thus the initial process is associated to a conservative quantum system. Conversely, if H is the Hamiltonian of a conservative quantum system the transition operators associated to it, according to theorem (6.1) are

$$Z(t, s)[a] = \exp(-i(t - s)H) \cdot a \cdot \exp(i(t - s)H)$$

hence they are stationary.

Remark that, as in the commutative case, the stationarity of the transition operators does not imply the stationarity or the process. For this the further condition: $\varphi_0 \cdot Z(t) = \varphi_0$; $\forall t \in \mathbb{R}^+$ is needed.

(6.4) A theorem of J. von Neumann, generalized by V. S. Varadarajan [30, Vol. 1, pg. 163] asserts that a set of quantum observables admits a family of joint distributions if and only if the observables commute. Theorem (6.1) shows that, even if one limits oneself to the consideration of joint distributions of observables at different times and allows the commutativity of these ones (inside the larger algebra corresponding, in the classical case, to the algebra of the continuous functionals on the paths of the process) then, under the requirement that the statistical correlation among these observables be of markovian type, the only joint expectations compatible with the quantum mechanical evolutions and the choice of $\mathfrak{B}(\mathscr{H})$ as algebra of the quantum observables at a given time, are the trivial ones: i.e. the joint expectations at different times are given by the product of the expectations at the single instants of time.

But for noncommutative, as well as for classical stochastic processes, a property of Markovian type, expressing the *local* character of the statistical correlation among observables at different times, is necessary in order to guarantee the determinism of the time-evolution of the states which, as already remarked (cf. no. 1) is a fundamental characteristic of quantum systems. Therefore one can conclude that the only joint expectations, for observables at different times, compatible with the following four assumptions:

determinism of the (time) evolution.

reversibility of the time-evolution

$\mathfrak{B}(\mathscr{H})$ as algebra of the observables at any time

commutativity of observables at different times (in $A \approx \bigotimes_{\mathbb{R}^+} \mathfrak{B}(\mathscr{H})$) are the trivial ones. The first three assumptions are well established in quantum mechanics. The fourth one arises from the consideration of a quantum process as a particular discrete stochastic process. Usually one considers observables at different times (i.e. in different A_t) mapped, through implicit use of isomorphisms, into the same $\mathfrak{B}(\mathscr{H})$ where of course in general they do not commute.

(6.5) Quantum systems have been characterized as those noncommutative Markof processes whose quasi-conditional expectations have the form:

$$\bar{E}_{t,s}(a_s \cdot a_t) = \varphi_s(a_s) \cdot \bar{Z}(t, s)[a_t], \tag{6.5.1}$$

where φ_s is a state on A_s and $\bar{Z}(t, s)$: $A_t \rightarrow A_s$, is a C^*-algebra isomorphism. A simple way for building noncommutative Markof processes which are not of quantum type, is the following: let, for every $s \in \mathbb{R}^+$ $(\varphi_s{}^{\iota})_{\iota \in F}$ a family of states on A_s ; $(Z_{\iota}(t, s))_{\iota \in F}$ a family of automorphism groupoids of $\mathfrak{B}(\mathscr{H})$ and $(l_\iota)_{\iota\in F}$ a family of projections in $\mathfrak{B}(\mathscr{H})$ such that

$$l_\iota \cdot l_J = \delta_{\iota J} l_\iota ; \qquad \sum_{\iota\in F} l_\iota = 1$$

$$Z_\iota(t, s)[l_\iota] = l_\iota ; \qquad \iota \in F.$$

Denote as in no. (4), $J_t : \mathfrak{B}(\mathscr{H}) \to A_t$ the tth immersion; J_t^* the left-inverse of J_t ; and

$$\bar{Z}_\iota(t, s) = J_s \cdot Z_\iota(t, s) \cdot J_t^*.$$

Define, for every $s < t$, $a_t \in A_t$, $a_s \in A_s$:

$$\bar{E}_{t,s}(a_s \cdot a_t) = \sum_{\iota\in F} \varphi_s{}^\iota(a_s) \cdot \bar{Z}_\iota(t, s) \cdot P^t_{l_\iota}[a_t] \tag{6.5.2}$$

where

$$P^t_{l_\iota}(a_t) = J_t(l_\iota) \cdot a_t \cdot J_t(l_\iota); \qquad a_t \in A_t .$$

From our assumptions it follows that $\bar{Z}_\iota(t, s) \cdot P^t_{l_\iota} = P^s_{l_\iota} \cdot \bar{Z}_\iota(t, s)$, $\iota \in F$. Each $\bar{E}_{t,s}$ is a completely positive linear map because it is a sum of such ones. Moreover:

$$\bar{E}_{t,s}(1_s \cdot 1_t) = \sum_{\iota\in F} J_s(l_\iota) = 1,$$

and, if $r < s < t$, $\forall a_r \in A_r$, $\forall a_t \in A_t$.

$$\begin{aligned}
&\bar{E}_{s,r}(a_r \cdot \bar{E}_{t,s}(1_s \cdot a_t)) \\
&\quad = \sum_{\iota J} \varphi_r{}^\iota(ar) \cdot \bar{Z}_\iota(s, r) \cdot P^s_{l_\iota} \cdot \varphi_s{}^J(1_s) \cdot P^s_{l_J} \cdot \bar{Z}_J(t, s)[a_t] \\
&\quad = \sum_{\iota} \varphi_r{}^\iota(a_r) \cdot \bar{Z}_\iota(s, r) \cdot \bar{Z}_\iota(t, s) \cdot P^t_{l_\iota}[a_t] \\
&\quad = \bar{E}_{t,r}(a_r \cdot a_t).
\end{aligned}$$

Therefore the conditions of Proposition (3.7) are satisfied without any restriction; hence for *any* state φ_0 on A_0 , the couple $\{\varphi_0 , (\bar{E}_{t,s})\}$ defines a unique noncommutative Markof chain on A. The transition operators of this chain are given by:

$$Z(t, s)[a] = \sum_{\iota\in F} Z_\iota(t, s)[l_\iota a l_\iota]; \qquad a \in \mathfrak{B}(\mathscr{H})$$

and, denoting $V \in T(\mathcal{H}) \mapsto V \cdot Z(t, s)$, the action induced by the adjoint of $Z(t, s)$ on the trace-class operators on $\mathcal{H}$, one has:

$$W \cdot Z(t, s) = \sum_{\iota \in F} l_\iota \cdot (W \cdot Z_\iota(t, s)) \cdot l_\iota .$$

Therefore, in general $Z(t, s)$ will map pure states into mixtures; consequently the non-commutative Markof process determined by $\{\varphi_0, (\bar{E}_{t,s})\}$ *does not correspond* to a quantum process.

The passage from quantum Markof processes (characterized by (6.5.1)) to processes characterized by (6.5.2) is the analogue, *for the dynamics*, of the passage from a pure state to a mixture of states and corresponds to the passage from a "pure dynamic" to a "mixture of dynamics."

(6.6) The fact that quantum Markof processes are characterized as those whose Kolmogorof operator is determined by the commutator with an "Hamiltonian function $H(t)$" lies at the root of the possibility of defining a "Schrödinger representation" for such processes and of establishing the equivalence with the original (Heisenberg) representation. From it, in fact, it follows that the transition operators of the process are of the form: $Z(t, s)[a] = U(t, s)^* \cdot a \cdot U(t, s)$ where the $U(t, s)$ are unitaries in $\mathfrak{B}(\mathcal{H})$ satisfying $U(s, r) \cdot U(t, s) = U(t, r)$. Hence the well known formulas:

$$a_S = u(t, 0) \cdot a_H(t) \cdot u(t, 0)^*; \qquad a_S, a_H(t) \in \mathfrak{B}(\mathcal{H})$$

$$\zeta_S(t) = u(t, 0)\, \zeta_H; \qquad \zeta_S(t), \zeta_H \in \mathcal{H}$$

define observables and states $a_s \mapsto \langle \zeta_s(t), a_s \zeta_s(t) \rangle$ of a new realization of the process unitarily isomorphic to the initial one.

For noncommutative Markof processes of more general type the transition operators determine an irreversible evolution, i.e. they do not map pure states into pure states, hence, for such processes a "Schrödinger representation" cannot be even defined, while, as the preceding discussion shows, the Heisenberg representation still makes sense for them, the Schrödinger equation in Heisenberg's form, generalizes into the non-commutative forward Kolmogorof equation.

For this reason such noncommutative Markof processes (which may be thought to correspond to "non-Hamiltonian" quantum systems) provide a mathematical model in support of Dirac's assertion on the inequivalence, in the context of a general quantum theory, between the

Schrödinger and the Heisenberg picture. More precisely Dirac asserts [5, pg. 6]) that: "... the situation that we have at one particular time is essentially the same whether we are thinking in terms of the Heisenberg picture or the Schrödinger picture. The difference between the Heisenberg picture and the Schrödinger picture comes into effect only when we vary t_0."

8. Conclusions

In the present work the following inclusions have been discussed: Quantum systems $\subseteq$ Markof Processes $\subseteq$ Symmetric stochastic processes.

Our restriction to discrete stochastic processes means that one can consider the algebra of the observables at each fixed time $t \in \mathbb{R}^+$. Clearly it is physically more meaningful to consider observables localized in intervals of time. With natural assumptions on the algebras of local observables, the theory carries out also to this case without difficulties. Yet, even with the above modification the theory is essentially non-relativistic, since time is considered as an exterior parameter. To make it such it is necessary to localize the observables not only in time, but also in space; i.e. one has to consider local algebras $A(B)$ where B is a region in Minkowski space, and the relations among them (which correspond to the relations among the random variables of a stochastic process) expressed by the Haag–Kastler axioms [12].

In the terminology of algebraic quantum field theory the results of the present paper might be synthetized in the assertion that a noncommutative Markof state on the algebra of "non-relativistic quasi-local observables" (i.e. the noncommutative analogue of the algebra of continuous functionals on the paths of stochastic process) of a quantum system contains all the information on the dynamic of such a system; that the Markof property expresses the *locality* (in time) of the interaction; and that, in its turn, the state is completely recovered (up to the initial state) by its local characteristic (i.e. the quasi-conditional expectations).

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