On the quantum Feynman–Kac formula

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Contents

1 Introduction

The Feynman–Kac formula is a technique, based on functional integration, which allows to perturb a markovian (i.e. positivity and identity preserving) semi–group $\exp tH$ with suitable continuity properties into a new positivity preserving semi–group whose formal generator has the form $H + V$ and to obtain a fairly explicit representation of the latter in terms of a functional integral.

The purpose of this paper is to show that the formalism on which the classical Feynman–Kac formula is based can be generalized to a quantum (i.e. non–commutative) context. The role played, in the classical case, by multiplicative functionals is played in the quantum case by localized 1–cocyles (markovian cocycles). In the algebraic (i.e. L^∞) theory, perturbations arising from unitary markovian cocycles are derivations (such perturbations have no classical analogue), those arising from hermitian markovian cocycles are dissipations. A more general form of markovian cocycle allows to obtain the full dissipative part of Lindblad's generator of a quantum markovian semi–group. As far as possible, we develop the formalism in a language common to the classical and quantum case (cf. §'s (1), (2), (4), (13), (14)).

The equivalence between the algebraic framework of the present approach and the usual probabilistic one is based on the equivalence between the theory of commutative local algebras with a state and the theory of classical stochastic processes. This equivalence lies at the basis of our definition of "quantum stochastic process" and is briefly reviewed in § (0), which means to provide a heuristic back–ground as well as a motivation for the definitions introduced later.

In §'s (8) to (11) we review some analytical properties of the perturbed semi–group, and in § (12) the beautiful Kac–Ray asymptotic estimates of the spectrum of generators.

We have not discussed here the important property of hypercontractivity; for this we refer to [16], [7], [22] and to the bibliography therein.

As far as the classical Feynman–Kac formula is concerned, our main result is the asymptotic estimate (90), obtained in Theorem (12.2), which generalizes the corresponding result of Ray [29] for the Wiener process. The interest of this result lies in the fact that it provides a rigorous foundation for the so–called WKB estimates.

Dealing with the classical case we develop the theory for a general state space S since we want our results to be applicable to the case in which S

is a differentiable manifold. However we consider only scalar valued functionals of the process motivated earlier approaches to the non–commutative Feynmann–Kac formula (cf. [17], [28], [34]), whose results have been applied by Malliavin [25] to the diffusion theory of differential forms.

In the particular contexts of Euclidean fermion quantum field theory, of Clifford algebras over real Hilbert spaces, and of the quantum Wiener process, non–commutative generalizations of the Feynmann–Kac formula have been discussed respectively by K. Osterwalder and R. Schrader [27], R. Schrader and D.A. Uhlenbrock [31], R. Hudson and P. Ion [19].

2 Stochastic processes and local algebras

Following J.L. Doob [10] we define a stochastic process indexed by a set T and with values in a measurable space $(\mathcal{S}, \mathcal{B})$ a s family of μ–equivalence classes of random variables $x_t : (\Omega, \theta, \mu) \to (\mathcal{S}, \mathcal{B})$ defined on a probability space (Ω, θ, μ) and with values on $(\mathcal{S}, \mathcal{B})$.

The space S is called the state space of the process and stochastic processes are classified according to their finite dimensional distributions. As shown in [2] this amounts to the following: let $\mathcal{F}$ be the family of finite subsets of T; denote, for $F \in \mathcal{F}$, θ_F the σ–algebra generated by the random variables x_t $(t \in F)$, i.e.

$$\theta_F = \bigvee_{t \in F} x_t^{-1}(\mathcal{B})$$

let μ_F be the restriction of μ on θ_F and denote

$$\mathcal{A}_F = L^\infty(\Omega, \theta_F, \mu_F)$$

$$\mathcal{A} = \text{ norm closure of } \bigcup \{\mathcal{A}_F : F \in \mathcal{F}\}$$

the norm closure being meant in the sense of the usual norm on $L^\infty(\Omega, \theta, \mu)$.

The measure μ induces a state (i.e. a positive normalized linear functional), still denoted μ, on the C^*–algebra $\mathcal{A}$, defined by:

$$\nu(f) = \int_\Omega f d\mu \; ; \quad f \in \mathcal{A}$$

(throughout the present paper we shall adopt, for what concerns C^*– and W^*–algebras, the notations and nomenclature of S. Sakai's monograph [20]).

The state μ on $\mathcal{A}$ is locally normal in the sense that for each $F \in \mathcal{F}$ the state $\mu_F = \mu|\mathcal{A}_F$ is normal.

Two stochastic processes indexed by the same set T are called stochastically equivalent if, denoting there exists an isomorphism of C^*–algebras

$$u : \mathcal{A} \to \mathcal{A}'$$

such that

$$u(\mathcal{A}_F) = \mathcal{A}'_F; \ F \in \mathcal{F}$$

$$u|\mathcal{A}_F \text{ is normal; } F \in \mathcal{F}$$

$$\mu' \cdot u = \mu$$

If (x_t), (x'_t) are the random variables defining the two processes, an explicit form of the isomorphism u is given by the map

$$F(x_{t_1}, \ldots, x_{t_n}) \mapsto F(x'_{t_1}, \ldots, x'_{t_n})$$

for any bounded measurable function $F : S^n \to C$, and for any $t_1, \ldots, t_n \in T$.

It is possible to characterize, up to equivalence, those triples $\{\mathcal{A}, (\mathcal{A}_F), \mu\}$ which come from stochastic processes in Doob's sense (cf. [2]) and, more generally, every triple $\{\mathcal{A}, (\mathcal{A}_F), \mu\}$ such that

– $\mathcal{A}$ is a C^*–algebra (commutative)

– $\mathcal{A}_F$ is a W^*–algebra $\subseteq \mathcal{A}$

– $\mathcal{A}$ = norm closure of $\bigcup\{\mathcal{A}_F : F \in \mathcal{F}\}$

– $F \subseteq G \Rightarrow \mathcal{A}_F \subseteq \mathcal{A}_G$

– μ is a locally normal state on $\mathcal{A}$

defines a stochastic process in the sense of I.E. Segal [32].

An element $\varphi \in \mathcal{A}_F$ $(F \in \mathcal{F})$ is a bounded measurable functional of the random variables x_t $(t \in F)$. Often one has to deal with bounded measurable functionals of infinitely many of the random variables x_t.

This leads to the consideration of local algebra $\mathcal{A}_I$ where I is no longer a finite subset of T. The following definitions are often used (when T is a topological space):

$$\mathcal{A}_I = L^\infty(\Omega, \theta_I, \mu_I)$$

$$\theta_I = \vee_{t \in I} x_t^{-1}(\mathcal{B}) \ ; \quad \mu_I = \mu|\theta_I$$

when I is an open set, and

$$\mathcal{A}_I = \cap\{\mathcal{A}_B : B \text{ open } \supseteq I\}$$

when I is an arbitrary set. Sometimes a more delicate definition is useful (cf. [4]), but in the present paper the general definition of the algebras $\mathcal{A}_I$ will not be discussed and we shall simply assume that the algebra $\mathcal{A}_I$ or, equivalently, the σ–algebras θ_I, for an arbitrary set $I \subseteq T$, are defined in such a way as to satisfy the conditions of Definition (1.1) (cf. § 1), and:

$$F(x_{t_1}, \dots, x_{t_n}) \in \mathcal{A}_I$$

whenever $F : S^n \to C$ is a bounded measurable function and $t_1, \dots, t_n$ are in I ($n \in \mathbb{R}$) (or, at least, for a weakly dense set of such functions).

The choice of a family of local algebras associated to a stochastic process is not canonical but depends on the process. One might consider bounded continuous functions if S is a topological space, bounded $\mathcal{C}^\infty$–functions if S is manifold, or even unbounded functions (for example the algebra of polynomials of a gaussian process). All these choices are sufficient to determine the class of stochastic equivalence of the process in the sense that their weak closure in the *GNS* representation associated to the state are algebraically isomorphic (cf. [3] for a precise formulation).

Two important classes of maps of the algebra $\mathcal{A}$ into itself are associated to stochastic processes:

i) conditional expectations

ii) automorphisms induced by symmetries of T.

The conditional expectations associated to μ

$$E_I : \mathcal{A} \to \mathcal{A}_I = L^\infty(\Omega, \theta_I, \mu_I)$$

are always defined and satisfy

$$I \subseteq J \Rightarrow E_1 E_J = E_1 \text{ (projectivity)}$$

An injective map $g : T \to T$ induces an endomorphism of $\mathcal{A}$ with a left inverse if and only if the maps

$$G(x_{t_1}, \dots, x_{t_n}) \mapsto G(x_{gt_1}, \dots, x_{gt_n}) \tag{1}$$

($G : S^n \to C$, bounded measurable, $t_1, \dots, t_n \in T$) are well defined as maps,

$$\mathcal{A}_F = L^\infty(\Omega, \theta_F, \mu_F) \to \mathcal{A}_{gF} = L^\infty(\Omega, \theta_{gF}, \mu_{gF})$$

($F = \{t_1, \dots, t_n\}$); i.e. if and only if the measure μ_F are quasi–invariant for such maps.

If, moreover, the measures μ_F are invariant for the maps above, then one easily verifies that

$$u_g^* \cdot E_{gI} \cdot u_g = E_I$$

where $u_g : \mathcal{A} \to \mathcal{A}$ is the endomorphism induced by (1) and u_g^* its left inverse.

In the following we will be mainly concerned with the case in which

$$T = \mathbb{R} \text{ or } \mathbb{R}^+$$

and the maps $g : T \to T$ are translations

$$gt = t + s \ ; \quad t \in T \text{ (for some } s; s \geq 0 \text{ if } T = \mathbb{R}^+)$$

or reflections

$$gt = s - t \text{ (for some } s; s \geq t \text{ if } T = \mathbb{R}^+)$$

3 Local algebras on $\mathbb{R}(\mathbb{R}^+)$

In this § we establish some notations which will be used throughout the paper.

Let T be $\mathbb{R}$ or $\mathbb{R}^+$; let $\mathcal{F}$ be a family of T containing the finite parts and the intervals (open, half–open, bounded or not) of T.

Definition 1 *A family of local algebras on T is a couple $\{\mathcal{A}, (\mathcal{A}_I)\}$, $(I \in \mathcal{F})$ such that*

i) $\mathcal{A}$, $\mathcal{A}_I$ are involutive complex algebras with unit.

ii) $\mathcal{A}_I \subseteq \mathcal{A}_J$ if $I \subseteq J$

iii) $\mathcal{A}_{\mathbb{R}} = \mathcal{A}$

iv) $\mathcal{A}_I = \vee\{\mathcal{A}_J : J \subseteq I; J \in \mathcal{F}\}$ if I is open.

A conditional expectation $\mathcal{A} \to \mathcal{A}_I$ is a linear map $E_I : \mathcal{A} \to \mathcal{A}_I$ such that

$$a \geq 0 \ ; \quad a \in \mathcal{A}, \ \Rightarrow E_I(a) \geq 0$$

$$E_I(1) = 1$$

$$E_I(a_I a) = a_I E_I(a) \ ; \quad a_I \in \mathcal{A}_I \ ; \quad a \in \mathcal{A}$$

$$E_I(a^*) = E_I(a)^* \ ; \quad a \in \mathcal{A}$$

A family of conditional expectations (E_I), $E_I : \mathcal{A} \to \mathcal{A}_I$, is called projective if

$$I \subseteq J \Rightarrow E_1 E_J = E_I \tag{2}$$

We shall assume that there is an action

$$t \in T \mapsto u_t \in \text{ End } (\mathcal{A})$$

of T on $\mathcal{A}$ by $*$–endomorphisms which satisfies

$$u_t \mathcal{A}_I = \mathcal{A}_{I+t} \quad \text{(covariance)} \tag{3}$$

$$u_t \text{ has a left inverse denoted } u_t^* \quad (u_t^* \text{ is the inverse of } u_t \text{ if } T = \mathbb{R}) \tag{4}$$

$$u_t u_s = u_{t+s} \tag{5}$$

A projective family (E_I) of conditional expectations is called covariant if

$$u_t^* E_{I+t} u_t = E_I \tag{6}$$

this is equivalent to

$$u_t E_I u_t^* | \mathcal{A}_{[t,+\infty[} = E_{I+t|\mathcal{A}[t,+\infty[} \tag{7}$$

Time reflections also will play an important role in our exposition.

Let $T = \mathbb{R}^+$; a 1–parameter family of time reflections is a family of automorphism (or anti–automorphisms)

$$r_t : \mathcal{A}_{[0,t]} \to \mathcal{A}_{[0,t]}, \ t > 0$$

such that

$$r_t^2 = id; \quad t > 0 \tag{8}$$

$$r_t u_s = u_{t-s} \ ; \quad 0 \leq s \leq t \tag{9}$$

If $T = \mathbb{R}$ a time reflection is an automorphism (anti–automorphism) $r : \mathcal{A} \to \mathcal{A}$ such that

$$r^2 = id \tag{10}$$

$$r u_t = u_{-t} r \tag{11}$$

If, moreover,

$$r | \mathcal{A}_0 = id \tag{12}$$

then we say that the system $\{\mathcal{A}, (\mathcal{A}_I), (u_t)\}$ enjoys the reflection property.

A triple $\{\mathcal{A}, (\mathcal{A}_I)_{I\in\mathcal{F}}, \mu\}$ where $\{\mathcal{A}, (\mathcal{A}_I)\}$ is a family of local algebras and μ is a state on $\mathcal{A}$ will be called a stochastic process (quantum stochastic process if $\mathcal{A}$ is not abelian). If the $\mathcal{A}_I$ are W^*–algebras μ is required to be locally normal. This definition is justified by the discussion in § (0) (cf. [2], [3] for a more detailed discussion). Remark that the triple $\{\mathcal{A}, (\mathcal{A}_I)_{I\in\mathcal{F}}, \mu\}$ determines an *equivalence class* of stochastic processes. In many circumstances it is usefull to single out a set of functions (or operators) which in some specified sense generate the local algebras $\mathcal{A}_I$. Usually in the literature it is such a set of generators which is called a stochastic process and, in such a case, one usually requires that the two sets of generators are identified by the relation of stochastic equivalence described above.

There are many examples of classical stochastic processes whose associated local algebras satisfy the above listed conditions. Using Clifford algebras, or the representation theory of the CCR, it is not difficult to construct examples of non–commutative local algebras with the above listed properties (cf., for example, [37]).

4 Markovianity and semi–groups

To avoid circumlocutions, we adopt the convection that, if $T = \mathbb{R}^+$, the symbols

$$E_{]-\infty,0]},\ E_{]-\infty,t]}\ ,\quad \mathcal{A}_{]-\infty,t]}, \ldots$$

stand for

$$E_{[0,t]}, \quad E_{\{0\}}, \quad \mathcal{A}_{[0,t]}, \ldots$$

respectively.

Definition 2 *The family (E_I) is said to be Markovian if $\forall\, t \in T$*

$$E_{]-\infty,t]}(\mathcal{A}_{[t,+\infty[}) \subseteq \mathcal{A}_t \tag{13}$$

The properties of the conditional expectations easily imply that (13) is equivalent to

$$E_{]-\infty,t]}(a) = E_{\{t\}}(a)\ ; \qquad \forall\, a \in \mathcal{A}_{[t,+\infty[} \tag{14}$$

There are many equivalent ways of formulating the Markov property. The formulation (13) (and its multi–dimensional analogues, cf. [1]) underlines the locality aspect of the Markov property and is particularly well suited for the quantum generalization.

Proposition 1 *In the above notations, let (E_I) be a projective, covariant, markovian family of conditional expectations, and define*

$$P^t = E_{]-\infty,0]}u_t|\mathcal{A}_0 \ ; \quad t \geq 0 \tag{15}$$

then P^t is a 1–parameter, positivity preserving semi–group $\mathcal{A}_0 \to \mathcal{A}_0$ such that

$$P^t(1) = 1 \ ; \qquad t \geq 0 \tag{16}$$

Proof. P^t is positivity preserving and $P^t(1) = 1$ since $E_{]-\infty,0]}$ and u_t have this properties; because of the Markov property

$$P^t = E_{\{0\}}u_t$$

hence

$$P^t\mathcal{A}_0 \subseteq \mathcal{A}_0$$

and

$$P^tP^s = E_{\{0\}}u_tE_{\{0\}}u_s = E_{\{0\}}E_{\{t\}}u_{t+s} = E_{\{0\}}E_{]-\infty,t]}u_{t+s} = E_{\{0\}}u_{t+s} = P^{t+s}$$

hence P^t is a semi–group.

A semi–group $P^t : \mathcal{A}_0 \to \mathcal{A}_0$, positivity preserving and such that $P^t(1) = 1$, is called a markovian semi–group (on $\mathcal{A}_0$).

The relation (13) can be called the "foreward" Markov property (the past conditioning the future). The "backward" Markov property (the future conditioning the past) is expressed by

$$E_{[t,+\infty[}(\mathcal{A}_{]-\infty,t]}) \subseteq \mathcal{A}_t \tag{17}$$

Reasoning as in the proof of Proposition (2.2) one verifies that, if (E_I) is backward Markovian, covariant, projective, then

$$P^t = u_t^*E_{[t,+\infty[}|\mathcal{A}_0; \ t \geq 0 \tag{18}$$

is a Markovian semi–group on $\mathcal{A}_0$.

The definitions of the semi–groups (1) and (18) can be schematically illustrated by the diagrams

If $T = \mathbb{R}$ and the system $\{\mathcal{A}, (\mathcal{A}_I), (E_I)(u_t)\}$ admits a reflection, then it is easy to verify that the two definitions coincide.

Remark 1. The proof of the semi–group property makes use only of covariance and projectivity, and the fact that P^t maps $\mathcal{A}_0$ into itself follows from the Markov property. Thus the construction above holds for any covariant, projective, markovian, normalized family $(E_{]-\infty,t]})$ of completely positive maps.

Remark 2. The relation (1) shows he deep connection between stationary (or, more generally, covariant) Markov processes and the theory of unitary dilations of semi–groups. We refer to [12] for a discussion of this topic and bibliographic references.

5 Semi–groups and markovianity

In the previous § we have seen that every covariant stochastic process, as defined in § (1), determines a markovian semi–group P^t. If the process has an initial distribution w_0 (resp. is stationary with invariant distribution w_0), then the couple $\{w_0, P^t\}$ uniquely determines the stochastic equivalence class of the process. It is important to remark that the equivalence class of the process is meant here with respect to the localization given by the *finite subsets* of the index set $T \subseteq \mathbb{R}$. Without this clarification the above assertion is, in general, false (this is the case, for example, for Markov fields — i.e. generalized processes — on the real line, for which the natural equivalence relation is not based on the finite subsets of $\mathbb{R}$ but on the open intervals).

In the following we shall use the term *process* to imply that the localization is based on the finite subsets if the set of indices, and the term *field* for the more general situation.

There is a well known procedure which allows to associate a stochastic process (resp. stationary stochastic process) with initial (resp. stationary) distribution w_0, uniquely determined up to equivalence, to a couple $\{w_0, P^t\}$, where w_0 is a probability distribution on a measurable space $(S, \mathcal{B})$, and P^t is a markovian semi–group acting on some subspace of $L^\infty(S, \mathcal{B})$ with appropriate continuity properties (cf. [10], [14], for example). The equivalence class of the process, i.e. the joint expectations, are determined by:

$$\mu_{0,t_1,\dots,t_n}((f_0 \circ x_0) \cdot (f_1 \circ x_{t_1}) \cdot \ldots \cdot (f_n \circ x_{t_n})) =$$
$$= w_0(f_0 \cdot [P^{t_1}[f_1 \cdot [P^{t_2-t_1} \cdot \ldots \cdot [P^{t_n-t_{n-1}} f_{t_n}]]] \ldots])$$

where $f_0, \dots, f_n \in L^\infty(S, \mathcal{B})$, (x_t) are the random variables of the process, $0 < t_1 < \ldots < t_n$, $n \in \mathbb{R}$ and the dot denotes pointwise multiplication.

Thus all classical covariant Markov process are determined up to the initial (resp. stationary) distribution and up to stochastic equivalence, by a markovian semi–group. As shown in [1], [2], [3], the situation in the quantum case is more delicate; in particular, the extrapolation of the above assertion to the quantum case is wrong.

Probably the most studied Markov process is the Wiener process (or Brownian motion) which is obtained when $S = \mathbb{R}^N$ with the Borel σ–algebra and

$$P^t f(x) = e^{1/2t\Delta} f(x) = \int_{\mathbb{R}^N} \frac{e^{-|x-y|^{2/2t}}}{(2\pi t)^{N/2}} f(y) dy$$

(dy–Lebesgue measure on $\mathbb{R}^N$, $|x|^2 = \sum_{J=2}^N x_J^2$, if $x = (x_1, \ldots, x_N)$), and w_0 is any probability measure on $\mathbb{R}^N$ (if $w_0 = \delta_x$ = Dirac measure on x, one speaks of Wiener process with initial condition x). All the results expounded in the present paper extend some results obtained in the framework of the Wiener process.

6 Local perturbations

In the notations of the preceedings §'s, let

$$P_0^t = E_{]-\infty,0]} u_t : \mathcal{A}_0 \to \mathcal{A}_0 \tag{19}$$

be a Markovian semi–group and let, for each $t \geq 0$

$$\tilde{M}_t : \mathcal{A} \to \mathcal{A} \tag{20}$$

be a positivity preserving operator (completely positive if $\mathcal{A}$ is a C^*–algebra). Define, of $t \geq 0$

$$P^t = E_{]-\infty,0]} \tilde{M}_t u_t | \mathcal{A}_0 \tag{21}$$

Remark that $P^t(\mathcal{A}_0) \subseteq \mathcal{A}_0$ if and only if $\forall\, a_0 \in \mathcal{A}_0$

$$P^t a_0 = E_{\{0\}} P^t a_0 = E_{\{0\}} \tilde{M}_t u_t a_0 = E_{]-\infty,0]} E_{[0,t]} \tilde{M}_t E_{[0,t]} u_t a_0$$

This means that if $P^t(\mathcal{A}_0) \subseteq \mathcal{A}_0$ then we can always assume, without changing the action of P^t, that

$$\tilde{M}_t = E_{[0,t]} \tilde{M}_t E_{[0,t]} \tag{22}$$

or, equivalently, that

$$\tilde{M}_t(\mathcal{A}_{[0,t]}) \subseteq \mathcal{A}_{[0,t]} \tag{23}$$

and that, conversely, if (23) holds, then P^t maps $\mathcal{A}_0$ into itself.

In the following we shall assume that (23) holds. Since $\forall\, a \in \mathcal{A}_0$:

$$\begin{aligned}
P^t P^s a_0 &= E_{]-\infty,0]}\tilde{M}_t u_t E_{]-\infty,0]}\tilde{M}_s u_s a_0 = \\
&= E_{]-\infty,0]}\tilde{M}_t E_{]-\infty,t]} u_t \tilde{M}_s u_t^* \cdot u_{t+s} a_0 = \\
&= E_{]-\infty,0]} E_{]-\infty,t]}\tilde{M}_t(u_t\tilde{M}_s u_t^*) u_{t+s} a_0 = \\
&= E_{]-\infty,0]}\tilde{M}_t(u_t\tilde{M}_s u_t^*) u_{t+s} a_0
\end{aligned}$$

we have that, if $P^t : \mathcal{A}_0 \to \mathcal{A}_0$ is a semi–group then we can always assume, without changing the action of P^t on $\mathcal{A}_0$, that

$$\tilde{M}_{t+s} = \tilde{M}_t(u_t\tilde{M}_s u_s^*) \tag{24}$$

Conversely, if (24) holds, then P^t is a semi–group on $\mathcal{A}_0$.

Remark 1. If $\mathcal{A}$ is abelian, a simple form for the operator $\tilde{M}_t$ is

$$\tilde{M}_t(a) = M_t \cdot a \ ; \qquad a \in \mathcal{A} \tag{25}$$

where M_t is a positive element in (or affiliated to) $\mathcal{A}$. In this case the conditions (23), (24) are equivalent to

$$M_t \in \mathcal{A}_{[0,t]} \tag{26}$$

$$M_{t+s} = M_t u_s(M_s) \tag{27}$$

Remark 2. If $\mathcal{A}$ is not abelian, the choice (25) for the operator $\tilde{M}_t$ will not give rise, in general, to a positivity preserving semi–group P^t. This is not the case, however, if M_t lies in the center of $\mathcal{A}$ (or, more generally, commutes with $\mathcal{A}_t$). In such a situation the derivation of the generator is exactly the same as in the classical case (cf. § (6)). This remark has been used by K. Osterwalder and R. Schrader to prove a Feynman–Kac formula for boson–fermion models in euclidean quantum field theory [27].

Remark 3. The proof of the semi–group property carries over, without difficulties, in the assumptions of Remark (1) after Proposition (2.2), provided the operator $\tilde{M}_t$ commutes with $E_{]-\infty,t]}$.

Definition 3 *A 1–parameter family* (M_t) *of operators in* $\mathcal{A}$ *(resp. affiliated to* $\mathcal{A}$*) satisfying (26) and (27) will be called a markovian cocycle.*

A markovian cocycle will be called positive (resp. hermitian, unitary,...) if for each tM_t *is positive (resp. hermitian, unitary,...).*

7 Markovian cocycles: classical case

The results of § (1), (2), (4) do not depend on the structure of the algebra $\mathcal{A}$. From now, till to § (13) included, we shall assume that the algebra $\mathcal{A}$ is abelian and, to fix the ideas, we shall assume that the algebras $\mathcal{A}_I$, $I \subseteq \mathbb{R}$ (or $\mathbb{R}^+$), the endomorphisms u_t and r_t, the conditional expectations E_I come from a Markov process (x_t)–$t \in \mathbb{R}$ or $\mathbb{R}^+$ — on a probability space (Ω, θ, μ) and with state space $(S, \mathcal{B})$, in the way described in § (0).

The action of u_t, r_t, E_I is extended to all positive measurable functions on Ω and to all measurable functions for which it makes sense.

We shall freely use the notations of § (0), and use the notation

$$F \hat{\in} \theta_I \text{ or equivalently } F \hat{\in} \mathcal{A}_I$$

to mean that the function F is θ_I–measurable.

The discussion in § (4) implies that, if M_t is a positive markovian cocycle, then

$$P^t = E_{]-\infty,0]} M_t u_t \tag{28}$$

is a positivity preserving semi–group on $\mathcal{A}_0$ and, conversely, if P^t, defined by (28), is a positivity preserving semi–group on $\mathcal{A}_0$ then we can assume, without modifying P^t, that M_t is a positive markovian cocycle.

Denote $\chi_{\Omega[0,t]}$ the support of M_t. Because of (26) $\chi_{\Omega[0,t]}$ is the characteristic function of a set $\Omega_{[0,t]} \subseteq \Omega$ and $\Omega_{[0,t]} \in \theta_{[0,t]}$.

One has

$$M^t = \chi_{\Omega_{[0,t]}} e^{-U_{[0,t]}} \tag{29}$$

where $U_{[0,t]} \hat{\in} \theta_{[0,t]}$ is a real valued function and one can assume that supp $U_{[0,t]} = \Omega_{[0,t]}$. Denoting

$$\begin{aligned} \chi_{\Omega_{[s,t+s]}} &= u_s(\chi_{\Omega_{[0,t]}}) \\ U_{[s,t+s]} &= u_s(U_{[0,t]}) \end{aligned} \tag{30}$$

the cocycle property (27) is easily seen to be equivalent to the relations

$$\Omega_{[0,t+s]} = \Omega_{[0,t]} \cap \Omega_{[t,t+s]} \tag{31}$$

$$U_{[0,t+s]} = U_{[0,t]} + U_{[s,t+s]} \tag{32}$$

Conversely, if $\chi_{\Omega[s,t]}$, $U_{[s,t]} \hat{\in} \Omega_{[s,t]}$ satisfy (30), (31), (32), then M_t, defined by (29), is a positive Markovian cocycle. A map

$$[s,t] \subseteq \mathbb{R} \mapsto U_{[s,t]} \hat{\in} \theta_{[s,t]}$$

satisfying (30) and (32) is called a covariant additive functional with respect to the family of σ–algebras $\theta_{[s,t]}$.

A typical example of additive functional is given by

$$U_{[0,t]} = \int_0^t V_s ds \tag{33}$$

where

$$V_0 \hat{\in} \mathcal{A}_0 \ ; \quad V_s = u_s(V_0) \tag{34}$$

and the integral is meant in measure.

The problem of classifying the additive functionals of a given Markov process has been widely studied in the literature (cf. for example, [1]).

An example of a family $\Omega_{[s,t]}$ of sets in $\theta_{[s,t]}$ satisfying (30) and (31) can be constructed as follows: choose a separable realization of the process (x_t) (cf. [35]) and, for some set $S_0 \subseteq S$ (the state space of the process) define

$$\Omega_{[s,t]} = \bigcap_{s \le r \le t} x_r^{-1}(S_0)$$

then $\Omega_{[s,t]}$ is in $\theta_{[s,t]}$ and, clearly, conditions (30) and (31) are satisfied.

8 Formal generators

Let $\{\mathcal{A}, (\mathcal{A}_I), (E_I), (u_t)\}$ be as in the preceeding paragraph, and (M_t) be a markovian cocycle. Denote

$$P_0^t = E_{]-\infty,t]} u_t : \mathcal{A}_0 \to \mathcal{A}_0 \tag{35}$$

$$P^t = E_{]-\infty,t]} M_t u_t : \mathcal{A}_0 \to \mathcal{A}_0 \tag{36}$$

In this § we show that there is a simple connection between the formal generators of P_0^t and P^t. This connection constitutes the essence of the "Feynmann–Kac formula". Under various analytical assumptions on P_0^t and on M_t, some of which will be considered later, this connection becomes a rigorous one. The main idea is contained in the following formal manipulations.

Let us denote:

$$H_0 = \lim_{t \downarrow 0} \frac{1}{t} \left(P_0^t - 1 \right) \tag{37}$$

$$H = \lim_{t \downarrow 0} \frac{1}{t} \left(P^t - 1 \right) \tag{38}$$

(here and in the following we shall not specify neither the topologies in which the limits are taken, nor the subsets of $\mathcal{A}_0$ in which such limits exist). Let $f_0 \in \mathcal{A}_0$, then

$$Hf_0 = \lim_{t\downarrow 0} \frac{1}{t} \{P^t f_0 - f_0\} = \lim_{t\downarrow 0} \frac{1}{t} \{E_{]-\infty,0]} M_t u_t f_0 = f_0\} =$$

$$= \lim_{t\downarrow 0} \frac{1}{t} \{E_{]-\infty,0]} u_t f_0 - f_0\} +$$

$$+ \lim_{t\downarrow 0} \frac{1}{t} \{E_{]-\infty,0]} M_t u_t f_0 - E_{]-\infty,0]} u_t f_0\} =$$

$$= \lim_{t\downarrow 0} \frac{1}{t} \{P_0^t f_0 - f_0\} + E_{]-]\infty,0]} \left\{ \lim_{t\downarrow 0} \frac{1}{t} [M_t - t] u_t f_0 \right\} =$$

$$= H_0 f_0 + A_0 f_0$$

where A_0 denotes the operator of multiplication by the function

$$A_0 = \lim_{t\downarrow 0} \frac{1}{t} (M_t - 1) = \frac{d}{dt}\Big|_{t-0} M_t \tag{39}$$

thus the required connection is given by

$$H = H_0 + A_0 \tag{40}$$

where A_0 is given by (39). For example, if

$$M_t = e^{-\int_0^t V_s ds} \tag{41}$$

a formal derivation of the right hand side gives

$$A_0 f = -V_0 f \ ; \qquad H = H_0 - V_0$$

One can prove that, whenever $H_0 - V_0$ makes sense as a well defined operator on a certain domain, the semi–group (36), with M_t given by (41), can be defined. There are, however, situations in which the semi–group is well defined even if the operator $H_0 - V_0$ is not. In such cases the right hand side of (40) is well defined by (38), while the left hand side is not and the above procedure can be considered as defining a “generalized sum” of the operators H_0 and $-V_0$.

9 Kernels

Let us show that if the semi–group P_0^t defined by (35) has an integral kernel then the perturbed semi–group P^t, defined by (36), has an integral kernel too, of which we give a functional representation. The existence of such a functional representation of the integral kernel of the perturbed semi–group is what makes the Feynman–Kac formula such a powerfull tool in estimates on eigenvalues or eigenvectors of the perturbed generator $H = H_0 + A_0$.

Let us consider the conditional expectation

$$E_{\{t,0\}} : \mathcal{A} \to \mathcal{A}_{\{0,t\}} = \mathcal{A}_0 \vee \mathcal{A}_t$$

If $f_0 \in \mathcal{A}_0$, then

$$P^t f_0 = E_{]-\infty,0]} M_t u_t f_0 = E_{\{0\}} M_t u_t f_0 = \tag{42}$$

$$= E_{\{0\}} E_{\{0,t\}} (M_t u_t f_0) = E_{\{0\}} E_{\{0,t\}} (M_t) u_t (f_0)$$

Recall, from § (5), that we have introduced the identification

$$\mathcal{A}_0 \cong L^\infty(\mathcal{S}, m_0) \tag{43}$$

where $S = (S, \mathcal{B})$ is a measurable space and m_0 is a positive, finite or σ–finite measure. Assume, moreover, that P_0^t has an integral kernel:

$$(P_0^t f_0)(x_0) = \int_S p_t(x_0, y) f_0(y) m_0(dy) \tag{44}$$

This implies that, if $F \hat{\in} \mathcal{A}^+_{\{0,t\}}$, then

$$E_{\{0\}}(F)(x_0) = \int_S p_t(x_0, y) F(x_0, y) m_0(dy)$$

in particular (42) implies that

$$(P^t f_0) = E_{\{0\}}(E_{\{0,t\}}(M_t) u_t(f_0)) = \tag{45}$$

$$= \int_S p_t(x_0, y) E_{\{0,t\}}(M_t)(x_0, y) f_0(y) m_0(dy)$$

thus, if P_0^t has a kernel $P_t(x, y)$, then P^t has a kernel $k_t(x, y)$ given by

$$k_t(x, y) = p_t(x, y) E_{\{0,t\}}(M_t)(x, y) \tag{46}$$

where, s remarked in § (5), we have identified the elements of $L^\infty(\Omega, \theta_{\{s,t\}}, \mu_{\{s,t\}})$ with ($\mu_{\{s,t\}}$–classes of) functions $S \times S \to C$. In the following we shall assume that

$$p_t(x,y) > 0 \qquad m_0 \otimes m_0 \quad \text{a.e.} \tag{47}$$

Since $E_{\{0,t\}}$ is the conditional expectation, on $\mathcal{A}_{\{0,t\}}$, of the measure $\mu = m_0.\ E_{\{0\}}$, one can easily verify that the action of $E_{\{0,t\}}$ on a functional $F = F(x_0, x_{t_1}, \dots, x_{t_n}, x_t)$ depending on the finite set of random variables x_0, $x_{t_1}, \dots, x_{t_n}, x_t$, with $0 < t_1 < \dots < t_n < t$ is given by

$$E_{\{0,t\}}(F)(y_0, y_t) = \frac{1}{p_t(y_0, y_t)} \int_S m_0(dy_{t_1}) \int_S \dots \int_S m_0(dy_{t_n}) \cdot$$

$$\cdot F(y_0, y_{t_1}, \dots, y_{t_n}, y_t) \cdot p_{t_1}(y_0, y_{t_1}) p_{t_2 - t_1}(y_{t_1}, y_{t_2}) \cdot \dots \cdot p_{t - t_n}(y_{t_n}, y_t)$$

for example, if $F = F(x_s)$ depends only on the random variable x_s, one has

$$E_{\{0,t\}}(F)(y_0, y_t) = \tag{48}$$

$$= \frac{1}{p_t(y_0, y_t)} \int_S m_0(dy_s) p_s(y_0, y_s) p_{t-s}(y_s, y_t) \cdot F(y_s)$$

Useful estimates on the kernel $k_t(x,y)$ can be obtained by using its explicit form, given by (46) and the considerations above. For example, let M_t be of the form

$$M_t = e^{-\int_0^t V_s ds} \ ; \qquad V_s = u_s(V)$$

for some measurable function $V : S \to \mathbb{R}$ (recall that

$$\mathcal{A}_0 \cong L^\infty(\mathcal{S}, \mathcal{B}))$$

We can always write V in the form

$$V = V^{(c)} - V^{(-c)} \ ; \quad \text{with} \quad V^{(c)} \geq -c \ ; \quad V^{(-c)} \geq c$$

where c is some constant. From (46) we obtain

$$k_t(x,y) \leq e^{ct} p_t(x,y) E_{\{0,t\}}(e^{\int_0^t V_s^{(-c)}})(x,y) \tag{49}$$

and, applying Jensen's inequality:

$$k_t(x,y) \leq e^{ct} p_t(x,y) \frac{1}{t} \int_0^t ds E_{\{0,t\}}(e^{t V_s^{(-c)}})(x,y) \tag{50}$$

or, equivalently:

$$K_t(x,y) \le e^{ct}\frac{1}{t}\int_0^t ds \int_S m_0(dz)p_s(x,z)p_{t-s}(z,y)e^{tV^{(-c)}(z)} \tag{51}$$

The estimates (50), (51) are useful to derive from the properties of $p_t(x,y)$ the corresponding, or slightly weaker, properties of $k_t(x,y)$.

Examples of the use of these estimates will be given in the following §'s. For example, if $p_t(x,y)$ is the Wiener kernel and $m_0(dy) = dx$ is the Lebesgue measure, then, if $V^{(-c)} \in L^p(\mathbb{R}, dx)$ for some $p \ge 1$, the right hand side of (51) is finite.

10 Action on $L^p(S, m_0)$

In the following we shall denote $\|\cdot\|_{P,q}$ the norm of a linear operator from $L^p(S,m_0)$ to $L^q(S,m_0)$. By construction, on $L^\infty(S,m_0)$:

$$P_0^t f = E_{\{0\}} u_t f \tag{52}$$

Hence $\|P_0^t\|_{\infty,\infty} = 1$. If $f \in L^1 \cap L^\infty(S,m_0)$, $f \ge 0$, then

$$\|P_0^t f\|_1 = \int\int p_t(x,y)f(y)m_0(dx)m_0(dy)$$

Therefore, if

$$\sup_{y\in S}\int_S p_t(x,y)m_0(dx) < +\infty \qquad (\text{resp. } \le 1) \tag{53}$$

by interpolation P_0^t can be extended to a bounded operator (resp. a contraction) $L^p(m_0) \to L^p(m_0)$ for each $p \in [1,+\infty]$. Remark that, if the kernel is symmetric, i.e.

$$p_t(x,y) = p_t(y,x)$$

then $\|P_0^t\|_{1,1} = \|P_0^t(1)\|_\infty = 1$.

We shall not discuss here the important property of hypercontractivity (resp. supercontractivity) of P_0^t, i.e. $\|P_0^t\|_{P,q} \le 1$ for appropriate t and $q > p > q$ (resp. for all $t > 0$ and all $q > p > 1$). These are a far–reaching generalization of the property of "instantaneous smoothing" of the Wiener semi–group ($P_0^t\{L^p(\mathbb{R},dx)\} \subseteq \mathcal{C}^\infty(\mathbb{R},dx)$) and have played a fundamental role in recent researches (cf. [7], [16], [22], and the bibliography therein).

Assuming $\|P_0^t\|_{1,1} < +\infty$, from the explicit form (46) of the kernel of P^t one immediately deduces that

$$\|E\|_{\{0,t\}}(M_t)\|_\infty < +\infty \tag{54}$$

is a sufficient condition for P^t to be a bounded operator $L^p(S,m_0) \to P^P(S,m_0)$ for every $p \in [1,+\infty]$. For example, if M_t has the form

$$M_t = e^{-\int_0^t V_s ds}$$

then, by Jensen's inequality

$$E_{\{0,t\}}(e^{-\int_0^t V_s ds}) \leq \frac{1}{t}\int_0^t ds E_{\{0,t\}}(e^{-tV}s) = \tag{55}$$

$$= \frac{1}{p_t(x_0,x_t)} \cdot \frac{1}{t}\int_0^t ds \int_S m_0(dx_s) e^{-tV(x_s)} p_s(x_0,x_s) p_{t-s}(x_s,x_t)$$

and the uniform boundedness of the right hand side of (55) (mod. m_0) gives a sufficient condition for (54) to be verified. If $S = \mathbb{R}$, $m_0(dx) = dx$ (Lebesgue measure) and $p_t(x,y)$ is the Wiener kernel, a simple computation shows that

$$e^{-tV(-c)} \in L^p(\mathbb{R},dx) \tag{56}$$

for some $t > 0$ and $p > 1$, is a sufficient condition for the uniform boundedness of the right hand side of (55). In some cases instead of (56) one can derive a weaker condition of the form

$$e^{-tV(-c)} \in L^P_{\text{loc}}(\mathbb{R},dx)$$

by coupling the above mentioned estimate with a probabilistic argument (with very low probability a particle goes outside a sufficiently large region in the finite interval $[0,t]$ (cf. [7] for the case of the Wiener measure).

If $m(S) < +\infty$ then one can deduce the continuity of the action of P_0^t (resp. P^t) from $P^P(S,m)$ to $L^q(S,m)$, using the explicit form of the kernels and a general criterion due to L.V. Kantorovich (cf. [20], pg. 417) according to which if, for τ, $\sigma > 0$, there are constants c_1, c_2 such that

$$\int_S |k(x,y)|^\tau m(dy) \leq c_1$$

$$\int_S |k(x,y)|^\sigma m(dx) \le c_2$$

then $p^t : L^p \to L^q$ is bounded of each p, $q \ge 1$ such that $q \ge p$; $q \ge \sigma$; $(1 - \sigma/q)p' \le r$ (here and in the following we shall use the notation: $p' = p/p - 1$ for $p > 1$, $p \ne \infty$).

11 Self–adjointness and reflections

Self — adjointness properties of the semi–groups P_0^t, P^t are related to time — reflections, as shown by the following results.

Proposition 2 *Let $T = \mathbb{R}^+$; assume that $\{\mathcal{A}, (\mathcal{A}_I)\}$ admits a 1–parameter family of reflections (r_t) which leave μ invariant; then*
i) P_0^t is self–adjoint
ii) $P^{t*} = E_{\{0\}} r_t(M_t) u_t$
in particular, if $r_t(M_t)^* = M_t$, then P^t is self–adjoint.

Remark. By self–adjointness of an operator $A : L^\infty(S, m_0) \to L^\infty(S, m_0)$, here we mean that

$$\langle f, Ag \rangle = \mu(f^* \cdot [A\ g]) = \mu([A\ f]^* \cdot g) = \langle Af, g \rangle$$

for each $f, g \in L^1 \cap L^\infty(S, m_0)$, ($*$ means complex conjugation).

Proof. Let a_0, $b_0 \in L^1 \cap L^\infty(S, m_0)$, then

$$\begin{aligned} \langle a_0, P^t b_0 \rangle &= \mu(a_0^* E_{\{0\}} M_t u_t(b_0)) = \mu(a_0^* M_t u_t(b_0)) = \\ &= \mu(r_t(a_0^*) r_t(M_t) b_0) = \mu(u_t(a_0^*) r_t(M_t) b_0) = \\ &= \langle E_{\{0\}} r_t(M_t)^* u_t a_0, b_0 \rangle \end{aligned}$$

which proves (ii). For $M_t = 1$ we have (i).

Proposition 3 *Let $T = \mathbb{R}$; assume that $\{\mathcal{A}, (\mathcal{A}_I)\}$ admits a reflection r which leaves μ invariant. Then*
i) P_0^t is self–adjoint
ii) $P^{t*} = E_{\{0\}} u_t r(M_t)^*$
in particular, if $u_t r(M_t) * = M_t$, then P^t is self–adjoint.

Proof. Let a_0, $b_0 \in L^\infty \cap L^1(S, m_0)$, then

$$\begin{aligned}\langle a, P^t b_0\rangle &= \mu(a_a^* M_t u_t(b_0)) = \mu(a_0^* r(M_t) u_{-t}(b_0) = \\ &= \mu([r(M_t)^* a_0]^* \cdot u_{-t}(b_0)) = \\ &= \langle E_{\{0\}} u_t[r(M_t)^* a_0], b_0\rangle\end{aligned}$$

which proves (ii) and, for $M = 1$, (i).

Remark 1. In the proofs above no use has been made of the commutativity of the algebras $\mathcal{A}_I$; r_t, r above are automorphisms of $\mathcal{A}$. In the classical case, M_t is a positive function, and $M_t^* = M_t$.

Remark 2. The typical example of M_t, in the classical case:

$$M_t = \exp - \int_0^t V_s ds = \exp - \int_0^t u_s V_0$$

satisfies both the conditions of Proposition (9.1) and (9.2).

Remark 3. Assertion (i) in both Proposition (9.1) and (9.2) admits a partial converse, in the sense that, if P_0^t is the semi–group associated to a Markov process and P_0^t is self–adjoint, then one can construct a 1–parameter family of reflections (resp. a reflection in case $T = \mathbb{R}$) for the local algebras of the process.

12 Strong continuity in $L^p(S, m)$

Let $(\Sigma, \mathcal{H}, \nu)$ be a finite or σ–finite measure space. A semi–group Q^t is said to be strongly continuous on $L^1 \cap L^\infty(\Sigma, \nu)$ if

$$\lim_{t\downarrow 0} Q^t f = f \quad \forall\, f \in L^1 \cap L^\infty(\Sigma, \nu) \tag{57}$$

the limit being meant in ν–measure. In the analysis of this notion, we shall follow [22].

Lemma 1 *If Q^t is strongly continuous in measure on $L^1 \cap L^\infty(\Sigma, \nu)$ and the map*

$$t \in]0, b[\to \|Q^t\|_{q,q} \tag{58}$$

is bounded for every $0 < b < +\infty$ and $q \in [1, +\infty]$, then Q^t is strongly continuous on $L^p(\Sigma, \nu)$ for every $p \in]1, +\infty[$. If, moreover, $m(S) < +\infty$, then the result holds for $p = 1$ too.

Proof. Let $p \in [1, +\infty[$. It will be sufficient to prove that $Q^t f \to f$ as $t \downarrow 0$ for every $f \in L^1 \cap L^\infty(\Sigma, \nu)$, since $t \mapsto \|Q^t\|_{p,p}$ is locally bounded.

Let $\varepsilon, \delta > 0$ be given. For $f \in L^1 \cap L^\infty(\Sigma, \nu)$ let $B \subseteq \Sigma$ be a measurable set such that:

$$\nu(\Sigma - B) < \delta \ ; \qquad |Q^t f(x) - f(x)| < \varepsilon$$

for $x \in B$ and t small enough. Such a B exists by strong continuity in measure. One has:

$$\begin{aligned} \|Q^t f - f\|_P &\leq \chi_{lB}(Q^t f - f)\|_P + \|\chi_B(P^t f - f)\|_P \leq \\ &\leq Q^t f - f\|_\infty \nu(\Sigma - B)^{1/P} + \left\{ \int_B |Q^t f - f|^P d\nu \right\}^{1/P} \leq \\ &\leq \{Q^t f\|_\infty + \|f\|_\infty\} \delta^{1/P} + \varepsilon^{\frac{P-1}{P}} \{\|Qt\|_1 + \|f\|_1\}^{1/P} \end{aligned}$$

Since ε, δ are arbitrary, (58) implies the result.

If $\nu(\Sigma) < +\infty$, the assertion for $p = 1$ follows from

$$\|Q^t f - f\|_1 \leq \{\|Q^t f\|_\infty + \|f\|_\infty\}\delta + \varepsilon\nu(S)$$

Remark. The result of Lemma (10.1) is false, in general, for $p = +\infty$, even if $M(S) < + + \infty$. The Wiener semi–group is a well–known conerexample. In this case, in fact,

$$P^t f(x) = \int_{\mathbb{R}} \frac{e^{-(x-y)^2/2t}}{(2\pi)^{1/2}} f(y) dy \ ; \quad x \in [0,1]$$

thus $P^t f$ is continuous for every $f \in L^1 \cap L^\infty([0,1], dx)$ and there–$\|P^t f - f\|_\infty \to 0$, as $t \downarrow 0$, if and only if f coincides with a continuous function outside a set of Lebesgue measure 0. This fact implies that P^t is not strongly continuous on $L^\infty([0,1], dx)$, but, being P^t a contraction semi–group on $L^p([0,1], dx)$ for $p \in [1, \infty]$, that it is strongly continuous on $L^p([0,1], dx)$ for $p \in [1, +\infty[$.

The assumptions made in § imply that (u_t) acting on $L^p(\Omega, \theta, \mu)$ satisfies the conditions of Proposition (10.1). Therefore also

$$P_0^t = E_{\{0\}} u_t$$

satisfies these conditions. Concerning the perturbed semi–group $P^t = E_{\{0\}} M_t u_t$, an obvious sufficient condition for strong continuity in measure on $L^1 \cap L^\infty(E, m_0)$ is that

$$\lim_{t \downarrow 0} |P^t f - P_0^t f| = \lim_{t \downarrow 0} |E_{\{0\}}([M_t - 1] u_t(f))| = 0 \tag{59}$$

in measure for each $f \in L^1 \cap L^\infty(S, m_0)$.

Lemma 2 *The condition (59) is satisfied if*

$$\lim_{t \downarrow 0} E_{\{0\}}(M_t) = 1 \ ; \qquad m_q \ a.l. \tag{60}$$

and if, for some $a > 0$ and $M = M(a) \in \mathbb{R}$, one of the following conditions is satisfied:

$$\|M_t\|_{L^\infty(\Omega,\mu)} \le M \ ; \quad \forall\, t \in]0, a[\tag{61}$$

$$\|E_{\{0\}}(M_t^P)\|_\infty \le M \ ; \quad \forall\, t \in]0, a[, \ \textit{for some } p \in]1, +\infty[\tag{62}$$

$$\|E_{\{0,t\}}(M_t)\|_{L^\infty(\Omega,\mu)} \le M \ ; \quad t \in]0, a[\tag{63}$$

$$\|E_{\{0\}}(M_t^P)\|_1 \le M, \ \forall\, t \in]0, a[, \ \textit{for some } p \in]1, +\infty[\tag{64}$$

Proof. Let $f \in L^1 \cap L^\infty(S, m_0)$; one has:

$$|P^t f - P_0^t f| = |E_{\{0\}}([M_t - 1] u_t(f))| \le \tag{65}$$

$$\le |E_{\{0\}}(M_t[u_t f -])| + |E_{\{0\}}(M_t) \cdot f - P_0^t f|$$

Now, $|E_{\{0\}}(M_t[u_t f - f])|$ is ≤ 1 than any of the following three quantities:

$$\|M_t\|^\infty \cdot E_{\{0\}}(|u_t f - f|)$$

$$E_{\{0\}}(M_t^P)^{1/P} E_{\{0\}}(|u_t - f|^{P'})^{1/P'}$$

$$\|E_{\{0,t\}}(M_t)\|_\infty \cdot E_{\{0\}}(|u_t f - f|)$$

hence, if any of the conditions (61), (62), (64), is satisfied, it tends to zero as $t \downarrow 0$.

Moreover $|E_{\{0\}}(M_t) f|$ is $\le$ then any of the following three quantities;

$$\|M_t\|_\infty \cdot |f| \ ; \quad E_{\{0\}}(M_t^P)^{1/P} |f| \ ; \qquad \|E_{\{0,t\}}(M_t)\|_\infty \cdot |f|$$

Thus, if any of the conditions (61), (62), (63) is satisfied, it tends to zero in $L^1(S,m)$ by (60) and dominated convergence; hence in measure.

Concerning the condition (64), we remark that, if $f \in L^1 \cap L^\infty(S,m)$, then

$$m|E_{\{0\}}(M_t[u_t f - f])| \le \mu(M_t|u_t f - f|) \le \|M_t\|_P \|u_t f - f\|_{P'} = \|E_{\{0\}}(M_t^P)\|_1^{1/P} \cdot \|u_t f - f\|_{P'}$$

Moreover, if $B \subseteq S$, then

$$m(E_{\{0\}}(M_t) f_{\chi_B}) \le \|E_{\{0\}}(M_t^P)\|_1^{1/P} \|f\|_\infty^{P'/1} m(|f|_{\chi_B})$$

therefore, if (64) holds, the family of functions

$$\{E_{\{0\}}(M_t) f : t \in]0, a[\}$$

is uniformly integrable with respect to m. Hence by Vitali's theorem and (60), $E_{\{0\}}\ (M_t) f \to f$ in $L^1(m)$ as $t \downarrow 0$.

Remark. The conditions (61), (62), (63) also imply that (58) is satisfied; hence that P^t is strongly continuous on $L^P(S, m_0)$.

There are many sufficient conditions which assure that the $H_0 + V_0$ is well defined as an operator (or as a quadratic form), and that the equality $H = H_0 - V_0$ holds on a core for these operators (resp. in the sense of quadratic forms). We shall not discuss these conditions here, and refer to the papers [7], [11], [22], and the bibliography therein.

13 Compactness conditions

The knowledge of he explicit form of the kernel $k_t(x,y)$ of the semi–group P^t allows one to apply to P^t the known compactness criteria for linear operators from $L^P(S, m_0)$ to $L^q(S, m_0)$. For example, if

$$m_0(S) < +\infty \tag{66}$$

then a sufficient condition for the compactness of $P^t : L^P(S, m_0) \to L^q(S, m_0)$ is that

$$\int \int |k_t(x,y)|^{r'} m_0(dx) m_0(dy) < +\infty \tag{67}$$

with $1 \le q < +\infty$, $r = \min(p, q')$, $1/r + 1/r' = 1$ (cf. [20], pg. 425).

Therefore, if $m_0(S) < +\infty$ and if

$$\int\int |p_t(x,y)|^{r'} m_0(dx)m_0(dy) < +\infty \qquad (68)$$

the same kind of conditions which guarantee the continuity of $P^t : L^P(S,m_0) \to L^q(S,m_0)$ (cf. § (8)) will guarantee the compactness. For example, under the assumptions (66) and (68, the condition

$$\|E_{\{0,t\}}(M_t)\|_\infty < +\infty$$

is a sufficient condition for the compactness of $P^t : L^P(S,m_0) \to L^q(S,m_0)$.

If $m_0(S) = +\infty$, the compactness of $P^t : L^p(S,m_0) \to L^q(S,m_0)$ is equivalent to compactness on regions of finite measure, plus a "tail condition" which is formalized as follows:

Lemma 3 *A linear operator $A : L^p(S,m_0) \to L^q(S,m_0)$ is compact if and only if there exists a sequence (S_n) such that*

$$S_n \subseteq S_{n+1} \subseteq S \ ; \quad m_0(S_n) < +\infty \ ; \quad \bigcup_n S_n = S \qquad (69)$$

$$\chi_{S_N} A \textit{ is compact for each } n;\ \chi_{S_n}(x) = \{\, 0\ x \notin S_n 1 x \in S_n \qquad (70)$$

$$\lim_n \|\chi_{S-S_n} A\|_{p,q} = 0 \qquad (71)$$

Proof. *Sufficiency*. $\chi_{S-S_n}A = A - \chi_{S_n}A$. Hence, if conditions (69), (70), (71) are satisfied, A is norm limit of compact operators, therefore compact.

Necessity. If A is compact, $\chi_{S'}A$ is compact for each $S' \subseteq S$. Let $\varepsilon > 0$; B_1–the unit ball in $L^P(S,m_0)$; $f_1,\dots,f_n$ an $\varepsilon/3$ net for $A(B_1) \subseteq L^q(S,m_0)$. Let $S_1 \subseteq S$ be such that $m_0(S_1) < +\infty$, and

$$\|chi_{S-S_1} f_j\|_q < \varepsilon/3 \ ; \quad j = 1,\dots,n$$

Then, for each $g \in B_1$, one has

$$\|\chi_{S-S_1} Ag\|_q \leq \varepsilon \qquad (72)$$

In fact, if $g \in B_1$ is such that (72) is false, then for some $j = 1,\dots,n$:

$$\varepsilon/3 \geq \|f_j - Ag\|_q \geq \|chi_{S-S_1}(f_j - Ag)\|_q \geq$$

$$\geq | \; \|\chi_{S-S_1} Ag\|_q - \|\chi_{S-S_1} f_j\|_q | \geq \varepsilon - \varepsilon/3 = 2/3\varepsilon$$

which is absurd. And this ends the proof.

Let us consider the "tail estimate" (71) for the semi–group P^t. We shall deal only with the estimate of $\|\chi_{S-S_1} P^t\|_{q,q}$; the method for the estimate of $\|\chi_{S-S_1} P^t\|_{p,q}$, $p \neq q$, is similar.

Let $f \in L^q(S, m_0)$. Then if $S_1 \subseteq S$

$$\|\chi_{S-S_1} P^t f\|_q^q \leq \int \chi_{S-S_1}(x) m_0(dx) \int k_t(x,y)^q |f(y)|^q m_0(dy) =$$

$$= \int |f(y)|^q m_0(dy) \int \chi_{S-S_1}(x) k_t(x,y)^q m_0(dx)$$

therefore

$$\|chi_{S-S_1} P^t\|_{q,q} \left\{ \sup_{y \in S} \int_{S-S_1} k_t(x,y)^q m_0(dx) \right\}^{1/q} \leq \tag{73}$$

$$\leq \left\{ \sup_{y \in S} \int_{S-S_1} p_t(x,y)^q E_{\{0,t\}}(M_t^q)(x,y) m_0(dx) \right\}^{1/q}$$

If M_t has the form

$$M_t = e^{-\int_0^t V_s ds} \tag{74}$$

or is majorized by a functional of this form, then, using Jensen's inequality as in formula (51) we obtain, using (48):

$$\|\chi_{S-S_1} P^t\|_{q,q}^q \leq \sup_{y \in S} \int_{S-S_1} m_0(dx) \frac{1}{t} \int_0 ds \int_S m_0(dz) \cdot p_s(x,z) p_{t-s}(z,y) e^{-tqV(z)} \tag{75}$$

In order to estimate

$$\int_{S-S_1} m_0(dx) \int_S m_0(dz) p_s(x,z) p_{t-s}(z,y) e^{-tqV(z)}$$

let us write it in the form:

$$\int_{S^2} m_0(dz) p_s(S - S_1, z) p_{t-s}(z,y) e^{-tqV(z)} +$$

$$+ \int_{S-S_2} m_0(dz) p_s(S - S_1, z) p_{t-s}(z,y) e^{-tqV(z)}$$

where $S_2 \subseteq S_1$ and we use the notation

$$p_s(S - S_1, z) = \int_{S-S_1} m_0(dx) p_s(x, z)$$

The first integral is majorized by

$$\varepsilon(S_2, S - S_1; s) \int_S m_0(dz) p_{t-s}(z, y) e^{-tqV(z)}$$

where

$$\varepsilon(S_2, S - S_1; s) = \sup\{p_s(S - S_1, z) : z \in S_2\}$$

and the second integral is majorized by

$$\delta(V, S - S_2) \int_0^t ds \int_S m_0(dz) p_s(S - S_1, z) p_{t-s}(z, y) =$$

$$= \delta(V, S - S_2) t \cdot p_t(S - S_1, y)$$

where

$$\delta(V, S - S_2) = \sup\{e^{tqV(z)} : z \in S - S_2\}\ .$$

Now, if $V(z) \to +\infty$ as $z \to +\infty$ (in the sense that $\forall \lambda > 0$ there exists $S_2 \subseteq S$, $m_0(S_2) < +\infty$, such that $V(x) > \lambda$ for $x \in S - S_2$), $\delta(V, S - S_2)$ can be made arbitrarily small by choosing S_2 large enough.

The condition that $\varepsilon(S_2, S - S_1; s) \to 0$ uniformly in $s \leq t$, for S_2 and S_1 sufficiently large has a simple probabilistic interpretation: let, for simplicity, S be a metric space and interpret the Markov process defined by m_0 and P_0^t (according to § (3)) as describing the motion of a particle in the "position space" S. Then $p_s(x, y)$ is the probability density that the particle jumps from position x at time 0 to position y at time s. If S_2 is the ball centered at an arbitrary point $x_0 \in S$ and with radius $a > 0$, and S_1 is the ball centered at x_0 and with radius $a + d$, then the condition

$$\sup_{s \leq t} \varepsilon(S_2, S - S_1, s) \to 0 \text{ as } a \to +\infty,\ d \to +\infty$$

means that the probability that in a time $\leq t$ the particle makes a jump of "lenght" $\geq d$ becomes negligible as$d \to +\infty$.

14 Asymptotic estimates

Assume that $P^t : L^2(S, m_0) \to L^2(S, m_0)$ is compact self–adjoint. Then there is an orthonormal basis (Φ_n) in $L^2(S, m_0)$ and an increasing divergent sequence $\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_n \ldots$ of real numbers such that

$$k_t(x, y) = \sum_n e^{-\lambda_n t \overline{\Phi_n(x)} \Phi_n(x)} \tag{76}$$

The fundamental idea of M. Kac (cf. [18], [29]) is to compare the classical expansion (76) to the representation of $k_t(x, y)$ in terms of functional integrals

$$k_t(x, y) = p_t(x, y) E_{\{0,t\}} \left(e^{-\int_0^t V_s ds} \right) (x, y) \tag{77}$$

in order to obtain informations on the asymptotic behaviour, as $\lambda \to +\infty$, of the quantity

$$N(\lambda) = \sum_{\lambda_n < \lambda} 1 \tag{78}$$

representing the number of eigenvalues of the generators of P^t.

Remark that

$$k_t(x, x) = \sum_n e^{-\lambda_n t} |\Phi_n(x)|^2 = \int_0^\infty e^{-\lambda^t} dN_x(\lambda) \tag{79}$$

where

$$N_x(\lambda) = \sum_{\lambda_n < \lambda} |\Phi_n(x)|^2 \tag{80}$$

and, because of (77)

$$k_t(x, y) = p_t(x, x) E_{\{0,t\}} (e^{-\int_0^t V_s ds})(x, x) \tag{81}$$

Thus, if

$$E_{\{0,t\}} (e^{-\int_0^t V_s ds})(x, x) \to 1, \text{ as } t \to 0 \tag{82}$$

and this is surely the case if, for example, $V_0 \in L^p(S, m_0)$ for some $p \in [1, +\infty]$ (cf. [22], Lemma (3.2)), then

$$\int_0^\infty e^{-\lambda t} dN_x(\lambda) \sim p_t(x, x), \text{ as } t \to 0$$

where, here and in the following, $a_t \sim b_t$, as $t \to 0$, means

$$\lim_{t \to 0} a_t / b_t = 1$$

Therefore, if $p_t(x, y)$ (i.e. the kernel of P_0^t) satisfies

$$p_t(x, x) \sim \frac{A}{t^y}\ , \ \text{ as } t \to 0^+ \tag{83}$$

for $A > 0$ and $\gamma > 0$, the Hardy–Littlewood–Karamata theorem ([36], pg. 192) implies that

$$N_x(\lambda) \sim \frac{A\lambda^\gamma}{\Gamma(\gamma+1)}\ , \ \text{ as } \lambda \to +\infty\ . \tag{84}$$

Now assume that $m_0(S) < +\infty$, and that the convergence in (82) is dominated (for example: V_0–bounded below), then (81) and (83) imply that

$$\int_S k_t(x, x) m_0(dx) = \sum_n e^{-\lambda_n t} = \int_0^\infty e^{-\lambda t} dN(\lambda) \sim \tag{85}$$

$$\sim \frac{A}{t^y} m_0(S) \text{ as } t \to 0$$

therefore, again by Hardy–Littlewood–Karamata's theorem

$$N(\lambda) \sim \frac{m_0(S) A}{\Gamma(y+1)} \lambda\gamma \text{ as } \lambda \to +\infty \tag{86}$$

In the case of he Wiener kernel

$$p_t(x, y) = \frac{e^{-|x-y|^2/2t}}{(2\pi t)^{N/2}}\ ; \quad x, y \in \mathbb{R}^N$$

$$p_t(x, x) \sim \frac{1}{(2\pi t)^{N/2}}\ , \ \text{ as } t \to 0$$

If $m_0(S) = +\infty$, the estimate of $N(\lambda)$, for $\lambda \to +\infty$, will depend on the behaviour at infinity of $V = V_0$. The form of this dependence is deduced from the estimates in the following Lemma:

Lemma 4 *In the above notations, one has:*

$$\int_S k_t(x,x)m_0(dx) \leq \tag{87}$$

$$\leq \int_S m_0(dx)p_t(x,x)\frac{1}{t}\int_0^t ds E_{\{0,t\}}(e^{-tV})(x,x)$$

$$\int_S k_t(x,x)m_0(dx) \leq \tag{88}$$

$$\leq \int_S m_0(dx)p_t(x,x)e^{-\int_0^t E_{\{0,x\}}(V_s)(x,x)ds}$$

Proof. Immediate consequence of (77) and Jensen's inequality.

A corollary of the Lemma above is that, if the kernel $p_t(x,y)$ (of P_0^t) is symmetric (i.e. if P_0^t is self–adjoint), then

$$\int_S k_t(x,x)m_0(dx) \leq \int_S m_0(dx)p_t(x,x)\frac{1}{t}\int_0^t ds E_{\{0,t\}}(e^{-tV_s})(x,x) =$$

$$= \int_S m_0(dx)\frac{1}{t}\int_0^t ds \int_S m_0(dz)p_s(x,z)p_{t-s}(z,x)e^{-t(z)} =$$

$$= \int_S m_0(dz)p_t(z,z)e^{-tV(z)}$$

Thus for symmetric $p_t(x,y)$ one has

$$\int_S k_t(x,x)m_0(dx) \leq \int_S m_0(dz)p_t(z,z)e^{-tV(z)} \tag{89}$$

In the following we shall always assume the $p_t(x,y)$ is symmetric. Our aim is now to prove the following basic estimate:

$$\int_S k_t(x,x)m_0(dx) \sim_{(t\to 0)} \int_S p_t(x,x)e^{-tV(x)}m_0(dx) \tag{90}$$

which is the main tool in the estimates of WKB type. This estimate will be established under certain assumptions on the "potential" V, which have a natural probabilistic interpretation. We will prove, using an idea of D. Ray [29], that

$$\int_S m_0(dx)p_t(x,x)e^{-\int_0^t E_{\{0,t\}}(V_s)(x,x)ds} \tag{91}$$

$$\sim_{(t\to 0)} \int_S m_0(dx) p_t(x,x) e^{-tV(x)}$$

and this, together with (88) and (87) implies (90). In the course of the proof we will assume that there is a $T > 0$ such that, for each $0 < t < T$

$$0 < \int_S m_0(dx) p_t(x,x) e^{-tV(x)} < +\infty$$

To prove (91), first remark that

$$\frac{\int_S m_0(dx) p_t(x,x) \exp\left\{-\int_0^t E_{\{0,t\}}(V_s)(x,x)ds\right\}}{\int_S m_0(dx) p_t(x,x) \exp\{-tV(x)\}} \leq 1 \tag{92}$$

because of (88) and (89). Denoting I_t the right hand side of (92), (91) is equivalent to

$$\lim_{t\to 0} I_t = 1 \tag{93}$$

Let us introduce the notation, for $A \subseteq S$

$$\nu_t(A) = \frac{\int_A p_t(x,x) e^{-tV(x)} m_0(dx)}{\int_S p_t(x,x) e^{-tV(x)} m_0(dx)} \tag{94}$$

The ν_t is a probability measure and

$$I_t = \int_S \nu_t(dx) e^{-\left\{\int_0^t ds E_{\{0,t\}} V_s)(x,x) - tV(x)\right\}} \tag{95}$$

Fix a number $0 < \alpha < 1$, and define, for $x \in S$

$$\mathcal{U}_t(x) = \{z \in S : |V(z) - V(x)| < 1/t^a\} \tag{96}$$

One has:

$$\int_S \nu_t(dx) \exp\left\{-(1/t^a)\int_0^t ds \int_{U_t(x)} m_0(dz) \frac{p_s(x,z)p_{t-s}(x,z)}{p_t(x,x)}\right\} \tag{97}$$

$$\cdot \exp\left\{-\int_0^t ds \int_{S-U_t(x)} \frac{p_s(x,z)p_{t-s}(z,x)}{p_t(x,x)} [V(z) - V(x)]\right\} \leq I_t$$

or, equivalently

$$\int_S \nu_t(dx) \exp\left[-t^{1-a}\left\{1 - 1/t \int_0^t ds \int_{S-\iota_t(x)} m_0(dz) \frac{p_s(x,z)p_{t-s}(z,x)}{p_t(x,x)}\right\}\right] \cdot \tag{98}$$

$$\cdot \exp\left[-\int_0^t ds \int_{S-U_t(x)} \frac{p_s(x,z)p_{t-s}(z,x)}{p_t(x,x)} [V(z)-V(x)]\right] \leq I_t$$

Since clearly

$$\int_0^t ds \int_{S-U_t(x)} m_0(dz) \frac{p_s(x,z)p_{t-s}(z,x)}{p_t(x,x)} \leq t$$

it follows that

$$e^{-t^1-a} \cdot \int_S \nu_t(dx) e^{-\int_0^t ds \int_{S-U_t(x)} \frac{p_s(x,z)p_{t-s}(z,x)}{p_t(x,x)} [V(z)-V(x)] m_0(dz)} \leq I_t \tag{99}$$

Let us introduce the assumption (we shall prove elsewhere that it can be weakened)

$$V(x) \geq 0 \ ; \quad m - \forall\, x \in S \tag{100}$$

Then (99) and (100) imply that

$$I_t \geq e^{-t^{1-a}} \int_S \nu_t(dx) e^{-\int_0^t ds \int_{S-U_t(x)} \frac{p_s(x,z)p_{t-s}(z,x)}{p_t(x,x)}} V(z) = \tag{101}$$

$$= e^{-t^{1-a}} + \int_S \nu_t(dx) \left[e^{-\int_0^t ds \int_{S-U_t(x)} m_0(dx) \frac{p_s(x,z)p_{t-s}(z,x)}{p_t(x,x)} V(z)} - 1 \right]$$

Therefore we see that, if V satisfies the "diffusion type" condition

$$label12.27 \sup_{x\in S} \int_0^t ds \int_{S-U_t(x)} m_0(dz) \frac{p_s(x,z)p_{t-s}(z,x)}{p_t(x,x)} V(z) \leq \varrho^{(t)} \tag{102}$$

with $\varrho^{(t)} \to 0$ as $t \to 0$, then

$$\lim_{t\to 0} \int_S \nu_t(dx) \left[e^{-\int_0^t ds \int_{S-U_t(x)} m_0(dx) \frac{p_s(x,z)p_{t-s}(z,x)}{p_t(x,x)} V(z)} - 1 \right] = 0$$

therefore

$$\lim_{t\to 0} \inf I_t \geq 1$$

hence, because of (**??**)

$$\lim_{t\to 0} I_t = 1$$

which is our thesis. Summing up

Theorem 1 *In the assumptions of this §, if*

i) $0 < \int_S m_0(dx) p_t(x,x) e^{-tV(x)} < +\infty$*;* $0 < t < T$

ii) $V \geq 0$

iii) $\sup_{x\in S} \int_0^t ds \int_{S-U_t(x)} m_0(dz) \frac{p_s(x,z)p_{t-s}(z,x)}{p_t(x,x)}\ V(z) \leq \varrho(t)\ (t \overset{\to 0}{\to} 0)$

then the estimate (91), hence (90), holds.

Proof. From the above discussion.

Condition iii) has a simple probabilistic interpretation. First of all, remark that it can be written in the form:

$$\sup_{x\in S} \int_0^t ds E_{\{0,t\}}(\chi^{(s)}_{S-U_t(x)} V_s)(x,x) \leq \varrho^{(t)}\ (t \overset{\to 0}{\to} 0)$$

where χ_A is the characteristic function of the set A, and $\chi^{(s)}_A = u_s(\chi_A)$ where u_s is the shift (cf. § (0) and (1)). Now, the quantity

$$E_{\{0,t\}}(\chi^{(s)}_{S-U_t(x)} V_s)(x,x)$$

defines the expectation value of the observable V_s computed along the trajectories which begin at x at time 0, end up in x at time t, and such that at time s the "particle" has a "position" z, whose "potential energy" satisfies:

$$|V(z) - V(x)| \geq 1/t^a \tag{103}$$

Thus, condition iii) means that these trajectories give contribution to the expectation value the more negligible, the smaller t.

In order words: it is very unlikely that, within an interval of time t, very small, a particle starting from x at time 0 reaches a level of "potential energy" $V(z)$ which differs from $V(x)$ more than $1/t^a$.

15 Quantum case: L^∞–theory

From now on $\{\mathcal{A}, (\mathcal{A}_I)\}$ will be a system of local C^*–algebras. We keep the notations and assumptions of § (1); then the results of § (2) yield a completely positive, identity preserving semi–group

$$Z_0(t) = E_{]-\infty,0]} u_t : \mathcal{A}_0 \to \mathcal{A}_0 \tag{104}$$

Such a semi–group will be called a quantum markovian semi–group.

Applying the perturbation theory of § (4) with a completely positive perturbation $\tilde{M}_t$, we obtain a completely positive semi–group

$$Z(t) = E_{]-\infty,0]}\tilde{M}_t u_t : \mathcal{A}_0 \to \mathcal{A}_0 \tag{105}$$

For a large class of algebras $\mathcal{A}_0$ the infinitesimal generators of completely positive semi–groups (also called quantum dynamical evolutions) have been characterized in a series of papers started with the important results of Gorini – Kossakowski – Sudarshan ($\mathcal{A}_0$–finite dimensional) [15] and Lindblad [23] ($\mathcal{A}_0 = \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and bounded generator), which have been generalized to more general algebras [24], and to the case of unbounded generator [9].

A simple example of completely positive map $M_t : \mathcal{A} \to \mathcal{A}$ is given by

$$\tilde{M}_t(a) = M_t a M_t^* \ , \qquad a \in \mathcal{A} \tag{106}$$

For such a map the conditions (23), (24) are easily seen to be equivalent to

$$M_t a M_t^* \in \mathcal{A}_{[0,t]} \ , \quad \forall\, a \in \mathcal{A}_{[0,t]} \tag{107}$$

$$M_t u_t (M_s a u_t M_s)^* M_t^* = M_{t+s} a M_{t+s}^* \ ; \qquad \forall\, a \in \mathcal{A}_{[0,t]} \tag{108}$$

Therefore, for such a map, if conditions (107) and (108) are satisfied, one can assume, without changing the action of $Z(t)$ on $\mathcal{A}_0$, that

$$M_t \in \mathcal{A}_{[0,t]} \tag{109}$$

$$M_{t+s} = M_t u_t(M_s) \tag{110}$$

Conversely it is obvious that (109) and (110) imply (23), (24) for $\tilde{M}_t$ given by (106). Remark that (109) and (110) are the conditions which, according to Definition (4.1), define a markovian cocycle.

Assuming M_s invertible for each t, we will consider two families of markovian cocycles:

– hermitian markovian cocycles: $M_t = e^{-V_{[0,t]}}$

– unitary markovian cocycles: $M_t = e^{iV_{[0,t]}}$

where in both cases $V_{[0,t]}$ is an hermitian operator in $\mathcal{A}_{[0,t]}$ (if $\mathcal{A}$ is realized as an algebra of operators on some Hilbert space $\mathcal{H}$, then we can allow $V_{[0,t]}$ to be an unbounded self–adjoint operator affiliated to $\mathcal{A}_{[0,t]}$. In the hermitian case we shall therefore add the regularity condition

$$E_{\{0\}}(e^{-2V_{[0,t]}}) \in \mathcal{A}_0$$

where the left hand side is defined by normality).

Introducing the notation

$$V_{[s,t+s]} = u_s(V_{[0,t]}) \tag{111}$$

the cocycle condition (110) becomes

$$e^{-V_{[0,t+s]}} = e^{iV_{[0,t]}} e^{-V_{[t,t+s]}} \tag{112}$$

in the hermitian case and, in the unitary case,

$$e^{iV_{[0,t+s]}} = e^{iV_{[0,t]}} e^{iV_{[t,t+s]}} \tag{113}$$

In the hermitian case the operator M_t and $u_t(M_s)$ must commute.

Therefore condition (112) is equivalent to

$$V_{[0,t+s]} = V_{[0,t]} + V_{[t,t+s]} \tag{114}$$

Thus, under suitable regularity conditions, the generic form of a hermitian markovian cocycle is

$$M_t = e^{-\int_0^t V_s ds} \ ; \quad V_s = u_s(V_0) \tag{115}$$

for some operator valued distributions V_s. More precisely we can say that the structure theory of hermitian markovian cocycles is reduced to the classical structure theory of multiplicative functionals. The situation is different for unitary markovian cocycles. Under suitable regularity conditions, their generic form is given by a time–ordered exponential

$$M_t = T(e^{-\int_0^t V_d s}) \ ; \quad V_s = u_s(V_0) \tag{116}$$

(for the definition and properties of time–ordered exponentials, cf. [6], [26]).

As in the classical case, one can express the formal generator of $Z(t)$ as a "perturbation" of the formal generator of $Z_0(t)$. In fact, let

$$\delta_0 = \lim_{t \downarrow 0} \frac{1}{t} \{Z_0(t) - 1\}$$

$$\delta = \lim_{t \downarrow 0} \frac{1}{t} \{Z(t) - 1\}$$

where, here and in the following, all the limit are meant in a formal way. Since, for $a_0 \in \mathcal{A}_0$,

$$\begin{aligned} Z(t)a_0 - a_0 &= E_{\{0\}}\tilde{M}_t u_t(a_0) - a_0 = \\ &= \{E_{\{0\}}u_t a_0 - a_0\} + E_{\{0\}}([\tilde{M}_t - 1]u_t(a_0)) = \\ &= \{Z_0(t)a_0 - a_0\} + E_{\{0\}}([\tilde{M}_t - 1]u_t(a_0)) \end{aligned}$$

it follows that

$$\delta(a_0) = \delta_0(a_0) + \tilde{A}_0(a_0) \tag{117}$$

where

$$\tilde{A}_0(a_0) = \frac{d}{dt}\Big|_{t=0} \tilde{M}_t(a_0) = \lim_{t\downarrow 0} \frac{1}{t}\{\tilde{M}_t(a_0) - a_0\}$$

In particular, if $\tilde{M}_t$ has the form

$$\tilde{M}_t a = M_t a M_t^* \tag{118}$$

one has

$$\tilde{A}_0(a_0) = A_0 a_0 + a_0 A_0^*$$

$$A_0 = \frac{d}{dt}\Big|_{t=0} M_t = \lim_{t\downarrow 0} \frac{1}{t}\{M_t - 1\}$$

For an hermitian (resp. unitary) markovian cocycle of the form (115) (resp. (116)) the formal derivative can be explicitly performed, giving:

$$A_0 a_0 = -V_0 a_0 \quad (\text{resp. } A_0 a_0 = iV_0 a_0)$$

therefore, in this cases, one has, for hermitian markovian cocycles of the form (115

$$\delta a_0 = \delta_0 a_0 - \{V_0, a_0\}$$

— where $\{\cdot,\cdot\}$ denotes the anti–commutator — and, for unitary markovian cocycles of the form (116)

$$\delta a_0 = \delta_0 a_0 + i[V_0, a_0]$$

where $[\cdot,\cdot]$ denotes the commutator.

Using a perturbation $\tilde{M}_t$ of more general form one can obtain the full Lindblad's form of generators of quantum dynamical evolutions, according to the equality (117).

16 Quantum case: L^2–theory

Let $\mathcal{A}$, $(\mathcal{A}_I)$, (u_t), (E_I) be as in § (1). We hall now assume that the $\mathcal{A}_I$ are W^*–algebras and that on $\mathcal{A}$ a locally normal state φ has been given which is u_t–invariant and E_I–invariant, namely:

$$\varphi \cdot u_t = \varphi, \ \forall\, t \tag{119}$$

$$\varphi \cdot E_I = \varphi \ , \quad \forall\, I \in \mathcal{F} \tag{120}$$

Let $\mathcal{H}$, π, 1_φ denote the GNS triple associated to $\mathcal{A}$ and φ (cf. [30]) and denote, for $I \in \mathcal{F}$

$$\mathcal{H}_I = [\pi(\mathcal{A}_I) \cdot 1_\varphi] = \text{ norm closure in } \mathcal{H} \text{ of } \pi(\mathcal{A}_I) \cdot 1_\varphi \ .$$

(119) implies that there is a 1–parameter unitary group (U_t) on $\mathcal{H}$ such that

$$\pi(u_t a) \cdot 1_\varphi = U_t \pi(a) \cdot 1_\varphi \ ; \quad a \in \mathcal{A} \tag{121}$$

and (120) implies that, denoting e_I the orthogonal projection $\mathcal{H} \to \mathcal{H}_I$, one has

$$\pi(E_I a) \cdot 1_\varphi = e_I \pi(a) \cdot 1_\varphi \ ; \quad a \in \mathcal{A} \tag{122}$$

In these notations, the covariance condition (7) is equivalent to

$$U_t \mathcal{H}_I = \mathcal{H}_{I+t} \Leftrightarrow u_t e_I u_t^* = e_{I+t} \tag{123}$$

and the Markov property (13) to:

$$e_{]-\infty,t]} \mathcal{H}_{[t,+\infty[} \subseteq \mathcal{H}_{\{t\}} \tag{124}$$

The locality condition $I \subseteq J \Rightarrow \mathcal{A}_I \subseteq \mathcal{A}_J$ implies that

$$I \subseteq J \Rightarrow e_I e_J = e_I \tag{125}$$

One easily verifies that conditions (123), (124), (125) imply that P_0^t, defined by

$$P_0^t = e_{\{0\}} U_t \rightharpoonup \mathcal{H}_{\{0\}} \ ; \quad t \geq 0 \tag{126}$$

is a semi–group $\mathcal{H}_{\{0\}}$, whose adjoint in $\mathcal{H}_{\{0\}} \to \mathcal{H}_{\{0\}}$ is given by:

$$P_0^{t*} = e_{\{0\}} U_t^* \rightharpoonup \mathcal{H}_{\{0\}} \tag{127}$$

The semi–group P_0^t is positivity preserving, in the sense that the positive cone $\mathcal{H}^+_{\{0\}}$ = closure of $\pi(\mathcal{A}^+_{\{0\}}) \cdot 1$ in $\mathcal{H}$ is mapped into itself by P_0^t. Moreover $P_0^t(1_\varphi) = 1_\varphi$. For the positivity of P_0^t in the Hilbert space sense, cf. the remark after Proposition (14.5).

Proposition 4 *The following conditions are equivalent:*
i) $P_0^t : \mathcal{H}_{\{0\}} \to \mathcal{H}_{\{0\}}$ *is self–adjoint;* $t > 0$
ii) $e_{\{0\}} U_t e_{\{0\}}$ *is self–adjoint;* $t > 0$
iii) $\varphi(a_0 u_t(b_0)) = \varphi(u_t(a_0) b_0)$*;* $\forall\, a_0, b_0 \in \mathcal{A}_0$.

Proof. The equivalence i) $\Leftrightarrow$ ii) follows from

$$\langle e_{\{0\}}, P_0^t e_{\{0\}} \eta \rangle = \langle \xi, e_{\{0\}} U_t e_{\{0\}} \eta \rangle \ ; \quad \xi, \eta \in \mathcal{H}$$

The equivalence ii) $\Leftrightarrow$ iii) follows from

$$\langle \pi(a_0) \cdot 1_\varphi, U_t \pi(b_0) \cdot 1_\varphi \rangle = \varphi(a_0^* U_t(b_0)) \ ; \quad a_0, b_0 \in \mathcal{A}_0$$

and the fact that $\pi(\mathcal{A}_0) \cdot 1_\varphi$ is dense in $\mathcal{H}_{\{0\}}$.

Remark that property iii) is a weak form of reflection invariance. Thus Proposition (14.1) implies that, also in the quantum case, reflection invariance is a sufficient condition for self–adjointness. We shall only consider the case in which the set of indices in $\mathbb{R}$ and the reflection is $t \mapsto -t$ (in the case of $\mathbb{R}^+$ one has to consider a 1–parameter family of reflections, but the proof is similar).

Corollary 1 *Let* $\mathbb{R}$ *be the set of indices. Assume that there is an automorphism* $r : \mathcal{A} \to \mathcal{A}$ *such that:*

$$r \rightharpoonup \mathcal{A}_{\{0\}} = id \tag{128}$$

$$r \cdot u_t = u_{-t} r \tag{129}$$

$$\varphi \cdot r = \varphi \tag{130}$$

Then P_0^t *is self–adjoint.*

Proof. Under our assumptions, $\forall\, a_0,\, b \in \mathcal{A}_0$

$$\varphi(a_0 u_t(b_0)) = \varphi(u_{-t}(a_0) b_0) = \varphi(r[u_{-t}(a_0) b_0]) = \varphi(u_t(a_0) b_0)$$

thus condition ii) of Proposition (14.1) is satisfied.

More generally, we have:

Corollary 2 *Assume that there exists an operator R:*

$$R : \mathcal{H} \to \mathcal{H} \textit{ such that:}$$

$$R^* U_t R = U_t^* \tag{131}$$

$$R e_{\{0\}} = e_{\{0\}} \tag{132}$$

then P_0^t is self–adjoint.

Proof. Under our assumptions, one has

$$(e_{\{0\}} U_t e_{\{0\}})^* = e_{\{0\}} U_t^* e_{\{0\}} = e_{\{0\}} R^* U_t R e_{\{0\}} = e_{\{0\}} U_t e_{\{0\}}$$

thus condition (ii) of Proposition (14.1) is satisfied.

In the classical case a markovian semi–group is unitary if and only if it is induced by a point transformation. The situation is similar in the quantum case.

Proposition 5 *The following assertions are equivalent:*
i) P_0^t is unitary
ii) $e_{\{0\}} U_t e_{\{0\}}$ is unitary
iii) $e_{\{0\}} e_{\{t\}} e_{\{0\}} = e_{\{0\}} = e_{\{0\}} e_{\{-t\}} e_{\{0\}}$.

Proof. Clearly i) $\Leftrightarrow$ ii) and

$$P_0^{t*} P_0^t = e_{\{0\}} U_t^* e_{\{0\}} U_t e_{\{0\}} = e_{\{0\}} e_{\{-t\}} e_{\{0\}}$$

$$P_0^t P_0^{t*} = e_{\{0\}} U_t e_{\{0\}} U_t^* e_{\{0\}} = e_{\{0\}} e_{\{t\}} e_{\{0\}}$$

thus i) $\Leftrightarrow$ iii).

Remark that, if the conditions of Corollary (14.3) are verified, then the two equalities in condition iii) are equivalent.

Perturbations of P_0^t can be introduced at a Hilbert space level. Let $\overline{M}_t : \mathcal{H} \to \mathcal{H}$ be a linear operator localized in $\mathcal{H}_{[0,t]}$, in the sense that

$$\overline{M}_t e_{[0,t]} = e_{[0,t]} \overline{M}_t \ ; \tag{133}$$

defining

$$P^t = e_{\{0\}} \overline{M}_t U_t \tag{134}$$

one again verifies that the cocycle property

$$\overline{M}_{t+s} = \overline{M}_t U_t (\overline{M}_s) U_t^* \tag{135}$$

is sufficient for P^t to be a semi–group and that, conversely, if P^t is a semi–group, one can assume that (135) holds without changing the action of P^t.

Remark. In general P^t will not be positivity preserving in the sense that the positive cone $\mathcal{H}^+_{\{0\}} = [\pi(\mathcal{A}^+_{\{0\}}) \cdot 1_\varphi]$ is mapped into itself. Moreover, while

$$P^t_0 \pi(a_0) 1_\varphi = \pi(Z_0(t)[a_0]) \cdot 1_\varphi$$

no such relation holds for P^t even if $\overline{M}_t$ has the form

$$\overline{M}_t = \pi(M_t) \tag{136}$$

for some $M_t \in \mathcal{A}_{[0,t]}$. In fact, as one easily verifies, if (136) holds, then, $\forall\, a_0 \in \mathcal{A}_0$

$$P^t \pi(a_0) P^{t*} = \pi(E_{\{0\}}(M_t u_t(a_0) M^*_t)) = \pi(Z(t)[a_0]) \tag{137}$$

However, if M_t is "well behaved" with respect to time reflections, then for each t, P^t is a positive operator in the Hilbert space sense. More precisely (cf. [28] and [31], theorem (5.4)):

Proposition 6 *Let R be as in Corollary (14.3). Then*
i) a sufficient condition for the self–adjointness of P^t is

$$RM^*_t R = U *_t M_t U_t \tag{138}$$

ii) if, moreover, R is an involution and

$$Re_I = e_{-I} R \ ; \qquad I - \textit{interval in } \mathbb{R} \tag{139}$$

then P^t is positive in the Hilbert space sense.

Proof. i) is obvious since, in this case

$$P^{t*} = e_{\{0\}} U^*_t M^*_t e_{\{0\}} = e_{\{0\}} R^* U_t R M^*_t R e_{\{0\}} =$$

$$= e_{\{0\}} U_t U^*_t M_t U_t e_{\{0\}} = P^t \ .$$

Under the assumption ii) one has, for $\xi \in \mathcal{H}_{\{0\}^2}$:

$$\begin{aligned}
\langle \xi, P^t \xi \rangle &= \langle \xi, M_t U_t \xi \rangle = \langle \xi, M_{t/2} (U_{t/2} M_{t/2} U^*_{t/2}) U_t \xi \rangle = \\
&= \langle U^*_{t/2} \xi, (U^*_{t/2} M_{t/2} U_{t/2}) M_{t/2} U_{t/2} \xi \rangle = \\
&= \langle R^* U_{t/2} R \xi, R M^*_{t/2} R (M_{t/2} U_{t/2}) \xi \rangle = \\
&= \langle M_{t/2} U_{t/2} \xi, R (M_{t/2} U_{t/2}) \xi \rangle = \| P^{t/2} \xi \|^2
\end{aligned}$$

the last equality being due to the fact that

$$e_{[0,t/2]}Re_{[0,t/2]} = Re_{[-t/2,0]}e_{[0,t/2]} =$$
$$= Re_{[-t/2,0]}e_{[-\infty,0]}e_{[0,t/2]} = Re_{\{0\}} = e_{\{0\}}$$

Remark 1. In particular, under the assumptions of Corollary (14.5), with $M_s = 1$ for each s, P_0^t is always a positive operator in $\mathcal{H}_{\{0\}}$.

Remark 2. If the process $\{\mathcal{A}, (\mathcal{A}_I), \varphi\}$ admits a time reversal, i.e. an automorphism or anti–automorphism) $r : \mathcal{A} \to \mathcal{A}$ such that: i) $\varphi \circ r = \varphi$; ii) $r \circ u_t = u_{-t} \circ r$; iii) $r \rightharpoonup \mathcal{A}_0 = id$; iv) $r^2 = id$; v) $r\mathcal{A}_I = \mathcal{A}_{-I} (I \subseteq \mathbb{R})$; then an involution $R : \mathcal{H} \to \mathcal{H}$ satisfying (131), (132) and (139) can be defined by $R\pi(a)1_\varphi = \pi(r(a))1_\varphi$ $(a \in \mathcal{A})$.

Remark 3. If $\overline{M}_t$ has the form (136) then the semi–group P^t can be characterized by the property

$$\varphi(a_0 M_t u_t(b)) = \varphi_0(a_0 P^t b_0) \tag{140}$$
$$\forall\, a_0, b_0 \in \mathcal{A}_0,\ (\varphi_0 = \varphi \rightharpoonup \mathcal{A}_0)$$

An equality of type (140) is frequently called a "Feynman–Kac–Nelson formula". R. Schrader and D.A. Unhlenbrock [31] prove such an equality in the context of Clifford algebras over real Hilbert spaces and for a particular choice of the perturbation M_t, using the Trotter product, or the Duhamel, formula.

The point of view advocated in the present paper is that, just as in the classical case, a quantum Feynman–Kac formula can be used to construct a perturbed semi–group, for a given local perturbation M_t, even in cases in which the above mentioned formulae are not applicable.

Since, for $y \in \mathcal{H}_{\{0\}}$

$$\begin{aligned} \frac{1}{t}\{P^t y - y\} &= \frac{1}{t}\{e_{\{0\}}M_t U_t y - y\} = \\ &= \frac{1}{t}\{e_{\{0\}}U_t y - y\} + \frac{1}{t}\{e_{\{0\}}[M_t - 1]U_t y\} \end{aligned}$$

denoting H_0 (resp. H) the generator of P_0^t (resp. P^t), one has the formal identity

$$Hy = H_0 y + V_0 y$$

where

$$V_0 = \lim_{t \downarrow 0} \frac{1}{t}\{M_t - 1\} = \frac{d}{dt}\Big|_{t=0} M_t$$

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