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Stopping Times for Quantum Markov Chains

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In the paper we introduce stopping times for quantum Markov states. We study algebras and maps corresponding to stopping times, give a condition of strong Markov property and give classification of projections for the property of accessibility. Our main result is a new recurrence criterium in terms of stopping times (Theorem 1 and Corollary 2). As an application of the criterium we study how, in Section 6, the quantum Markov chain associated with the one-dimensional Heisenberg (usually non-Markovian) process, obtained from this quantum Markov chain by restriction to a diagonal subalgebra, is such that all its states are recurrent. We were not able to obtain this result from the known recurrence criteria of classical probability.

KEY WORDS: Von Neumann algebra; quantum Markov chain; strong Markov property; recurrence; Heisenberg potential.

1. NOTATIONS

Stopping times in Fock spaces have been considered by several authors.[3, 5–8] In a more general context they have been studied in Ref. 4 and, under the assumption that the relevant subalgebras are expected, in Ref. 9.

In classical probability stopping times play a major role in the solution of problems like the recurrence of states, first exit times, etc. It is therefore natural to expect that quantum stopping times could be used to study the analogue problems in quantum probability.

The recurrence problem for quantum Markov chains was studied in Ref. 2 using the quantum analogue of the usual potential theory for classical Markov chains.

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In the present paper we prove a recurrence criterium for quantum Markov chains using stopping times techniques. The advantage of the present criterium over the previous one is that it is much easier to check. We illustrate this statement by proving a general recurrence result on the quantum Markov chains associated to the Heisenberg potential, which does not seem to be easily obtained with direct methods.

In the present paper we use the notations of Ref. 2, which we recall briefly.

Let $\mathscr{B}$ be the algebra of all bounded operators on some complex Hilbert space $\mathscr{H}$ (or some von Neumann subalgebra of it). Let $\mathscr{A} := \bigotimes_N \mathscr{B}$ be the tensor product of countably many copies of $\mathscr{B}$ and

$$j_n \colon \mathscr{B} \to \bigotimes_N \mathscr{B} = \mathscr{A} \tag{1.1}$$

the natural embedding of $\mathscr{B}$ onto the nth factor of $\mathscr{A}$. In the following, in order to simplify the notations, we shall assume that the algebra $\mathscr{A}$ acts on the space $\mathscr{H}$ of its GNS representation with respect to the state ϕ; the cyclic vector, corresponding to ϕ, shall be denoted Φ and $\mathscr{A}''$ shall denote the von Neumann algebra generated by $\mathscr{A}$. Let ϕ be a locally normal faithful state on $\mathscr{A}$ and ϕ_0 a normal faithful state on $\mathscr{B}$.

Definition 1. A state ϕ is called a *homogeneous quantum Markov chain* (QMC)[2] if

$$\phi(j_1(a_1) \otimes \cdots \otimes j_n(a_n)) = \phi_0(\mathscr{E}(a_1 \otimes \mathscr{E}(a_2 \otimes \cdots \otimes \mathscr{E}(a_n \otimes \mathbf{1}) \cdots) \tag{1.2}$$

where $\mathscr{E} \colon \mathscr{B} \otimes \mathscr{B} \to \mathscr{B}$ is a *transition expectation*, i.e., a completely positive map satisfying the condition $\mathscr{E}(\mathbf{1} \otimes \mathbf{1}) = 1$ and ϕ_0 is called the *initial state*.

To every transition expectation we associate two Markovian operators (i.e., completely positive identity preserving maps of $\mathscr{B}$ into itself):

$$P(a) = \mathscr{E}(\mathbf{1} \otimes a) \qquad \text{(backward transition operator)} \tag{1.3}$$

$$T(a) = \mathscr{E}(a \otimes \mathbf{1}) \qquad \text{(forward transition operator)} \tag{1.4}$$

As shown in Ref. 2, in the classical case T is the identity operator and P coincides with the usual Markov transition operator.

In the following we shall denote $\phi \equiv (\mathscr{E}, \phi_0)$ the quantum Markov chain defined by Eq. (1.2).

We shall denote $\mathscr{A}_{n]} := (\mathscr{B})^{\otimes n+1}$ the subalgebra of $\mathscr{A}$, generated by the first $(n+1)$ factors, i.e., by the elements of the form

$$a_{n]} := a_0 \otimes \cdots \otimes a_n \otimes 1_{[n+1} = j_0(a_0)\, j_1(a_1) \cdots j_n(a_n)$$

with $a_0, a_1, \ldots, a_n \in \mathscr{B}$.

It is known from Ref. 1 that for each $n \in \mathbb{N}$, there exists a unique completely positive identity preserving map $E_{n]}: \mathscr{A} \mapsto \mathscr{A}_{n]}$ characterized by the property

$$E_{n]}(a_{m]}) = a_0 \otimes \cdots \otimes a_{n-1} \otimes \mathscr{E}(a_n \otimes \mathscr{E}(a_{n+1} \otimes \cdots \otimes \mathscr{E}(a_m \otimes 1) \cdots)) \qquad \forall m > n \tag{1.5}$$

Similarly, for any $m < n$ there exists an operator $E_{n,m}: \mathscr{A}_n \mapsto \mathscr{A}_m$ characterized by the property

$$E_{n,m}(b_0 \otimes b_1 \otimes \cdots \otimes b_n) := b_0 \otimes \cdots \otimes b_{m-1} \otimes \mathscr{E}(b_m \otimes \mathscr{E}(b_{m+1} \otimes \cdots \otimes \mathscr{E}(b_{n-1} \otimes b_n) \cdots)$$

It is easily seen that

$$E_{m,h} \cdot E_{n,m} = E_{n,h} \qquad \text{for} \quad h < m < n \tag{1.6}$$

2. STOPPING TIMES

Definition 2. A (discrete) *stopping time* associated with e is a sequence $\{\tau_k\}_{k \geq 0}$ with the following properties:

(i) $\tau_k \in \mathscr{A}_{k]} \qquad \forall k \geq 0$.

(ii) τ_k is a projection of $\mathscr{A} \qquad \forall k \geq 0$.

(iii) The τ_k are mutually orthogonal.

One can canonically associate a stopping time to any projection $e \in \mathscr{B}$ defining:

$$\tau_0 = e \otimes 1_{[1} = j_0(e)$$

$$\tau_1 = e^{\perp} \otimes e \otimes 1_{[2} = j_0(e^{\perp}) \, j_1(e)$$

$$\tau_k = (e^{\perp})^{\otimes k} \otimes e \otimes 1_{[k+1} = j_0(e^{\perp}) \cdots j_{k-1}(e^{\perp}) \, j_k(e) \tag{2.1}$$

Identifying the projection e to an event E and the index $n \in \mathbb{N}$ to a discrete time, the projection τ_k corresponds to the fact that the event E happens for the first time at the instant k. It is easy to see that $(\tau_k)_{k \geq 0}$ is a stopping time in the sense of Definition 2.

Denote

$$\tau_\infty^n := (e^{\perp})^{\otimes (n+1)} \otimes 1_{[n+1} \tag{2.2}$$

We shall identify $\mathscr{B}$ to the subalgebra $j_0(\mathscr{B})$ of $\mathscr{A}''$. Since the sequence of projections (τ_∞^n) is decreasing, its strong limit exists in $\mathscr{A}''$ and we shall denote it

$$\tau_\infty := \lim_{n\to\infty} \tau_\infty^n = \bigotimes_{N} e^\perp \tag{2.2a}$$

where the last equality in (2.2a) should be considered as a suggestive notation.

The projection τ_∞^n corresponds to the fact that the event E does not take place in the first n instants, and the projection τ_∞ to the fact that the event E never takes place. Keeping this interpretation in mind, we introduce the following:

Definition 3. A projection e is called *completely accessible* if

$$E_{0]}(\tau_\infty) := \lim_{n\to\infty} E_{0]}(\tau_\infty^n) = 0 \tag{2.3}$$

Notice that the limit in (2.3) always exists in the strong topology on $\mathscr{A}''$, being a decreasing sequence of positive operators.

Proposition 1. In the above notations:

$$\sum_{k\geqslant 0} \tau_k = 1_{\mathscr{A}} - \tau_\infty \tag{2.4}$$

where the sum is meant in the strong topology in $\mathscr{A}''$. Moreover, a projection e is completely accessible if and only if

$$E_{0]}\left(\sum_{k\geqslant 0} \tau_k\right) = 1_{\mathscr{B}} \tag{2.5}$$

Proof. We have

$$\begin{aligned}\tau_0 + \tau_1 &= e\otimes 1_{[1} + e^\perp \otimes e \otimes 1_{[2} = e\otimes 1_{[1} + e^\perp \otimes (1 - e^\perp) \otimes 1_{[2}\\ &= 1_{\mathscr{A}} - e^\perp \otimes e^\perp \otimes 1_{[2}\end{aligned}$$

Continuing this procedure, we have

$$\tau_0 + \cdots + \tau_k = 1_{\mathscr{A}} - (e^\perp)^{k+1} \otimes 1_{[k+1} = 1_{\mathscr{A}} - \tau_\infty^n \qquad \forall k \geqslant 0 \tag{2.5a}$$

So taking the limit for $k\to\infty$, we get (2.4). The equivalence between (2.3) and (2.5) follows now easily by taking the $\mathscr{E}_{0]}$ expectation of both sides of (2.5a) then the limit for $k\to\infty$. □

3. ALGEBRAS ASSOCIATED WITH A STOPPING TIME

In this section we suppose e to be completely accessible. Define the von Neumann subalgebras of $\mathscr{A}$:

$$\tau' := \{a \in \mathscr{A} : a\tau_k = \tau_k a,\ \forall k \geqslant 0\}$$

$$\mathscr{A}_{\tau]} := \{a \in \tau' : a\tau_k \in \mathscr{A}_{k+1]},\ \forall k \geqslant 0\}$$

The algebra $\mathscr{A}_{\tau]}$ introduced by J.-L. Sauvageot in Ref. 9 is the discrete version of the "pre-τ von Neumann algebra" by R. Hudson.[8]

Proposition 2. For any $a \in \tau'$,

$$E_{0]}\left(\sum_{k \geqslant 0} \tau_k a\right) = E_{0]}(a) \tag{3.1}$$

if and only if e is completely accessible.

Proof. It is sufficient to prove (3.1) for a positive a. By Proposition 1 one finds

$$\sum_{k \geqslant 0} \tau_k a = a - \tau_\infty a \tag{3.2}$$

Since τ_∞ is a projection in $\mathscr{A}''$ and $\tau' \subseteq \mathscr{A}'$ = the commutant of $\mathscr{A}$, it follows, using (2.3), that

$$E_{0]}(\tau_\infty a) = E_{0]}(\tau_\infty a \tau_\infty) \leqslant \|a\|\, E_{0]}(\tau_\infty) = 0 \tag{3.2a}$$

Hence, taking $E_{0]}$ of both sides of (3.2), we get (3.1).

Conversely, if (3.1) takes place, then (3.2a) takes place for any τ'. Taking $a = 1$, we have that e is completely accessible.

Introduce for all $a \in \tau'$:

$$E_{\tau]}(a) := \sum_{n \geqslant 0} E_{n+1]}(a) \cdot \tau_n = \sum_{n \geqslant 0} E_{n+1]}(a\tau_n) \tag{3.3}$$

where, in the last equality, we have used the construction of τ'. $E_{\tau]}$ is clearly a completely positive map $E_{\tau]} : \tau' \mapsto \mathscr{A}_{\tau]}$. □

Proposition 3. The projection $\sigma_\tau := 1_{\mathscr{A}} - \tau_\infty \subset \mathrm{Center}(\mathscr{A}_{\tau]})$ has the following properties: $E_{\tau]}(1_{\mathscr{A}}) = \sigma_\tau$ and for all $a \in \tau'$

$$E_{\tau]}(a)\sigma_\tau = \sigma_\tau E_{\tau]}(a) = E_{\tau]}(a) \tag{3.4}$$

Proof. The first statement of (3.4) follows directly from (3.3). By orthogonality of $\{\tau_n\}_{n\geqslant 0}$ we have

$$E_{\tau]}(a)\sigma_\tau = E_{\tau]}(a)\, E_{\tau]}(1_{\mathscr{A}}) = \sum_{n\geqslant 0} E_{n+1]}(a)\tau_n \sum_{n\geqslant 0} \tau_n = E_{\tau]}(a)$$

which proves (3.4). □

4. STRONG MARKOV PROPERTY

Let us first remind that a quantum Markov chain $(\mathscr{E}, \phi_0)$ has the following (quantum) Markov property:

$$E_{n]}(\mathscr{A}_{[n}) \subseteq \mathscr{A}_n \tag{4.1}$$

which in the classical case [that is $\mathscr{B}$ is Abelian and also $\mathscr{E}(a\otimes b) = a\cdot P(b)$] coincides with the usual classical Markov property.

In order to generalize (4.1) for the case of local times, let us introduce

$$\mathscr{A}_{[\tau} := \{a \in \tau' : a\tau_k \in \tau_k \mathscr{A}_{[k+1}, \ \forall k \geqslant 0\}$$

and

$$\mathscr{A}_\tau := \{a \in \tau' : a\tau_k \in \tau_k \mathscr{A}_{k+1}, \ \forall k \geqslant 0\}$$

Lemma 1. If e is completely accessible, then in the previous notations

$$E_{\tau]}(\mathscr{A}_{[\tau}) \subseteq \mathscr{A}_\tau \tag{4.2}$$

Proof. It is clear that $E_{\tau]}(\mathscr{A}_{[\tau}) \subseteq \tau'$. Now let $a \in \mathscr{A}_{[\tau}$, then we have for each $k \geqslant 0$

$$\tau_k \sum_{n\geqslant 0} E_{n+1]}(a\tau_n) = \tau_k E_{k+1]}(a\tau_k) = E_{k+1]}(a\tau_k) \in \tau_k \mathscr{A}_{k+1}$$

5. CLASSIFICATION OF PROJECTIONS

Lemma 2. Let $S: \mathscr{A} \to \mathscr{B}$ be a completely positive normalized map $[S(1) = 1]$. There exist $0 \leqslant x = x^* \in \mathscr{A}_n$ such that $S(x) = 0$ if and only if its support e satisfies $S(e) = 0$.

Proof. Let e be the support of x and suppose that $S(e) = 0$. Then from the Kadison–Schwarz inequality

$$S(y)^*\, S(y) \leqslant S(y^* y) \tag{5.1}$$

we have that

$$|S(x)|^2 = |S(xe)|^2 = S(xe)^* S(xe) \leqslant S(ex^*xe) \leqslant \|x\|^2 S(e) = 0$$

Conversely, if $x \geqslant \lambda e$, then clearly $S(x)=0$ implies $S(e)=0$. The thesis then follows by a density argument. □

Corollary 1. For S and e as in Lemma 2, one has $S(e)=0$ if and only if $S(xe)=S(ex)=0 \quad \forall x \in \mathscr{A}_n$.

Proof. If $S(e)=0$, then for any $x \in \mathscr{A}$, $S(xe)=0$ by (5.1). The converse is clear.

Definition 4. A projection e is called *$\mathscr{E}$-recurrent* (or simply *recurrent*)[(2)] if

$$f(e) := \frac{1}{\nu_e} \operatorname{Tr}\left(E_{0]}\left[\sum_{n \geqslant 0} e \otimes \left(\otimes\, e^{\perp}\right)^n \otimes e\right]\right) = 1 \tag{5.2}$$

where $\nu_e := \operatorname{Tr}(\mathscr{E}(e \otimes 1))$ and assume the condition

$$0 < \operatorname{Tr}(\mathscr{E}(e \otimes 1)) < +\infty$$

Definition 5. A projection e is called *$\mathscr{E}$-n-transient* (or simply *n-transient*) if there exist $n \geqslant 0$ such that

$$E_{0]}(\tau_k) = 0 \qquad \forall k \geqslant n+1 \qquad \text{and} \qquad E_{0]}(\tau_n) > 0 \tag{5.3}$$

Definition 6. A transition expectation $\mathscr{E}$ on $\mathscr{B}$ is called *reducible* if there exist a projection $\rho \in \mathscr{A}$ such that

$$E_{0]}(\rho a \rho) = E_{0]}(a) \qquad \forall a \in \mathscr{A} \tag{5.4}$$

Otherwise the chain is called *irreducible.*

Definition 7. Let $e, f \in \operatorname{Proj}(\mathscr{B})$, $e, f \neq 0$. We say that f is *accessible* from e (and write $e \to f$) if there exists $n \in \mathbf{N}$ such that

$$\mathscr{E}(e \otimes P^n T f) = E_{0]}(e \otimes 1_{n-1]} \otimes f) \neq 0 \tag{5.5}$$

If $e \to f$ and $f \to e$, then we say that e and f *communicate* and write $e \leftrightarrow f$. We say that a projection e is *unessential* if there exist a projection f such that

$$e \to f \qquad \text{and} \qquad \mathscr{E}(f \otimes P^n T e) = 0 \qquad \forall n \geqslant 0$$

Otherwise e is called *essential.*

Definition 8. Fix a partition of the identity $I=(e_i)_i$ ($\sum e_i=1$; $e_ie_j=\delta_{ij}$) and denote $I^0=\{\text{all unessential projections of } I\}$ and $I^+=I\backslash I^0$. The projections

$$I^u := \sum_{e_i\in I^0} e_i \qquad \text{and} \qquad I^e := \sum_{e_i\in I^+} e_i$$

will be called the *unessential part* of I and the *essential part* of I correspondingly. For any $e\in I^+$ the projection

$$f := \sum_{\{e_j\in I^+ \text{ and } e_j\leftrightarrow e\}} e_j$$

will be called *maximal projection* communicated with e.

Proposition 4. The relation $\leftrightarrow$ of communication divides the projection I^e into jointly orthogonal subprojections $e_1^+, e_2^+, e_3^+,...$ such that for any $e\leqslant e_i^+, f\leqslant e_j^+$ $(e, f\in I^+)$ one has

$$\begin{aligned} & e\leftrightarrow f \qquad \text{if} \quad i=j \\ e\not\leftrightarrow f \qquad \text{and} \qquad & f\not\leftrightarrow e \qquad \text{if} \quad i\neq j \end{aligned} \tag{5.6}$$

Proof. Choose some $e_{k_1}\in I^+$ and consider e'_{k_1} – a maximal projection communicating with e_{k_1} (that is the sum of all those projections from I^+). Put $e_1^+ := e'_{k_1}$. Choosing some $e_{k_2}\leqslant I^e-e_1^+$ put $e_2^+ := e'_{k_2}$. Choosing some $e_{k_3}\leqslant I^e-e_1^+-e_2^+$ put $e_3^+ := e'_{k_3}$ and so on.

For any $f, g\in I^+$ if $e\leftrightarrow f$ $(e, f\in I)$ then $f\leqslant e'$, and therefore $f'=e'$, which implies (5.6). The orthogonality of e_i^+ is clear.

Remark. In the classical case [i.e., $\mathscr{B}$ is Abelian and $\mathscr{E}(a\otimes b)=a\cdot P(b)$] the condition (5.5) becomes

$$\operatorname{supp} P^n(f)=1 \qquad \forall f\in \operatorname{Proj}(\mathscr{B}) \qquad f\neq 0 \qquad \forall n\geqslant 1 \tag{5.7}$$

because

$$\mathscr{E}(e\otimes P^{n-1}f)=eP^nf\neq 0 \qquad \forall e, f\neq 0$$

if and only if (5.6) takes place.

Remark. Notice the obvious implication:

$$\{\text{all states of } \mathscr{E} \text{ communicate}\} \Rightarrow \{\mathscr{E} \text{ is reducible}\}$$

Theorem 1.

(i) $\mathscr{E}(e \otimes 1) = 0$ if and only if $E_{0]}(\tau_\infty) = 1$.

(ii) If $\mathscr{E}$ is irreducible then no e can be n-transient.

(iii) e is $\mathscr{E}$-recurrent if and only if

$$\mathscr{E}(e \otimes E_{0]}(\tau_\infty)) = 0 \tag{5.8}$$

In particular, if e is completely accessible then e is $\mathscr{E}$-recurrent.

(iv) If $\mathscr{E}$ is faithful then e is completely accessible if and only if e is $\mathscr{E}$-recurrent

(v) If all projections of $\mathscr{B}$ communicate and e is $\mathscr{E}$-recurrent then e is completely accessible.

Proof.

(i) Let $\mathscr{E}(e \otimes 1) = 0$. Then, by definition of τ_k, $E_{0]}(\tau_n) = 0$, $\forall n \geqslant 0$. So, taking $E_{0]}$ of both sides of (2.5*a*), we have $E_{0]}(\tau_\infty) = 1$.

Conversely, if $E_{0]}(\tau_\infty) = 1$, then by (2.5*a*) $E_{0]}(\tau_n) = 0$, $\forall n \geqslant 0$, so $E_{0]}(\tau_0) = 0 = \mathscr{E}(e \otimes 1)$.

(ii) By (2.5*a*) and (5.3) we have

$$1 - E_{0]}(\tau_\infty) = E_{0]}\left(\sum_{k \geqslant 0} \tau_k\right) = E_{0]}\left(\sum_{k \leqslant n} \tau_k\right) = E_{0]}(1_{\mathscr{A}} - \tau_\infty^n)$$

so

$$E_{0]}(\tau_\infty) = E_{0]}(\tau_\infty^n) \tag{5.9}$$

and by the obvious relation

$$\tau_\infty^n + \tau_n = \tau_\infty^{n-1}$$

we have

$$E_{0]}(\tau_\infty^n) = E_{0]}(\tau_\infty^{n-1}) - E_{0]}(\tau_n) \tag{5.10}$$

If e is n-transient, then, by definition of τ_n and τ_∞^{n-1}, we have

$$E_{0]}((e^\perp)^{\otimes n} \otimes e) = E_{0]}((e^\perp)^{\otimes n} \otimes 1)$$

so $\mathscr{E}$ is reducible by the Corollary 1 of Lemma 2.

(iii) We shall use the following necessary and sufficient condition for e to be $\mathscr{E}$-recurrent (Proposition 2.1 of Ref. 2):

$$E_{0]} \sum_{n \geqslant 0} \left(e \otimes \left(\bigotimes e^\perp\right)^n \otimes e\right) = \mathscr{E}(e \otimes 1) \tag{5.11}$$

which can be rewritten as

$$E_{0]}\left(e\otimes\sum_{n\geqslant 0}\tau_n\right)=E_{0]}(e) \tag{5.12}$$

and by (2.4) as (5.8).

(iv) Since $\mathscr{E}$ is faithful, (5.12) becomes

$$E_{0]}\left(\sum_{n\geqslant 0}\tau_n\right)=1 \tag{5.13}$$

which by (2.5) means that e is completely accessible.

The converse is clear.

(v) Suppose that all the projections of $\mathscr{B}$ communicate and e is $\mathscr{E}$-recurrent, but e is not completely accessible; this is means that $E_{0]}(\tau_\infty)\neq 0$. By the criteria of recurrence (5.11)

$$E_{0]}(e\otimes\tau_\infty)=E_{0]}(e\otimes E_{0]}(\tau_\infty))=0 \tag{5.14}$$

in the sense that

$$E_{0]}(e\otimes E_{0]}(\tau_\infty^n))\to 0 \qquad \text{as} \quad n\to\infty$$

Therefore

$$E_{0]}(e\otimes e\otimes E_{0]}(\tau_\infty^n))\to 0 \qquad \text{as} \quad n\to\infty$$

and also

$$E_{0]}(e\otimes e^\perp\otimes E_{0]}(\tau_\infty^n))\to 0 \qquad \text{as} \quad n\to\infty$$

so

$$E_{0]}(e\otimes 1\otimes E_{0]}(\tau_\infty^n))\to 0 \qquad \text{as} \quad n\to\infty \tag{5.15}$$

Iterating (5.15), we have

$$E_{0]}(e\otimes 1_{n-1]}\otimes E_{0]}(\tau_\infty))=0 \qquad \forall n\geqslant 1 \tag{5.16}$$

Since $E_{0]}(\tau_\infty)\geqslant 0$, we can consider its spectral decomposition and get the inequality

$$E_{0]}(\tau_\infty)=\int_0^\infty \lambda\, dG(\lambda)\geqslant\int_{[a,b]\subset(0,+\infty)}\lambda\, dG(\lambda)$$
$$\geqslant a\cdot G([a,b])=:ap \qquad p\in\operatorname{Proj}(\mathscr{B})$$

for some $b>a>0$ such that $p=G([a,b])\neq 0$.

From (5.16) we have

$$0=E_{0]}(e\otimes 1_{n-1]}\otimes E_{0]}(\tau_{\infty}))\geqslant aE_{0]}(e\otimes 1_{n-1]}\otimes p) \qquad \forall n\geqslant 1$$

therefore for these $e, p\in \mathrm{Proj}(\mathscr{B})$ we have $e, p>0$ and $e\not\to p$ which contradicts the condition that all projections of $\mathscr{B}$ communicate.
This contradiction gives us $E_{0]}(\tau_{\infty})=0$, which is our thesis. □

Corollary 2. Let $S_e:\mathscr{B}\to\mathscr{B}$ denote the linear map defined by

$$S_e(b)=\mathscr{E}(e^{\perp}\otimes b) \qquad b\in\mathscr{B}$$

Then the limit, in the strong operator topology,

$$\lim_{n\to\infty} S_e^n(1)=:S_e^{\infty}(1)\in\mathscr{B} \tag{5.17}$$

exists and e is $\mathscr{E}$-recurrent if and only if

$$\mathscr{E}(e\otimes S_e^{\infty}(1))=0 \tag{5.18}$$

In particular, if $\mathscr{E}$ is faithful, then (5.18) is equivalent to

$$S_e^{\infty}(1)=0 \tag{5.19}$$

Proof. The limit (5.17) exists since $S_e^n(1)$ is a decreasing sequence of positive operators. Moreover

$$\begin{aligned}\mathscr{E}(e\otimes E_{0]}(\tau_{\infty})) &= \lim_{n\to\infty} \mathscr{E}(e\otimes E_{0]}(\tau_{\infty}^{n}))\\ &= \lim_{n\to\infty} \mathscr{E}(e\otimes \mathscr{E}(e^{\perp}\otimes(e^{\perp}\otimes \cdots \otimes\mathscr{E}(e^{\perp}\otimes 1)\cdots)\\ &= \lim_{n\to\infty} \mathscr{E}(e\otimes S_e^n(1))\end{aligned}$$

from which the thesis follows. □

Corollary 3. If all states of $\mathscr{E}$ communicate then the following statements are equivalent:

- e is recurrent.
- e is completely accessible.
- $E_{0]}(\sigma_{\tau})=1$.
- $E_{0]}E_{\tau}(1_{\mathscr{A}})=1$.

6. THE QUANTUM MARKOV CHAIN ASSOCIATED WITH THE HEISENBERG POTENTIAL

Consider the case of the so called two-level systems $\mathscr{B} = M_2(\mathbf{C})$. Define

$$h := \frac{1}{2} \sum_{j=0}^{3} j_0(\sigma_j)\, j_1(\sigma_j) = \frac{1}{2} \sum_{j=0}^{3} \sigma_j \otimes \sigma_j$$

where the Pauli matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

together with the identity matrix

$$\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

form a linear basis of $M_2(\mathbf{C})$.

Now define on the "chain algebra" $\mathscr{A} := \bigotimes_N \mathscr{B}$, the shift endomorphism u_n, characterized by $u_n j_k = j_{n+k}$ and $h_n := u_n(h)$, $n \geqslant 0$.

The formal Hamiltonian of the Heisenberg model

$$H = \sum_{n \geqslant 0} h_n \tag{6.1}$$

defines a Gibbs state on the algebra $\mathscr{A}$ formally:

$$\phi(\cdot) = \mathrm{Tr}(e^{-\beta H/N} \cdot) \tag{6.2}$$

where β is a constant, called the inverse temperature in the physical literature.

One can associate with this formal Hamiltonian a quantum Markov chain with the transition operator

$$\mathscr{E}(a \otimes b) = S^{-1/2}\, \overline{\mathrm{Tr}_2}((1 \otimes W)(c + sh)(a \otimes b)(c + sh))\, S^{-1/2} \tag{6.3}$$

where $S := \overline{\mathrm{Tr}_2}((1 \otimes W)(c + sh)^2)$, W is a density operator in $M_2(\mathbf{C})$, $c := \cosh(\beta)$, and $s := \sinh(\beta)$.

We shall restrict our considerations to the case $W = \frac{1}{2} \cdot \mathbf{1}$. Then (6.3) becomes

$$\mathscr{E}(a \otimes b) = \frac{1}{2\gamma} \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \tag{6.4}$$

where $\gamma := c^2 + s^2 + cs$ and for $a = \sum_{\alpha=0}^{3} a_\alpha \sigma_\alpha$, $b = \sum_{\alpha=0}^{3} b_\alpha \sigma_\alpha$, one has

$$z_{11} = (c^2+s^2)(2a_0b_0 + a_0b_3 + a_3b_0) + 2cs[(a_0+a_3)(b_0+b_3) + a_1b_1 + a_2b_2] + a_3b_0 - a_0b_3$$

$$z_{12} = (c^2+cs)\cdot 2b_0(a_1 - a_2 i) + (cs+s^2)\cdot 2a_0(b_1 - b_2 i)$$

$$z_{21} = \overline{z_{12}}$$

$$z_{22} = (c^2+s^2)(2a_0b_0 - a_0b_3 - a_3b_0) + 2cs[(a_0-a_3)(b_0-b_3) + a_1b_1 + a_2b_2] + a_0b_3 - a_3b_0$$

It is easy to see that any projection in $M_2(\mathbf{C})$ has the form

$$e = \frac{1}{2}\left(1 + \sum_{j=1}^{3} x_j\sigma_j\right) \qquad x \in \mathbf{R} \qquad x_1^2 + x_2^2 + x_3^2 = 1$$

that is,

$$e = \frac{1}{2}\begin{pmatrix} 1+x_3 & x_1 - x_2 i \\ x_1 + x_2 i & 1 - x_3 \end{pmatrix} \qquad \text{and} \qquad e^\perp = \frac{1}{2}\begin{pmatrix} 1-x_3 & -x_1 + x_2 i \\ -x_1 - x_2 i & 1 + x_3 \end{pmatrix} \tag{6.5}$$

Theorem 2. For the Heisenberg quantum Markov chain (6.4), any projection on a direction $\bar{x} = (x_1, x_2, x_3)$ given in (6.5) is $\mathscr{E}$-recurrent.

Proof. In the above notations

$$\alpha := \frac{s^2}{2\gamma} \qquad \beta := \frac{c(c+s)}{\gamma} \qquad \sigma := \frac{(c+s)^2}{2\gamma} \qquad \gamma := c^2 + s^2 + cs$$

one can calculate that

$$T(e^\perp) = \mathscr{E}(e^\perp \otimes 1) = E_{0]}(\tau_\infty^0) = \alpha\cdot 1 + \beta e^\perp \qquad T(e) = \alpha\cdot 1 + \beta e$$

$$\mathscr{E}(e^\perp \otimes e^\perp) = \sigma e^\perp$$

and then it is not difficult to get the recurrent formula

$$E_{0]}(\tau_\infty^n) = \alpha^{n+1}\cdot 1 + \beta\frac{\alpha^n - \sigma^n}{\alpha - \sigma} e^\perp$$

Therefore

$$E_{0]}(e\otimes\tau_\infty^n) = \frac{\alpha^{n+1}}{2\gamma}[s^2\cdot 1 + 2c(c+s)e] + \frac{\beta}{2\gamma}\frac{\alpha^n - \sigma^n}{\alpha - \sigma}[s^2 1 + e]$$

Since $|\alpha| < 1$ and $|\sigma| < 1$, we have

$$E_{0]}(e\otimes\tau_\infty) = \lim_{n\to\infty} E_{0]}(e\otimes\tau_\infty^n) = 0$$

so the criterium of Section 5 is fulfilled and the thesis follows. □

7. FIRST EXIT (HITTING) TIMES

Let us fix $e, l \in \text{Proj}(\mathscr{B})$ and consider the sequence of numbers

$$P_{le}^{n} := \frac{1}{v_l} \text{Tr}(E_{0]}(l \otimes \tau_{n-1}^{e})) \qquad n \geqslant 1 \qquad v_l := \text{Tr}(Tl) \tag{7.1}$$

It is a classical probability distribution (not necessarily proper) because

$$\sum_{n \geqslant 1} P_{le}^{n} = \frac{1}{v_l} \text{Tr}\left(E_{0]}\left(l \otimes \sum_{n \geqslant 0} \tau_n^e\right)\right) = \frac{1}{v_l} \text{Tr}(E_{0]}(l \otimes (1 - \tau_{\infty}^{e})))$$

$$= \frac{1}{v_l} \text{Tr}(Tl) - \frac{1}{v_l} \text{Tr}(E_{0]}(l \otimes \tau_{\infty}^{e})) = 1 - P_{le}^{\infty} \tag{7.2}$$

So we can introduce the classical integer valued random variable (not necessarily proper) (since l, e are fixed, omitting them does not create confusions):

$$\theta_{le} = \theta = \begin{cases} n+1 & \text{with probability } P_{le}^{n} \qquad (n \geqslant 1) \\ \infty & \text{with probability } P_{le}^{\infty} \end{cases} \tag{7.3}$$

It is clear that θ is the hitting time of a projection e for the chain $(\mathscr{E}, \varphi_l)$, where

$$\varphi_l(a) := \frac{1}{v_l} \text{Tr}(E_{0]}(l \otimes a)) \qquad \forall a \in \mathscr{A}_{n]} \qquad n \geqslant 0$$

Now we can define the algebra associated with the hitting time θ:

$$\mathscr{A}_{\theta]} := \begin{cases} \mathscr{A}_{n]} & \text{if } \theta = n \geqslant 0 \\ \mathscr{A} & \text{if } \theta = \infty \end{cases} \tag{7.4}$$

And the corresponding conditional expectation:

$$E_{\theta]}(a) := \sum_{n \geqslant 1} P_{le}^{n} E_{n]}(a) \qquad a \in \mathscr{A} \tag{7.5}$$

In the last formula and in the sequel in similar situations, the expression $a \in \mathscr{A}$ should be understood as $a \in \mathscr{A}_n$, $\forall n \geqslant 0$.

Proposition 5. For any $a \in \mathscr{A}$ one has

$$E_{0]} E_{\theta]}(a) = E_{0]}(a)(1 - P_{le}^{\infty}) \tag{7.6}$$

Proof. Let $a \in \mathscr{A}_{n]}$, i.e., of the form $a = a_{n]} = a_0 \otimes \cdots \otimes a_n \otimes 1 \otimes 1 \otimes \cdots$. Then

$$
\begin{aligned}
E_{\theta(a)]} &= \sum_{n \geq 1} P^n_{le} \cdot E_{n]}(a) \\
&= P^1_{le} \cdot a_0 \otimes E_{0]}(a_{1]}) + P^2_{le} \cdot a_0 \otimes a_1 \otimes E_{0]}(a_{2]}) + \cdots \\
&\quad + P^n_{le} \cdot a_0 \otimes \cdots \otimes a_{n-1} \otimes E_{0]}(a_{n]}) + \sum_{k \geq n+1} P^k_{le} \cdot a_0 \otimes \cdots \otimes a_n
\end{aligned} \tag{7.7}
$$

So we have

$$
\begin{aligned}
E_{0]} E_{\theta]}(a) &= P^1_{le} E_{0]}(a_0 \otimes E_{0]}(a_{1]})) + P^2_{le} E_{0]}(a_0 \otimes a_1 \otimes E_{0]}(a_{2]})) + \cdots \\
&\quad + P^n_{le} E_{0]}(a_0 \otimes \cdots \otimes a_{n-1} \otimes E_{0]}(a_{n]})) \\
&\quad + \sum_{k \geq n+1} P^k_{le} E_{0]}(a_0 \otimes \cdots \otimes a_n) \\
&= E_{0]}(a_{n]}) \sum_{k \geq 1} P^k_{le} = E_{0]}(a_{n]})(1 - P^\infty_{le})
\end{aligned} \tag{7.8}
$$

Since (7.8) takes place for any $n \geq 0$, the thesis follows. □

Corollary 4. If θ is the proper random variable then

$$
E_{0]} E_{\theta]}(a) = E_{0]}(a) \qquad \forall a \in \mathscr{A} \tag{7.9}
$$

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