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A QUANTUM FORMULATION OF THE FEYNMAN – KAC FORMULA

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1.

We discuss a formulation, in the general setting of W^*- (or C^*)-algebras, of the classical Feynman – Kac formula. The equivalence, in the commutative case, of the present formulation and the usual one is based on the identification between stochastic processes and local algebras discussed in [1], [2].

By an algebraic realization of a stochastic process indexed by a set T we mean a triple $\{\mathscr{A}, (\mathscr{A}_\alpha)_{\alpha \in \mathscr{F}}, \mu\}$, where $\mathscr{F}$ is a family of subsets of T, directed by inclusion, and

- $\mathscr{A}$ is a C^*-algebra,
- for each $\alpha \in \mathscr{F}$, $\mathscr{A}_\alpha$ is a C^*-algebra and $\mathscr{A}_\alpha \subseteq \mathscr{A}$,
- if $\alpha \subseteq \beta$ then $\mathscr{A}_\alpha$ is a C^*-algebra contained in $\mathscr{A}_\beta$,
- $\mathscr{A}$ is the norm closure of $\bigcup_{\alpha \in \mathscr{F}} \mathscr{A}_\alpha$,
- μ is a state on $\mathscr{A}$,

(by C^*-algebra we will always mean C^*-algebra with unity). Two triples $\{\mathscr{A}^{(j)}, (\mathscr{A}_\alpha^{(j)})_{\alpha\in\mathscr{F}}, \mu^{(j)}\}$ $(j = 1, 2)$ are said to be *stochastically equivalent* if, denoted $\{\mathscr{H}_j, \pi_j, 1_{(j)}\}$ the GNS triple associated to $\{\mathscr{A}^{(j)}, \mu^{(j)}\}$ $(j = 1, 2)$ there exists a unitary operator U: $\mathscr{H}_1 \to \mathscr{H}_2$ such that

$$U 1_{(1)} = 1_{(2)},$$
$$U\pi_1(\mathscr{A}_\alpha^{(1)})'' U^* = \pi_2(\mathscr{A}_\alpha^{(2)})'', \qquad \forall \alpha \in \mathscr{F}.$$

By a *stochastic process* indexed by T we mean an equivalence class of triples $\{\mathscr{A}, (\mathscr{A}_\alpha), \mu\}$ for the relation described above.

Remark 1. The choice of this equivalence relation is motivated by the fact that, in the Abelian case, it reduces to the usual equivalence relation among stochastic processes (i.e. equality of the joint probabilities).

Remark 2. Often the local algebras $\mathscr{A}_\alpha$ are defined by a set of generators with algebraic relations among them and, in the definition of stochastic equivalence, one adds the condition that the map $a \longmapsto UaU^*$ establishes a one-to-one correspondence between the sets of generators corresponding to the two triples. This is the point of view adopted by I.E. Segal in [5]. The slightly more general definition of equivalence adopted here is motivated by the fact that the probability law of classical stochastic process (i.e. its joint distributions) can be determined by the expectation values of different algebras of localized functionals of the process, among the generators of which there might not be a natural correspondence.

To every realization of a classical stochastic process one can associate a triple as above, in which the algebra $\mathscr{A}$ is Abelian, in such a way that equivalent realizations correspond to stochastically equivalent triples. The multiplicity of equivalent triples associated to a given stochastic process reflects both the multiplicity of equivalent realizations of the process, and the fact that the probability law of the process can be determined by the expectation values of different classes of functionals of the process (continuous, step functions, . . .). Conversely one can show that to every triple $\{\mathscr{A}, (\mathscr{A}_\alpha)_{\alpha\in\mathscr{F}}, \mu\}$ there corresponds a stochastic process in the generalized sense of I.E. Segal [4]. It is also possible to characterize those triples which correspond to stochastic processes in the usual sense (cf. [1]). In the present note we are concerned only with the abstract algebraic formulation of the Feynman – Kac formula. Analytical properties, examples and applications will be dealt with elsewhere. For a more detailed discussion of the results expounded here, and their relations to previous papers, we refer to [3].

2.

The context in which we shall discuss the algebraic (i.e. L^∞) quantum Feynman – Kac formula is defined by:

– a family of local algebras $\{\mathscr{A}, (\mathscr{A}_I)_{I\in\mathscr{F}}\}$ on $\boldsymbol{R}$ (respectively $\boldsymbol{R}^+$),

– a projective family of conditional expectations (E_I),

– a one-parameter automorphism (respectively endomorphism) group (u_t) on $\mathscr{A}$.

$\mathscr{F}$ is a family of subsets of $\boldsymbol{R}$ (respectively $\boldsymbol{R}^+$) containing the finite sets and the intervals (open, half-open, closed, bounded or not); the algebras $\mathscr{A}_I$ are C^*-algebras satisfying the locality conditions

$$I \subseteq J \Rightarrow \mathscr{A}_I \subseteq \mathscr{A}_J$$

$$\mathscr{A} = \text{norm closure of } \bigcup\{\mathscr{A}_J : J \in \mathscr{F}\}.$$

For each $I \in \mathscr{F}$, $E_I : \mathscr{A} \to \mathscr{A}_I$ is a conditional expectation, i.e. a norm one projection, and the projectivity condition means that

$$I \subseteq J \Rightarrow E_I E_J = E_I.$$

The action of (u_t) on $\mathscr{A}$ is covariant in the sense that

$$u_t \mathscr{A}_I = \mathscr{A}_{I+t}; \qquad I \in \mathscr{F},\ t \in \boldsymbol{R} \text{ (respectively } \boldsymbol{R}^+)$$

and the system of conditional expectations (E_I) satisfies the covariance condition

$$u_t^* E_{I+t} = E_I; \qquad I \in \mathscr{F},\ t \in \boldsymbol{R} \text{ (respectively } \boldsymbol{R}^+),$$

where $u_t^* = u_{-t}$ is the inverse of u_t (left-inverse if the index set is $\boldsymbol{R}^+$).

We adopt the convection that, if the index set is $\boldsymbol{R}^+$, symbols like $E_{]-\infty,t]}, E_{]-\infty,0]}, \mathscr{A}_{]-\infty,t]} \cdots$ stand respectively for $E_{[0,t]}, E_{\{0\}}, \mathscr{A}_{[0,t]} \cdots$.

The family (E_I) is supposed to be Markovian; more precisely, we shall only use the property

$$E_{]-\infty,t]}(\mathscr{A}_{[t,+\infty[}) \subseteq \mathscr{A}_t, \qquad t \geqslant 0.$$

The Markov property implies that

$$P_0^t = E_{]-\infty,0]} u_t \restriction \mathscr{A}_0, \qquad t > 0$$

maps $\mathscr{A}_0$ into itself. Moreover, if $s, t > 0$

$$P_0^t P_0^s = E_{]-\infty,0]} u_t E_{]-\infty,0]} u_s = E_{]-\infty,0]} E_{]-\infty,t]} u_{s+t} =$$

$$= E_{]-\infty,0]} u_{s+t} = P^{s+t}.$$

Thus (P_0^t) is a semi-group acting on $\mathscr{A}_0$. Clearly (P_0^t) is completely positive and identity preserving. A completely positive identity preserving semi-group will be called a Markovian semi-group.

Remark that the construction of P_0^t does not make use of the fact that the $E_{]-\infty,t]}$ are *projection* operators; any projective, covariant, Markovian family of completely positive maps (in particular of quasi-conditional expectations) will do.

The idea of the Feynman – Kac formula is to introduce perturbations of the semi-group P_0^t by means of local perturbations of the conditional expectations (E_I). We will be interested in perturbations which preserve the complete positivity of the semi-group, but not necessarily the property of being identity preserving.

A completely positive map $\widetilde{M}_t \colon \mathscr{A} \to \mathscr{A}$ is said to be localized in $\mathscr{A}_{]0,t]}$ if

$$\widetilde{M}_t(\mathscr{A}_{[0,t]}) \subseteq \mathscr{A}_{[0,t]}, \qquad t > 0$$

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$$\widetilde{M}_t = E_{[0,t]}\widetilde{M}_t E_{[0,t]}.$$

If $(\widetilde{M}_t)_{t>0}$ is a family of such maps, the Markov property implies that, for each $t > 0$, the completely positive operator

$$P^t = E_{]-\infty,0]}\widetilde{M}_t u_t \restriction \mathscr{A}_0 \tag{1}$$

maps $\mathscr{A}_0$ into itself. Moreover, for each $a_0 \in \mathscr{A}_0$

$$\begin{aligned} P^t P^s a_0 &= E_{]-\infty,0]}\widetilde{M}_t u_t E_{]-\infty,0]}\widetilde{M}_s u_s a_0 = \\ &= E_{]-\infty,0]}\widetilde{M}_t E_{]-\infty,t]}(u_t\widetilde{M}_s u_t^*)u_{t+s}a_0 = \\ &= E_{]-\infty,0]}E_{]-\infty,t]}\widetilde{M}_t(u_t\widetilde{M}_s u_t^*)u_{t+s}a_0 = \\ &= E_{]-\infty,0]}\widetilde{M}_t(u_t\widetilde{M}_s u_t^*)u_{t+s}a_0; \end{aligned}$$

in fact $u_t\widetilde{M}_s u_s a_0 \in u_t\mathscr{A}_{[0,s]} = \mathscr{A}_{[t,t+s]}$, and therefore

$$\begin{aligned} \widetilde{M}_t E_{]-\infty,t]}(u_t\widetilde{M}_s u_t^*)u_{t+s}a_0 &= \\ &= \widetilde{M}_t E_{[0,t]}(u_t\widetilde{M}_s u_t^*)u_{t+s}a_0 = \\ &= E_{[0,t]}\widetilde{M}_t(u_t\widetilde{M}_s u_t^*)u_{t+s}a_0 = \\ &= E_{]-\infty,t]}\widetilde{M}_t(u_t\widetilde{M}_s u_t^*)u_{t+s}a_0 \end{aligned}$$

Thus the condition

$$\widetilde{M}_{t+s} = \widetilde{M}_t(u_t\widetilde{M}_s u_t^*) \tag{2}$$

is sufficient for the 1-parameter family (P^t), defined by (1) to be a semigroup.

Denoting δ_0 (respectively δ) the formal generator of P_0^t (respectively P^t) one has, for each $a_0 \in \mathscr{A}_0$

$$\begin{aligned} P^t a_0 - a_0 &= E_{]-\infty,0]}\widetilde{M}_t u_t a_0 - a_0 = \\ &= (E_{]-\infty,0]}u_t a_0 - a_0) + E_{]-\infty,0]}([\widetilde{M}_t - 1]u_t a_0) = \\ &= (P_0^t a_0 - a_0) + E_{]-\infty,0]}([\widetilde{M}_t - 1]u_t a_0) \end{aligned}$$

which implies the formal identity

$$\delta = \delta_0 + \widetilde{A}_0$$

where, formally,

(3) $$\widetilde{A}_0(a_0) = \frac{d}{dt}\Big|_{t=0} \widetilde{M}_t(a_0) = \lim_{t \downarrow 0} \{\widetilde{M}_t a_0 - a_0\}.$$

There are several analytical conditions which make the above formal argument a rigorous one. We shall not discuss them here.

A simple example of completely positive map $\widetilde{M}_t\colon \mathcal{A} \to \mathcal{A}$ localized on $\mathcal{A}_{[0,t]}$ is given by

(4) $$\widetilde{M}_t a = M_t a M_t^*, \qquad a \in \mathcal{A}$$

where

(5) $$M_t \in \mathcal{A}_{[0,t]}.$$

If $(M_t)_{t>0}$ is a family of such maps, satisfying

(6) $$M_{t+s} = M_t u_t(M_s),$$

then the family $(\widetilde{M}_t)$ defined by (4) satisfies (2) and therefore P^t, defined by (1), is a semi-group.

A family $(M_t)_{t>0}$ satisfying (5) and (6) will be called a *Markovian cocycle.*

Remark. Unbounded Markovian cocycles can be easily handled when each $\mathcal{A}_{[0,t]}$ is a W^*-algebra acting on some Hilbert space $\mathcal{H}$ (independent of t). We shall discuss this case in the following paragraph.

In the classical case a Markovian cocycle is just a covariant multiplicative functional of the process. In the quantum case one can easily construct examples of Markovian cocycles using time-ordered exponentials but, even for the simplest known families of local algebras, a detailed analysis of the structure of the associated Markovian cocycles is still lacking. In the perturbation $\widetilde{M}_t$ is given by (4), then the corresponding operator $\widetilde{A}_0$, defined by (3), has the form

$\widetilde{A}_0(a_0) =$

where

$A_0 = \frac{d}{dt}\Big|$

Thus, for example,

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Denote $\mathcal{H}, \pi, 1_\varphi$
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Moreover, for each
that

$$\widetilde{A}_0(a_0) = A_0 a_0 + a_0 A_0^*$$

where

$$A_0 = \frac{d}{dt}\Big|_{t=0} M_t = \lim_{t \downarrow 0} \frac{1}{t}\{M_t - 1\}.$$

Thus, for example, if M_t is a Hermitian Markovian cocycle of the form

$$M_t = e^{-\int_0^t V_s ds} \quad ; \quad V_s = u_s(V_0); \quad V_0 \in \mathscr{A}_0; \quad V_0 = V_0^*$$

the formal generator of P^t is

$$\delta = \delta_0 - \{V, \cdot\}$$

where $\{V_0, a\} = V_0 a + a V_0$ denotes the anti-commutator, while, if M_t is a unitary Markovian cocycle of the form

$$M_t = T(e^{-\int_0^t V_s ds})$$

where T denotes the time-ordered exponential and V_s is as above, then the formal generator of P^t is

$$\delta = \delta_0 + i[V_0, \cdot]$$

where $[V_0, a] = V_0 a - a V_0$ denotes the commutator.

3.

Assume now that on the algebra $\mathscr{A}$ a state φ is given, with the properties

$$\varphi = \varphi \circ u_t, \quad \forall t \in \boldsymbol{R}; \quad \varphi = \varphi \circ E_I, \quad \forall I \in \mathscr{F}.$$

Denote $\mathscr{H}, \pi, 1_\varphi$ the GNS triple associated to $\mathscr{A}$ and φ; $\mathscr{H}_I = [\pi(\mathscr{A}_I) 1_\varphi]$ the norm closure in $\mathscr{H}$ of $\pi(\mathscr{A}_I) 1_\varphi$ $(I \in \mathscr{F})$; and e_I: $\mathscr{H} \to \mathscr{H}_I$ the corresponding orthogonal projection. One has

$$e_I \pi(a) 1_\varphi = \pi(E_I(a)) 1_\varphi, \qquad a \in \mathscr{A}.$$

Moreover, for each u_t there exists a unitary operator U_t: $\mathscr{H} \to \mathscr{H}$ such that

$$U_t \pi(a) 1_\varphi = \pi(u_t(a)) 1_\varphi .$$

One has

$$U_t e_I U_t^* = e_{I+t}; \quad I \subseteq J \Rightarrow e_I e_J = e_I$$

and the Markov property implies that

$$e_{]-\infty,t]} \mathscr{H}_{[t,+\infty[} \subseteq \mathscr{H}_{\{t\}} \Leftrightarrow e_{]-\infty,t]} e_{[t,+\infty[} = e_{\{t\}}.$$

The above listed properties imply that

$$P_0^t = e_{\{0\}} U_t \restriction \mathscr{H}_{\{0\}}$$

is a semi-group on $\mathscr{H}_{\{0\}}$ whose adjoint is

$$P_0^{t*} = e_{\{0\}} U_t^* \restriction \mathscr{H}_{\{0\}}.$$

The semi-group P_0^t is self-adjoint if and only if the process $\{\mathscr{A}, (\mathscr{A}_I), \varphi\}$ has the following weak time-reflection property

(7) $$\varphi(a_0 u_t(b_0)) = \varphi(u_t(a_0) b_0), \quad \forall a_0, b_0 \in \mathscr{A}_0.$$

In particular (7) is always true if the process admits a time reflection, i.e. if there is an automorphism $r\colon \mathscr{A} \to \mathscr{A}$ such that

$$r \restriction \mathscr{A}_{\{0\}} = \mathrm{id}; \quad r u_t = u_{-t} r; \quad \varphi r = \varphi$$

(if the index set is $\boldsymbol{R}^+$, one has to introduce a 1-parameter family of time reflections $r_t\colon \mathscr{A}_{[0,t]} \to \mathscr{A}_{[0,t]}$). More generally, if $R\colon \mathscr{H} \to \mathscr{H}$ is any operator such that

(8) $$R e_{\{0\}} = e_{\{0\}},$$

(9) $$R^* U_t R = U_t^*,$$

then P_0^t is self-adjoint. Similarly one can find criteria for the unitarily of P_0^t.

L^2 Markovian cocycles are defined, in analogy with the L^∞ case, by the properties

(10) $$M_t(\mathscr{H}_{[0,t]}) \subseteq \mathscr{H}_{[0,t]}, \qquad t > 0$$

(11) $M_{t+s} = M$

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(12) $P^t = e_{\{0\}}$

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(11) $\qquad M_{t+s} = M_t (U_t M_s U_t^*)$

where M_t is an operator on $\mathscr{H}$ (if M_t is unbounded, the equalities (10), (11) shall be considered on a dense set). The 1-parameter family

(12) $\qquad P^t = e_{\{0\}} M_t U_t \restriction \mathscr{H}_{\{0\}}$

is then a semi-group on $\mathscr{H}_{\{0\}}$ whose adjoint is

$$P^{t*} = e_{\{0\}} U_t^* M_t^* e_{\{0\}}.$$

In particular, if $R\colon \mathscr{H} \to \mathscr{H}$ is a linear operator satisfying (8), (9) and

(13) $\qquad R M_t^* R = U_t^* M_t U_t$

then P^t is self-adjoint. The semi-group P_0^t is positive in the sense that it maps the positive cone $\mathscr{H}_+$ (= closure in $\mathscr{H}_{\{0\}}$ of $\pi(\mathscr{A}_0^+) 1_\varphi$) into itself but, in general, not in the Hilbert space sense. The semi-group P^t will usually not have even that kind of positivity. If, however, there is a linear involution $R\colon \mathscr{H} \to \mathscr{H}$ satisfying (8), (9), (13) and

(14) $\qquad R e_I = e_{-I} R; \qquad I$ interval $\subseteq \boldsymbol{R}$

then P^t (and, a fortiori, P_0^t) is positive in the Hilbert space sense. In fact, if $\xi \in \mathscr{H}_{\{0\}}$, then

$$\begin{aligned} \langle \xi, P^t \xi \rangle = \langle \xi, M_t U_t \xi \rangle &= \langle \xi, M_{\frac{t}{2}} (U_{\frac{t}{2}} M_{\frac{t}{2}} U_{\frac{t}{2}}^*) U_t \xi \rangle = \\ &= \langle U_{\frac{t}{2}}^* \xi, (U_{\frac{t}{2}}^* M_{\frac{t}{2}} U_{\frac{t}{2}}) M_{\frac{t}{2}} U_{\frac{t}{2}} \xi \rangle = \\ &= \langle R^* U_{\frac{t}{2}} R \xi, R M_{\frac{t}{2}}^* R (M_{\frac{t}{2}} U_{\frac{t}{2}}) \xi \rangle = \\ &= \langle M_{\frac{t}{2}} U_{\frac{t}{2}} \xi, R M_{\frac{t}{2}} U_{\frac{t}{2}} \xi \rangle = \| P^{\frac{t}{2}} \xi \|^2 \end{aligned}$$

the last equality being due to the fact that

$$\begin{aligned} e_{[0,\frac{t}{2}]} R e_{[0,\frac{t}{2}]} &= R e_{[-\frac{t}{2},0]} e_{[0,\frac{t}{2}]} \equiv \\ &= R e_{[-\frac{t}{2},0]} e_{]-\infty,0]} e_{[0,\frac{t}{2}]} = R e_{\{0\}} = e_{\{0\}}. \end{aligned}$$

If $r\colon \mathscr{A} \to \mathscr{A}$ is an automorphism (or anti-automorphism) such that

(i) $ru_t = u_{-t}r$,

(ii) $r \upharpoonright \mathscr{A}_0 = \mathrm{id}$,

(iii) $r\mathscr{A}_I = \mathscr{A}_{-I}$ $(I \subseteq R)$,

(iv) $r^2 = \mathrm{id}$,

(v) $\varphi r = \varphi$

(i.e. a time reversal), then an operator $R\colon \mathscr{H} \to \mathscr{H}$ satisfying (8), (9), (14), can be defined by

$$R\pi(a)1_\varphi = \pi(r(a))1_\varphi, \qquad a \in \mathscr{A}.$$

The computation of the formal generator H of P^t is carried out in analogy with the L^∞ case, and yields $H = H_0 + V_0$, where H_0 is the generator of P_0^t, and

$$V_0 = \frac{d}{dt}\Big|_{t=0} M_t .$$

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