Quantum Random Walks

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Contents

Abstract

After a short review of the notion of a quantum Markov chain, a particular class of such chains, generalizing in a natural way the usual random walks, is introduced. In Section (5) a limit theorem for quantum random walks is proved showing that the diffusion limit of the continuous coherent chain is an abelian extension of the Fock quantum Brownian motion on $L^2(\mathbf{R}_+)$.

1 Introduction

Let $S = \{1, ..., d\}$ be a finite set . If we fix a basis in $\mathbf{C}^d$ the functions $f : S \longrightarrow \mathbf{C}$ can be identified to diagonal $d \times d$ matrices. A probability measure $p = (p_1, ..., p_d)$ on S also determines a diagonal matrix $w = diag(p_1, ..., p_d)$ and if E denotes expectation with respect to p and j(f) is the diagonal matrix corresponding to f in a fixed basis of $\mathbf{C}^d$, then one has:

$$E(f) = Tr(w \cdot j(f)) \tag{1}$$

where $Tr(\ \cdot\ )$ denotes the (non normalized) trace on the $d \times d$ complex matrices. If we change the basis in $\mathbf{C}^d$ then the new basis will be related to the old one by a unitary trasformation U (all bases are supposed orthonormal for the usual scalar product in $\mathbf{C}^d$) and the matrices associated to j(f) and will change according to the rules:

$$U^* j(f) U = j\prime(f) \qquad ; \qquad UwU^* = w\prime \tag{2}$$

All the maps $j\prime : C(S) \longrightarrow M_d = M(d; \mathbf{C})$ are embeddings of the functions $S \longrightarrow \mathbf{C}$ into the diagonal $d \times d$ matrices over the complex numbers. Now fix w as above. In a basis independent language this means that we have chosen a positive matrix of unit trace and with eigenvalues $p_1, ..., p_1$. Trace-one positive matrices are called density matrices and are the analogue of probability densities with respect to the counting measure (1,...,1) on S. The analogue in M_d of the counting measure is the trace. If we fix w and let j(f) vary in all possible ways according to (2), then the expectation value on the left hand side of (1) will vary and it is easy to verify that it will vary among all the probability measures on S.

The pair $\{w, j\prime\}$, where w is a density matrix and $j\prime$ varies among all the embeddings $C(S) \longrightarrow M_d$ (C(S) denoting the continuous functions on

S), is the simplest example of a quantum stochastic process (with index set given by all the functions $f \to \mathbf{C}$). The term quantum refers to the fact that usually $j\prime(f)$ and $j\prime\prime(f)$ do not commute. Here the $j\prime$ represent the random variables in the sense made precise by equation (9) in the following. For each fixed $j\prime$ the expectation functional on C(S) defined by (1) defines a unique probability measure p_j , on S. To deal simultaneously with all the probability spaces $(S, p_{j'})$ in a classical probabilistic framework, one would need a huge sample space, for example $\prod_{j'} (S, p_{j'})$. In a quantum probabilistic framework we only need a single $d \times d$ density matrix: the price payed for finite dimensionality is noncommutativity.

In classical probability repeated samples are described by product spaces. Thus for example,two samples are described by the spaces $S \times S$ and their distribution by a probability measure on $S \times S$ or equivalently,by Riesz theorem, by a positive normalized linear functional on $C(S \times S) \cong C(S) \otimes C(S)$. Similarly in quantum probability two samples on a system (or two copies of the same system) are described by a positive normalized linear functional on $M_d \otimes M_d$. Such a functional is called a state and for matrix algebras, the formula

$$\varphi(x) = Tr(w \cdot x) \qquad ; \qquad \forall x \in M \tag{3}$$

establishes a one-to-one correspondence between states φ on M and density matrices w in M.

The quantum analogue of the sample path corresponding to countably many repetitions of the same experiment

$$\Omega = \prod_{n \in \mathbf{N}} S \tag{4}$$

or, better,of the continuous fuctions on it:

$$\mathcal{C}(\Omega) = \mathcal{C}(\prod_n S) \cong \bigotimes_n \mathcal{C}(S) \tag{5}$$

is the infinite tensor product of matrix algebras

$$\mathcal{A} = \bigotimes_n M_d \tag{6}$$

(in both case the C^*-norm on the tensor product is uniquely defined). **Example**. The following example shows in a simple concrete case how, using a non commutative structure, one can deal simultaneously with infinitely many classical stochastic process. Fix a unit vector $\Phi \in \mathbf{C}^2$ and a unitary 2×2 matrix $U \in M(2; \mathbf{C})$; let (ψ_j) $(j = 0, 1)$ be an orthonormal basis in $\mathbf{C}^2$ and denote e_j the rank one projection on the direction ψ_j. One easily verifies that for each $n \in N$ the numbers

$$\| U \cdot e_{j_n} \cdot U \cdot e_{j_n-1} \cdot U \cdot U e_{j_1} \Phi \|^2 = P_{j_1,\ldots,j_n}$$

define a probability measure on $\{0, 1\}^n$ and that the sequence of probability measures thus obtained satisfies Kolmogorov' s compatibility condition, hence it defines a unique probability measure on the sample space $\Omega = \prod_n \{0, 1\}$. An explicit description of this probability measure can be obtained as follows: denote

$$\varphi_j = U\psi_j$$

$$P_{ij} = |< \psi_j, \varphi_j >|^2 = |< \psi_j, U\psi_j >|^2$$

$$P_j^o = |< \psi_j, \Phi >|^2$$

then a simple calculation shows that

$$P_{j_1,\ldots,j_n} = P_{j_1}^o \cdot P_{j_1,j_2} \cdot P_{j_2,j_3} \cdots P_{j_n-1,j_n}$$

i.e. we obtain the classical Markov chain with bistochastic transition matrix p_{ij} .In the 2×2 case all bistochastic matrices can arise, with an appropriate choice of U. This will not be the case for matrices of order $n \times n$ and the characterization of those bistochastic matrices which can arise in this way is an open problem for $n \geq 4$ (cf. [8] for partial results in this direction). It is sometimes convenient to look at M_d as the algebra of all operators on the Hilbert space C^d and at the infinite tensor product (1.6) as an algebra of operators on the Hilbert space

$$\mathcal{H} = \bigotimes_{\mathbf{N}} C^d \tag{7}$$

However , as shown by von Neumann [11], the infinite tensor product (7) makes sense as a separable Hilbert space only if one singles out a sequence of unit vectors in C^d, and therefore it depends on this arbitrary choice, while the infinite tensor product (6) has an intrinsic meaning. For this reason we prefer the the algebraic approach to the Hilbert space one. Recall that a classical stochastic process (ξ_n) on Ω with probability distribution P can be characterized by the triple:

$$\{\mathcal{A}, (j_n)_{n\in N}, \varphi\} \tag{8}$$

where

$$\mathcal{A} = \mathcal{C}(\Omega)$$

$$j_n : \mathcal{C}(S) \longrightarrow \mathcal{C}(\Omega)$$

is the embedding characterized by

$$j_n(f)(\omega) = f(\xi_n(\omega)) \quad ; \quad f \in \mathcal{C}(S) \quad ; \quad \omega \in \Omega \quad ; \quad n \in N \tag{9}$$

$$\varphi(a) = \int_\Omega a(\omega)dP(\omega) \tag{10}$$

In [1], [2], [3] a symmetric quantum stochastic process was defined as a triple (8) where $\mathcal{A}$ is the infinite tensor product of matrix algebras (6), φ is any state on $\mathcal{A}$ and $j_n : M_d \longrightarrow \mathcal{A}$ is the embedding (sometimes also called ampliation) of M_d into $\mathcal{A}$ consisting in letting M_d operate on the n-th factor of (7) and trivially on all the other ones i.e. :

$$j_n(b) = 1 \bigotimes \cdots \bigotimes 1 \bigotimes b \bigotimes 1 \bigotimes 1 \bigotimes \cdots\cdots \quad ; \quad b \in M_d \tag{11}$$

The term symmetric here refers to the fact that here observables at different times commute, i.e.

$$[j_m(f), j_n(g)] = 0 \quad ; \quad m \neq n \; ; \; f, g \in M_d \tag{12}$$

Quantum stochastic process without this restriction were studied in [4]. Independent repeated trials are described in quantum , as in classical , prob-

ability by product states. A product state φ on $\mathcal{A}$ is characterized by the property:

$$\varphi(j_1(f_1) \cdot j_2(f_2) \cdot .... \cdot j_n(f_n)) = \varphi_1(f_1) \cdot .... \cdot \varphi_n(f_n) \tag{13}$$

(for every $n \in N$; $f_1, ..., f_n \in M_d$) where the φ_j are states on M_d. Such a state will be denoted

$$\varphi = \bigotimes_{j\in\mathbf{N}} \varphi_j \tag{14}$$

If $\mathcal{A} = \mathcal{C}(\Omega) \cong \bigotimes_N \mathcal{C}(S)$ and j_n and φ have the from (9) and (10) respectively, one recovers the usual notion of independence of the random variables (ξ_n) and φ_j is the (expectation with respect to the) distribution of the j-th random variable.

For quantum independent processes one can prove central limit theorems [6] and invariance principles [5]. The next step after independent sequences is that of Markovian sequences. Since the deepest difference (both conceptual and technical) between classical and quantum probability lies in the notion of conditional expectation, one can expect that the notion of quantum Markov chain will not be a simple traslation in a noncommutative language of the corresponding classical notion. In the following we briefly review the notion of quantum Markov chain and produce some examples of physical significance.

2 Generalized Markov Chains, Stationarity, Ergodicity

Let $\mathcal{B}$ be a C^*-algebra. The basic examples of $\mathcal{B}$ that we will have in mind are:

$$\mathcal{B} = \mathcal{C}(S)$$

the algebra of continuous functions on a compact Hausdorff space. Or

$$\mathcal{B} = L^\infty(S, m) = L^\infty(S, \mathcal{F}, m)$$

the algebra of all bounded complex valued measurable functions on some measure space $S = (S, \mathcal{F}, m)$ with the supremum norm, or

$$\mathcal{B} = \mathcal{B}(H_o)$$

the algebra of all bounded operators on a separable complex Hilbert space H_o.

If $\mathcal{B}$ is commutative then there is only one norm on $\mathcal{B} \otimes \mathcal{B}$ making it a C^*-algebra ([10] ,pg.62). If $\mathcal{B}$ has the form $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, then we define the norm on $\mathcal{B} \otimes \mathcal{B}$ as the one coming from the natural action of $\mathcal{B} \otimes \mathcal{B}$ on $\mathcal{H} \otimes \mathcal{H}$. For a general C^*-algebra , we fix a C^*-algebra norm on the tensor product $\otimes_{\mathbf{N}}\mathcal{B}$.

We will denote $\mathcal{A}$ the C^*-tensor product $\otimes_{\mathbf{N}}\mathcal{B}$ of a countable set of copies of $\mathcal{B}$. This is a C^*-algebra characterized by the following properties: i) For each natural integer n there exists an embedding

$$j_n : \mathcal{B} \longrightarrow \mathcal{A} = \otimes_{\mathbf{N}} \mathcal{B} \tag{15}$$

such that, for each n , the map

$$j_{[0,n]} = j_o \otimes j_1 \otimes ... \otimes j_n \ : \ a_o \otimes a_1 \otimes ... \otimes a_n \in (\otimes \mathcal{B})^{n+1} \longrightarrow j_o(a_o) j_1(a_1) \cdot ... \cdot j_n(a_n) \in \otimes_N \mathcal{B} \tag{16}$$

is an isomorphism. ii)For each natural integer n, and for each $a_o, a_1, ...a_n$ in $\mathcal{B}$, one has

$$\| \ j_o(a_o) j_1(a_1) \ ... \ j_n(a_n) \ \| = \| \ a_o \ \| \cdot \| \ a_1 \ \| \cdot ... \cdot \| \ a_n \ \|$$

iii)The algebra $\otimes_{\mathbf{N}}\mathcal{B}$ is the norm closure of the algebra $\mathcal{A}^o$ generated by the elements $j_n(b)$ where n is any natural integer and b is any element of $\mathcal{B}$. We will often use the symbolic notation

$$j_o(a_o) j_1(a_1) \cdot \cdots \cdot j_n(a_n) = a_o \otimes a_1 \otimes ... \otimes a_n \otimes 1 \otimes \cdots \tag{17}$$

For any sub-set I of the natural integers $\mathbf{N}$ we denote

$$\mathcal{A}_I = \qquad \text{algebra spanned by } j_n(\mathcal{B}) \qquad ; \qquad n \in \mathbf{N}$$

and for any finite set F , $j_F \cong \otimes_{n \in F} \ j_n$ denotes the isomorphism of $(\otimes \mathcal{B})^{|F|}$ with the algebra $\mathcal{A}_F$. Thus

$$\mathcal{A} = \qquad \text{closure of } \bigcup_n \mathcal{A}_{[0,n]} = \mathcal{A}_{\mathbf{N}} = \otimes_{\mathbf{N}} \mathcal{B} \tag{18}$$

For each finite n, the elements of each local algebra $\mathcal{A}_{[o,n]}$ are naturally identified (using the isomorphism (16)) to operators acting on the (n + 1)-st tensor power of H_o. In the following we will freely use this identification. If I is reduced to a single point n, we use the notation

$$\mathcal{A}_n = j_n(\mathcal{B}) \tag{19}$$

Notice that, due to condition (i) above, the algebras $\mathcal{A}_I, \mathcal{A}_J$ commute, if the sets I and J are disjoint. The algebras $\mathcal{A}_J$ are called the local algebras. The algebra $\mathcal{A} = \mathcal{A}_{\mathbf{N}}$ is also called the algebra of quasi-local observables. We will say that an element a of $\mathcal{A}$ is localized in I if a belongs to $\mathcal{A}_I$. If φ is any state on $\mathcal{A}$, its restriction on $\mathcal{A}_I$ ($I = [0, n]$) will be denoted φ_I. Such a state is completely determined by its values on the elements of the form

$$a_o \otimes a_1 \otimes \cdots \otimes a_n \otimes 1 \otimes \cdots \tag{20}$$

In case $\mathcal{B} = \mathcal{B}(H_o)$, a state φ on $\mathcal{A}$ is called locally normal if for each natural integer n there exists a density operator $W_{[o,n]}$ acting on $(\otimes H_o)^n$ such that

$$\varphi(a_o \otimes a_1 \otimes \cdots \otimes a_n \otimes 1 \otimes \cdots) = Tr_{[o,n]}(W_{[o,n]} \cdot a_o \otimes a_1 \cdots \otimes a_n) \tag{21}$$

In the following, unless explicitly stated otherwise, by "state on $\mathcal{A}$ " we will mean "locally normal state on $\mathcal{A}$ ".

Lemma 1 *Let be given, for each integer n, a state $\psi_{[0,n]}$ on $(\otimes\mathcal{B})^n$ such that*

$$\psi_{[0,n+1]}(a \otimes 1) = \psi_{[0,n]}(a \otimes 1) \qquad \forall n \in \mathbf{N} \qquad \forall a \in (\otimes\mathcal{B})^n \tag{22}$$

Then there exists a unique state φ on $\mathcal{A} = \otimes_{\mathbf{N}}\mathcal{B}$ such that

$$\varphi(j_{[0,n]}(a)) = \psi_{[0,n]}(a) \qquad \forall n \in \mathbf{N} \qquad \forall a \in (\otimes\mathcal{B})^n \tag{23}$$

Proof. Because of (22), the family $(\phi_{[0,n+1]})$ is projective.

Definition 1 *The* **shift** *on $\mathcal{A}$ is the unique endomorphism u of $\mathcal{A}$ into itself satisfying*

$$u(j_o(a_o)j_1(a_1) \cdot \cdots \cdot j_n(a_n)) = j_1(a_o)j_2(a_1) \cdot \cdots \cdot j_{n+1}(a_n) \qquad \forall a_o, \cdots, a_n \in \mathcal{B}$$

or equivalently

$$u \circ j_n = j_{n+1} \qquad \forall n \in \mathbf{N} \tag{24}$$

Clearly for each natural integer n and each subset I of $\mathbf{N}$:

$$u^n(\mathcal{A}_I) = \mathcal{A}_{I+n} \tag{25}$$

This property is called the **covariance** *of the family of local algebras $(\mathcal{A}_I)$ with respect to the shift. A state φ on $\mathcal{A}$ is called* **stationary** *if it is invariant for the shift, i.e. if*

$$\varphi(u(a)) = \varphi(a) \ ; \ a \in \mathcal{A} \tag{26}$$

Definition 2 *A* **transition expectation** *from* $\mathcal{B}\otimes\mathcal{B}\longrightarrow\mathcal{B}$ *is a completely positive map* $\mathcal{E}:\mathcal{B}\otimes\mathcal{B}$ *satisfying*

$$\mathcal{E}_n(1\otimes 1)=1 \qquad \forall n\in\mathbf{N} \tag{27}$$

If $\mathcal{E}$ *is a transition expectation, the operator* $P:\mathcal{B}\longrightarrow\mathcal{B}$ *defined by*

$$P(b)=\mathcal{E}(1\otimes b) \qquad b\in\mathcal{B} \tag{28}$$

is called the **Markovian** *(or* **transition** *) operator associated to* $\mathcal{E}$. *In general, any completely positive identity preserving operator* $P:\mathcal{B}\longrightarrow\mathcal{B}$ *will be called a* **Markovian operator**. *While an operator* $P:\mathcal{B}\longrightarrow\mathcal{B}$ *satisfying the condition*

$$P(1)\leq 1 \tag{29}$$

will be called **sub-Markovian**.

Example If $\mathcal{B}=\mathcal{B}(H)$ for some Hilbert space H, then the most general transition expectation has the form

$$\mathcal{E}(x)=\sum_j \bar{Tr}_2(K_j^* x K_j) \qquad x\in\mathcal{B}\otimes\mathcal{B} \tag{30}$$

for some operators $K_j\in\mathcal{B}\otimes\mathcal{B}$, where $\bar{T}r_2$ denotes the operator valued weight, in the sense of Haagerup, defined by

$$\bar{T}r_2 : a\otimes b\in(\mathcal{B}\otimes\mathcal{B})_+\to a\cdot Tr(b)$$

Let $(\mathcal{E}_n)_{n\geq 0}$ be any sequence of completely positive normalized maps from $\mathcal{B}\otimes\mathcal{B}$ to $\mathcal{B}$. Then for each integer n and for each state φ_o on $\mathcal{B}$ the map

$$a_o\otimes a_1\otimes\cdots\otimes a_n \ \in(\otimes\mathcal{B})^{n+1}\mapsto\varphi_o(\mathcal{E}_o(a_o\otimes\mathcal{E}_1(a_1\otimes\cdots\otimes\mathcal{E}_n(a_n\otimes 1)))) \tag{31}$$

defines a unique state $\psi_{[0,n]}$ on $(\otimes\mathcal{B})^{n+1}$ which is normal if each $\mathcal{E}_n$ is normal. Letting $a_n=1$ in (4), one finds that

$$\psi_{[0,n+1]}\mid(\otimes\mathcal{B})^{n+1}\otimes 1\subseteq(\otimes\mathcal{B})^{n+2}=\psi_{[0,n]} \tag{32}$$

Therefore , by Lemma 1, there exists a unique state φ on $\otimes_{\mathbf{N}}\mathcal{B}$ satisfying (2).

Definition 3 *The state φ, characterized by (4) will be called the* **generalized Markov chain** *associated to the pair $\{\varphi_o, (\mathcal{E}_n)\}$. If for each n*

$$\mathcal{E}_n = \mathcal{E}_o =: \mathcal{E} \tag{33}$$

then we speak of an **homogeneous** *generalized Markov chain. The completely positive, identity preserving, normal maps $\mathcal{E}_n$ are called the* **transition expectations** *of the generalized Markov chain φ.*

Remark. For a reader not familiar with the language of quantum probability, it might be useful to describe the classical analogue of the construction through which the generalized quantum Markov chains are defined. This leads to a class of processes strictly larger than the classical Markov chains: these are recovered through a particular choice of the (classical) transition expectation. Let S be a compact Hausdorff space ; denote $\mathcal{C}(S)$ the space of

continuous complex valued functions on S and let $\mathcal{E} : \mathcal{C}(S \times S) \longrightarrow \mathcal{C}(S)$ be an integral operator with kernel

$$\mathcal{E}(f)(x) = \int_{S\times S} K(x; dy, dz) f(y, z) \qquad f \in \mathcal{C}(S \times S) \tag{34}$$

where, for each $x \in S$

$$K(x; dy, dz) \geq 0 \qquad ; \qquad \int_{S\times S} K(x; dy, dz) = 1 \tag{35}$$

(i.e. K can be looked at as a Markovian kernel on $S \times S$). The operator $\mathcal{E}$ satisfies the conditions

$$f \in \mathcal{C}(S \times S) \qquad ; f \geq 0 \Longrightarrow \mathcal{E}(f) \geq 0 \qquad ; \qquad \mathcal{E}(1_{S\times S}) = 1_S \tag{36}$$

(1_S -resp.$1_{S\times S}$ - is the constant function equal to one on S -resp. $S \times S$). Now let Ω denote the space of sequences on S

$$\Omega = \prod_{\mathbf{N}} S$$

with the product topology. By Tychonov 's theorem Ω is a compact Hausdorff space and by the Stone-Weierstrass theorem the complex valued functions on Ω which depend only on a finite number of variables are dense in $\mathcal{C}(\Omega)$

in the supremum norm. If m_o is any probability measure on S there exists a unique state φ on the sub-algebra of the functions depending only on a finite number of variables such that for each integer n and for each function $f = f(x_o, x_1, \cdots, x_n) \in \mathcal{C}(S^{n+1})$ one has

$$\varphi(f) = \tag{37}$$

$$\int_S \cdots \int_S f(x_o, x_1, \cdots, x_n) dm_o(x_o) K(x_o; dx_1, dy_1) K(y_1; dx_2, dy_2) \cdots K(y_{n-2}; dx_{n-1}, dy_{n-1}) K(y_{n-1}$$

Since $\mid \varphi(f) \mid \leq \| f \| \quad ; \forall f \in \mathcal{C}(S \times S)$ this state uniquely determines a state on $\mathcal{C}(\Omega)$ still denoted φ. By Riesz 's theorem [12] there exists a unique Baire probability measure on Ω such that

$$\varphi(f) = \int_\Omega f(\omega) dP(\omega)$$

If the kernel K has the form

$$K(x; dz, dy) = P(x; dy)\delta_x(dy) \tag{38}$$

where P(x;dy) is a Markovian kernel on S and δ_x is the Dirac measure concentrated at $x \in S$, then the expression (37) reduces to the familiar expression for the expectation of the function f relatively to the Markov chain with initial distribution m_o and transition kernel P(x,dy) i.e.

$$\varphi(f) = \int_S \cdots \int_S f(x_o, y_1, \cdots, y_n) dm_o(x_o) P(y_1, dy_2) \cdots P(y_{n-1}, dy_n)$$

Notice that if f has the form $f = f_o \otimes f_1 \otimes \cdots \otimes f_n$ for some $f_o, \cdots, f_n \in \mathcal{C}(S)$ i.e

$$f(x_o, \cdots, x_n) = f_o(x_o) f_1(x_1) \cdots f_n(x_n) \qquad x_o, \cdots, x_n \in S$$

then the expectation value $\varphi(f)$ can be written

$$\varphi(f) = \int_S dm_o(x_o) \mathcal{E}\Bigg(f_o \otimes \Bigg(f_1 \otimes \cdots \otimes bigl(f_{n-1} \otimes \mathcal{E}(f_n \otimes 1)) \cdots \Bigg) \Bigg)(x_o)$$

or, denoting as usual the integral with the same symbol as the corresponding measure

$$\varphi(f) = m_o\Bigg(\mathcal{E}\Bigg(f_o \otimes \mathcal{E}(f_1 \otimes \cdots \otimes (f_{n-1} \otimes \mathcal{E}(f_n \otimes 1)) \cdots) \Bigg) \Bigg) \tag{39}$$

while, in terms of the operator P, defined by

$$Pf(x) = \int_S P(x; dy) f(y) \qquad x \in S \tag{40}$$

(39) becomes:

$$\varphi(f_1 \otimes \cdots \otimes f_n) = m_o\Big(f_o \cdot P\Big(f_1 \cdot P\Big(f_2 \cdot \cdots \cdot P(f_n) \cdots\Big)\Big)\Big) \tag{41}$$

which is the usual formula for the Markov expectations associated to the homogeneous Markovian kernel (40). Notice that, in terms of the operators $\mathcal{E}$, defined by (34), (38) and P defined by (40), the identity (34) can be expressed as:

$$\mathcal{E}(f \otimes g) = f \cdot P(g) \qquad f, g \in \mathcal{C}(S) \tag{42}$$

($\cdot$ denoting the pointwise product). When an arbitrary C^*-algebra is substituted for $\mathcal{C}(S)$, the right hand side of (41) is no longer positive. For this reason the obvious generalization of formula (42) to a quantum context, does not lead to a state, but to a linear functional which is usually non positive. Notice that this construction is a trivial generalization of the classical Markov chains since it reduces to a usual Markov chain on a larger space $(S \times S)$, however the classical processes obtained by restriction of a generalized quantum Markov chain to a diagonal sub-algebra of $\otimes_{\mathbf{N}} M$ of the form $\otimes_{\mathbf{N}} D$, where D is a commutative sub-algebra of M, will not be in general of this type since the transition expectation $\mathcal{E}$ in general will not map $D \otimes D$ into itself. Hence these processes represent a new class of classical processes whose joint probabilities at any order are explicitly known. Moreover, the results at the end of this Section show that many of their properties (such as the structure of the invariant distributions, periodic states, ergodic and mixing properties, ...) are determined, like for the usual Markov chains, by a Markovian transition operator.

Proposition 1 *The generalized Markov chain φ, determined by the pair $\{\varphi_o, (\mathcal{E}_n)\}$ is stationary if*
i) it is homogeneous (i.e. $\mathcal{E}_n = \mathcal{E}$ independently of n)
ii) Denoting $P : \mathcal{B} \to \mathcal{B}$ the Markovian operator associated to $\mathcal{E}$ i.e.

$$P(b) = \mathcal{E}(1 \otimes b) \qquad ; \; b \in \mathcal{B}$$

one has :

$$\varphi_o \circ P = \varphi_o \tag{43}$$

Proof Clear from (31) and (26). **Remark**. In the conditions of Proposition (1), if $\mathcal{E}$ is given by

$$\mathcal{E}(x) = \overline{Tr}_2(H^* x H) \tag{44}$$

and if w_o is the density matrix of the state φ_o then the stationarity condition (37) becomes

$$\overline{Tr}_1(H(w_o \otimes 1)K^*)H = 1 \otimes w_o \cong u(w_o) \cong 1 \otimes w_o \otimes 1 \otimes \cdots \tag{45}$$

The following result is useful to produce examples of stationary Markov chains.

Theorem 1 *Let $\mathcal{B} = \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and let $\mathcal{L} : \mathcal{B} \otimes \mathcal{B} \longrightarrow \mathcal{B}$ be a completely positive map such that the map*

$$b \in \mathcal{B}_+ \mapsto Tr(\mathcal{L}(1 \otimes b)) = \varphi(b) \in [0, \infty) \tag{46}$$

is a faithful weight on $\mathcal{B}$. Let h be the Radon-Nikodym derivative of the weight (40) with respect to the trace on $\mathcal{B}$ i.e.

$$Tr(\mathcal{L}(1 \otimes b)) = Tr(hb) \qquad \forall b \in \mathcal{B} \tag{47}$$

and denoting

$$h = \mathcal{L}(1)$$

Then the map $\mathcal{E}$, formally defined by

$$\mathcal{E}(x) = h^{-1/2} \mathcal{L}(x) h^{-1/2} \qquad x \in \mathcal{B} \otimes \mathcal{B} \tag{48}$$

(cf. the proof below for the precise definition) is a transition expectation with invariant weight φ i.e.

$$Tr\Big(h\mathcal{E}(1 \otimes b)\Big) = Tr(hb) \qquad \forall b \in \mathcal{B} \tag{49}$$

Proof. By assumption the state (40) is faithful, hence h is invertible on a dense set D. Notice that for all $x \in (\mathcal{B} \otimes \mathcal{B})_+$, the sesquilinear form

$$q_x(\xi, \eta) =< \mathcal{L}(x)^{1/2} h^{-1/2} \xi, \mathcal{L}(x)^{1/2} h^{-1/2} \eta > \qquad ; \ \xi, \eta = \in D$$

is positive and

$$q_x(\xi, \xi) \leq \parallel x \parallel 2 \cdot \parallel \xi \parallel^2$$

therefore there exists a map

$$\mathcal{E} : x \in \mathcal{B} \otimes \mathcal{B} \longrightarrow \mathcal{B}$$

characterized by the property

$$q_x(\xi, \xi) =< \xi, \mathcal{E}(x) \xi > \qquad ; \ x \in \mathcal{B} \qquad ; \ \xi \in D$$

We shall use the notation

$$\mathcal{E}(x) := h^{-1/2} \cdot \mathcal{L}(x) \cdot h^{-1/2}$$

It is clear that $\mathcal{E}$ satisfies the condition

$$\mathcal{E}(x) \leq || x || \qquad \forall x \in \mathcal{B} \otimes \mathcal{B}$$

Since clearly $\mathcal{E}(1) = 1$ and $\mathcal{E}$ is completely positive, it follows that $\mathcal{E}$ is a transition expectation from $\mathcal{B} \otimes \mathcal{B}$ to $\mathcal{B}$. Finally , if $b \in \mathcal{B}_+$, then

$$Tr\Big(h\mathcal{E}(1 \otimes b)\Big) = Tr(\mathcal{L}(1 \otimes b)) = Tr(hb)$$

Example. Let H be any operator in $\mathcal{B} \otimes \mathcal{B}$. If $\mathcal{L} : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$ is defined by

$$\mathcal{L}(x) = \bar{T}r_2(H^* x H) \qquad x \in \mathcal{B} \otimes \mathcal{B}$$

then the operator h in the above Proposition becomes

$$h = \bar{T}r_1(HH^*)$$

Definition 4 *Let V be a real Banach space. A bounded linear operator $T : V \longrightarrow V$ is called a* **Perron operator** *with maximal eigenvalue λ, eigenvector b and invariant state ψ if there exists a nonzero vector b in V and a nonzero continuous linear functional ψ on V such that for any x in V*

$$\lim_{n\to\infty} \frac{T^n}{\lambda} x = \psi(x)b \tag{50}$$

It is well known that, if T is a Perron operator and let λ, ψ and b as in Definition (4) above. Then

$$Tb = \lambda b \qquad ; \qquad \psi \circ T = \lambda\psi \tag{51}$$

$$\psi(b) = 1 \tag{52}$$

Moreover λ is a simple eigenvalue and any other eigenvalue of T has modulus strictly less than λ.

It is also known (cf. [2]) that, if $\varphi = \{\varphi_o, \mathcal{E}\}$ is a homogeneous Markov chain and if the Markovian operator P, associated to $\mathcal{E}$ is a Perron operator, then φ is mixing and φ is a factor state.

3 Quantum Random Walks

In this section we introduce the notion of a quantum random walk as a simple example of quantum Markov chain. Consider a classical random walk on the integers with probabilities

$$p(+1) = \text{Prob } \{ \text{ unit jump on the right}\} \qquad , \qquad p(-1) = 1 - p(+1) \tag{53}$$

If u_{+1} , u_{-1} denote the shift operators on the right and on the left, defined on functions $f : \mathbf{Z} \longrightarrow \mathbf{C}$ by:

$$(u_{+1}f)(j) = f(j+1) \qquad ; \qquad (u_{-1}f)(j) = f(j-1) \tag{54}$$

then the transition operator P of the random walk is given by

$$Pf = p(+1)u_{+1}(f) + p(-1)u_{-1}(f) \tag{55}$$

According to the analysis of the previous section,the operator P is not sufficient, in the quantum case, to determine the joint probabilities: to this goal the transition expectation $\mathcal{E}$ is needed. A natural candidate for the transition expectation of a quantum random walk is:

$$\mathcal{E}(a \bigotimes b) = \bar{T}r_2(\sum_{k=\pm 1} U_k^* b U_k \otimes f_k a f_k^*) \tag{56}$$

where the U_k are unitary operators; $U_{-1} = (U_{+1})^{-1}$, and the f_k are operators such that:

$$\sum_{k=\pm 1} f_k f_k^* = 1 \qquad ; \qquad Tr(f_k f_k^*) = p(k) \qquad ; \qquad k = \pm 1 \tag{57}$$

The Markovian operator associated to $\mathcal{E}$ is:

$$P(b) = \mathcal{E}(1 \otimes b) = p(+1) U_{+1}^* \cdot b \cdot U_{+1} \ + \ p(-1) U_{-1}^* b U_{-1} \tag{58}$$

which is the natural quantum generalization of (55). There is no problem in extending (**??**) to an arbitrary state space (rather than $\{\pm 1\}$) and to an arbitrary Hilbert space H rather than $\mathbf{C}^2$. In fact, if (S, μ) is a (σ-finite) measure space, $x \in S \longmapsto U_x$ is a $*$-strongly measurable unitary operator valued map, $x \in S \longmapsto f_x$ is a (Hilbert-Schmidt) operator valued map one can define

$$\mathcal{E}(a \otimes b) = Tr_2(\int U_x^* b U_x \otimes f_x a f_x^* d\mu(x)) = \int T_r(| f_x |^2 a) U_x^* b U_x d\mu(x) \tag{59}$$

If

$$\int Tr(| f_x |^2) d\mu(x) = 1 \tag{60}$$

then the operator

$$P(b) = \mathcal{E}(1 \otimes b) = \int U_x^* b U_x p(x) d\mu(x) \tag{61}$$

where the integral in (9) is a Bochner integral for the $*$-strong topology on $\mathcal{B}(\mathcal{H})$ [sak] and

$$p(x) = Tr(| f(x) |^2) \tag{62}$$

is Markovian. Its dual action on the density matrices is

$$P'(w_o) = \int U_x w_o U_x^* p(x) d\mu(x) \tag{63}$$

which can be interpreted as follows: the state w_o of the system evolves according to a random quantum dynamics. In a unit time interval, the Heisenberg dynamics $w_o \longmapsto U_x w_o U_x^*$ occurs with probability $p(x)\mu(dx)$. The evolution of the random walk is the average over all the reversible quantum dynamics.

4 The coherent chain

In this section we construct a particular example of a quantum random walk with a nice physical interpretation. For lack of space we do not introduce the notions of coherent states, Weyl operators, ... and refer for them to any book on quantum theory (*e.g.*[9]). In the notation of the previous section, let us choose:

$$S = \mathbf{C} \cong \{\psi(z)\ :\ z \in \mathbf{C}\} = \{\text{the set of coherent vectors on} \Gamma(\mathbf{C}) \cong L^2(\mathbf{R})\}$$

$$\mu = \frac{1}{\pi}\{\text{ the Lebesgue measure on } \mathbf{C}\} \cong \mathbf{R}^2$$

$$U_z = W(z) = \text{the Weyl operator corresponding to } z \in \mathbf{C}$$

$$f_z = \frac{\mid \psi(z) >< \psi(z) \mid}{\parallel \psi(z) \parallel^2} = \mid z >< z \mid = \text{the rank one projection onto } \ \mathbf{C} \cdot \psi(z)$$

Thus in this case (3.7) becomes:

$$\mathcal{E}(a \otimes b) = \int_{\mathbf{C}} < \psi(z), a\psi(z) > W(z)^* \cdot b \cdot W(z) \frac{dz}{\pi} \tag{64}$$

and the associated Markovian operator is:

$$P(b) = \int_{\mathbf{C}} W(z)^* \cdot b \cdot W(z)\ \frac{e^{-|z|^2}}{\pi} dz \tag{65}$$

whose dual action on the density matrix w_o is:

$$P'(W_o) = \int_{\mathbf{C}} W(z) \cdot W_o \cdot W(z)^* \frac{e^{-|z|^2}}{\pi} dz \tag{66}$$

Equation (66) gives the evolution of the state W_o of the field in one unit time. The interpretation of equation (66) is simple: in a time unit the coherent monochromatic signal, represented by the coherent vector $\mid z >= W(z)\psi(0)$ impinges on a receiver, in the state W_o, with probability density $\exp\{- \mid z \mid^2 \}$. The effect of the single signal $\mid z >$ would be the transition $W_o \longmapsto W_(z) \cdot W_o \cdot W(z)^*$. For example if originally the receiver was in the vacuum state $\mid \psi(0) >< \psi(0) \mid$ then the transition above brings it into the coherent state $\mid z >< z \mid$, as expected. If it were in the Gibbs state at inverse temperature β, i.e. $exp\{\beta a^+ a\}/Z_\beta$ then it would have shifted into the displaced thermal operator (or shifted Gibbs state [9]) $exp\{-\beta(a+z)^+(a+z)/Z_\beta$. If we do not know precisely which signal is sent, but we only know that the signal $\mid z>$ is sent with probability $(\exp - \mid z\mid^2)/\pi$, then the state of our quantum random walk is obtained by averaging over all these transitions according to the formula (66).

It is interesting to compute the dual action of transition operator $\mathcal{E}$ on the density matrices. One finds:

$$\mathcal{E}(w_o) = \int_{\mathbf{C}} \frac{e^{-|z|^2}}{\pi}\, dz\ W_z W_o W_z^* \otimes \mid z >< z \mid \tag{67}$$

According to (38) we obtain the density matrix after n units time by iterative application of the trasformation (67) in such a way that, after each step the operator $\mathcal{E}'$ acts only on the last factor of the tensor product. With the notation

$$p(z) = \frac{e^{-|z|^2}}{\pi} \tag{68}$$

and with the choice of the initial state to be the vacuum:

$$W_o = \mid 0 >< 0 \mid \tag{69}$$

we obtain for the joint density matrix after n units of time:

$$W_{[o,n]} = \int_{\mathbf{C}} \cdots \int_{\mathbf{C}} dz_o \cdot \cdots \cdot dz_n p(z_o) \cdot p(z_1 - z_o) \cdot \cdots \cdot p(z_n - z_{n-1}) \mid z_o >< z_o \mid \otimes \mid z_1 >< z_1 \mid \otimes \cdots . \otimes \tag{70}$$

The identity (70) is particularly revealing: it shows that the coherent quantum chain admits an alternative description purely in classical terms i.e. as a classical stochastic process with values in the states of a quantum system. To show this let us denote, for each $z \in C$,

$$\varphi_z(a) =< z, az > \qquad ; \qquad a \in \mathcal{B}(H)$$

Then the space S, introduced at the beginning of this section can be identified to the set of states:

$$\{\varphi_z : \in \mathbf{C}\}$$

Let (ξ_n) be the classical Markov chain with state space $\mathbf{C}$, the transition density $p(z - z\prime)$, where $p(z)$ is given by (68), and initial distribution - the Dirac delta at the origin. If $(\Omega, \mathcal{F}, P)$ is the probability space of the chain then it is easy to verify that the state φ of the quantum coherent chain, whose sequence of density matrices is given by (70) can be represented in the form

$$\varphi = \int_\Omega \otimes_{n\in\mathbf{N}} \varphi_{\xi_n} \, dP \tag{71}$$

i.e φ is the convex combination, with respect to the measure P, of the "random product states" $\otimes_{n\in\mathbf{N}}\varphi_{\xi_n}$. Given the explicit form (71) one can now calculate the distributions of the various classical processes which correspond to the physical quantities of interest. For example, for the field process

$$A_n(u) = ua_n^+ + u^* a_n \qquad ; \qquad n \in \mathbf{N} \qquad ; \qquad u \in \mathbf{C} \tag{72}$$

we find:

Proposition 2 *The field random variables* $A_o(u), \cdots, A_n(u)$ *have the form*

$$A_o = R_o + 2\sigma Y_o \tag{73}$$

$$A_1 = R_1 + 2\sigma(Y_o + Y_1)$$

$$A_n = R_n + 2\sigma(Y_o + \cdots + Y_n)$$

where all the R_j, Y_j *are i.i.d. Gaussians with mean* 0 *and variance* $\mid u \mid^2$

Remark. In (73) we have introduced a variance σ^2 in the density (68). The R_j represent the signal contribution and the Y_j the noise contribution at the j-th instant. For the number process (N_n) we find that each (N_n), has a geometric distribution with parameter

$$\frac{2n\sigma^2}{1+2n\sigma^2}$$

5 The continuous coherent chain

In conclusion, let us outline how to construct a continuons version of the discrete coherent chain introduced in this section. Let $H_o \subseteq L^2(\mathbf{R})$ be the pre-Hilbert space of the continuons complex valued functions on $\mathbf{R}$ with compact support , let $\{\mathcal{H}, W, \Phi\}$ be the associated Fock representation and $W(H_o)$ the Weyl C^*-algebra. Let $(\Omega, \mathcal{F}, P)$ be the Wiener probability space with

$$\Omega = \mathcal{C}(\mathbf{R}_+ \; ; \; \mathbf{R}^2) \cong \mathcal{C}(\mathbf{R}_+ \; ; \; \mathbf{C})$$

Even if the generic Wiener trajectory ω is not in H_o ,the state

$$\varphi_\omega \ : \ W(f) \in W_(H_o) \longrightarrow \varphi_\omega(W_(f)) = \lim_{t\to\infty} < W(\omega_{[0,t]}) \cdot \Phi, W(f) W(\omega_{[0,t]}) \cdot \Phi > \tag{74}$$

where $\omega_{[0,t]}$ denotes the restriction of ω on the interval $[0,t]$, is well defined on $W(H_o)$ and in fact one has

$$\varphi_\omega(W(f)) =< W(\omega_{[0,T]})\Phi, W(f)W(\omega_{[0,T]}) \cdot \Phi >$$

for any T on the right of the support of f. The map $\omega \in \Omega \mapsto \varphi_\omega \in \{$ the set of states on $\Omega\}$ is clearly measurable, hence

$$\varphi = \int_\Omega \varphi_\omega dP(\omega) \tag{75}$$

is a state on $W(H_o)$. As the following considerations show, this state can be considered as a continuum limit of the discrete coherent chains considered in the previous section.

In fact , for $n \in N$ and $f \in H_o$ with $suppf \subseteq [0,T]$, we can define

$$f_n(j) = \frac{1}{\sqrt{n}} f(\frac{j}{n}T) \qquad ; \qquad j = 0, 1, \cdots, n \tag{76}$$

$$p^{(n)}(x) = \frac{e^{-|x|^2/2(T/n)}}{2\pi(T/n)} \qquad ; \ x \in \mathbf{C} \cong \mathbf{R}^2 \tag{77}$$

and then form the coherent chain associated to the transition density $p^{(n)}(x)$ according to the construction explained in Section (4.). This leads to the coherent chain :

$$\varphi^{(n)}(\ \cdot\ ) = \int_{\Omega^{(n)}} dP^{(n)} \bigotimes_{j\in\mathbf{N}} \varphi_{\xi_j^{(n)}/\sqrt{n}}(\ \cdot\ ) \tag{78}$$

where $(\Omega^{(n)}, \mathcal{F}^{(n)}, P^{(n)})$ is the probability space of the classical $\mathbf{C}$-valued Markov chain $(\xi_j^{(n)})$ with transition density (77). Since f has compact support, for large j one has $W(f_n(j)) = 1$, therefore the infinite product

$$\bigotimes_{j\in\mathbf{N}} W(f_n(j)) = W_n(f)$$

makes sense and one has, according to (71) and (72):

$$\varphi^{(n)}(W_n(f)) = \int_{\Omega^{(n)}} dP^{(n)} \prod_{n\in\mathbf{N}} < \xi_j^{(n)}/\sqrt{n}, W(f_n(j)) \cdot \xi_j^{(n)}/\sqrt{n} >=$$

$$= \int_{\Omega^{(n)}} dP^{(n)} \prod_{j\in\mathbf{N}} e^{-iImf_n(j)\xi_j^{(n)}/\sqrt{n}-\frac{1}{2}|f_n(j)|^2} =$$

$$= \int_{\Omega^{(n)}} dP^{(n)} e^{-iIm\sum_{j\in\mathbf{N}} f(\frac{jT}{n})\xi_j^{(n)}\cdot\frac{1}{n} - \frac{1}{2}\sum_{j\in\mathbf{N}}|f(\frac{jT}{n})|^2\frac{1}{n}} \tag{79}$$

But,denoting $B(t)$ the t-th random variable of the Wiener process, the expression (79) is, by our construction, equal to

$$E\left(e^{-iIm\sum_{j\in\mathbf{N}} f(\frac{jT}{n})\cdot[B(\frac{(j+1)T}{n})-B(\frac{jT}{n})]-\frac{1}{2}\sum_{j\in\mathbf{N}}|f(\frac{jT}{n})|^2\frac{1}{n}}\right)$$

where E denotes Wiener expectation. By dominated convergence on then finds:

$$\lim_{n\to\infty} \varphi^{(n)}(W_n(f)) = \varphi(W(f))$$

with φ given by (75) which gives the required approximation of the continuons coherent chain by discrete ones.

References

[1] L.Accardi Noncommutative Markov chains. In: international School of Mathematical Physics, Camerino (1974) 268-295.

[2] L.Accardi On the noncommutative markovian property (in russian). Funkt. Anal. and its Appl. 9 (1975) 1-8.

[3] L.Accardi Topics in quantum probability. Physics Reports 77 (1981) 169-192.

[4] L.Accardi, A.Frigerio , J.Lewis Quantum stochastic processes Publications of the Research institute for Mathematical Sciences Kyoto University 18 (1982) 97-133.

[5] L.Accardi ,Bach A. The harmonic oscillator as quantum central limit of Bernoulli processes. submitted to Prob. Th. and Rel. Fields , November 1987

[6] Giri Von Waldenfels

[7] Haagerup U.

[8] Kruszynski

[9] Louisell W.H. Radiation and noise in quantum electronics. McGraw-Hill (1964)

[10] Sakai S. Springer 1971

[11] von Neumann J.

[12] Yoshida K. Functional analysis Springer 1965