**Non homogeneous quantum Markov states and quantum Markov fields**

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**Abstract**
The program relative to the investigation of quantum Markov states for general one–dimensional spin models is carried on, following the strategy developed in the last years. In such a way, the emerging structure is fully clarified. This analysis is a starting point for the solution of the basic (still open) problem concerning the construction of a theory of quantum Markov fields, i.e. quantum Markov processes with multi–dimensional indices.

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# 1 Introduction

One of the basic open problems in quantum probability is the construction of a theory of quantum Markov fields, that is quantum Markov processes with a (possibly) multi–dimensional index set. This program concerns the generalization of the theory of classical Markov fields (see e.g. [10, 11]) to the non commutative setting, naturally arising in quantum statistical mechanics and quantum field theory. We would like to mention some relevant applications of the classical theory to classical statistical mechanics ([12, 16]), and constructive quantum field theory ([15]). On the other hand, the original definition of quantum Markov chains ([1]), which is the study of the quantum Markov property for the one–side forward spin chain, was strongly dependent on the totally ordered structure of the index set, as in the classical case.

However, the investigation of classes of states on the spin chain subjected to Ising–type Hamiltonians (i.e. Hamiltonians relative to pairwise interactions), was vastly developed in the last decades. Concerning this point, the reader is referred to [6, 9, 13, 17] and the literature cited therein.

The approach followed in [4] was rather different. Namely, classes of states of physical interest on the spin chain were studied through the Markov property. Among other results, it was pointed out that there is an intrinsic characterization of the Markov property in terms of the local modular groups of the state under consideration. In such a way, one obtains a nice connection with classes of local Hamiltonians satisfying certain commutation relations.

All the matter was reconsidered in [5] where further progress were obtained. In particular, relatively to the homogeneous one–side forward spin chain, starting from a Markov state $\varphi$, a classical Markov process on a standard probability space $(\Omega, \mu)$ (more precisely a usual Markov chain), together with a field $\{\varphi_\omega\}_{\omega\in\Omega}$ of "elementary" Markov states are recovered. The pair $((\Omega, \mu), \{\varphi_\omega\}_{\omega\in\Omega})$ describes $\varphi$ at level of finite–dimensional distributions. Moreover, further connections with local Hamiltonians were pointed out.

In the present paper, we deal with the most general one–dimensional case. Namely, our framework is the study of the class of Markov states on the quasi–local algebra $\mathfrak{A}$ naturally associated to the non homogeneous one–side backward or forward, or two–side spin chain (see below for the definitions).[1] Following the strategy developed in [4, 5], we show that, even in our general situation, one can recover for a Markov state $\varphi$, a non homogeneous (classical) Markov chain described by the law $\mu$ on the space $\Omega$ of all trajectories, together with a measurable field $\{\varphi_\omega\}_{\omega\in\Omega}$ of states on the quasi–local algebra $\mathfrak{A}$. The states $\varphi_\omega$ are canonically recovered by states $\psi_\omega$ on suitable $C^*$–inductive limits $\mathfrak{B}_\omega$, which are Markov states w.r.t. sequences of (Umegaki) conditional expectations whose ranges are all von Neumann factors. The pair $((\Omega, \mu), \{\varphi_\omega\}_{\omega\in\Omega})$ uniquely determines the Markov state $\varphi$ under consideration.

Namely, as a first result we obtain (Section 3) a disintegration

$$\varphi = \int_\Omega \varphi_\omega \mu(d\omega)$$

of the Markov state $\varphi$ into elementary Markov states (in the sense explained above). In such a way, we get a splitting between the commutative (classical) part, and the non commutative (quantum) part which "lives on the fibres". Further (Section 4), we prove a reconstruction theorem for Markov states on (non homogeneous) spin chains.

Another result of interest is the following (Section 5): the connection with one–dimensional models of statistical mechanics is fully clarified. Namely, the Markov property for a locally faithful state $\varphi$, is characterized by the existence of a very explicit Ising–type Hamiltonian (canonically associated to $\varphi$) which generates on the quasi–local algebra $\mathfrak{A}$, a one–parameter group of automorphisms admitting $\varphi$ as a KMS–state.

[1] In order to extend the theory of Markov processes to situations with multi–dimensional indices, the non homogeneous cases should be necessarily taken into account as the "interacting degrees of freedom" localized in a finite volume, increase with the volume, see below.

All the mentioned characterizations of the Markov property for the state $\varphi$, are again equivalent to the Markov property defined only by properties of generalized conditional expectations defined in [3], which are in our situation, canonical objects intrinsically associated to the local structure of the quasi–local algebra $\mathfrak{A}$, and the state $\varphi$ under consideration. Then we obtain a complete description of the deep connection between the Markov property defined by (Umegaki) transition expectations, the same property stated in terms of generalized conditional expectations, and finally one–dimensional models of statistical mechanics with pairwise interaction Hamiltonians.

In such a way, the structure emerging from the Markov property is fully understood, at least for any one–dimensional model.

As these intrinsic characterizations do not deeply depend on the total order of the involved index set, we have some hope that the theory, or at least the essential part of it, could be extended to the multi–dimensional case, which was the original motivation for this work.

We consider a *quasi–local algebra* $\mathfrak{A}$ obtained in the following way. For each $j$ in an index set $I$, a finite–dimensional $C^*$–algebra $M^j$ is assigned and, for each finite subset $\Lambda \subset I$, we define

$$\mathfrak{A}_\Lambda := \otimes_{j\in\Lambda} M^j$$

The local algebra $\mathfrak{A}$ is the $C^*$–inductive limit associated to the directed system $\{\mathfrak{A}_\Lambda\}_{\Lambda\in I}$ with the natural embeddings

$$\iota_{\Lambda,\widehat{\Lambda}} : A_\Lambda \in \mathfrak{A}_\Lambda \to A_\Lambda \otimes I_{\widehat{\Lambda}\setminus\Lambda} \in \mathfrak{A}_{\widehat{\Lambda}}\,, \quad \Lambda \subset \widehat{\Lambda}$$

In this situation we write

$$\mathfrak{A} := \otimes_{j\in I} M^j$$

where the infinite tensor product is defined w.r.t. the unique $C^*$–cross norm. We often denote by $\iota_\Lambda : \mathfrak{A}_\Lambda \mapsto \mathfrak{A}$ the canonical injection of $\mathfrak{A}_\Lambda$ into $\mathfrak{A}$ and refer to [7, 8, 23] for further details.

By a (Umegaki) *conditional expectation* $E : \mathfrak{A} \mapsto \mathfrak{B} \subset \mathfrak{A}$ we mean a norm–one projection of the $C^*$–algebra $\mathfrak{A}$ onto a $C^*$–subalgebra (with the same identity $I$) $\mathfrak{B}$. The map $E$ is automatically a completely positive identity–preserving $\mathfrak{B}$–bimodule map, see [19], Section 9. When $\mathfrak{A}$ is a matrix algebra, the structure of a conditional expectation is well–known, see [5], Lemma 3 (see also [14], Proposition 5 for more general cases when the centre of the range of $E$ is infinite–dimensional and atomic). Namely, suppose that $\mathfrak{A}$ is a full matrix algebra and consider the (finite) set $\{P_i\}$ of minimal central projections of the range $\mathfrak{B}$ of $E$, we have

$$E(x) = \sum_i E(P_i x P_i) P_i$$

Then $E$ is uniquely determined by its values on the reduced algebras

$$\mathfrak{A}_{P_i} := P_i \mathfrak{A} P_i = N_i \otimes \bar{N}_i$$

where $N_i \sim \mathfrak{B} P_i$ and $\bar{N}_i \sim (\mathfrak{B}' \wedge \mathfrak{A}) P_i$. In fact, there exist states $\phi_i$ on $\bar{N}_i$ such that

$$E(P_i(a \otimes \bar{a})P_i) = \phi_i(\bar{a}) P_i (a \otimes I) P_i \tag{1}$$

For the general theory of operator algebras the reader can consult [8, 18, 19, 21].

# 2 Non homogeneous Markov states: general properties

A necessary step for the construction of quantum Markov states on multi–dimensional lattices (Markov fields), is to extend the strategy developed in [4, 5].

The main example we have in mind is the following. We consider the standard lattice $\mathbb{Z}^d$ in the $d$–dimensional space $\mathbb{R}^d$, together with a quasi–local algebra of observables defined as the infinite $C^*$–tensor product $\mathfrak{A} = \otimes_{x \in \mathbb{Z}^d} M$ with $M$ a fixed full matrix algebra. Suppose further that an increasing sequence $\{R_k\}$ of bounded regions exhausting $\mathbb{Z}^d$ is kept fixed, and consider for $k > l + 1$, the local subalgebra given by

$$\mathfrak{A}_{k,l} := \otimes_{x \in R_k \setminus \overset{\circ}{R}_l} M$$

where $\overset{\circ}{R}_k = \{x \in R_k : \mathrm{dist}(x, R_k^c) > 1\}$.[2]

[2]We denote, as usual, by $R^c$ the complement of the set $R$.

If such a picture is the framework for our analysis, we can point out two procedures of conditioning.

The first one is in connection with a fixed sequence $\{F_{k,l}\}$ of conditional expectations from the *outside* or *future* to the *inside* or *past*

$$F_{k,l} : \mathfrak{A}_{k,l} \mapsto \mathfrak{A}_{k-1,l}$$

satisfying

$$\mathfrak{A}_{k-2,l} \subset \mathcal{R}(F_{k,l})$$

where $\mathcal{R}(F_{k,l})$ denotes the range of $F_{k,l}$.

Such a situation corresponds in the terminology of [5], to the localization $\mathcal{A}_o = \mathfrak{A}_{k-2,l}$, $\mathcal{A}_i = \mathfrak{A}_{R_k \setminus R_{k-1}}$ and $\mathcal{A}_b = \mathfrak{A}_{R_{k-1} \setminus R_{k-2}}$.

The second one is given by considering a fixed sequence $\{E_{k,l}\}$ of conditional expectations from the *inside* to the *outside*

$$E_{k,l} : \mathfrak{A}_{k,l} \mapsto \mathfrak{A}_{k,l+1}$$

satisfying

$$\mathfrak{A}_{k,l+2} \subset \mathcal{R}(E_{k,l})$$

Such a situation corresponds in the terminology of [5], to the localization $\mathcal{A}_o = \mathfrak{A}_{k,l+2}$, $\mathcal{A}_i = \mathfrak{A}_{\overset{\circ}{R}_{l+1} \setminus \overset{\circ}{R}_l}$ and $\mathcal{A}_b = \mathfrak{A}_{\overset{\circ}{R}_{l+2} \setminus \overset{\circ}{R}_{l+1}}$.

Summarizing, in both situations we obtain a chain $\{M^j\}_{j \in I}$ of finite–dimensional factors given by $M^j := \mathfrak{A}_{R_j \setminus R_{j-1}}$ or $M^j := \mathfrak{A}_{\overset{\circ}{R}_{-j+1} \setminus \overset{\circ}{R}_{-j}}$ respectively.[3]

Taking into account the last examples, we start by considering a totally ordered countable discrete set $I$ containing, possibly a smallest element $j_-$ and/or a greatest element $j_+$. Namely, if $I$ contains neither $j_-$, nor $j_+$, then $I \sim \mathbb{Z}$. If just $j_+ \in I$, then $I \sim \mathbb{Z}_-$, whereas if only $j_- \in I$, then $I \sim \mathbb{Z}_+$. Finally, if both $j_-$ and $j_+$ belong to $I$, then $I$ is a finite set and the analysis becomes easier. If $I$ is order–isomorphic to $\mathbb{Z}$, $\mathbb{Z}_-$ or $\mathbb{Z}_+$, we put simbolically $j_-$ and/or $j_+$ equal to $-\infty$ and/or $+\infty$ respectively. In such a way, the objects with indices $j_-$ and $j_+$ will be missing in the computations.

We consider the quasi–local algebra $\mathfrak{A}$ obtained when full matrix algebras with possibly different dimensions

$$M^j := \mathbb{M}_{d_j}(\mathbb{C})$$

[3]In the last case we are using the reverse order, that is $I \sim \mathbb{Z}_-$, the negative integers. This choice is a pure matter of convenience, see below.

describe the observables relative to $j \in I$. If $\Lambda \subset I$ is finite, $\mathfrak{A}_\Lambda$ has an obvious meaning whereas, for arbitrary $\Lambda$, the algebra $\mathfrak{A}_\Lambda$ will be the $C^*$–inductive limit of the algebras associated with all finite subsets of $\Lambda$. We always write

$$\mathfrak{A}_\Lambda = \otimes_{j\in\Lambda} M^j$$

as this causes no confusion.

For every $k \leq l \leq j$, let $\{E_{k,j}\}$ be a sequence of conditional expectations defined on the algebras $\{\mathfrak{A}_{\Lambda_{k,j+1}}\}$ (where $\Lambda_{k,j} := [k, j]$), and satisfying

$$E_{k,j} \quad (\mathfrak{A}_{\Lambda_{k,j+1}}) \subset \mathfrak{A}_{\Lambda_{k,j}} \tag{2}$$

$$E_{k,j} \quad {}_{\mathfrak{A}_{\Lambda_{k,j-1}}} = \mathrm{id} \tag{3}$$

$$E_{k,j} \quad {}_{\mathfrak{A}_{\Lambda_{l,j}}} = E_{l,j} \tag{4}$$

**Definition 1** *A state $\varphi$ on the quasi–local algebra $\mathfrak{A}$ is said to be a Markov state w.r.t. a sequence $\{E_{k,j}\}$ of conditional expectations satisfying conditions (2), if the restictions of $\varphi$ to $\mathfrak{A}_{\Lambda_{k,j}}$ are invariant under the $E_{k,j}$:*

$$\varphi_{\mathfrak{A}_{\Lambda_{k,j}}} \circ E_{k,j} = \varphi_{\mathfrak{A}_{\Lambda_{k,j+1}}} \tag{5}$$

By restriction of $\{E_{k,l}\}$, we recover a sequence $\{\mathcal{E}^j\}_{j_-\leq j<j_+}$ of *transition expectations*

$$\mathcal{E}^j : M^j \otimes M^{j+1} \mapsto M^j$$

defined by the identity

$$E_{k,j}(A_k \otimes \cdots \otimes A_{j-1} \otimes A_j \otimes A_{j+1}) = A_k \otimes \cdots \otimes A_{j-1} \otimes \mathcal{E}^j(A_j \otimes A_{j+1})$$

It is immediate to check that the sequence $\{E_{k,j}\}$ of conditional expectations is uniquely determined by the sequence $\{\mathcal{E}^j\}$ of transition expectations and vice–versa. Therefore, in the sequel we will often use exchangeably the symbols $E_{k,j}$ or $\mathcal{E}^j$.

The situation described above can be fitted into the framework of [2]. Namely, we can start with a totally ordered set $\{\alpha_j\}_{j_-\leq j<j_+}$ where the "bounded set" $\alpha_j$ is given by $\alpha_j := \Lambda_{j,j_+}$ with "boundary" $\partial\alpha_j := \Lambda_{j,j}$. Suppose further we have a sequence of transition expectations $\{\mathcal{E}^j\}_{j_-\leq j<j_+}$ satisfying for each $j_- < j < j_+$,

$$\mathcal{E}^{j-1}(A \otimes B) = \mathcal{E}^{j-1}(A \otimes \mathcal{E}^j(B \otimes I)) \tag{6}$$

It can be shown (see (25)) that, if $\varphi$ is a locally faithful Markov state w.r.t. the sequence of transition expectations $\{\mathcal{E}^j\}$, then $\{\mathcal{E}^j\}$ must satisfy (6).

Identifying $\mathfrak{A} \cong \mathfrak{A}_{\Lambda_{j_-,j-1}} \otimes \mathfrak{A}_{\Lambda_{j,j_+}}$ and considering generators $A \otimes B$ with $A \in \mathfrak{A}_{\Lambda_{j_-,j-1}}$ and $B := B_j \otimes \cdots \otimes B_{l-1} \otimes B_l$ any localized element of $\mathfrak{A}_{\Lambda_{j,j_+}}$, we define

$$E_{\alpha'_j}(A \otimes B) := A \otimes \mathcal{E}^j(B_j \otimes \cdots \otimes \mathcal{E}^{l-1}(B_{l-1} \otimes B_l) \cdots)))$$

Because of condition (6), the $E_{\alpha'}$ can be extended to conditional expectations (which we continue to call $E_{\alpha'}$) on all of $\mathfrak{A}$. Such extensions satisfy the projectivity condition

$$E_{\alpha'_k} E_{\alpha'_j} = E_{\alpha'_k}$$

if $k \leq j$, and the Markov property

$$E_{\alpha'_k}(\mathfrak{A}_{\alpha_k}) \subset \mathfrak{A}_{\partial\alpha_k}$$

The above analysis suggests how one can prove the existence of non trivial examples of Markov states on one–dimensional chains. Namely, suppose we have a sequence $\{\mathcal{F}^j\}_{j_- \leq j < j_+}$ of transition expectations and define

$$\mathcal{E}^j(A \otimes B) := \mathcal{F}^j(A \otimes \mathcal{F}^{j+1}(B \otimes I)) \tag{7}$$

Then it is easy to verify that the new sequence $\{\mathcal{E}^j\}_{j_- \leq j < j_+}$ is made of conditional expectations and satisfies (6). Hence, taking into account the above considerations, we can conclude that the set of Markov states w.r.t. $\{\mathcal{E}^j\}$ is nonvoid by a simple application of Theorem 1.1 of [2].

The following result is essentially contained in [5]. We include its proof for completeness.

**Proposition 2.1** *Let $\varphi$ be a state on the quasi–local algebra $\mathfrak{A}$.*

*Then $\varphi$ is a Markov state w.r.t. the sequence $\{\mathcal{E}^j\}_{j_- \leq j < j_+}$ of transition expectations if and only if*

$$\varphi(\iota_{k,l}(A)) = \varphi(\iota_{k,k}(\mathcal{E}^k(A_k \otimes \cdots \otimes \mathcal{E}^{l-1}(A_{l-1} \otimes A_l) \cdots))) \tag{8}$$

*for every $k, l \in I$ with $k < l$, and $A := A_k \otimes \cdots \otimes A_{l-1} \otimes A_l$ any linear generator of $\mathfrak{A}_{\Lambda_{k,l}}$.*

**Proof**. Suppose that $\varphi$ is a Markov state w.r.t. the sequence $\{\mathcal{E}^j\}_{j_-\leq j<j_+}$ of transition expectations, and $A\in\mathfrak{A}_{\Lambda_{k,l}}$ is as above. Then the Markov property leads to

$$\begin{aligned}&\varphi(\iota_{k,l}(A))\equiv\varphi_{\Lambda_{k,l}}(A)\\ =\;&\varphi_{\Lambda_{k,l-1}}(A_k\otimes\cdots\otimes\mathcal{E}^{l-1}(A_{l-1}\otimes A_l))\end{aligned}$$

Then (8) follows by a repetead application of the Markov property.

Conversely, suppose that $\varphi$ satisfies the chain of conditions (8) and fix a generator $A:=A_k\otimes\cdots\otimes A_{l-1}\otimes A_l$ of $\mathfrak{A}_{\Lambda_{k,l}}$. Then, by (8) we get

$$\begin{aligned}\varphi_{\Lambda_{k,l}}(A)\equiv\;&\varphi(\iota_{k,l}(A))\\ =\;&\varphi(\iota_{k,k}(\mathcal{E}^k(A_k\otimes\cdots\otimes\mathcal{E}^{l-1}(A_{l-1}\otimes A_l)\cdots)))\end{aligned}$$

But, again by (8),

$$\begin{aligned}&\varphi(\iota_{k,k}(\mathcal{E}^k(A_k\otimes\cdots\otimes\mathcal{E}^{l-1}(A_{l-1}\otimes A_l)\cdots)))\\ =\;&\varphi_{\Lambda_{k,l-1}}(A_k\otimes\cdots\otimes\bar{A}_{l-1})=\varphi_{\Lambda_{k,l-1}}(E_{k,l}(A))\end{aligned}$$

where $\bar{A}_{l-1}=\mathcal{E}^{l-1}(A_{l-1}\otimes A_l)$ and $A_k\otimes\cdots\otimes\bar{A}_{l-1}$ is precisely $E_{k,l}(A)$.

We have just proved that

$$\varphi_{\Lambda_{k,l}}(A)=\varphi_{\Lambda_{k,l-1}}(E_{k,l}(A))$$

which is the Markov property for $\varphi$, as the $A$ as above linearly generate all of $\mathfrak{A}_{\Lambda_{k,l}}$, and $k<l$ is arbitrary.

# 3 Disintegration of Markov states

In this section we study the structure of Markov states. As final result we obtain a disintegration of a Markov state into "elementary Markov states" in a sense we are going to explain.

We start by considering a Markov state $\varphi$ on the quasi–local algebra $\mathfrak{A}$ w.r.t. the sequence $\{\mathcal{E}^j\}_{j_-\leq j<j_+}$ of transition expectations. As the structure of such expectations is well-known, we consider the centre $Z^j$ of the range

$\mathcal{R}(\mathcal{E}^j)$ of $\mathcal{E}^j$, together with the generating family $\{P^j_{\omega_j}\}_{\omega_j\in\Omega_j}$ of atomic projections, which in the finite–dimensional case is in one–to–one correspondence with its spectrum $\Omega_j$. We set

$$B^j := \oplus_{\omega_j\in\Omega_j} P^j_{\omega_j} M^j P^j_{\omega_j}$$

and define

$$\mathfrak{B} := \left(\otimes_{j_-\leq j<j_+} B^j\right)\otimes M^{j_+} \tag{9}$$

Then we obtain in a canonical way, a conditional expectation

$$E : \mathfrak{A} \mapsto \mathfrak{B}$$

defined to be the (infinite) tensor product of the following conditional expectations

$$a\in M^j \mapsto \sum_{\omega_j\in\Omega_j} P^j_{\omega_j} a P^j_{\omega_j} \tag{10}$$

together with the identity map on $M^{j_+}$. The projections $P^j_{\omega_j}$ generate also the centre of $B^j$, which in such a way coincides with $Z^j \equiv \mathcal{Z}(\mathcal{R}(\mathcal{E}^j))$. Furthermore, the reduced algebra

$$M^j_{P^j_{\omega_j}} \equiv P^j_{\omega_j} M^j P^j_{\omega_j}$$

can be written as

$$M^j_{P^j_{\omega_j}} = N^j_{\omega_j} \otimes \bar{N}^j_{\omega_j} \tag{11}$$

with $N^j_{\omega_j}$ and $\bar{N}^j_{\omega_j}$ all finite–dimensional factors. Again, the states $\phi^j_{\omega_j}$ on $\bar{N}^j_{\omega_j}\otimes M^{j+1}$ are uniquely recovered by the transition expectation $\mathcal{E}^j$ according to Formula (1).

Following [5], we can recover

(a) a classical Markov process on the compact space

$$\Omega := \prod_{j_-\leq j<j_+} \Omega_j \equiv \prod_{j_-\leq j<j_+} \operatorname{spec}(\mathcal{Z}(\mathcal{R}(\mathcal{E}^j))) \tag{12}$$

whose law $\mu$ is uniquely determined by the initial distribution and transition probabilities given respectively by

$$\begin{aligned} \pi^{j_-}_{\omega_{j_-}} &:= \varphi(\iota_{\Lambda_{j_-,j_-}}(P^{j_-}_{\omega_{j_-}})) \\ \pi^j_{\omega_j,\omega_{j+1}} &:= \phi^j_{\omega_j}(I\otimes P^{j+1}_{\omega_{j+1}})\,, \end{aligned} \tag{13}$$

see [24], Theorem 7.2.

As we are dealing with the measure space $(\Omega, \mu)$ obtained as the projective limit of compatible measure spaces $\{(\Omega_\Lambda, \mu_\Lambda)\}_{\Lambda\subset I}$, we denote by $q_\Lambda : \Omega \mapsto \Omega_\Lambda$ the canonical projection of $\Omega$ onto $\Omega_\Lambda$. For details relative to measure on infinite dimensional spaces, the reader can consult [24] and the literature cited therein.

Let $\Omega_0 \subset \Omega$ be the set consisting of those $\omega \in \Omega$ such that all $\pi^j_{q_{\Lambda_{j,j}}(\omega)}$ together with $\pi^j_{q_{\Lambda_{j,j}}(\omega),q_{\Lambda_{j+1,j+1}}(\omega)}$ are nonvanishing. It is straightforward to verify that $\Omega_0$ is a measurable set of full $\mu$–measure.

Consider, for each $\omega \in \Omega$, the (infinite) tensor product $\mathfrak{B}_\omega$ given by

$$\begin{aligned}\mathfrak{B}_\omega := & \qquad (\otimes_{j_-\le j<j_+} M^j_{P^j_{\omega_j}}) \otimes M^{j_+} \\ \equiv & \quad N^{j_-}_{\omega_{j_-}} \otimes (\otimes_{j_-\le j<j_+-1}(\bar{N}^j_{\omega_j} \otimes N^{j+1}_{\omega_{j+1}})) \otimes (\bar{N}^{j_+-1}_{\omega_{j_+-1}} \otimes M^{j_+})\end{aligned} \tag{14}$$

We remark that, in non trivial cases (i.e. when $I$ is infinite), $\mathfrak{B}_\omega$ cannot be viewed in a canonical way as a subalgebra of $\mathfrak{A}$ ([23]). However, a completely positive identity–preserving map $E_\omega : \mathfrak{A} \mapsto \mathfrak{B}_\omega$ is uniquely defined as the (infinite) tensor product of the maps

$$a \in M^j \mapsto P^j_{q_{\Lambda_{j,j}}(\omega)} a P^j_{q_{\Lambda_{j,j}}(\omega)}\,, \tag{15}$$

together with the identity map on $M^{j_+}$. We have trivially

$$E_\omega \circ E = E_\omega \tag{16}$$

where $E$ is obtained by the (infinite) tensor product of the maps given in (10).

Denoting (with an abuse of notation) by $\omega_j$ the canonical projection $q_{\Lambda_{j,j}}(\omega)$ of $\omega$ in $\Omega_j$, we further recover for $\omega \in \Omega_0$

(b) the state $\psi_\omega$ on $\mathfrak{B}_\omega$ given by

$$\psi_\omega := \eta^{j_-}_{q_{\Lambda_{j_-,j_-}}(\omega)} \otimes (\otimes_{j_-\le j<j_+-1} \eta^j_{q_{\Lambda_{j,j}}(\omega),q_{\Lambda_{j+1,j+1}}(\omega)}) \otimes \eta^{j_+-1}_{q_{\Lambda_{j_+-1,j_+-1}}(\omega)} \tag{17}$$

determined by the initial distribution, which is the state on $N^{j_-}_{\omega_{j_-}}$ given by

$$\eta^{j_-}_{\omega_{j_-}}(a) := \frac{\varphi(\iota_{\Lambda_{j_-,j_-}}(P^{j_-}_{\omega_{j_-}}(a\otimes I)P^{j_-}_{\omega_{j_-}}))}{\pi^{j_-}_{\omega_{j_-}}}\,; \tag{18}$$

by the states $\eta^j_{\omega_j,\omega_{j+1}}$ on $\bar{N}^j_{\omega_j}\otimes N^{j+1}_{\omega_{j+1}}$, given by

$$\eta^j_{\omega_j,\omega_{j+1}}(\bar{a}\otimes b):=\frac{\phi^j_{\omega_j}(\bar{a}\otimes P^{j+1}_{\omega_{j+1}}(b\otimes I)P^{j+1}_{\omega_{j+1}})}{\pi^j_{\omega_j,\omega_{j+1}}}\,; \tag{19}$$

and by the final distribution which is the state on $\bar{N}^{j_+-1}_{\omega_{j_+-1}}\otimes M^{j_+}$ given by

$$\eta^{j_+-1}_{\omega_{j_+-1}}(\bar{a}\otimes B):=\phi^{j_+-1}_{\omega_{j_+-1}}(\bar{a}\otimes B) \tag{20}$$

Finally, we recover

(c) A sequence $\{\mathcal{E}_\omega{}^j\}_{j_-\leq j<j_+}$ of conditional expectations

$$\begin{aligned}\mathcal{E}_\omega{}^j:\quad & M^j_{P^j_{\omega_j}}\otimes M^{j+1}_{P^{j+1}_{\omega_{j+1}}}\mapsto M^j_{P^j_{\omega_j}}\,,\\ \mathcal{E}_\omega{}^{j_+-1}:\quad & M^{j_+-1}_{P^{j_+-1}_{\omega_{j_+-1}}}\otimes M^{j_+}\mapsto M^{j_+-1}_{P^{j_+-1}_{\omega_{j_+-1}}}\end{aligned}$$

given by

$$\begin{aligned}\mathcal{E}_\omega{}^j((a\otimes\bar{a})\otimes(b\otimes\bar{b}))= &\quad \eta^j_{\omega_j,\omega_{j+1}}(\bar{a}\otimes b)\eta^{j+1}_{\omega_{j+1},\omega_{j+2}}(\bar{b}\otimes I)a\otimes I\,,\\ \mathcal{E}_\omega{}^{j_+-1}((a\otimes\bar{a})\otimes B)= &\quad \eta^{j_+-1}_{\omega_{j_+-1}}(\bar{a}\otimes B)a\otimes I\end{aligned} \tag{21}$$

The proof of the following proposition follows by an elementary application of Proposition 2.1 and is left to the reader.

**Proposition 3.1** *The state $\psi_\omega$ satisfies the Markov property w.r.t. the sequence of transition expectations $\{\mathcal{E}_\omega{}^j\}_{j_-\leq j<j_+}$ given by (21).*

Finally, we note that the map

$$\omega\in\Omega_0\mapsto\psi_\omega\circ E_\omega\in\mathcal{S}(\mathfrak{A})$$

is $\sigma(\mathfrak{A}^*,\mathfrak{A})$–measurable.

We are ready to prove the announced result concerning the disintegration of a Markov state into elementary Markov states which are minimal in the sense that the ranges of the associated transition expectations factorize as in (11), that is they have a trivial centre .

**Theorem 3.2** *Let $\varphi$ be a Markov state on the quasi–local algebra $\mathfrak{A}$ w.r.t. the sequence $\{\mathcal{E}_j\}_{j_-\leq j<j_+}$ of transition expectations.*

*Define the set $\Omega$ by (12); the probability measure $\mu$ on $\Omega$, by (13); the quasi–local algebra $\mathfrak{B}_\omega$ by (14), the map $E_\omega$ by the projections (15); the state $\psi_\omega$ on $\mathfrak{B}_\omega$ by (17).*

*Then $\varphi$ admits a disintegration*

$$\varphi=\int_\Omega \varphi_\omega \mu(d\omega)$$

*where $\omega\in\Omega\mapsto\varphi_\omega\in\mathcal{S}(\mathfrak{A})$ is a $\sigma(\mathfrak{A}^*,\mathfrak{A})$–measurable map satisfying, for $\mu$–almost all $\omega\in\Omega$,*

$$\varphi_\omega=\psi_\omega\circ E_\omega$$

**Proof**. If the state $\varphi$ satisfies the Markov property w.r.t. $\{\mathcal{E}_j\}_{j_-\leq j<j_+}$, we can find a non homogeneous Markov process on $\Omega$ with law $\mu$ as above. Consider the abelian $C^*$–subalgebra $\mathfrak{Z}$ of $\mathfrak{B}$ given by

$$\mathfrak{Z}:=(\otimes_{j_-\leq j<j_+}Z^j)\otimes I\,,$$

together with the GNS representation $\pi$ of $\mathfrak{B}$ relative to $\varphi_{\mathfrak{B}}$. Then $\pi(\mathfrak{Z})''\subset\pi(\mathfrak{B})'\cap\pi(\mathfrak{B})''$. As $\pi(\mathfrak{Z})''\sim L^\infty(\Omega,\mu)$ (see [24], Theorem 7.2 and [21], Section III.2), we have for $\pi$ the direct–integral disintegration

$$\pi=\int_\Omega^\oplus \pi_\omega\mu(d\omega)$$

where $\omega\mapsto\pi_\omega$ is a weakly measurable field of representations of $\mathfrak{B}$, see [21], Theorem IV.8.25.

Further, by mimicking the proof of Proposition IV.8.34 of [21], we find a measurable field $\omega\mapsto\xi_\omega$ of unit vectors such that, for each $a\in\mathfrak{B}$, we get

$$\varphi(a)=\int_\Omega\langle\pi_\omega(a)\xi_\omega,\xi_\omega\rangle\mu(d\omega)$$

As $\varphi$ is a Markov state, it is invariant w.r.t. $E$. Then

$$\varphi=\int_\Omega \varphi_\omega \mu(d\omega)$$

for the $\sigma(\mathfrak{A}^*,\mathfrak{A})$–measurable field $\varphi_\omega$ defined as

$$\varphi_\omega:=\langle\pi_\omega(E(\,\cdot\,))\xi_\omega,\xi_\omega\rangle$$

Let now $A \in \mathfrak{A}_{\Lambda_{k,l}}$ be given by

$$A = P^k_{\bar{\omega}_k}(a_{\bar{\omega}_k} \otimes \bar{a}_{\bar{\omega}_k})P^k_{\bar{\omega}_k} \otimes \cdots \otimes P^l_{\bar{\omega}_l}(a_{\bar{\omega}_l} \otimes \bar{a}_{\bar{\omega}_l})P^l_{\bar{\omega}_l}\,, \tag{22}$$

and consider

$$Z = P^k_{\hat{\omega}_k} \otimes \cdots \otimes P^l_{\hat{\omega}_l}$$

Then, taking into account that $\pi(\iota_{\Lambda_{k,l}}(Z))$ belongs to the diagonal algebra, we obtain

$$\varphi(\iota_{\Lambda_{k,l}}(AZ)) = \int_\Omega z(\omega)\varphi_\omega(\iota_{\Lambda_{k,l}}(A))\mu(d\omega)$$

where

$$z(\omega) = \delta_{\omega_k,\hat{\omega}_k} \cdots \delta_{\omega_l,\hat{\omega}_l}$$

is the function on $\Omega$ representing the operator $\pi(\iota_{\Lambda_{k,l}}(Z))$.

Now $\varphi(\iota_{\Lambda_{k,l}}(AZ))$ can be also computed by the Markov property obtaining

$$\begin{aligned}
\varphi(\iota_{\Lambda_{k,l}}(AZ)) = & \quad \textstyle\sum_{\omega_{k-1},\omega_{l+1}} \pi^{k-1}_{\omega_{k-1}} \pi^{k-1}_{\omega_{k-1},\bar{\omega}_k} \cdots \pi^l_{\bar{\omega}_l,\omega_{l+1}} \\
\times & \quad \delta_{\bar{\omega}_k,\hat{\omega}_k} \cdots \delta_{\bar{\omega}_l,\hat{\omega}_l} \eta^{k-1}_{\omega_{k-1},\bar{\omega}_k}(I \otimes a_{\bar{\omega}_k}) \cdots \eta^l_{\bar{\omega}_l,\omega_{l+1}}(\bar{a}_{\bar{\omega}_l} \otimes I) \\
\equiv & \quad \textstyle\int_\Omega z(\omega)\psi_\omega(E_\omega(\iota_{\Lambda_{k,l}}(A)))\mu(d\omega)
\end{aligned}$$

If the localization of $A$ contains $j_-$, $j_+ - 1$ or $j_+$, it is easy to show by analogous computations, that the last result holds as well.

Namely, bearing in mind (16), we have just shown that

$$\int_\Omega z(\omega)\varphi_\omega(a)\mu(d\omega) = \int_\Omega z(\omega)\psi_\omega(E_\omega(a))\mu(d\omega)$$

for each fixed localized operator $a \in \mathfrak{A}$ and each function $z \in C(\Omega)$ depending only on finitely many variables. As such functions are dense in $C(\Omega)$, we conclude by the uniqueness of the Radon–Nikodym derivative, that for each localized element $a \in \mathfrak{A}$, there exists a measurable set $\Omega_a \subset \Omega_0$ of full $\mu$–measure such that, when $\omega \in \Omega_a$, we have,

$$\varphi_\omega(a) = \psi_\omega(E_\omega(a)) \tag{23}$$

By considering linear combinations with rational coefficients, we can select a measurable set $F \subset \Omega_0$ of full $\mu$–measure and a dense subalgebra $\mathfrak{A}_0 \subset \mathfrak{A}$ of localized operators such that (23) continues to be true on $F$, for each element of $\mathfrak{A}_0$.

Consider now a sequence $a_n \in \mathfrak{A}_0$ converging to $a \in \mathfrak{A}$. If $\omega \in F$ we obtain

$$\varphi_\omega(a) = \lim_n \varphi_\omega(a_n) = \lim_n \psi_\omega(E_\omega(a_n)) = \psi_\omega(E_\omega(a)),$$

that is (23) holds on $F \subset \Omega_0$ for each $a \in \mathfrak{A}$.

We have just proved that the Markov state $\varphi$ under consideration admits the disintegration

$$\varphi(a) = \int_{\Omega_0} \psi_\omega(E_\omega(a))\mu(d\omega) \tag{24}$$

by "elementary" Markov states $\psi_\omega$, where $\Omega_0$ is the measurable set of full $\mu$–measure on which $\psi_\omega \circ E_\omega$ is well–defined.

**Corollary 3.3** *Let*

$$\varphi = \int_\Omega \varphi_\omega \mu(d\omega)$$

*be the disintegration of a Markov state $\varphi$ as in Theorem 3.2.*

*Then $\varphi_\omega$ is a factor state for $\mu$–almost all $\omega \in \Omega$.*

**Proof**. It is enough to show that $\psi_\omega \circ E_\omega$ is a factor state for all $\omega \in \Omega_0$.

As $\psi_\omega$ is an infinite product state on $\mathfrak{B}_\omega$ w.r.t the factorization pointed out in (14), the double commutant $\pi_{\psi_\omega}(\mathfrak{B}_\omega)''$ of the GNS representation of $\psi_\omega$ gives rise to an Araki–Woods factor, see [19], Section A.17. The proof easily follows from [8], Theorem 2.6.10, by noticing that, if $A, B \in \mathfrak{A}$ are localized in separated regions of $I$, then

$$E_\omega(AB) = E_\omega(A)E_\omega(B)$$

We note that the *type* of the factor $\pi_{\psi_\omega \circ E_\omega}(\mathfrak{A})''$ is determined by the *eigenvalue list* associated to the states given in (18), (19) and (20); see [19], Section A.17. Furthermore, the disintegration (24) for the Markov state $\varphi$, even if it is made of factor states, does not correspond to the central disintegration, see [18], Section 3. The central decomposition of $\varphi$ could be connected to the problem of the ergodic decomposition of the non homogeneous Markov chain associated with $\mu$.

# 4 A reconstruction theorem

In this section we analyze the possibility to obtain the converse of the disintegration result contained in the previous section.

The following theorem can be also regarded as a reconstruction result for the class of non commutative Markov processes treated in the sequel.

**Theorem 4.1** *Consider, for* $j_- \leq j < j_+$*, a sequence* $Z^j$ *of commutative subalgebras of* $M^j$ *with spectra* $\Omega_j$ *and generators* $\{P^j_{\omega_j}\}_{\omega_j \in \Omega_j}$*; a Markov process on the product space*

$$\Omega := \prod_{j_- \leq j < j_+} \Omega_j$$

*with law* $\mu$ *determined, for* $\omega_j \in \Omega_j$*,* $\omega_{j+1} \in \Omega_{j+1}$*, by all marginal distributions* $\pi^j_{\omega_j}$*, and all transition probabilities* $\pi^j_{\omega_j,\omega_{j+1}}$*.*

*For* $\omega_j \in \Omega_j$ *such that* $\pi^j_{\omega_j} > 0$*, fix a splitting as (11)*

$$M^j_{P^j_{\omega_j}} = N^j_{\omega_j} \otimes \bar{N}^j_{\omega_j}$$

*by finite–dimensional factors.*

*For* $\omega_{j_-} \in \Omega_{j_-}$ *such that* $\pi^{j_-}_{\omega_{j_-}} > 0$*, choose a initial distribution* $\eta^{j_-}_{\omega_{j_-}}$ *on* $N^{j_-}_{\omega_{j_-}}$*.*

*For each pair* $(\omega_j, \omega_{j+1}) \in \Omega_j \times \Omega_{j+1}$ *such that* $\pi^j_{\omega_j,\omega_{j+1}} > 0$*, consider a state* $\eta^j_{\omega_j,\omega_{j+1}}$ *on* $\bar{N}^j_{\omega_j} \otimes N^{j+1}_{\omega_{j+1}}$*.*

*For* $\omega_{j_+-1} \in \Omega_{j_+-1}$ *such that* $\pi^{j_+-1}_{\omega_{j_+-1}} > 0$*, consider a final distribution* $\eta^{j_+-1}_{\omega_{j_+-1}}$ *on* $N^{j_+-1}_{\omega_{j_+-1}} \otimes M^{j_+}$*.*

*Then there exists a measurable set* $\Omega_0$ *of full* $\mu$*–measure such that, for each* $\omega \in \Omega_0$*, the state* $\psi_\omega$ *in (17) is a well–defined Markov state on the quasi–local algebra* $\mathfrak{B}_\omega$ *given in (14).*

*Moreover, defining* $E_\omega : \mathfrak{A} \mapsto \mathfrak{B}_\omega$ *by (15), and the* $\sigma(\mathfrak{A}^*, \mathfrak{A})$*–measurable map*

$$\omega \in \Omega_0 \mapsto \psi_\omega \circ E_\omega =: \varphi_\omega \in \mathcal{S}(\mathfrak{A}) \,,$$

*the state* $\varphi$ *on* $\mathfrak{A}$ *given by*

$$\varphi := \int_{\Omega_0} \varphi_\omega \mu(d\omega)$$

*is a Markov state w.r.t. any sequence* $\{\mathcal{E}^j\}_{j_-\leq j<j_+}$ *of transition expectations with* $\mathcal{Z}(\mathcal{R}(\mathcal{E}^j)) = Z^j$*, determined according to (1), by states* $\phi^j_{\omega_j}$ *satisfying, for each* $j_- \leq j < j_+$ *and* $\omega_j \in \Omega_j$*,*

$$\begin{aligned}
\pi^j_{\omega_j}\phi^j_{\omega_j} & (\bar{a}\otimes P^{j+1}_{\omega_{j+1}}(b\otimes\bar{b})P^{j+1}_{\omega_{j+1}}) \\
= & \textstyle\sum_{\omega_{j+2}} \pi^j_{\omega_j}\pi^j_{\omega_j,\omega_{j+1}}\pi^{j+1}_{\omega_{j+1},\omega_{j+2}}\eta^j_{\omega_j,\omega_{j+1}}(\bar{a}\otimes b)\eta^{j+1}_{\omega_{j+1},\omega_{j+2}}(\bar{b}\otimes I)\,, \\
\pi^{j_+-2}_{\omega_{j_+-2}} & \phi^{j_+-2}_{\omega_{j_+-2}}(\bar{a}\otimes P^{j_+-1}_{\omega_{j_+-1}}(b\otimes\bar{b})P^{j_+-1}_{\omega_{j_+-1}}) \\
= & \pi^{j_+-2}_{\omega_{j_+-2}}\pi^{j_+-2}_{\omega_{j_+-2},\omega_{j_+-1}}\eta^{j_+-2}_{\omega_{j_+-2},\omega_{j_+-1}}(\bar{a}\otimes b)\eta^{j_+-1}_{\omega_{j_+-1}}(\bar{b}\otimes I)\,, \\
\pi^{j_+-1}_{\omega_{j_+-1}} & \phi^{j_+-1}_{\omega_{j_+-1}}(\bar{a}\otimes B) \\
= & \pi^{j_+-1}_{\omega_{j_+-1}}\eta^{j_+-1}_{\omega_{j_+-1}}(\bar{a}\otimes B)
\end{aligned} \tag{25}$$

**Proof**. The state $\psi_\omega$ is well–defined on the measurable set $\Omega_0$ of full $\mu$–measure consisting of sequences $\omega$ such that all the $\pi^j_{q_{\Lambda_{j,j}}(\omega)}$, and $\pi^j_{q_{\Lambda_{j,j}}(\omega),q_{\Lambda_{j+1,j+1}}(\omega)}$ are nonvanishing. Moreover, $\psi_\omega$ is a Markov state on $\mathfrak{B}_\omega$ w.r.t. the sequence $\{\mathcal{E}_\omega{}^j\}_{j_-\leq j<j_+}$ of transition expectations

$$\begin{aligned}
\mathcal{E}_\omega{}^j : & \quad M^j_{P^j_{\omega_j}} \otimes M^{j+1}_{P^{j+1}_{\omega_{j+1}}} \mapsto M^j_{P^j_{\omega_j}}\,, \\
\mathcal{E}_\omega{}^{j_+-1} : & \quad M^{j_+-1}_{P^{j_+-1}_{\omega_{j_+-1}}} \otimes M^{j_+} \mapsto M^{j_+-1}_{P^{j_+-1}_{\omega_{j_+-1}}}
\end{aligned}$$

given by (21), see Proposition 3.1. Further, as we have pointed out above, the map

$$\omega\in\Omega_0 \mapsto \psi_\omega\circ E_\omega \in \mathcal{S}(\mathfrak{A})$$

is $\sigma(\mathfrak{A}^*,\mathfrak{A})$–measurable.

Next, by Proposition 2.1, it is enough to show that, for each elementary tensor $A = A_k\otimes\cdots\otimes A_l$ localized in $\mathfrak{A}_{\Lambda_{k,l}}$,

$$\varphi(\iota_{k,l}(A)) = \varphi(\iota_{k,k}(B)) \tag{26}$$

where $B$ is given by

$$B := \mathcal{E}^k(A_k\otimes\cdots\cdots\otimes\mathcal{E}^k(A_{l-1}\otimes A_l)\cdots)$$

To simplify, we suppose also that $\Lambda_{k,l}$ does not contain $j_-$ or $j_+ - 1$, otherwise the result can be obtained by quite similar computations left to the reader. Furthermore, we can choose $A$ as in (22), restricting ourselves to the case when $(\bar\omega_k,\dots,\bar\omega_l)\in q_{\Lambda_{k,l}}(\Omega_0)$, otherwise (26) is trivially satisfied. In such a situation we get, for $\omega\in\Omega_0$,

$$\begin{aligned}\varphi_\omega(\iota_{k,l}(A)) = & \quad \delta_{q_{\Lambda_{k,k}}(\omega),\bar\omega_k}\cdots\delta_{q_{\Lambda_{l,l}}(\omega),\bar\omega_l}\\ \times & \quad \eta^{k-1}_{q_{\Lambda_{k-1,k-1}}(\omega),\bar\omega_k}(I\otimes a_{\bar\omega_k})\cdots\eta^{l}_{\bar\omega_l,q_{\Lambda_{l+1,l+1}}(\omega)}(\bar a_{\bar\omega_l}\otimes I)\end{aligned}$$

Then

$$\varphi(\iota_{\Lambda_{k,l}}(A)) = \textstyle\int_{\Omega_0}\varphi_\omega(\iota_{\Lambda_{k,l}}(A))\mu(d\omega)$$
$$=\sum_{\omega_{k-1},\omega_{l+1}}\pi^{k-1}_{\omega_{k-1}}\quad\pi^{k-1}_{\omega_{k-1},\bar\omega_k}\cdots\pi^{l}_{\bar\omega_l,\omega_{l+1}}\eta^{k-1}_{\omega_{k-1},\bar\omega_k}(I\otimes a_{\bar\omega_k})\cdots\eta^{l}_{\bar\omega_l,\omega_{l+1}}(\bar a_{\bar\omega_l}\otimes I)$$

Conversely, we have for $B$

$$B=\Gamma P^k_{\bar\omega_k}(a_{\bar\omega_k}\otimes I)P^k_{\bar\omega_k}$$

where $\Gamma$ is the number given by

$$\begin{aligned}\Gamma = & \quad \phi^k_{\bar\omega_k}(\bar a_{\bar\omega_k}\otimes P^{k+1}_{\bar\omega_{k+1}}(a_{\bar\omega_{k+1}}\otimes I)P^{k+1}_{\bar\omega_{k+1}})\times\cdots\\ \times & \quad \phi^{l-2}_{\bar\omega_{l-2}}(\bar a_{\bar\omega_{l-2}}\otimes P^{l-1}_{\bar\omega_{l-1}}(a_{\bar\omega_{l-1}}\otimes I)P^{l-1}_{\bar\omega_{l-1}})\phi^{l-1}_{\omega_{l-1}}(\bar a_{\bar\omega_{l-1}}\otimes P^{l}_{\bar\omega_l}(a_{\bar\omega_l}\otimes \bar a_{\bar\omega_l})P^{l}_{\bar\omega_l})\end{aligned}$$

Next, we get by (25),

$$\begin{aligned}\Gamma = & \quad \textstyle\sum_{\omega_{l+1}}\pi^k_{\bar\omega_k,\bar\omega_{k+1}}\cdots\pi^l_{\bar\omega_l,\omega_{l+1}}\\ \times & \quad \eta^k_{\bar\omega_k,\bar\omega_{k+1}}(\bar a_{\bar\omega_k}\otimes a_{\bar\omega_{k+1}})\cdots\eta^l_{\bar\omega_l,\omega_{l+1}}(\bar a_{\bar\omega_l}\otimes I)\end{aligned}$$

Finally, collecting the last calculations, we obtain

$$\begin{aligned}\varphi(\iota_{\Lambda_{k,k}}(B)) = & \quad \textstyle\int_{\Omega_0}\varphi_\omega(\iota_{\Lambda_{k,k}}(B))\mu(d\omega)\\ = & \quad \textstyle\sum_{\omega_{k-1}}\pi^{k-1}_{\omega_{k-1}}\pi^{k-1}_{\omega_{k-1},\bar\omega_k}\Gamma\eta^{k-1}_{\omega_{k-1},\bar\omega_k}(I\otimes a_{\bar\omega_k})\\ = & \quad \textstyle\sum_{\omega_{k-1},\omega_{l+1}}\pi^{k-1}_{\omega_{k-1}}\pi^{k-1}_{\omega_{k-1},\bar\omega_k}\cdots\pi^l_{\bar\omega_l,\omega_{l+1}}\\ \times & \quad \eta^{k-1}_{\omega_{k-1},\bar\omega_k}(I\otimes a_{\bar\omega_k})\cdots\eta^l_{\bar\omega_l,\omega_{l+1}}(\bar a_{\bar\omega_l}\otimes I)\\ \equiv & \quad \varphi(\iota_{\Lambda_{k,l}}(A))\end{aligned}$$

Taking into account the definition of $E:\mathfrak{A}\mapsto\mathfrak{B}$, the assertion follows as such $A$ linearly generate all of $E(\mathfrak{A}_{\Lambda_{k,l}})$.

# 5 Connection with statistical mechanics

In this section we investigate links between Markov states and Ising potentials on chains. We have also a natural connection with the definition of the Markov property in terms of quasi–conditional expectations.

Suppose we have a locally faithful state on the quasi–local algebra $\mathfrak{A}$, then a potential $h_\Lambda$ is canonically defined for each finite subset $\Lambda$ of the index set $I$ as

$$\varphi_{\mathfrak{A}_\Lambda} = \mathrm{Tr}_{\mathfrak{A}_\Lambda}(e^{-h_\Lambda}\,\cdot\,) \tag{27}$$

Such a set of potentials $\{h_\Lambda\}_{\Lambda\subset I}$ satisfies normalization conditions

$$\mathrm{Tr}_{\mathfrak{A}_\Lambda}(e^{-h_\Lambda}) = 1 \tag{28}$$

together with compatibility conditions

$$(\mathrm{Tr}_{\mathfrak{A}_{\widehat{\Lambda}\setminus\Lambda}} \otimes \mathrm{id}_{\mathfrak{A}_\Lambda})(e^{-h_{\widehat{\Lambda}}}) = e^{-h_\Lambda} \tag{29}$$

for finite subsets $\Lambda \subset \widehat{\Lambda}$.

As the structure of Markov states is fully understood, the set of potentials related to $\varphi$ by (27) satisfies some nice properties. Namely, we find for a Markov state $\varphi$, sequences of selfadjoint operators $\{H_j\}_{j_-\leq j\leq j_+}$, $\{\widehat{H}_j\}_{j_-\leq j\leq j_+}$ localized in $\mathfrak{A}_{\Lambda_{j,j}}$, and $\{H_{j,j+1}\}_{j_-\leq j<j_+}$ localized in $\mathfrak{A}_{\Lambda_{j,j+1}}$ respectively. Such opertors satisfy the following commutation relations

$$\begin{aligned}
[H_j, H_{j,j+1}] = 0, \quad [H_{j,j+1}, \widehat{H}_{j+1}] = 0\,,\\
[H_j, \widehat{H}_j] = 0, \quad [H_{j,j+1}, H_{j+1,j+2}] = 0
\end{aligned} \tag{30}$$

and, for $\Lambda = [k, l]$, they give rise to the potentials $\{h_\Lambda\}$ by

$$h_{\Lambda_{k,l}} = H_k + \sum_{j=k}^{l-1} H_{j,j+1} + \widehat{H}_l \tag{31}$$

for each $k \leq l$, see below.

We have

**Theorem 5.1** *Let $\varphi \in \mathcal{S}(\mathfrak{A})$ be locally faithful.*

*Then the following assertions are equivalent.*

*(i) $\varphi$ is a Markov state.*

*(ii)* *The set of potentials* $\{h_{\Lambda_{k,l}}\}$ *associated to* $\varphi$ *by (27), can be recovered by (31), from sequences* $\{H_j\}_{j_-\leq j\leq j_+}$, $\{\widehat{H}_j\}_{j_-\leq j\leq j_+}$ *and* $\{H_{j,j+1}\}_{j_-\leq j<j_+}$ *of selfadjoint operators localized in* $\mathfrak{A}_{\Lambda_{j,j}}$ *and* $\mathfrak{A}_{\Lambda_{j,j+1}}$ *respectively, and satisfying commutation relations (30).*

*(iii)* *For each* $k<l$, *the generalized conditional expectation ( [3])*

$$\epsilon_{k,l}:\mathfrak{A}_{\Lambda_{k,l+1}}\mapsto\mathfrak{A}_{\Lambda_{k,l}}$$

*leaving fixed* $\varphi_{\mathfrak{A}_{\Lambda_{k,l}}}$, *acts identically on* $\mathfrak{A}_{\Lambda_{k,l-1}}$.

**Proof**. $(i)\Rightarrow(ii)$ As $\varphi$ is a locally faithful Markov state w.r.t. the sequence $\{\mathcal{E}^j\}$ of transition expectation, for each $j_-\leq j<j_+$ and every $\omega_j\in\Omega_j$ we recover, taking into account Formulae (13), (18), (19) and (20), the following set of potentials: $\{h^j_{\omega_j}\}$, $H_{j_+}$, $\{h^j_{\omega_j,\omega_{j+1}}\}$, $h^{j_+-1}_{\omega_{j_+-1},j_+}$, and finally $\{\hat{h}^j_{\omega_j}\}$, related to the following positive faithful functionals.

The potentials $h^{j_-}_{\omega_{j_-}}$ are related to the initial distribution $\pi^{j_-}_{\omega_{j_-}}\eta^{j_-}_{\omega_{j_-}}$ on $N^{j_-}_{\omega_{j_-}}$; the $h^j_{\omega_j}$ are related to the functionals

$$\sum_{\omega_{j-1}}\pi^{j-1}_{\omega_{j-1}}\pi^{j-1}_{\omega_{j-1},\omega_j}\eta^{j-1}_{\omega_{j-1},\omega_j}(I\otimes\,\cdot\,)$$

on $N^j_{\omega_j}$; whereas $H_{j_+}$ is related to the final distribution

$$\sum_{\omega_{j_+-1}}\pi^{j_+-1}_{\omega_{j_+-1}}\eta^{j_+-1}_{\omega_{j_+-1}}(I\otimes\,\cdot\,)$$

on $M^{j_+}$.

Further, the potentials $h^j_{\omega_j,\omega_{j+1}}$ are related to the functionals $\pi^j_{\omega_j,\omega_{j+1}}\eta^j_{\omega_j,\omega_{j+1}}$ on $\bar{N}^j_{\omega_j}\otimes N^{j+1}_{\omega_{j+1}}$; and $h^{j_+-1}_{\omega_{j_+-1},j_+}$ to the final distribution $\eta^{j_+-1}_{\omega_{j_+-1}}$ on $\bar{N}^{j_+-1}_{\omega_{j_+-1}}\otimes M^{j_+}$.

Finally, the potentials $\hat{h}^j_{\omega_j}$ are related to the functionals

$$\sum_{\omega_{j+1}}\pi^j_{\omega_j,\omega_{j+1}}\eta^j_{\omega_j,\omega_{j+1}}(\,\cdot\,\otimes I)$$

on $\bar{N}^j_{\omega_j}$; and $\hat{h}^{j_+-1}_{\omega_{j_+-1}}$ to the final distribution $\eta^{j_+-1}_{\omega_{j_+-1}}(\,\cdot\,\otimes I)$ on $\bar{N}^{j_+-1}_{\omega_{j_+-1}}$.

Now we define

$$
\begin{aligned}
H_j := & \sum_{\omega_j} P^j_{\omega_j}(h^j_{\omega_j}\otimes I)P^j_{\omega_j}\\
H_{j,j+1} := & \sum_{\omega_j,\omega_{j+1}} (P^j_{\omega_j}\otimes P^{j+1}_{\omega_{j+1}})(I\otimes h^j_{\omega_j,\omega_{j+1}}\otimes I)(P^j_{\omega_j}\otimes P^{j+1}_{\omega_{j+1}})\\
H_{j_+-1,j_+} := & \sum_{\omega_{j_+-1}} (P^{j_+-1}_{\omega_{j_+-1}}\otimes I)(I\otimes h^{j_+-1}_{\omega_{j_+-1},j_+})(P^{j_+-1}_{\omega_{j_+-1}}\otimes I)\\
\widehat{H}_j := & \sum_{\omega_j} P^j_{\omega_j}(I\otimes \hat{h}^j_{\omega_j})P^j_{\omega_j}\\
\widehat{H}_{j_+-1} := & \sum_{\omega_{j_+-1}} P^{j_+-1}_{\omega_{j_+-1}}(I\otimes \hat{h}^{j_+-1}_{\omega_{j_+-1}})P^{j_+-1}_{\omega_{j_+-1}}
\end{aligned}
$$

Putting $\widehat{H}_{j_+}=0$, it is straightforward to verify that the $H_j$, the $H_{j,j+1}$ and the $\widehat{H}_j$ satisfy commutation relations (30), and give rise to $h_{\Lambda_{k,l}}$ through (31).
*(ii)* $\Rightarrow$ *(iii)* If $\{h_{\Lambda_{k,l}}\}$ satisfies all the properties listed above, the generalized conditional expectation $\epsilon_{k,l}$ can be obtained as

$$\epsilon_{k,l}(a)=(\mathrm{id}_{\mathfrak{A}_{\Lambda_{k,l}}}\otimes \mathrm{Tr}_{\mathfrak{A}_{\Lambda_{l+1,l+1}}})(k^*_{k,l}ak_{k,l})$$

where $k_{k,l}$ is the transition cocycle given, in such a situation, by

$$k_{k,l}=e^{-\frac{1}{2}(H_{l,l+1}+\widehat{H}_{l+1})}e^{\frac{1}{2}\widehat{H}_l}\,,$$

see [3], Theorem 3.5, see also [4], pag. 260. Hence, $\epsilon_{k,l}$ acts as the identity on $\mathfrak{A}_{\Lambda_{k,l-1}}$.
*(iii)* $\Rightarrow$ *(i)* As the fixed point of $\epsilon_{k,l}$ is a $*$–algebra ([3], pag. 260), we can take the $L^2$–ergodic limit of $\epsilon_{k,l}$ obtaining a conditional expectation $\varepsilon_{k,l}$, see [19], Theorem 9.1. Such a conditional expectation $\varepsilon_{k,l}$ leaves $\varphi_{\mathfrak{A}_{\Lambda_{k,l}}}$ invariant by construction, and contains $\mathfrak{A}_{\Lambda_{k,l-1}}$ in its range by assumption. Hence, it can be written as

$$\varepsilon_{k,l}=\mathrm{id}_{\mathfrak{A}_{k,l-1}}\otimes {\mathcal{E}^l}_k$$

where ${\mathcal{E}^l}_k$ a transition expectation

$${\mathcal{E}^l}_k : M^l\otimes M^{l+1}\mapsto M^l$$

In such a way, for every $j < k$, we find by restrictions of $\varepsilon_{k,l}$, other conditional expectations of of $\mathfrak{A}_{\Lambda_{k,l+1}}$ into $\mathfrak{A}_{\Lambda_{k,l}}$ leaving $\varphi_{\mathfrak{A}_{\Lambda_{k,l}}}$ invariant. A simple application of Theorem 5.1 of [3], together with Takesaki existence Theorem [20] (also reported in [3]), leads to

$$\varepsilon_{k-1,l}(\mathfrak{A}_{\Lambda_{k,l+1}}) \subset \varepsilon_{k,l}(\mathfrak{A}_{\Lambda_{k,l+1}})$$

Namely, $\{\mathcal{E}^l{}_j\}_{j_- \leq j \leq l}$ is, as $j \to j_-$, a decreasing sequence of conditional expectations which converges by a standard martingale convergence Theorem (see e.g. [22] Theorem 3), to the conditional expectation

$$\mathcal{E}^l : M^l \otimes M^{l+1} \mapsto M^l$$

given by

$$\mathcal{E}^l := \lim_{j \to j_-} \mathcal{E}^l{}_j$$

Putting

$$E_{k,l} := \mathrm{id}_{\mathfrak{A}_{\Lambda_{k,l-1}}} \otimes \mathcal{E}^l ,$$

the set $\{E_{k,l}\}$ satisfies all the properties listed in (2), and $\varphi$ is a Markov state w.r.t. $\{E_{k,l}\}$.

A set $\{h_{\Lambda_{k,l}}\}$ of positive selfadjoint operators is called, following the terminology of [4], an *Ising potential* if

$$e^{-\frac{1}{2}h_{\Lambda_{k,l+1}}} e^{\frac{1}{2}h_{\Lambda_{k,l}}} \in \mathfrak{A}_{\Lambda_{l,l+1}}$$

Then any Markov state on the chain leads to a normalized Ising potential. Moreover, taking into account (30) and (31), the potentials $\{h_{\Lambda_{k,l}}\}$ arise from a pairwise interaction.

Further, Theorem 5.1 provides also the equivalence between the definition of the Markov property by *quasi–conditional expectations* ([4], Definition 2.1 or [5], Definition 2) and that given directly by conditional expectations (our Definition 1).

As an immediate consequence of the above results, we get

**Corollary 5.2** *Let $\varphi \in \mathcal{S}(\mathfrak{A})$ be a locally faithful Markov state w.r.t. the sequence $\{\mathcal{E}^j\}_{j_- \leq j < j_+}$ of transition expectations.*

*Then there exists another sequence $\{\mathcal{F}^j\}_{j_- < j \leq j_+}$ of transition expectations*

$$\mathcal{F}^j : M^{j-1} \otimes M^j \mapsto M^j$$

*such that $\varphi$ is also a Markov state (relative to the reverse order of the index set $I$) w.r.t. $\{\mathcal{F}^j\}_{j_- < j \leq j_+}$.*

**Proof**. As $\varphi$ is a locally faithful Markov state w.r.t. the sequence $\{\mathcal{E}^j\}_{j_-\leq j<j_+}$, the implication $(i) \Rightarrow (ii)$ of Theorem 5.1 tells us that the potentials $\{h_{\Lambda_{k,l}}\}$ relative to $\varphi$, have the form (31) for sequences $\{H_j\}$, $\{\widehat{H}_j\}$ and $\{H_{j,j+1}\}$ of selfadjoint operators satisfying all the properties listed above. Then $(ii) \Rightarrow (i)$ of Theorem 5.1 (by passing through Property $(iii)$) gets that $\varphi$ satisfies the Markov property, relative to the reverse order of the index set $I$, w.r.t a suitable sequence $\{\mathcal{F}^j\}_{j_-<j\leq j_+}$ of transition expectations as above.

The last result leads to a kind of *reflection symmetry* on the chain.[4]

We end this section with the generalization to our situation of Corollary 14 of [5].

**Theorem 5.3** *Let $\varphi \in \mathcal{S}(\mathfrak{A})$ be a locally faithful Markov state.*

*Then the pointwise–norm limit*

$$\lim_{\substack{k\to j_-\\ l\to j_+}} e^{-ith_{\Lambda_{k,l}}}ae^{ith_{\Lambda_{k,l}}}$$

*defines a one–parameter automorphisms group $t\mapsto \alpha_t$ on the quasi–local algebra $\mathfrak{A}$ which admits $\varphi$ as a KMS state. Further, $\varphi$ has a normal faithful extension on all of $\pi_\varphi(\mathfrak{A})''$.*

*In particular, any locally faithful Markov state is faithful.*

**Proof**. Thanks to the properties of $h_{\Lambda_{k,l}}$, the cocycle $e^{ith_{\Lambda_{k-1,l+1}}}e^{-ith_{\Lambda_{k,l}}}$ commutes with each element $a\in\mathfrak{A}$ localized in $\mathfrak{A}_{\Lambda_{k+1,l-1}}$.

Then $e^{-ith_{\Lambda_{k,l}}}ae^{ith_{\Lambda_{k,l}}}$ becomes asymptotically constant ($t$ fixed) on the localized elements $a\in\mathfrak{A}$, that is it trivially converges, pointwise in norm, on the localized elements of $\mathfrak{A}$. Next, by a standard 3–$\varepsilon$ trick, it converges on all of $\mathfrak{A}$ and defines an isometry $\alpha_t$. It is straigthforward to show that $t\mapsto\alpha_t$ is actually a group of automorphisms of $\mathfrak{A}$, which is also pointwise–norm continuous in $t$, that is a strongly continuous group of automorphisms of $\mathfrak{A}$. By constuction, $\varphi$ is automatically a KMS state for $\alpha_t$ at inverse temperature $\beta=-1$. The last assertions follow by [9], Corollary 5.3.9, taking into account that $\mathfrak{A}$ is a simple $C^*$–algebra ([8], Proposition 2.6.17).

[4]If $I$ is regarded as a discrete "time", the above symmetry is precisely a time–reversal one.

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