Measures on product spaces

Part I

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1 Introduction

Purpose of the present paper is the study of probability measures on countable[1] products of measurable spaces. We will discuss two problems: 1) equivalence of measures on product spaces; classification[2] of measures on such spaces.

Accordingly, the work is divided into two parts.

Definition 1 *A family of (probability) measure $(\Psi_k)_{k\in\mathbb{N}}$ on the product of measurable spaces $\prod_{i=1}^{k}(\Omega_i,\mathcal{B}_i)$ will be called a "cylindrical measure" on the product space $\prod_{i=1}^{\infty}(\Omega_i,\mathcal{B}_i)$ if there exists a single measure on the product $\prod_{i=1}^{\infty}(\Omega_i,\mathcal{B}_i)$ which extends each of the Ψ_k's.*

The measure will be called measure "induced" by the cylindrical measure $(\Psi_k)_{k\in\mathbb{N}}$. A necessary condition for (Ψ_k) to be a cylindrical measure is that the equality

$$\Psi_{k+1}(E_1;\ldots;E_k;\Omega_{k+1}) = \Psi_k(E_1;\ldots;E_k)\ ; \quad E_i\in\mathcal{B}_i\ ; \quad (1\le i\le k)$$

be satisfied.

We recall that two measures m and m' on a measurable space $(X,\mathcal{B})$ are said to be "$\mathcal{B}$–equivalent" $(m\sim m')$ if for every $E\in\mathcal{B}$, $m(E)=0$ is equivalent to $m'(E)=0$. m is called "$\mathcal{B}$–absolutely continuous" with respect to m', $(m\prec m')$ if $m'(E)=0$ implies $m(E)=0$. m and m' are said to be "orthogonal" $(m\perp m')$ if there are two disjoint sets B, B' such that $m(B)=m'(B')=1$.

When the σ–algebra with respect to which equivalence (resp. absolute continuity) is considered will be evident from the context we will omit it in the notation.

[1]All the results still hold without this hypothesis, which is done to keep evident the analogy with states of uniformly hyperfinite algebras (cfr. [6], [10]).

[2]For example $(\Psi_k)_{k\in\mathbb{N}}$ are always cylindrical measures when $\Psi_k=\prod_{i=1}^{k} m_i$ (m_i, a measure on $(\Omega_i;\mathcal{B}_i)$) or when $(\Omega_i,\mathcal{B}_i)=(\Omega,\mathcal{B})$ for every $\iota\in\mathbb{N}$ and the Ψ_k's are the measures induced by a Markov chain (this follows from Ionesco Tulcea's theorem; (cfr. [5], pg. 162). Or, finally, because of Kolmogorov's extension theorem, when the $(\Omega_i;\mathcal{B}_i)$ are standard Borel space (cfr. [5], pg. 83).

For two probability measures, m, m' on $(X;\mathcal{B})$ their "Hellinger integral" is defined as:

$$\rho(m,m') = \int_X \sqrt{dm \cdot dm'} = \int_X \sqrt{\frac{dm}{dn} \cdot \frac{dm'}{dn}} dn$$

where n is an arbitrary measure which dominates m and m' $(m \prec n; m' \prec n)$, and $\frac{dm}{dn}$ is the Radon–Nikodym derivative of m with respect to n. This having been set, the "problem of the equivalence" is set out as follows: "When do two cylindrical measures on the product $\prod_{i=1}^{\infty}(\Omega_i;\mathcal{B}_i)$ induce two measures equivalent with respect to the σ–algebra $\mathcal{B} = \prod_{i=1}^{\infty} \mathcal{B}_i$?".

If $(\Psi_k)_{k\in\mathbb{N}}$ and $(\Psi'_k)_{k\in\mathbb{N}}$ are cylindrical measures it is clear that the condition $\Psi_k \sim \Psi'_k$ for every $k \in \mathbb{N}$ is necessary for the equivlaence of the induced measures. Thus the problem of the equivalence can be reformulated as follows.

"If (Ψ_k) and (Ψ'_k) are cylindrical measures such that $\Psi_k \sim \Psi'_k$ when are the induced measures equivalent?".

In the case of product measures, this problem has been completely solved by Kakutani [2] who proved that if m_n and m'_n are probability measures on $(\Omega_n,\mathcal{B}_n)$ such that $m \sim m'_n$ for every n, then the product measurs: $\prod_{i=1}^{\infty} m_i$ and $\prod_{i=1}^{\infty} m'_i$ are orthogonal if and only if $\prod_{i=1}^{\infty} \rho(m_i;m'_i) = 0$ and equivalent if and only if $\prod_{i=1}^{\infty} \rho(m_i;m'_i) > 0$.

2 Asymptotic independence

Given a cylindrical measure $(\Psi_k)_{k\in\mathbb{N}}$ on the product $\prod_{i=1}^{\infty}(\Omega_i;\mathcal{B}_i)$ we set, for $k \in \mathbb{N}$ and $l \geq k$

$$\Psi_{k,l}(E_{k+1};\ldots;E_l) = \Psi_l(\Omega_1;\ldots;\Omega_k;E_{k+1};\ldots;E_l)$$

Furthermore, we denote:

$$(\tilde{\Omega}_l;\tilde{\mathcal{B}}_l) = \prod_{i=1}^{l}(\Omega_i;\mathcal{B}_i) \ ; \quad (\tilde{\Omega}_{k,l};\tilde{\mathcal{B}}_{k,l}) = \prod_{i=k+1}^{l}(\Omega_i;\mathcal{B}_i)$$

Then $(\tilde{\Omega}_l; \tilde{\mathcal{B}}_l)$ is naturally identified with $(\tilde{\Omega}_k \times \tilde{\Omega}_{k,l}; \tilde{\mathcal{B}}_k \times \tilde{\mathcal{B}}_{k,l})$ $(k \leq l)$.

$$\Psi_k \cdot \Psi_{k,l} : (E_1; \ldots; E_l) \in \tilde{\mathcal{B}}_l \to \Psi_k(E_1; \ldots; E_k) \cdot \Psi_{k,l}(E_{k+1}; \ldots; E_l)$$

is, thus, a measure on $(\tilde{\Omega}_l; \tilde{\mathcal{B}}_l)$, for every $k \leq l$, and one has $\Psi_l \prec \Psi_k \cdot \Psi_{k,l}$, so that it makes sense to speak of $\frac{d\Psi_l}{d\Psi_k \cdot \psi_{k,l}}$ the Radon–Nykodim derivative of Ψ_l with respect to $\Psi_k \cdot \Psi_{k,l}$

Lemma 1 *If* (Ψ_k) *and* (Ψ'_k) *are cylindrical measures on* $\prod_{i=1}^{\infty}(\Omega_i; \mathcal{B}_i)$ *the following formula holds, for* $k \in \mathbb{N}$; $k \leq l$

$$\rho(\Psi_l; \Psi'_l) - \rho(\Psi_k; \Psi'_k) = \rho(\Psi_k; \Psi'_k) \cdot [\rho(\Psi_{k,l}; \Psi'_{k,l}) - 1] + J_k(\Psi_l \cdot \Psi'_l)$$

where

$$\begin{aligned} J_k(\Psi_l; \Psi'_l) &= \int_{\tilde{\Omega}_l} \left[\sqrt{\frac{d\Psi_l}{d\Psi_k \cdot \Psi_{k,l}} \cdot \frac{d\Psi'_l}{d\Psi'_k \cdot \Psi'_{k,l}}} - 1 \right] \cdot \sqrt{d\Psi_k \cdot \Psi_{k,l} \cdot d\Psi'_k \cdot \Psi'_{k,l}} \\ &= \rho(\Psi_l; \Psi'_l) - \rho(\Psi_k \cdot \Psi_{k,l}; \Psi'_k \cdot \Psi'_{k,l}) \end{aligned}$$

Proof. From [2] Lemma 2, one has:

$$\begin{aligned} \rho(\Psi_l; \Psi'_l) &- \rho(\Psi_k; \Psi'_k) = \rho(\Psi_k; \Psi'_k) \cdot [\rho(\Psi_{k,l}; \Psi'_{k,l}) - 1] + \\ &+ [\rho(\Psi_l; \Psi'_l) - \rho(\Psi_k \cdot \Psi_{k,l}; \Psi'_k \cdot \Psi'_{k,l}) \end{aligned}$$

Furthermore

$$\begin{aligned} \rho(\Psi_l; \Psi'_l) &- \rho(\Psi_k \cdot \Psi_{k,l}; \Psi'_k \cdot \Psi'_{k,l}) = \\ &= \int_{\tilde{\Omega}_l} \sqrt{d\Psi_l \cdot d\Psi'_l} - \int_{\tilde{\Omega}_l} \sqrt{d\Psi_k \cdot \Psi_{k,l} \cdot d\Psi'_k \cdot \Psi'_{k,l}} \\ &= \int_{\tilde{\Omega}_l} \left[\sqrt{\frac{d\Psi_l}{d\Psi_k \cdot \Psi_{k,l}} \cdot \frac{d\Psi'_l}{d\Psi'_k \cdot \Psi'_{k,l}}} - 1 \right] \cdot \sqrt{d\Psi_k \cdot \Psi_{k,l} \cdot d\Psi'_k \cdot \Psi'_{k,l}} \end{aligned}$$

Remarks: 1) If the sequence $(\rho(\Psi_k; \Psi'_k))$ converges to zero, then $\lim_{\substack{k\to\infty \\ k\leq l}} J_k(\Psi_l; \Psi'_l) = 0$.

2) If $\lim_{\substack{k\to\infty \\ k\leq l}} J_k(\Psi_l; \Psi'_l) = 0$, then the sequence $(\rho(\Psi_k; \Psi'_k))$ tends to a limit $\alpha > 0$ if and only if $\lim_{\substack{k\to\infty \\ k\leq l}} \rho(\Psi_{k,l}; \Psi'_{k,l}) = 1$. Conversely, if the second equality

holds, the first equality is a necessary and sufficient condition in order that the sequence $(\rho(\Psi_k; \Psi'_k))$ converges.

If $(X, \mathcal{B})$ is a measurable space it is known that $\sigma(m, m') = -\lg \rho(m, m')$ is a symmetric function on the set of all probability measures on $(X, \mathcal{B})$, such that $\sigma(m; m') = 0$ if and only if $m = m'$. In general σ is not a distance, but the following lemma shows that a weak form of the triangular inequality holds (properly: σ induces a uniform structure).

Lemma 2 *Let $(X, \mathcal{B})$ be a measurable space and m_i; $(\angle = 1, 2, 3)$ probability measures on $(X, \mathcal{B})$. Then:*

$$|\rho(m_1, m_2 0 - \rho(m_2, m_3)| \leq \sqrt{2(1 - \rho(m_1, m_3))}$$

Proof. Let ν be a measure dominating m_i; $(i = 1, 2, 3)$, then

$$\begin{aligned} |\rho(m_1, m_2) \quad - \quad \rho(m_2, m_3)| &\leq \int_X \left| \sqrt{\frac{dm_1}{d\nu} \cdot \frac{dm_2}{d\nu}} - \sqrt{\frac{dm_2}{d\nu} \frac{dm_3}{d\nu}} \right| d\nu \\ &\leq \left\{ \int_X \left| \sqrt{\frac{dm_1}{d\nu}} - \sqrt{\frac{dm_3}{d\nu}} \right|^2 d\nu \right\}^{1/2} \\ &= \sqrt{2(1 - \rho(m_1, m_3))} \end{aligned}$$

Definition 2 *The cylindrical measure (Ψ_k) on the product space $\prod_{i=1}^{\infty} (\Omega_i; \mathcal{B}_i)$ is said to be "asymptotically independent" if*

$$\lim_{\substack{k \to \infty \\ k \leq l}} \rho(\Psi_l; \Psi_k \cdot \Psi_{k,l}) = 1$$

From Lemma (2) it follows thata if (Ψ_k) and (Ψ'_k) are both asymptotically independent, then $\lim_{\substack{k \to \infty \\ k \leq l}} J_k(\Psi_l; \Psi'_l) = 0$. In fact one has, from Lemma (1)

$$|J_k(\Psi_l; \Psi'_l)| \leq \rho(\Psi_l; \Psi'_l) - \rho(\Psi'_l; \Psi_k \cdot \Psi_{k,l})| +$$

$$+ |\rho(\Psi'_l; \Psi_k \cdot \Psi_{k,l}) - \rho(\Psi_k \cdot \Psi_{k,l}; \Psi'_k \cdot \Psi'_{k,l})|$$

$$\leq \sqrt{2(1 - \rho(\Psi_l; \Psi_k \cdot \Psi_{k,l}))} + \sqrt{2(1 - \rho(\Psi'_l; \Psi'_k \cdot \Psi'_{k,l}))}$$

Lemma 3 *The following conditions are equivalent:*

$$\lim_{\substack{k\to\infty\\ k\leq l}} \int_{\tilde{\Omega}_l} \left| \sqrt{\frac{d\Psi_l}{d\Psi_k \cdot \psi_{k,l}}} - 1 \right| d\Psi_k \cdot \psi_{k,l} = 0 \tag{1}$$

The cylindrical measure (Ψ_k) *is asymptotically independent* (2)

$$\lim_{\substack{k\to\infty\\ k\leq l}} \int_{\tilde{\Omega}_l} \left| \frac{d\Psi_l}{d\Psi_k \cdot \Psi_{k,l}} - 1 \right| d\Psi_k \cdot \Psi_{k,l} = 0 \tag{3}$$

$$\lim_{\substack{k\to\infty\\ k\leq l}} \int_{\tilde{\Omega}_l} \sqrt{\left| \frac{d\Psi_l}{d\Psi_k \cdot \Psi_{k,l}} - 1 \right|} d\Psi_k \cdot \Psi_{k,l} = 0 \tag{4}$$

Proof. Obviouslu [A11] $\to$ [A12]. If $\rho(\Psi_l; \Psi_k \cdot \Psi_{k,l})$ tends to 1, from the inequality

$$1 - \left(\frac{d\Psi_l}{d\Psi_k \cdot \psi_{k,l}} \right) \left(1 + \sqrt{\frac{d\Psi_l}{d\Psi_k \cdot \Psi_{k,l}}} \right)$$

one deduces

$$\int_{\tilde{\Omega}_l} \left| \frac{d\Psi_l}{d\Psi_{k'} \Psi_{k,l}} - 1 \right| d\Psi_k \cdot \Psi_{k,l} \leq 2^{3/2} \cdot \sqrt{1 - \rho(\Psi_l; \Psi_k \cdot \Psi_{k,l})}$$

so (2) $\to$ (3).

Since, clearly (3) $\to$ (4) it will be sufficient to prove [A14] $\to$ (1). Let

$$\lim_{\substack{k\to\infty\\ k\leq l}} \int_{\tilde{\Omega}_l} \sqrt{\left| \frac{d\Psi_l}{d\Psi_k \cdot \Psi_{k,l}} - 1 \right|} d\Psi_k \cdot \Psi_{k,l} = 0$$

Let us consider the sequence

$$\int_{\tilde{\Omega}_l} |\sqrt{1 + f_{k,l}} - 1| d\Psi_k \cdot \Psi_{k,l}$$

where $f_{k,l} = \frac{d\Psi_l}{d\Psi_k \cdot \Psi_{k,l}} - 1$. Let us introduce the sets:

$$E^+_{k,l} = \{x_l \in \tilde{\Omega}_l : f_{k,l}(x_l) \geq 0\} \; ; \quad E^-_{k,l} = \tilde{\Omega}_l - E^+_{k,l}$$

And let us consider separately the two addenda of the sum:

$$\int_{\tilde{\Omega}_l} \left| \sqrt{\frac{d\Psi_l}{d\Psi_k \cdot \Psi_{k,l}}} - 1 \right| d\Psi_k \cdot \Psi_{k,l} = \int_{E^+_{k,l}} |\ldots| d\Psi_k \cdot \Psi_{k,l} + \int_{E^-_{k,l}} |\ldots| d\Psi_k \cdot \Psi_{k,l}$$

On $E^+_{k,l}$ by construction $f_{k,l} \geq 0$, so

$$\int_{E^+_{k,l}} |\sqrt{1 + f_{k,l}} - 1| d\Psi_k \cdot \Psi_{k,l} \leq \int_{E^+_{k,l}} \sqrt{f_{k,l}} d\Psi_k \cdot \Psi_{k,l}$$

But if $f_{k,l} < 0$, then necessarily $-1 \leq f_{k,l} < 0$, except for a set of null $(\Psi_k \cdot \Psi_{k,l})$–measure. So if $F^-_{k,l} = \{x_l \in E^-_{k,l} : f_{k,l} = -1\}$, $G_{k,l} = E^-_{k,l} - F^-_{k,l}$, one has

$$\int_{E^-_{k,l}} |1 - \sqrt{1 - |f_{k,l}|}| d\Psi_k \cdot \Psi_{k,l} \leq \int_{G_{k,l}} \sqrt{|f_{k,l}|} d\Psi_k \cdot \Psi_{k,l} + \Psi_k \cdot \Psi_{k,l}(F^-_{k,l})$$

Summing up the last two inequalities, one finds

$$\int_{\tilde{\Omega}_l} \left| \sqrt{\frac{d\Psi_l}{d\Psi_k \cdot \Psi_{k,l}}} - 1 \right| d\Psi_k \cdot \Psi_{k,l} \leq \int_{\tilde{\Omega}_l} \sqrt{\left| \frac{d\Psi_l}{d\Psi_k \cdot \Psi_{k,l}} - 1 \right|} d\Psi_k \cdot \Psi_{k,l}$$

so (3) $\to$ (1) and this ends the proof.

The preceding Lemma justifies the following definition:

Definition 3 *Let γ be a real number such that $1 \leq t \leq \infty$. We shall say that the cylindrical measure (Ψ_k) is $L^+(\Psi_k \cdot \Psi_{k,l})$ asymptotically independent if*

$$\lim_{\substack{k \to \infty \\ k \leq l}} \int_{\tilde{\Omega}_l} \left| \frac{d\Psi_l}{d\Psi_k \cdot \Psi_{k,l}} - 1 \right|^t d\Psi_k \cdot \Psi_{k,l} = 0$$

When no confusion can arise we shall simply write L^t–asymptotically independent.

L^∞–asymptotic independence is called "uniform asymptotic independence". Lemma (3) can be stated: asymptotic independence is equivalent to L^1–asymptotic independence (resp. $L^{1/2}$).

Definition 4 *A cylindrical measure* (Ψ_k) *on the product* $\prod_{i=1}^{\infty}(\Omega_i, \mathcal{B}_i)$ *will be said to be "strongly* L^+*–asymptotically independent" if*

$$\lim_{\substack{k\to\infty \\ k\le l}} \sup_{\omega_k\in\tilde{\Omega}_k} \int_{\tilde{\Omega}_{k,l}} \left| \frac{d\Psi_l}{d\Psi_k \cdot \Psi_{k,l}}(\tilde{\omega}_k, \tilde{\omega}_{k,l}) - 1 \right|^t d\Psi_{k,l}(\tilde{\omega}_{k,l}) = 0$$

For $t = 1$*, we shall say that* (Ψ_k) *is strongly asymptotically independent.*

Lemma 4 *Let* (Ψ_k) *and* (Ψ'_k) *be two cylindrical measure on* $\prod_{i=1}^{\infty}(\Omega_i, \mathcal{B}_i)$ *such that* $\Psi_k \sim \Psi'_k$*, and* $\lim_{k\to\infty} \rho(\Psi_k; \Psi'_k) = \alpha > 0$*. Suppose the following condition is satisfied:*

[i1] The measures (Ψ_k)*,* (Ψ'_k) *are strongly asymptotically independent.*

Then:

$$\lim_{\substack{k\to\infty \\ k\le l}} \int_{\tilde{\Omega}_l} \sqrt{\frac{d\Psi'_k}{d\Psi_k}} \cdot \sqrt{\frac{d\Psi'_l}{d\Psi_l}}\, d\Psi_l = 1 \tag{5}$$

Proof. The cylindrical measures (Ψ_k); (Ψ'_k) are asymptotically independent, so from Lemma (1), (2) and from the hypothesis it follows that $\lim_{\substack{k\to\infty \\ k\le l}} \rho(\Psi_{k,l}; \Psi'_{k,l}) = 1$. Therefore, because of the equality:

$$\int_{\tilde{\Omega}_l} \sqrt{\frac{d\Psi'_k}{d\Psi_k}} d\Psi_l = \rho(\Psi_{k,l}; \Psi'_{k,l}) + \int_{\tilde{\Omega}_l} \sqrt{\frac{d\Psi_{k,l}}{d\Psi'_{k,l}}} \left[\sqrt{\frac{d\Psi'_l}{d\Psi'_k \cdot \Psi'_{k,l}} \cdot \frac{d\Psi_l}{d\Psi_k \cdot \Psi_{k,l}}} - 1 \right] d\Psi'_k \cdot \Psi'_{k,l}$$

the thesis is equivalent to the assertion that the second addendum of the right member of the equality tend to zero. From Fubini–Tonelli's theorem, this addendum can be expressed as

$$\int_{\tilde{\Omega}_{k,l}} \sqrt{\frac{d\Psi_{k,l}}{d\Psi'_{k,l}}} \cdot d\Psi'_{k,l} \int_{\tilde{\Omega}_k} \left[\sqrt{\frac{d\Psi'_l}{d\Psi'_k \cdot \Psi'_{k,l}} \cdot \frac{d\Psi_l}{d\Psi_k \cdot \Psi_{k,l}}} - 1 \right] d\Psi'_k$$

Let us first consider the integral: $\int_{\tilde{\Omega}_k} [\ldots] d\Psi'_k$.

Writing the term under square root in the form $1 + f_{k,l}$, with an argument similar to that used in Lemma (3) one finds

$$\int_{\tilde{\Omega}_k} \left[\sqrt{\frac{d\Psi'_l}{d\Psi'_k \cdot \Psi'_{k,l}} \cdot \frac{d\Psi_l}{d\Psi_k \cdot \Psi_{k,l}}} - 1 \right] d\Psi'_k \le \int_{\tilde{\Omega}_k} \sqrt{|f_{k,l}|} \cdot d\Psi'_k$$

So one sees that, after easy manipulations, the initial integral is majorized by

$$\int_{\tilde{\Omega}_l} \sqrt{\frac{d\Psi_{k,l}}{d\Psi'_{k,l}}} \cdot \sqrt{\left| \frac{d\Psi'_l}{d\Psi'_k \cdot \Psi'_{k,l}} - 1 \right|} \cdot d\Psi'_k \cdot \Psi'_{k,l} +$$

$$+ \int_{\tilde{\Omega}_l} \sqrt{\frac{d\Psi'_l}{d\Psi_k \cdot \Psi_{k,l}}} \cdot \sqrt{d\Psi'_k \cdot d\Psi_k} \cdot \sqrt{\left| \frac{d\Psi_l}{d\Psi_k \cdot \Psi_{k,l}} - 1 \right|} d\Psi_k \cdot \Psi_{k,l}$$

From the hypothesis and Lemma (1.4), the first addendum goes to zero for $k \to \infty$; $k \leq l$. Concerning the second, we observe that, applying repeatedly Cauchy inequality, it can be majorized by

$$\left\{ \int_{\tilde{\Omega}_k} I_{k,l}(\tilde{\omega}_k) d\Psi_k \cdot \int_{\tilde{\Omega}_{k,l}} \frac{d\Psi'_l}{d\Psi_k \cdot \Psi_{k,l}} \cdot d\Psi_{k,l} \right\}^{1/2}$$

where

$$I_{k,l}(\tilde{\omega}_k) = \int_{\tilde{\Omega}_{k,l}} \left| \frac{d}{d\Psi_k \cdot \Psi_{k,l}} (\tilde{\omega}_k; \tilde{\omega}_{k,l}) - 1 \right| \cdot d\Psi_{k,l}$$

In hypothesis [1], given $\varepsilon > 0$, $ak(\varepsilon)$ can be found such that, for $l \geq k \geq k(\varepsilon)$

$$\sup_{\tilde{\omega}_k \in \tilde{\Omega}_k} I_{k,l}(\tilde{\omega}_k) \leq \varepsilon$$

Therefore also the second integral tends to zero for $k \to \infty$; $k \leq \varepsilon$, and this proves the thesis.

It is known (cfr. [12]), that condition (5) is equivalent to $\Psi' < \Psi$. Thus, from Lemma (4), and [1], pg. 181, one deduces:

Theorem 1 *Two cylindrical measures* (Ψ_k) *and* (Ψ'_k) *on the product* $\prod_{i=1}^{\infty} (\Omega_i; \mathcal{B}_i)$, *strongly asymptotically independent and such that* $\Psi_k \sim \Psi'_k$ *for every* $k \in \mathbb{N}$, *are either orthogonal or equivalent according to the* $\lim_{k\to\infty} \rho(\Psi'_k; \Psi_k)$, *is equal or bigger than zero.*

Definition 5 *Two cylindrical measures* (Ψ_k), (Ψ'_k) *on the product* $\prod_{i=1}^{\infty} (\Omega_i; \mathcal{B}_i)$ *are said to be "weakly equivalent" if* $\lim_{\substack{k\to\infty \\ k \leq l}} \rho(\Psi_{kl}; \Psi'_{k,l}) = 1$.

In the case of product measures "weak equivalence" coincides with the equivalence relation between C_0–sequences introduced in (**??**) and from Kakutani's theorem it follows that two weakly equivalent product measures are either equivalent or there exists a finite subset of indices such that the finite product measures relative to this set are orthognal and the infinite product measures on the complement of this set are equivalent.

It can also be proved that weak equivalence is a measure theoretical generalization of Power's condition of "quasi–equivalence" of two factor states on a uniformly hyperfinite algebra (cfr. [6]). In case of asymptotically independent cylindrical measures, the property $\lim_{k\to\infty} \rho(\Psi_k; \Psi'_k) > 0$ and thus, in particular the equivalence of the induced measures implies weak equivalence.

Proposition 1 *Let* (Ψ_k) *and* (Ψ'_k) *be two weakly equivalent cylindrical measures on the product* $\prod_{i=1}^{\infty} (\Omega_i; \mathcal{B}_i)$*, such that* $\Psi_k \sim \Psi'_k$*;* $\forall\, k \in \mathbb{N}$*, and the sequences* $\left(\frac{d\Psi'_k}{d\Psi_k}\right)$ *and* $\left(\frac{d\Psi_k}{d\Psi'_k}\right)$ *are almost everywhere bounded. Then the induced measures are equivalent.*

Proof. From Lemma (1) and [4] pg. 172, weak equivalence implies

$$\lim_{\substack{k\to\infty \\ k\leq l}} J_k(\Psi_l; \Psi'_l) = 0$$

From the equality:

$$\int_{\tilde{\Omega}_l} \sqrt{\frac{d\Psi'_k}{d\Psi_k}} \cdot \sqrt{\frac{d\Psi'_l}{d\Psi_l}} \cdot d\Psi_l = \rho(\Psi_{k,l}; \Psi'_{k,l}) + \int_{\tilde{\Omega}_l} \sqrt{\frac{d\Psi_{k,l}}{d\Psi'_{k,l}}} \left[\sqrt{\frac{d\Psi'_l}{d\Psi'_k \cdot \Psi'_{k,l}} \cdot \frac{d\Psi_l}{d\Psi_k \cdot \Psi_{k,l}}} - 1 \right] d\Psi'_k \cdot \Psi'_{k,l}$$

and the fact that

$$\int_{\tilde{\Omega}_l} \sqrt{\frac{d\Psi_{k,l}}{d\Psi'_{k,l}}} \left[\sqrt{\frac{d\Psi'_l}{d\Psi'_k \cdot \Psi'_{k,l}} \frac{d\Psi_l}{d\Psi_k \cdot \Psi_{k,l}}} - 1 \right] d\Psi'_k \cdot \Psi'_{k,l} \leq \lambda \cdot J_k(\Psi_l; \Psi'_l)$$

it follows that, in our hypothesis the relatain (5), holds and the thesis follows by simmetry and the remark after Lemma 4.

3 Semi–Stationary Measures

Let $(\Omega, \mathcal{B})$ be a measure space. Set $(\hat{\Omega}, \hat{\mathcal{B}}) = \prod_{\mathbb{N}}(\Omega, \mathcal{B})$ (the product of $\mathbb{N}$–replicas of $(\Omega, \mathcal{B})$. The cylindrical measure (Ψ_k) on $(\hat{\Omega}, \hat{\mathcal{B}})$ is said to be "stationary" if $\Psi_{k,l} = \Psi_{l-k}$ for every, k, $l \in \mathbb{N}$, $k \leq l$, (cfr. [5] pg. 214). That is, in the notations of (§ 1), a cylindrical measure (Ψ_k) is stationary when

$$\Psi_l(\Omega; \ldots; \Omega; E_{k+1}; \ldots E_l) = \Psi_{l-k}(E_{k+1}; \ldots; E_l)$$

for every $E_i \in \mathcal{B}$, $k+1 \leq i \leq l$.

Theorem 2 *Two stationary, weakly equivalent cylindrical measures coincide.*

Two stationary cylindrical measures such that: $\lim\limits_{\substack{k\to\infty \\ k\leq l}} J_k(\Psi_l; \Psi'_l) = 0$ *are either orthogonal or coincident.*

Proof. The first assertion is obvious. Let (Ψ_k) and (Ψ'_k) now be satisfying the condition of the theorem. If the two measures are not orthogonal $\lim\limits_{k\to\infty} \rho(\Psi_k; \Psi_k) > 0$ and, from Lemma (1) their weak equivalence follows.

Thus if the two cylindrical measures are stationary the first part of the theorem proves their coincidence, and this concludes the proof. In particular the condition of theorem II is always satisfied when the two measures are asymptotically independent (cfr. remark after Lemma (2)), and in this case the thesis is a well known property of stationary, strongly mixing measures.

Definition 6 *The cylindrical mesure* (Ψ_k) *on the product is said to be "semi–stationary" if* $\Psi_{k,k+1} = \Psi_1$*. This, in the notation of 1) is equivalent to:*

$$\Psi_{k+1}(\Omega; \ldots; \Omega; E_{k+1}) = \Psi_1(E_{k+1})$$

Every Markov chain with a stationary distribution induces a semi–stationary cylindrical measure which is not, in general, stationary, unless the initial Markov chain is homogeneous.

Corollary (2.3). If two semi–stationary cylindrical measures (Ψ_k) and (Ψ'_k) are weakly equivalent then $\psi_1 \equiv \Psi'_1$.

4 An Application

Let $\Omega = \{s_1, \ldots, s_n\}$ be a set containing n points. $\mathcal{P}(\Omega) = \mathcal{B}$ the family of parts of Ω; $(P_{(k)})_{k\in\mathbb{N}}$ a sequence of n–dimensional stochastic matrices, ${}^t a = (a_1, \ldots, a_n)$, a stochastic vector. It is well known (cfr. [11]) that the Markov chain $\{(P_{(k)}); a\}$ induces on the product $(\hat{\Omega}, \hat{\mathcal{B}})$ a cylindrical measure (Ψ_k) defined by the equalities

$$\Psi_k(s_{J_1}; \ldots ; s_{J_k}) = a_{J_1} t^{(1)}_{J_1,J_2} \ldots t^{(k-1)}_{J_{k-1},J_k}$$

$$s_{J_i} \in \Omega \ ; \quad (1 \le i \le k) \ ; \quad P_{(k)} = (t^{(k)}_{i,J}) \ ; \quad k \in \mathbb{N}$$

In all the paragraph, with the term "Markov–chain" we shall always denote a finite Markov chain with discrete time, and such that: $t^{(k)}_{i,J} > 0$; $a_J > 0$.

Lemma 5 *Let $\{(P_{(k)}); a\}$ be a Markov chain. The cylindrical measure induced by it on the product $(\hat{\Omega}, \hat{\mathcal{B}})$ is semi–stationary if and only if ${}^t a P_{(k)} = {}^t a$; for every k.*

Proof. The sufficiency of the conditions is clear. Conversely, suppose (Ψ_k) semi–stationary. Then:

$$\Psi_2(\Omega, E_2) = \sum_{J_2\in E_2} \sum_{J_1\in\Omega} a_{J_1} t^{(1)}_{J_1,J_2} = \Psi_1(E_2) = \sum_{J_2\in E_2} a_{J_2}$$

from the arbitrariness of E_2 it follows that ${}^t a P_{(1)} = {}^t a$. Suppose ${}^t a P_{(h)} = {}^t a$ for $1 \le h \le k-1$, then

$$\begin{aligned}\Psi_{k,k+1}(E_{k+1}) &= \sum_{J_{k+1}\in E_{k+1}} \left(\sum_{J_k\in\Omega} \cdots \sum_{J_1\in\Omega} a_{J_1} \cdot t^{(1)}_{J_1,J_2} \ldots t^{(k)}_{J_k,J_{k+1}} \right) \\ &= \Psi_1(E_{k+1}) = \sum_{J_{k+1}\in E_{k+1}} a_{J_{k+1}}\end{aligned}$$

the arbitrariness of E_{k+1} and the inductive hypothesis yield ${}^t a P_{(k)} = {}^t a$ which ends the proof.

In particular we have the well known fact:

Lemma 6 *A Markov chain $\{(P_{(k)});a\}$ induces a stationary cylindrical measure on the product $(\hat{\Omega},\hat{\mathcal{B}})$ if and only if*

$$taP_{(k)} = \ {}^{t}a \ ; \quad \forall\, k \in \mathbb{N} \tag{6}$$

$$P_{(k)} = P \ ; \quad \forall\, k \in \mathbb{N} \tag{7}$$

(i.e., a Markov chain induces a stationary cylindrical measure if and only if it is an homogeneous Markov chain with an invariant distribution in the usual sense (cfr. [11]).

Both lemmata extend "verbatim" to the case of a continuous state space.

Lemma 7 *The cylindrical measure induced by the Markov chain $\{(P_{(k)});a\}$ is asymptotically indpeendent if and only if*

$$\lim_{k\to\infty} |t^{(k)}_{i,J} - [{}^{t}aQ_{(k)}]_J| \cdot [{}^{t}aQ_{(k)}]_i = 0$$

where $Q_{(k)} = P_{(1)} \cdot \ldots \cdot P_{(k)}$, and $[{}^{t}aQ_{(k)}]_J$ denotes the j–th component of ${}^{t}aQ_{(k)}$.

Proof. Let (Ψ_k) be the cylindrical measure induced by $\{(P_{(k)};a\}$. Then:

$$\Psi_{k,l}(s_{J_{k+1}},\ldots,s_{J_l}) = \sum_{J_1,\ldots,J_k\in\Omega} a_{J_1} t^{(1)}_{J_1,J_2}\ldots\gamma^{(l-1)}_{J_{l-1},J_l}$$

Thus: $\frac{d\Psi_l}{d\Psi_k\cdot\Psi_{k,l}}\,(s_1,\ldots,s_l)$, has the expression

$$\frac{\gamma^{(k)}}{\sum\limits_{J_1\ldots J_k\in\Omega} a_{J_1}\gamma^{(1)}_{J_1,J_2}\cdot\ldots\cdot\gamma^{(k)}_{J_k,J_{k+1}}} = \frac{\gamma^{(k)}_{J_k,J_{k+1}}}{[{}^{t}aQ_{(k)}]_{J_{k+1}}}$$

From Lemma (1.4) and the preceding equality the condition of asymptotic independence can be written, in this case

$$\lim_{k\to\infty}\sum_{i,J\in\Omega}\left|\frac{\gamma^{(k)}_{i,J}}{[{}^{t}aQ_{(k)}]_J} - 1\right| \cdot \Psi_{k-1,k}(s_i)\cdot\Psi_{k,k+1}(s_J) = 0$$

which is equivalent to:

$$\lim_{k\to\infty}|\gamma^{(k)}_{i,J} - [{}^{t}aQ_{(k)}]_J|\cdot|[{}^{t}aQ_{(k)}]_i = 0$$

and this concludes the proof.

Corollary (3.4). The cylindrical measure induced by the Markov chain $\{(P_{(k)}); a\}$ is semi–stationary and asymptotically independent if and only if

$$\lim_{k\to\infty} \gamma_{i,J}^{(k)} = a_J \ ; \quad 1 \leq i, J \leq n$$

Proof. Immediate from Lemma (5) and (7).

Proposition 2 *Let $\{(P_{(k)}); a\}$ be a Markov chain on. Then the following assertions are equivalent:*
[t1] The cylindrical measure induced by $\{(P_{(k)}); a\}$ on the product $(\hat{\Omega}, \hat{\mathcal{B}})$, is asymptotically independent and stationary.
[t2] $\gamma_{i,J}^{(k)} = a_J$; $(1 \leq i, J \leq n)$, for every k.
[t3] The measure induced by $\{(P_{(k)}); a\}$ on the product $(\hat{\Omega}, \hat{\mathcal{B}})$ coincides with the product mesure $\prod_{\mathbb{N}}^{t} a$.

Proof. Obviously [t3] $\to$ [t1], thus it will be sufficient to prove [t1] $\to$ [t2] $\to$ [t3]. For Lemma (6), [t1] implies $P_{(k)} = P$.

Therefore the condition of Corollary (3.4) becomes $\gamma_{i,J} = a_J$: $(1 \leq i, J \leq n)$ which is [t2].

Finally, if (Ψ_k) is the measure induced by $\{P, a\}$, [t2] implies $\rho\left(\Psi_k; \prod_{1}^{k} {}^t a\right) = 1$, which is [t3] and the proof is completed.

From Proposition (2) one deduces a characterization of the family of mutually orthogonal measures built by Kakutani [2]; Section 10). They are exactly those measures which arise from stationary, asymptotically independent, two–dimensional Markov chains. Consider, for example, the particular case when $\Omega = \{0, 1\}$; $\mathcal{B} = \mathcal{P}(\Omega) =$ the family of the parts of Ω; $0 < \gamma < 1$; $0 < \alpha < 1$. Define

$$P = \begin{pmatrix} 1-\gamma & \gamma \\ \alpha & 1-\alpha \end{pmatrix} \ ; \quad {}^t a = \left(\frac{\alpha}{\alpha+\gamma} ; \frac{\gamma}{\alpha+p} \right) \ ; \quad P_{(k)} = P^k$$

The Markov chain $\{P_{(k)}; a\}$ satisfies the conditions of Lemmata (5) and (**??**), therefore the cylindrical measure $(\Psi_k^{\alpha,\gamma})$ induced by it on $(\hat{\Omega}, \hat{\mathcal{B}})$ is semi–stationary and asymptotically independent. Denote $\Psi^{\alpha,\beta}$ the measure induced by $(\Psi_k^{\alpha,\beta})$.

From Lemma (1.1) and Corollary (2.3) it follows that $\lim_{k\to\infty} \rho(\Psi_k^{(\alpha,\beta)}; \Psi_k^{(\beta,q)})$ can be > 0 only if $\frac{\alpha}{\alpha+\beta} = \frac{\beta}{\beta+q}$. Thus the family $\Psi^{\alpha,\beta}$ $0 < \gamma < 1$; $0 < \alpha < 1$;

contains a two parameter family of mutually orthogonal measures. From Lemma (**??**), the product measures $\prod_{\mathbb{N}}^{t} a$ correspond to the case $\Psi^{\beta,1-\beta}$; $P_{(k)} = P$. Thus $\Psi^{\alpha,\beta} \perp \Psi^{\beta,1-s}$ for $\beta = \frac{\alpha}{\alpha+\beta}$.

With this class of measures one can built a two parameter family of mutually orthogonal factor states of "Markov type" on uniformly hyperfinite algebras, which reduce to the states considered by Powers in [6] when one takes $\gamma = 1 - \alpha$.

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