Non commutative Markov chains

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Contents

1 The Markov property: commutative case

Let $(\Omega, \mathcal{B}, \mu)$ be a measure space, $(\mathcal{B}_F)_{F\in\mathcal{F}}$–a family of sub–$\sigma$–algebras of $\mathcal{B}$ indexed by a family of parts of a set S and with the following properties:

$$F \subseteq G \Rightarrow \mathcal{B}_F \subseteq \mathcal{B}_G \tag{1}$$

$$S \in \mathcal{F} \ ; \quad \mathcal{B}_S = \mathcal{B} \tag{2}$$

Denote, for $F \in \mathcal{F}$, $A_F = L^\infty_{\mathbb{C}}(\Omega, \mathcal{B}_F, \mu)$ the algebra of (μ–classes of) bounded functions on Ω which are $\mathcal{B}_F$–measurable. If $F \subseteq G$, there exists a unique linear operator $E^\mu_{G,F} : A_G \subseteq A_F$, defined by the property:

$$\mu_G(f_F \cdot f_G) = \mu_F(f_F \cdot E^\mu_{G,F}(f_G)) \ ; \quad \forall\, f_F \in A_F \ ; \quad \forall\, f_G \in A_G \tag{3}$$

(μ_F denotes the restriction of μ on $\mathcal{B}_F$). The operator $E^\mu_{G,F}$ is called conditional expectation, with respect to μ, from A_G to A_F.
Let now S be a topological space, $\mathcal{F}$ a family of closed subsets of S containing S and for every $F \in \mathcal{F}$, the boundary of F, denoted hereinafter ∂F. For any $H \subseteq S$, denote $\mathcal{B}_H$ the σ–algebra on Ω spanned by the family $\{\mathcal{B}_F : F \in \mathcal{F}, \in H\}$.

Definition 1 *The conditional expectation $E^\mu_{G,F}$, $F \subseteq G$ is said to enjoy the Markov property*[1] *if:*

$$E^\mu_{G,F}(A_{G\setminus F}) \subseteq A_{\partial F} \tag{4}$$

A measure μ on $(\Omega, \mathcal{B})$ will be called a Markov measure with respect to the family of sub–σ–algebras $\{\mathcal{B}_F\}_{F\in\mathcal{F}}$, if for every $F \in \mathcal{F}$:

$$E^\mu_{S,F}(A_{F^c}) \subseteq A_{\partial F} \tag{5}$$

(F^c denotes the complement of F with respect to A).

Remark that, if μ is a Markov measure with respect to $\{\mathcal{B}_F\}$, then for every $F \subseteq G$, $E^\mu_{G,F}$ enjoys the Markov property. Moreover property (5) is easily seen to be equivalent to:

$$E^\mu_{S,F}(f_{|F}) = E^\mu_{S,\partial F}(f_{|F}) \ ; \quad \forall\, f_{|F} \in A_{|F} \tag{6}$$

Equality (6) is the characterization of te Markov property as first formulated by E. Nelson (see for example [7]).

[1] A sub–σ–algebra $\mathcal{B}_0$ of $\mathcal{B}$ will be thought, unless specified the contrary, as a σ–algebra on Ω, i.e. $\Omega \in \mathcal{B}_0$.

2 Conditional expectations

Put $A_S = A$, $A_F = B$; if $E : A \to B$ is a conditional expectation defined as in 1), then E enjoys the following properties:

[CE1)] $E(a) \geq 0$, if $a \geq 0$; $a \in A$ [CE2)] $E(ba) = b\ E(a)$; $b \in B$; $a \in A$ [CE3)] $E(a^*) = E(a)^*$; $a \in A$; [CE4)] $E(1) = 1$ [CE5)] $\|E(a)\| \leq \|a\|$; $a \in A$ [CE6)] $E(a) \cdot E(a)^* \leq E(aa^*)$

Properties CE1) – CE6) [2] are in fact a characterization of the conditional expectations (see S.c. Moy). This fact lies at the root of the following (due to Umegaki [12]).

Definition 2 *Let A, B– $C^{(*)}$–algebras, $B \leq A$. A conditional expectation from A to B is a linar map $E : A \to B$ satisfying CE1) – CE6).*

Properties CE1) – CE6) are not independent. In fact, the following theorem holds (Tomijama [11]):

Theorem 1 *Every projector of norm 1, $E : A \to B$ enjoys properties CE1) – CE6). Therefore, by Tomijama's theorem a conditional expectation from A to B is a linear norm 1 projection $E : A \to B$.*

3 Quasi–Conditional expectations

The definition of Markov property and Markov measure given in section (1), does not depend on the structure of the algebras A but only on the property:

$$F \subseteq G \to A_F \subseteq A_G \tag{7}$$

For an arbitrary family $\{A_F\}_{F\in\mathcal{F}}$ of C^*–algebras (where $\mathcal{F}$ is as in section (1) and a family of conditional expectations (in the sense of Definition ()) $(E_{G,F})$, one could define the Markov property for $E_{G,F}$ by (4), and a Markov state μ on A as a state for which there exists a family $(E_{G,F})$ of conditional expectations satisfying equality (3).

However, in case of arbitrary C^*–algebras, the mere existence of a family $(E_{G,F})$ of conditional expectations satisfying (3) for a given state μ on A_S undergoes severe restrictions, as one can deduce from a theorem of Takesaki [10]. Moreover, even if such a family exists, in many cases, the Markov

[2] Plus the continuity condition: if (a_k) is an increasing sequence of positive functions converging μ–almost everywhere to a, then $\lim E(a_k) = E(a)$, μ–almost everywhere.

property implies the triviality of these conditional expectations.
Assume for example, that the family $\{A_F\}$ of "local algebras", satisfies, in addition to (7), the following relations:

$$A_{F^c} = A'_F = \{a \in A_S : ab = ba \ , \quad \forall\, b \in A_F\} \tag{8}$$

$$A_F \cap A_{F^c} = \mathbb{C} \cdot 1 \tag{9}$$

Then, if a conditional expectation $E_{S,F} : A_S \to A_F$ enjoys the Markov property (see (5)), one easily verifies that

$$E_{S,F}(A_{F^c}) \subseteq \mathbb{C} \cdot 1 \tag{10}$$

i.e. the conditional expectation of every $a \in A_{F^c}$ is a scalar. In the commutative case this means that the observables in the algebras A_F and A_{F^c} are stochastically independent (with respect to the measure μ satisfying (3)).
In the non commutative case if S is discrete (10) implies, that any state μ satisfying (3) is a product state on A_S. Thus for the local algebras $\{A_F\}$ satisfying (7), (8), (9), with S–discrete, the family of "Markov states" obtained by direct generalization of (3) and (5) to the non–commutative case reduces to the class of product states. Since systems of "local algebras" of the type described above are very frequent, for instance, in quantum statistical mechanics, it is desirable to have a class of "Markov states" which is strictly larger, in such systems too, than the class of prodcut states. With this aim we introduce the following:

Definition 3 *Let $d(B) \subseteq B \subseteq A$, be C^*–algebras. A quasi–conditional expectation with respect to the triple $d(B) \subseteq B \subseteq A$, is a linear map $E : A \to B$ with the following properties:*

[i1)] $E(a) \geq 0$; if $a \geq 0$; $a \in A$ [i2)] $E(ca) = cE(a)$; $c \in d(B)$; $a \in A$ [i3)] $\|E(c')\| \leq \|c'\|$; $c' \in d(B)'$
where the symbol ? denotes the commutant with respect to A. A quasi–conditional expectation will be called normalized if [i4)] $E(1) = 1$.

If $d(B) = \mathbb{C} \cdot 1$, E is simply a positive linear map $A \to B$. Every quasi–conditional expectation, E with respect to the triple $d(B) \subseteq B \subseteq A$, satisfies:

$$E(d(B)' \cap A) \subseteq d(B)' \cap B \tag{11}$$

The relation (11) can be considered as a non–commutative formulation of the Markov property.

Example 1 *Let* $\{A_F\}_{F\in\mathcal{F}}$ *be a family of local algebras satisfying (7), (8). Let* $d : F \to d(F)$*, be a map such that*

$$F \subseteq G \to d(F) \subseteq d(G) \ ; \quad d(F) \subseteq F \ ; \quad F, G \in \mathcal{F} \ .$$

Then, if $E_{S,F}$ *is a quasi–conditional expectation with respect to the triple* $A_{d(F)} \subseteq A_F \subseteq A_G$*, the inclusion (11) takes the form:*

$$E_{S,F}(A_{d(F)^c}) \subseteq A_{d(F)^c} \cap A_F \tag{12}$$

in particular, if $d(F) = \overset{\circ}{F} =$ *interior of* F*, and the family* $\{A_F\}$ *satisfies*[3]

$$A_F \cap A_G = A_{F\cap G} \tag{13}$$

the Markov property for $E_{S,F}$ *is expressed by:*

$$E_{S,F}(A_{(\overset{\circ}{F})^c}) \subseteq A_{\partial F} \tag{14}$$

In section (5.) we prove that the strengthening (12) (or (13)) of the Markov property is sufficient, through a non–commutative analogue of (3), to supply a class of states of rather simple structure and strictly larger than the class of product states.
In the following section an elementary and meaningful example will be discussed in some detail in order to show that, in case of matrix algebras, even if a conditional expectation compatible with a state exists only in case of a product state, a quasi–conditional expectation compatible with a given state always exists.

4 The conditional expectation as "transfer operator"

Let $(S, \mathcal{B})$ be a measurable space, μ_0–a measure on $(S, \mathcal{B})$. Then the equality

$$\mu(f) = \mu_0(E(f)) \ ; \quad f \in L^\infty(S \times S, \mathcal{B} \times \mathcal{B}) \tag{15}$$

[3]The local algebras of quantum lattice systems, in particular, the local algebras which will be considered in section (5.), satisfy (13).

establishes a one–to–one correspondence between conditional expectations[4]

$$E : L^\infty(S \times S, \mathcal{B} \times \mathcal{B}) \to L^\infty(S \times S, \pi_1^{-1}(\mathcal{B})) \approx L^\infty(S, \mathcal{B}) \tag{16}$$

and measures μ on $S \times S$, whose restriction to $\pi_1^{-1}(\mathcal{B})$ is equal to μ_0. This correspondence reduces the study of all measures on $S \times S$ whose restriction to $\pi_1^{-1}(\mathcal{B})$ equals μ_0 to the study of the transfer operators

$$\nu \in \mathcal{S}(\mathcal{B}) \mapsto \nu \circ E \in S(\mathcal{B} \times \mathcal{B})^5$$

In the non commutative case the above correspondence does not subsist as one can see from the following, simple, example: let M be a matrix algebra, φ a state on $M \otimes M$, φ_0–the restriction of φ on $M \otimes 1$; then a conditional expectation $E : M \otimes M \to M \otimes 1$ satisfying

$$\varphi(x) = \varphi_0(E(x)) \ ; \quad \forall\, x \in M \otimes M \tag{17}$$

exists if and only if φ is a product state, i.e. $\varphi = \varphi_0 \otimes \varphi_1$, for some $\varphi_1 \in S(M)$[6] and in this one has only the one to one correspondence: conditional expectations $M \otimes M \to M \otimes 1 \to$ *product* states on $M \otimes M$. Thus one is led to the following problem: is it possible to define a class of operators $E : M \otimes M \to M \otimes 1$ such that when, in equality (17), E runs over this class the linear functional φ scans exactly the set of states on $M \otimes M$ whose restrictions to $M \otimes 1$ coincides with φ_0?
In our example assuming the state φ faithful, such a class always exists and it is given by all the operators E of the following type:

$$E(x) = \overline{\tau}_2(K^* x K) \ ; \ \lambda \in M \otimes M \tag{18}$$

where $\overline{\tau}_2$ is the unique linear map $M \otimes M \to M$, with the property

$$\overline{\tau}_2(a \otimes b) = a \cdot \tau(b) \ , \quad a, b \in M \tag{19}$$

(τ is the trace on M) and, if W_0 is the density matrix of φ_0, then $K = W^{1/2}(W_0 \otimes 1)^{-1/2}$ where W is any density matrix such that $\overline{\tau}_2(W) = W_0$. Therefore we are led to the following:

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[4] π_1 (resp. π_2) denotes the natural projection $S \times S \to S$ onto the first (resp. second) factor.

[5] $S(\mathcal{B})$ denotes the set of all probability measures on $(S, \mathcal{B})$.

[6] $S(M)$ is the set of states on M.

Definition 4 *An operator $K \in M \otimes M$ will be called a (square root of a) conditional density of the state φ with respect to the state φ_0, if*

$$\varphi(x) = \varphi_0(E(x))/\varphi_0(E(1)) \ ; \quad \varphi_0(E(1)) > 0 \tag{20}$$

where E is given by (18).

The conditional density K is called "normalized" if

$$E(1) = 1 \tag{21}$$

Every conditional expectation gives rise to a normalized conditional density of the type $K = 1 \otimes H$.
Assume now that φ is a state on $M \otimes M$ whose density matrix W is diagonal in a system $(e_{ij} \otimes f_{\alpha\beta})$ of matrix units for $M \otimes M$; i.e. in such a coordinate system $W = (P_{ii,jj} \cdot \delta_{ik} \cdot \delta_{jl})$. Then W_0 has coordinates $(q_{ii} \cdot \delta_{ik})$ in the system (e_{ij}) of matrix units for M, where $q_{ii} = \sum_j p_{ii,jj}$.
Therefore, if K is defined as described after (19) one finds $K^* = (W_0 \otimes 1)^{-1/2} W^{1/2} = (\sqrt{p_{ii,jj}/q_{ii}} \cdot \delta_{ik} \cdot \delta_{jl})$, and, if $x = (x(i,j) \cdot \delta_{ik} \cdot \delta_{jl})$ is a diagonal matrix, then $E(x) = (y_{ii} \cdot \delta_{ij})$: $y_{ii} = \sum_j x(i,j) p_{ii,jj}/q_{ii}$. That is, if W is diagonal then, for every diagonal matrix x, $E(x)$ coincides with the classical conditional expectation of x, even when the corresponding non–commutative expectation does not exist.
The conditional density K will be called compatible with the state φ_0 on M, if the restriction of the state φ, defined by (20) on $M \otimes 1$, coincides with φ_0. This is the case if and only if

$$W_0 = \overline{\tau}_2(K \cdot (W \otimes 1) \cdot K^*) \tag{22}$$

and, if (22) holds then another conditional density H, compatible with φ_0, defines through (20) the same state φ as K if and only if

$$H \cdot (W_0 \otimes 1) \cdot H^* = K \cdot (W_0 \otimes 1) \cdot K^* \tag{23}$$

Thus, in general, equality (20) does not determine the conditional density in a unique way.
The fact that (20) defines a state on $M \otimes M$ even if E is not compatible with φ_0, lies at the root of the distinction which will be done, in the non–commutative case, between Markov states and Markov chains (see section (5.)). A conditional density is compatible with every state φ_0 if and only if the map E defined by (15) is a conditional expectation.

5 One dimensional Markov states

Let M be $q \times q$ matrix algebra on $\mathbb{C}$, denote $A = \bigotimes_{\mathbb{N}} M$ the tensor product of $\mathbb{N}$ copies of A, $j_k : M \to j_k(M) \subset A$–the natural immersion of M onto the "k–th factor" of the product $\bigotimes_{\mathbb{N}} M$, and $M_{[m,n]}$ the sub–algebra of A spanned by $\bigcup_{k=m}^{n} j_k(M)$. For a quasi–conditional expectation $E_{n+1,n}$ with respect to the triple $M_{[0,n-1]} \subset M_{[0,n]} \subseteq M_{[0,n+1]}$, the Markov property is expressed as;

$$E_{n+1,n}(M_{[n,n+1]}) \subseteq M \tag{24}$$

Let φ be a state on A; denote $\varphi_{[0,n]}$ the restriction of φ on $M_{[0,n]}$; it is well known (see [9]) that φ is uniquely determined by the family $(\varphi_{[0,n]})_{n\in\mathbb{N}}$.

Definition 5 *A state φ on A will be called a Markov state with respect to the family $(M_{[0,n]})$, of subalgebras of A, if there exists a family $(E_{[0,n],[0,m]m<n}$ of quasi–conditional expectations with respect to the triples $M_{[0,m-1]} \subseteq M_{[0,m]} \subseteq M_{[0,n]}$ such that*

$$\varphi_{[0,n]} = \varphi_{[0,m]} \circ E_{[0,m],[0,m]} \tag{25}$$

Theorem 2 *Let φ be a Markov state on A. Then φ defines a pair $\{(\sigma_n); \varphi_0$ such that*

[i)] φ_0 is a state on M [ii)] $\forall\, n \in \mathbb{N}$; $\sigma_n : M \to \mathcal{L}(M)$ is a linear mapping such that the operator $a \otimes b \in M\otimes \mapsto \sigma_n(a)[b]$, is q^n–positive with norm ≤ 1 [iii)] For every $n \in \mathbb{N}$, $a_i \in M$, $0 \leq i \leq n$, the equality:

$$\varphi_{[0,n]}(j_0(a_0) \cdot \ldots \cdot j_n(a_n)) = (\sigma_{n-1}(a_{n-1})^* \cdot \ldots \cdot \sigma_i(a_0)^* \varphi_0)(a_{n-1}) \tag{26}$$

($\sigma_k(a_k)^* : M^* \to M^*$, denotes the adjoint of $\sigma_k(a_k)$), completely defines the projective family $(\varphi_{[0,n]})$.

Conversely, every such a pair defines a unique Markov state on A.
The proof of Theorem (2) is purely algebraic. The main point is the positivity of the linear functional defined by (26), and this is an almost immediate consequence of positivity properties of the operators σ_k.
The operators σ_k are connected with the quasi–conditional expectations $E_{k+1,k} = E_{[0,k+1,[0,k]}$, by the equalities:

$$E_{k+1,k}(j_k(a_k) j_{k+1}(a_{k+1})) = j_k(\sigma_k(a_k)[a_{k+1}]) \tag{27}$$

$$E_{k+1,k}(j_0(a_0) \cdot \ldots \cdot j_k(a_k)) = j_0(a_0) \cdot \ldots \cdot j_k(a_k) \cdot E_{k+1,k}(j_k(a_k) \cdot E_{k+1,k}(j_k(a_k) \cdot j_{k+1}(a_{k+1})) \tag{28}$$

and the state φ defined by the family $(\varphi_{[0,n]})$ will be such that

$$\varphi(j_0(a_0)\cdot\ldots\cdot j_n(a_n)) = \varphi_0(E_{1,0}(j_0(a_0)\cdot E_{2,1}(\ldots\cdot E_{2,1}(\ldots\cdot E_{n,n+1}(j_{n-1}(a_{n-1})j_n(a_n)$$

that is:

$$\varphi_{[0,n]} = \varphi_0 \cdot E_{1,0} \cdot \ldots \cdot E_{n,n-1} \tag{29}$$

Moreover, the projectivity condition (iii) insures that the quasi–conditional expectation $E_{k,k+1}$ is compatible with $\varphi_{[0,k]}$, in the sense that

$$\varphi_{[0,k]} = \varphi_{[0,k]} \circ E_{k,k-1}|_{M_{[0,k]}} \tag{30}$$

Condition (30) shows that, starting from an arbitrary pair $\{\varphi_0, (J_n)\}$ satisfying i), ii) of Theorem (2) the family of positive linear functionals defined by (26) in general is not projective. However a little analysis of the projectivity condition (26) shows that:

Lemma 1 *Let* $\{\varphi_0, (\sigma_n)\}$ *be a pair satisfying i) and ii) of Theorem (2). Assume that there is a sequence* (b_n) *of positive operators in* M *such that* $\varphi_0(b_0) > 0$ *and:*

$$\sigma_n(1)[b_{n+1}] = b_n \tag{31}$$

Then the equality:

$$\varphi_{[0,n]}(J_0(a_0) \cdot \ldots \cdot J_n(a_n)) = \frac{1}{\varphi_0(b_0)} \cdot (\sigma_n(a_n)^* \cdot \ldots \cdot \sigma_0(a_0)^*\varphi_0)(b_{n+1}) \tag{32}$$

defines a projective family of states $(\varphi_{[0,n]})$.

Definition 6 *A Markov chain on* $A := \bigotimes_N M$ *is a triple* $\{\varphi_0, (\sigma_n), (b_n)\}$ *which satisfies the conditions of Lemma (1). The state* φ *on* A*, uniquely determined by such a triple by means of formula (32) will also be called a Markov chain.*

Remark. The most frequent Markov chains are those for which either of the following conditions is satisfied.

$$b_n = 1\ , \quad \forall\, n \in \mathbb{N} \tag{33}$$

$$b_n = b > 0\ ; \quad \forall\, n \in \mathbb{N} \tag{34}$$

the former corresponds to *normalized* quasi–conditional expectations. Later on we shall give examples of Markov chains corresponding to both situations, see sections 7)–8). The connection between quasi–conditional expectations of Markov states and conditional densities (see section 4) is given by the following:

Lemma 2 *Let $\varphi = \{\varphi_0, (j_n)\}$ be a Markov state. Assume that, for every $n \in \mathbb{N}$ there is a $K_n \in M \otimes M$, $\|k_n\| \leq 1$ such that:*

$$\sigma_n(a)[b] = \overline{\tau}_2(K_n^*(a \otimes b)K_n) \tag{35}$$

Then, denoting $W_{[0,n]}$ the density matrix of $\varphi_{[0,n]}$ and

$$K_{[0,n]} = \underbrace{1 \otimes \ldots \otimes 1}_{n-1} \otimes K_n \in M_{[n-1,n]} \tag{36}$$

the family $(K_{[0,n]})$ is an agreeing family of conditional densities compatible with φ_0. Conversely every such a family defines, through (35), a unique Markov state.

The Lemma above characterizes Markov states as those states $\varphi \equiv (\varphi_{[0,n]})$ such that for each n there exists a conditional density $K_{[0,n]}$ of $\varphi_{[0,n]}$, compatible with $\varphi_{[0,n-1]}$ and such that $K_{[0,n]} \in M_{[n-1,n]}$. In particular every state such that

$$W_{[0,n]}^{1/2} \cdot (W_{[0,n-1]} \otimes 1)^{-1/2} \in M_{[n-1,n]}$$

is a Markov state.

6 The transfer matrix

Let $\varphi \equiv \{\varphi_0, (\sigma_n)\}$ be a Markov chain on $A = \bigotimes_{\mathbb{N}} M$, where M is a $q \times q$ complex matrix algebra. Assume the σ_n–normalized, that is:

$$\sigma_n(1)[1] = 1 \tag{37}$$

and let ψ be the state on A defined, according to Lemma (5.3), by

$$\psi_{[0,n]}(j_0(a_0) \cdot \ldots \cdot j_n(a_n)) = (\sigma_n(a_n)^* \cdot \ldots \cdot \sigma_0(a_0)^* \varphi_0)(1) \tag{38}$$

Define, for every n, the operator $Z_n : M \to M$, by

$$Z_n(a) = \sigma_n(1)[a] \tag{39}$$

Lemma 3 *For every n, Z_n is a q^n–positive linear map of M into itself such that $Z_n(1) = 1$. If (e_{ij}), (f_{ij}), $(f_{\alpha\beta})$ are systems of matrix units for M and*

$(\xi^{(n)}_{ij,\alpha\beta})$ *are the coefficients of* Z_n*, considered as a linear map from* $(M, (f_{\alpha\beta}))$ $(M, (e_{ij}))$*, then*

$$\xi^{(n)}_{ij,\alpha\beta} = \overline{\xi^{(n)}}_{ji,\beta\alpha} \tag{40}$$

$$\sum_{\alpha} \xi^{(n)}_{ij,ii} = \delta_{ij} \tag{41}$$

$$\sum_{ij,\alpha\beta} \xi^{(n)}_{ij,\alpha\beta} \bar{a}_i \bar{b}_\alpha a_j b_\beta \geq 0 \qquad ; \quad \forall\, (a_1, \ldots, a_j), (b_1, \ldots, b_y) \in \mathbb{C}^q \tag{42}$$

Conversely every family $(\xi^{(n)}_{ij,\alpha\beta})$ *satisfying (40), (41), (42), defines a positive linear map* $S : M \to M$*, such that* $S(1) = 1$*.*

In the commutative case a stochastic matrix P is defined as one which maps positive vectors into positive and such that $P1 = 1$. Therefore the matrices Z_n defined above are the natural non commutative analogous of the stochastic matrices. Moreover, if ψ is a Markov state, then

$$\psi_{n+1} = \psi_n \circ Z_n = Z^*_n \psi_n \tag{43}$$

which is the analogue of the well known “evolution equation” $w_{n+1} = P^*_n w_n$, for the one–dimensional densities of a Markov measure.
However we stress here two main differences between commutative and non commutative Markov chains:
1) In the commutative case, a Markov chains is completely determined by the initial distribution w_0 and the sequence (P_n) of transfer matrices. In the non commutative case this is not true because one needs to know the sequence (σ_n).
2) Equation (43) in the non commutative case holds only if ψ is a Markov state. If ψ is only a Markov chain (see Def. (6)) one can only assert that:

$$\psi_{n+1} = \varphi_n \cdot Z_n$$

$$\psi_n(a) = \psi_n(\sigma_n(a)[1])$$

On the algebra A the shift operator is the algebra–endomorphism T defined by the property $T \circ j_k = j_{k+1}$, $k \geq 0$. A state φ on A is called stationary if $\varphi \cdot T = \varphi$.

Lemma 4 *A Markov state* φ *is stationary if and only if it is determined by a pair* $\{\varphi_0, \sigma\}$*; i.e.* $\sigma_n = \sigma$*, for each* n*, and*

$$Z^* \varphi_0 = \varphi_0 \qquad ; \quad Z = \sigma(1) \tag{44}$$

In the commutative case the eigenvalues of the transfer matrix determine the ergodic behaviour of a stationary Markov chain. The same is true in the non commutative case.

Theorem 3 *Let $\varphi = \{\varphi_0, \sigma\}$ be a stationary Markov chain with transfer matrix $\sigma(1) = Z$. Then if 1 is the only unitary eigenvalue of Z and simple, φ is a factor state. Conversely, if φ is a factor state, then*

$$\lim_{k\to\infty} Z^k = 1 \otimes \varphi_0 \ ; \quad ((1 \otimes \varphi_0)(a) = 1 \cdot \varphi_0(a)) \tag{45}$$

the lmit in the left hand side of (45) being assumed in a certain topology on (M) which depends only on φ_0 and σ.

Define inductively $S_0 = \sigma(M)[1]$; $S_{n+1} = \sigma(M)[S_n]$; $T_0 = \sigma(M)^*[\psi_0]$; $T_{n+1} = \sigma(M)^*[T_n]$.
Put $S = \bigcup_n S_n$; $T = \bigcup_n T_n$. Then the limit (45) must be understood in the sense that:

$$\lim_{k\to\infty} \chi(Z^k(a)) = \chi(1) \cdot \varphi_0(a) \ ; \quad \forall\, \chi \in T \ ; \quad \forall\, a \in S$$

7 Examples of Markov chains

In this section we show that every classical Markov chain admits infinitely many non commutative extensions.
Consider the family of linear functionals on $\bigotimes_{\mathbb{N}} M = A$ defined by:

$$\psi_{[0,n]}(j_0(a_0) \cdot \ldots \cdot j_n(a_n)) = (\sigma_n(a_n)^* \cdot \ldots \cdot \sigma_0(a_0)^* \varphi_0)(1) \tag{46}$$

$$\sigma_n(a)[b] = \overline{\tau}_2(K_n^*(a \otimes b)K_n) \tag{47}$$

$$K_n = \sum_i e_{ii} \otimes W_i(n)^{1/2} \tag{48}$$

where (e_{ij}) is a system of matrix units for M and, for every i and n, $W_i(n)$ is a density matrix. One has:

$$\sigma_n(a)[b] = \sum_{ij} e_{ii} a e_{jj} \cdot \tau(W_i(n)^{1/2} \cdot b \cdot W_j(n)^{1/2}) \tag{49}$$

in particular

$$Z_n(b) = \sigma_n(1)[b] = \sum_i e_{ii} \cdot \tau(W_i(n)^{1/2} b W_i(n)^{1/2}) \tag{50}$$

$$Z_n(1) = 1 \tag{51}$$

Since the right hand side of (7) defines a completely positive linear map, the pair $\{\varphi_0, (\sigma_n)\}$ defines a Markov chain. Now assume that the $W_i(n)$ are diagonal

$$[W_i(n)]_{jj} = P_{ij}^{(n)} \tag{52}$$

then clearly, $(P_{ij}^{(n)})$ is a stochastic matrix for every n and the conditional density K_n is completely determined by the stochastic matrix $P_n = (P_{ij}^{(n)})$. Assume that φ_0 too has a diagonal density matrix $W_0 = (\delta_{ij} w_i)$. Then, if

$$a_k = \sum_{ij} a_{ij}^{(k)} \cdot e_{ij} \ ; \quad 0 \leq k \leq n$$

one finds

$$\psi_{[0,n]}(j_0(a_0) \cdot \ldots \cdot j_n(a_n)) = \sum_{i_0,\ldots,i_n} a_{i_0 i_0}^{(0)} \cdot a_{i_1 i_1}^{(1)} \cdot \ldots \cdot a_{i_n i_n}^{(n)} \cdot w_{i_0} \cdot P_{i_0 i_1}^{(1)} \cdot \ldots \cdot P_{i_{n-1} i_n}^{(n)} \tag{53}$$

In case of diagonal matrices a_k, the right hand side of equality (53) is nothing but the expectation value of the function $a_0 \otimes \ldots \otimes a_n$, with respect to the Markov chain $\{w_0, (p_n)\}$. Summing up:

Proposition 1 *Given a Markov chain $\mu \equiv \{w_0, (P_n)\}$ there exists a non commutative Markov chain $\psi \equiv \{\psi_0, (\sigma_n)\}$ on $A = \bigotimes_{\mathbb{N}} M$, such that its restriction to any preassigned algebra $B = \bigotimes_{\mathbb{N}} D$ of diagonal matrices coincides with μ. If μ is a stationary chain, the same holds for φ. If $\mu = \{w_0, P\}$ and $P > 0$, then φ is a factor state.*

8 One–dimensional nearest neighbour Ising model

In this section we prove that some (very simple) Gibbs states are Markov chains. For a one–dimensional quantum lattice system one $A = \bigotimes_{\mathbb{N}} M$, given the finite volume hamiltonians $H_{[0,n]} \in M_{[0,n]}$, the Gibbs state at the finite volume $[0, n]$ is defined by

$$\varphi_{[0,n]}(Q_{[0,n]}) = \tau_{[0,n]}(e^{-2H_{[0,n]}} \cdot Q_{[0-k]})/\tau_{[0,n]}(e^{-2H_{[0,n]}}) \qquad (54)$$

Denoting, as above, T the shift endomorphism in $\bigotimes_{\mathbb{N}} M$, and j_n the natural immersion of M into the "n–th factor" of $\bigotimes_{\mathbb{N}} M$, we shall consider hamiltonians of the following type:

$$H_{[0,n]} = \sum_{i=0}^{n-1} T'(\Phi_{[0,1]}) \qquad (55)$$

were $\phi_{[0,1]}$ is such that:

$$\Phi_{[0,1]} = (j_0 \otimes j_1)(\Phi) \ ; \quad \Phi = \Phi^* \in M \otimes M \qquad (56)$$

$$[\Phi_{[0,1]} \ , \quad T\Phi_{[0,1]}] = 0 \qquad (57)$$

where $[\cdot, \cdot]$ denotes the commutator.
These hypotheses are verified in the Ising model with nearest neighbour interaction (for which $\Phi = h \otimes h$; $h = h^* \in M$).
Applying to such states Araki's non–commutative modification of the transfer matrix method, one finds, using (57), the following expression for $\varphi_{[0,n]}$

$$\begin{aligned} \varphi_{[0,n]} \qquad & (j_0(a_0) \cdot \ldots \cdot j_k(a_k)) = \quad ; \quad (k \leq n) \\ = \quad & \tau(\mathcal{L}(a_0 \otimes \mathcal{L}(a_1 \otimes \ldots \mathcal{L}(a_k \otimes S^{n-k}(1)) \ldots)/\tau(S^n(1)) \end{aligned} \qquad (58)$$

where we have used the notations (see (19)):

$$\mathcal{L}(x) = \overline{\tau}_2(e^{-\Phi} \cdot x \cdot e^{-\Phi}) \ ; \quad x \in M \otimes M \qquad (59)$$

$$S(a) = \mathcal{L}(1 \otimes a) \ ; \quad a \in M \qquad (60)$$

Using a slight modification of a theorem of L. Gross (see [5]) one deduces the existence of a real $\lambda > 0$, $ab \in M$, $b > 0$, and a (faithful) state φ_0 on M, such that

$$S(b) = \lambda b \ ; \quad S^* \varphi_0 = \lambda \varphi_0 \qquad (61)$$

and:

$$\lim_{k \to \infty} \|\lambda^{-k} S^k - b \otimes \varphi_0\| = 0 \ ; \quad ((b \otimes \varphi_0)(a) := b \cdot \varphi_0(a))$$

$$\lim_{n\to\infty} \psi_{[0,n]}(j_0(a_0)\cdot\ldots\cdot j_k(a_k)) = \tau\left(\frac{\mathcal{L}}{\lambda}\left(a_0\otimes\ldots\otimes\frac{\mathcal{L}}{\lambda}(a_k\otimes b)\ldots\right)\right)/\tau(b) \tag{62}$$

Defining $\psi_0 = \tau/\tau(b)$; $\sigma(a)[c] = \frac{\mathcal{L}}{\lambda}(a\otimes c)$ the triple $\{\psi_0, \sigma, b\}$ enjoys the properties listed in Lemma (5.3), and therefore the formula

$$\psi_{[0k]}(j_0(a_0)\cdot\ldots\cdot j_k(a_k)) = (\sigma_k(a_k)^*\cdot\ldots\cdot\sigma_0(a_0)^*\psi_0)(b) \tag{63}$$

defines a non commutative Markov chain on $\bigotimes_{\mathbb{N}} M$.
Summing up: the infinite volume limit of the Gibbs states (54) with Hamiltonian satisfying (55), (56), (57), is a non commutative Markov chain, whose explicit expression is given by (63), where B and are uniquely determined by (61).

9 Approximation of Gibbs states with Markov chains

For one–dimensional quantum lattice systems with finite potentials, the infinite volume Gibbs state always exists, as proven in [2]. In the commutative case every Gibbs state is limit in nor Markov chains. In the non–commutative case the situation is essentially the same, at least in the one–dimensinal case.

Definition 7 *Let d be a natural integer ≥ 0. An inverse d–Markov chain on the algebra $A = \bigotimes_{\mathbb{N}} M$, is a triple $\{\varphi_{[0,d]}, \sigma, \rho\}$ such that*

[i)] $\varphi_{[0,d]}$ is a state on $M_{[0,d]}$, [ii)] $\sigma : M \to \mathcal{L}(M_{[0,d]})$ is a linear map such that the operator

$$a\otimes a_{[0,d]} \in M\otimes M_{[0,d]} \to \sigma(a)[a_{[0,d]}] \in M_{[0,d]}$$

is a completely positive linear contraction [iii)] $\rho : M_{[0,d]} \to M_{[0,d]}$ is a completely positive linear map such

$$\varphi_{[0,d]}(\rho(1)) > 0$$

[iv)] $\sigma(1)^*\varphi_{[0,d]} = \varphi_{[0,d]}$

Proposition 2 *An inverse d–Markov chain on $A = \bigotimes_{\mathbb{N}} M$ uniquely determines a state $\psi = (\psi_{[0,n]})$ on A by means of the equalities:*

$$\begin{aligned}\psi_{[0,n]} \quad & (J_0(a_0)\cdot\ldots\cdot J_n(a_n)) = \\ = \quad & \frac{1}{\varphi_{[0,d]}(\rho(1))}(\sigma(a_{d+1})^*\cdot\ldots\cdot\sigma(a_n)^*\varphi_{[0,d]})(\rho[J_0(a_0)\cdot\ldots\cdot J_d(a_d)])\end{aligned} \tag{64}$$

Putting, in (64), $d = 0$, and comparing the result thus obtained with (32), it is clear in which sense (64) defines an "inverse" Markov chain. A detailed analysis of the appearence of inverse Markov chains requires the conept of non–commutative conditional Gibbs distribution and will be done in another paper. Roughly speaking, the situation is the following: in classical probability theory, the parameter $t \in \mathbb{N}$ is interpreted as "time" and the conditioning goes from the past (inside), to the future (outside) therefore the family of local algebras on which one takes the conditioning is $(M_{[0,n]})$ i.e. the one considered in Definition (6). In statistical mechanics, the parameter $t \in \mathbb{N}$ is interpreted as a "position", and the conditioning takes place from the outside to the inside; therefore the appropriate family of local algebras is $(M_{[n,+\infty[})$. But, for each n, there is a natural isomorphism $T_c^{(n)} : M_{[n,+\infty[} \to A$ which allows to consider the quasi–conditinal expectations not as mappings $E : A \to M_{[n,+\infty[}$, but as maps $T_c^{(n)} E : A \to A$. By means of these considerations one can give a definition of an "inverse Markov state", similar to Def. (5.1), and prove (see [1]), Theorem (4.)) that the structure of an inverse d–Markov state is given by (64) with ρ–the identity operator and $\varphi_{[0,d]} = \psi_{[0,d]}$.

Theorem 4 *Every one–dimensional Gibbs state in the sense of Araki [2] is limit, in norm, of inverse* (d)*–Markov states for* $d \to \infty$.

10 Markov chains: continuous time

Let $(M_t)_{t\in\mathbb{R}_+}$ a family of type I factors (M_t is a factor of type I_{q_t} with $q_t \in \mathbb{N}$ or $q = +\infty$), let $A = \bigotimes_{\mathbb{R}_+} M$, $J_t : M_t \to A$, the natural immersion, and for $I \subseteq R$

$$M_I = \bigvee_{t\in I} j_t(M) = \text{sub-algebra of } A \text{ spanned by the } j_t(M),\ t \in I \tag{65}$$

A state φ on A will be called a Markov state with respect to the family $(M_{[0,t]})$ if there exists a family $(E_{t,s})_{s<t}$ of quasiconditional expectations with respect to the triples $M_{[0,s[} \subseteq M_{[0,s]} \subseteq M_{[0,t]}$, such that

$$\varphi_{[0,t]}(a_{[0,t]}) = \varphi_{[0,s]}(E_{t,s}(a_{[0,t]}))\ ;\quad \forall\, a_{[0,t]} \in M_{[0,t]} \tag{66}$$

in such cases the Markov property is expressed by:

$$E_{t,s}(M_{[s,t]}) \subseteq \overline{M}_s = j_s(M_s) \tag{67}$$

Theorem 5 *Every Markov state φ on A defines a pair $\{\varphi_0, (\sigma_s^t)\}$ such that*

[i)] φ_0 is a state on M [ii)] for every $s < t$, $\sigma_s^t : \overline{M}_s \to \mathcal{L}(\overline{M}_t, \overline{M}_s)$ is a linear map such that the map
$a_t \cdot a_s \in M_s \vee M_t \to \sigma_s^t(a_s)[a_t]$ extends to a completely positive linear contraction from $\overline{M}_s \vee \overline{M}_t$ to $\overline{M}_s$ [iii)] The linear extensions of the maps

$$\varphi_{\{t_0,\dots,t_n\}}(a_0 \cdot \dots \cdot a_{t_n}) = (\sigma_{t_{n-1}}^{t_n}(a_{t_{n-1}})^* \cdot \dots \cdot \sigma_0^{t_0}(a_0)^* \varphi_0)(a_{t_n}) \tag{68}$$

completely define the projective family of states $\{\varphi_{t_0,\dots,t_n}\}_{t_0<\dots<t_n}$. Conversely, each such a pair uniquely defines a Markov state on A.

Also in the continuous case we distinguish between Markov states and Markov chains. A Markov chain on A is a triple $\{\varphi_0, (\sigma_s^t), (b_s)\}$, where φ_0 and (σ_s^t) satisfy respectively i) and ii) in Theorem (10.1), and (b_s) is a family of operators such that

$$b_s \in \overline{M}_s \ ; \quad b_s > 0 \ ; \quad \sigma_r^s(1)[b_s] = b_r \qquad ; \quad r < s \ ; \quad \varphi_0\ (b_0) > 0 \tag{69}$$

$$\sigma_r^t(a_r) = \sigma_r^s(a_r) \cdot \sigma_s^t(1) \ ; \quad r < s < t \tag{70}$$

Under the conditions above the family

$$\psi_{[t_0,\dots,t_n]}(a_0 \cdot \dots \cdot a_{t_n}) - (\sigma_{t_n}^t(a_{t_n})^* \cdot \dots \cdot \sigma_0^{t_0}(a_0)^* \varphi_0)(b_t)/\varphi_0(b_0) \tag{71}$$

$0 < t_0 < \dots < t_n$, is a projective family of states on $A = \bigotimes_{t\in\mathbb{R}_+} M_t$ and therefore defines a unique state ψ on A.
Let now all M_t be isomorphic to a single type I factor M. Then R_+ has a natural action T by endomorphisms on A, determined by the property $T_s \circ J_t = J_{s+t}$. A state on A, is called stationary if it is invariant under this action. For a stationary Markov chain one has:

$$\sigma_s^t = \sigma_{t-s} \ ; \quad b_t = b \ ; \quad s < t \ , \quad s, t \in \mathbb{R}_+ \tag{72}$$

Stationary Markov chains with continuous time posses very strong ergodic properties.

Theorem 6 *Let $\psi \equiv \{\psi_0, (\sigma_s), b\}$ be a stationary Markov chain on $A = \bigotimes_{\mathbb{R}_+} M$ where M is a type I_q–factor ($q < \infty$). Then if for each s, $Z(s)$ is self–adjoint (as an operator from $L^2(M,\tau)$ into itself) and the eigenvalue* 1 *is simple, then ψ is a factor state.*

11 Kolmogorov and Schrödinger equations

Let us consider a Markov chain $\psi \equiv \{\varphi_0, (\sigma_s^t), 1\}$, i.e. a Markov chain with normalized quasi–conditional expectations. Define, for $s \in \mathbb{R}_+$

$$\varphi_s = Z(0,s)^*\varphi_0 \tag{73}$$

where $\sigma_s^t(1) = Z(s,t)$. Then

$$\varphi_t = Z(s,t)^*\varphi_s \ ; \quad s < t \tag{74}$$

$$Z(r,t) = Z(r,s) \cdot Z(s,t) \ ; \quad r < s < t \tag{75}$$

equations (74), (75) are the non commutative analogue respectively of the evolution equation and the Chapman–Kolmogorov equation for a classical Markov chain. The well known isomorphism between the predual M_* of M and the trace class operators TC on $\mathcal{H}$, (see [9], pg. 39) induces an action of $Z(s,t)$ on $T(\mathcal{H})$ which will be denoted $v \in T(\mathcal{H}) \to V \cdot Z(s,t) \in T(\mathcal{H})$.
Let us assume that the following limits, both in the weak topology for the duality $\langle M_*, M\rangle$, exist (J being a projection operator in (M)):

$$\lim_{\varepsilon\to 0^+} Z(t, t+\varepsilon) = J \tag{76}$$

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}(Z(t,t+\varepsilon) - J) = S(t) \tag{77}$$

Under these assumptions one easily verifies that the density matrix W_t of φ_t satisfies the equation (the derivative in the left–hand side is a weak derivative for the duality above).

$$\frac{d}{dt} W_t = W_t \cdot S(t) \tag{78}$$

in the sense that the left hand side exists and the quality holds.
Equation (78) is the non–commutative forward Kolmogorov equation.
Remark that, as $V \mapsto V \cdot Z(s,t)$ maps density matrices into density matrices, $V \mapsto V \cdot S(t)$ maps hermitian operators (in $T(\mathcal{H})$) into hermitians with null trace.
More precisely we have that $Z(s,t)$ is the "Green function" of eq. (78) thus, since $V \mapsto V \cdot Z(s,t)$ maps density matrices into density matrices, then if the initial data of (78) is a density matrix, the whole trajectory lies in the convex set $\mathcal{M}(\mathcal{H})$ of the density matrices on $\mathcal{H}$.

Conversely, if the equation (78) has the property above, then its Green function maps density matrices into density matrices.
Every operator $S(t)$ with the property that the Green function of (78) maps $\mathcal{M}(\mathcal{H})$ into itself will be called a "Kolmogorov operator" (see Doob [4]). In particular, if $H(t)$ is a hermitian operator on $\mathcal{H}$, then the operator

$$V \mapsto V \cdot S(t) = i[V, H(t)] = i(V \cdot H(t) - H(t) \cdot V) \tag{79}$$

is a Kolmogorov operator. Thus

$$\frac{d}{dt} W_t = i[W_t, H(t)] \tag{80}$$

is a particular case of (78). We thus conclude that the Schrödinger equation (80) can be considered as a particular case of the non commutative forward Kolmogorov equation.
However, we again remark that, while in the classical case the Kolmogorov equation (and the initial distribution) completely determine the Markov measure, in the non commutative case, the initial state φ_0 and the transfer matrices $(Z(s,t))$ are not sufficient, in general to determine the Markov chain $\{\varphi_0, (\sigma_s^t); 1\}$ not even in the stationary case.
However, let us consider the case when $Z(s,t) = Z(t-s)$, and the non commutative forward Kolmogorov equation is given by (80). In such a case the operator $S(t)$ and, therefore, $H(t)$ does not depend on t, and the Green function of equation (80) is a semigroup of inner automorphisms which clearly maps $\mathcal{M}(\mathcal{H})$ into itself. In this case the transfer semigroup $(Z(t))$ completely determines the state, once given the density matrix at time $t = 0$, due to the following:

Proposition 3 *Let ψ be a Markov chain with transfer matrices $(Z(s,t))$. Then $Z(s,t)$ is invertible for every s and t if and only if ψ factor into*

$$\psi = \bigotimes_{t \in \mathbb{R}_+} \varphi_t \ ; \quad \varphi_t = Z(s,t)^* \varphi_s$$

In particular, the hypothesis of Proposition (11.1) is satisfied when $Z(s,t) = Z(t-s)$ and, for $s \to 0$, $Z(s)$ tends to the identity matrix.

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