Fractal Quantum Brownian motions

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ABSTRACT

The notion of fractional quantum Brownian motion is introduced and the conditions of its existence (for a given pair of parameters $\alpha \in [0,2]$, $a \in \mathbb{R}$ and a given symplectic form σ) is derived. Some examples, involving representations of the CCR with "random Planck's constant" are produced.

Fractional Brownian Motions were discussed by Mandelbrot and more generally, in a series of papers, Mandelbrot has shown the potential wealth of applications of the stable laws of probability theory [1], [2], [3], [4]. Since the defining property of the stable laws is self–similarity, these laws can be considered as forfathers of the fractals and their samples as typical examples of them. In the present paper we begin to investigate the possible significance of the stable laws in quantum theory. More precisely we introduce the notion of stable (and more generally, infinitely divisible) state on the CCR $C^{*}-$ algebra. The associated cyclic representation should be called a stable (resp. infinitely divisible) representation of the CCR.

Let σ be a real symplectic form on $L^2(\mathbb{R})$ (i.e. a real valued symplectic form on $L^2(\mathbb{R})$ considered as a real space) and let W_σ denote the Weyl $C^{*}-$ algebra over $(L^2(\mathbb{R}), \sigma)$ (cf. [6]).

Let, for $I \subseteq \mathbb{R}$, χ_I denote the characteristic function χ_I of $I(\chi_I(x) = 0$ if $x \notin I$, $= 1$ if $x \in I)$) and let e_I denote the multiplication operator by χ_I. We shall identify $e_I L^2(\mathbb{R})$ with $L^2(I)$.

Suppose that

$$\sigma(f,\ g) = 0 \tag{1}$$

whenever $f \in L^2(I)$, $g \in L^2(J)$ with $\dot{I} \cap \dot{J} = \Phi$, ($\dot{I}$ denoting the interior of I).

The typical example, with which we shall be mainly concerned, is

$$\sigma(f,\ g) = \xi Im < f,\ g > \tag{2}$$

where ξ is a constant. For each $z \in \mathbb{C}$, the C^*–algebra generated by

$$\{W(z\chi_{(s, t)}) : 0 \leq t - s < +\infty\} \tag{3}$$

is abelian. A state φ on W_σ will be called **stationary** if, for each $z \in \mathbb{C}$ and $0 \leq t - s < +\infty$, one has

$$\varphi(W(z\chi_{(s, t)})) = \varphi(W(z\chi_{(0, t-s)})) \tag{4}$$

A state φ on W_σ will be called **infinitely divisible** if:

(i) φ is stationary

(ii) $t \in \mathbb{R} \mapsto \varphi(W(z\chi_{(0, t)})) =: \varphi(z, t)$ continuous for every $z \in \mathbb{C}$.

(iii) For each $z \in \mathbb{C}, t \mapsto \varphi(z, t)$ is an infinitely divisible function, i.e.

$$\varphi(z, nt) = \varphi(z, t)^n ; \ \forall n \in \mathbb{N} \tag{hi}$$

Conditions (ii) and (iii) imply

$$\varphi(z, t) = \varphi(z)^t ; \ \forall z \in \mathbb{C} ; \ \forall t \in \mathbb{R} \tag{5}$$

where $\varphi(z) := \varphi(z, 1)$.

Let $\psi(z)$ be defined by

$$\varphi(z) = e^{\psi(z)} \tag{6}$$

The state φ is called **stable** if, for every $a > 0$ there exists a $c(a) \in \mathbb{R}$ such that

$$\psi(z) = a\psi(z/c(a)) \tag{7}$$

In this case it is known [5] that $c(a)$ must have the form

$$c(a) = a^\alpha; \ a > 0 \tag{8}$$

for some $0 \leq \alpha \leq 2$.

The problem we want to investigate is the following: given a symplectic form σ on $L^2(\mathbb{R})$ satisfying (1), **classify all the infinitely divisible (resp. stable) states on W_σ.**

For a stable state φ one has

$$\varphi(W(z\chi_{[s, t)})) = e^{(t-s)\psi(z)} \tag{9}$$

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with $\psi(z)$ satisfying, for some $\alpha \in [0, 2]$,

$$\psi(\lambda z) = \lambda^\alpha \psi(z) \; ; \; \forall \lambda > 0$$

The functions ψ with this property are classified [5]. They have the form

$$\psi(z) = \left(-1 + i\frac{z}{|z|}a\right)|z|^\alpha$$

for some $a \in \mathbb{R}$ and α as above.

Let $q : L^2(\mathbb{R}_+) \to \mathbb{C}$ denote the functional defined by

$$\varphi(W(f)) = e^{-\frac{1}{2}q(f)} \tag{10}$$

It is known [6] that, denoting

$$q_s(f, g) := \frac{1}{2}(q(f + g) - q(f) - q(g)) \; ; \; f, g \in L^2(\mathbb{R}_+ \tag{11}$$

the functional

$$(f, \ g) \mapsto q_s(f, \ g) + i\sigma(f, \ g) \tag{12}$$

is positive definite and that, conversely, every functional $q : L^2(\mathbb{R}_+) \to \mathbb{C}$ with this property defines, via (11), a unique state on $W_\sigma(L^2(\mathbb{R}_+))$.

From (6) and (10), we deduce that

$$-\frac{1}{2}q(z\chi_{[s, \ t)}) = (t - s)\psi(z) \tag{13}$$

so that, for some

$$0 < \alpha < 2$$

one should have

$$q(z\chi 0_{[s, \ r)}) = -2(t - s)\left(-1 + i\frac{z}{|z|}a\right)|z|^\alpha \tag{1hi}$$

for some $a \in \mathbb{R}$.

If σ has the standard form

$$\sigma(f, \ g) = Im \int \overline{f}(s)g(s)ds \tag{14}$$

then

$$\sigma(z\chi_{[s, \ t)}, z^1\chi_{[s, \ t)}) = (t - s)Im\overline{z}z^1 \tag{15}$$

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In this case the positivity condition for the functional (12) implies that

$$(z, \, z^1) \in \mathbb{C}^2 \mapsto \left(1 - i\frac{z + z^1}{|z + z^1|}a\right)|z + z^1|^\alpha - \left(1 - i\frac{z}{|z|}a\right)|z|^\alpha$$

$$- \left(1 - i\frac{z^1}{|z^1|}a\right)|z^1|^\alpha + iIm\overline{z}z^1 =: Q(z, \, z^1)\alpha; \, a \qquad (16)$$

is positive definite.

Conversely, since the property of being positive definite is preserved under sums and pointwise limits, it is clear that, if $Q_{\alpha, \, a}(z, \, z^1)$, defined by (18), is positive definite, then if $q(f)$ has the form

$$q(f) = 2 \int dt \left(1 - i\frac{f(t)}{|f(t)|}a\right)|f(t)|^\alpha \qquad (17)$$

with the conventions $0/0 = 0$, $0^\alpha = 0$ (notice that $q(f)$ is well defined for a step function f), and if σ has the form (15), then $q(\cdot)$ defines, via (10), a state on $W_\sigma(\mathcal{S})$ ($\mathcal{S}$ being the space of finite valued step functions) if and only if the Kernel $Q_{\alpha, \, a}(z, \, z^1)$ is positive definite.

The classification of the parameters $\alpha \in [0, \, 2]$ and $a \in \mathbb{R}$, for which this positive definiteness takes place, is an open problem. The following construction, which extends to the quantum case Bochner's methods of obtaining the fractional Brownian motions from the standard Brownian motion, shows that some examples of infinitely divisible, in particular stable, states can be produced.

Let (ξ_t) be the classical increasing symmetric stable process with parameter $\alpha \in (0, \, 1]$ and let $A(\chi_{[0, \, t]})$, $A^+(\chi_{[0, \, t]})$ be the Fock or a finite temperature quantum Brownian motion as defined in [7]. Performing the random time charge

$$A^\pm(\chi_{[0,t]}) \to A^\pm(\chi_{[0, \, \xi_t]}) =: a_t^\pm$$

we obtain a representation of the CCR with random Planck's constant:

$$[a_t, \, a_t^+] = \xi_t \qquad (18)$$

Denoting E_ξ the expectation for the ξ–process, E the Fock (or a finite temperature) state on the Weyl algebra $W(L^2(\mathbb{R}_+))$. Then the state $\varphi = E_\xi \otimes E$, defined on the algebra

$$L^\infty(\Omega, \, \mathcal{F}, \, \mathcal{P}_\xi) \otimes \mathcal{W}(\mathcal{L}^\in(\mathbb{R}_+))$$

satisfies the identity

$$
\begin{aligned}
\varphi(W(z\chi_{(0,\,\xi_t)})) &= E_\xi(E(z\chi_{(0,\,\xi_t)})) = \\
&= E_\xi(\{E(W(z\chi_{(0,\,1)}))\}^{\xi_t}) \\
&= E_\xi(e^{\xi_t(-\frac{A}{2}|z|^2)}) = e^{-\frac{Ac}{2}|z|^{2\alpha}}
\end{aligned}
$$

where c is a constant and $A = 1$ in the Fock case, $A > 1$ in the finite temperature case.

Summing up: for any $\alpha \in (0,\,1]$, the process $A^\pm(\chi_{[0,\,\xi_t]})$ is a symmetric stable quantum process of parameter 2α satisfying the random CCR (18).

In conclusion let me mention that K. R. Parthasarathy has shown that any infinitely divisible random variable can be realized as a quantum stochastic integral on the Fock space of $L^2(\mathbf{R}_+)$, however there is no Weyl representation associated to this realization.

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