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L. ACCARDI
Y. G. LU

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The weak coupling limit without rotating wave approximation

by

L. ACCARDI and Y. G. LU (*)

Centro Matematico V. Volterra,
Dipartimento di Matematica, Universitá di Roma II, Italy

ABSTRACT. – We investigate the behaviour, in the weak coupling limit, of a system interacting with a Boson reservoir without assuming the rotating wave approximation, *i.e.* we allow the system Hamiltonian to have a finite set of charactristic frequencies rather than a single one [*cf.* equation (1.4)]. Our main result is the proof that the weak coupling limit of the matrix elements with respect to suitable **collective vectors** of the solution of the Schrödinger equation in interaction representation (*i.e.* the wave operator at time t) exists and is the solution of a quantum stochastic differential equation driven by a family of independent quantum Brownian motions, one for each characteristic frequency of the system Hamiltonian.

RÉSUMÉ. – Nous étudions le comportement dans la limite de couplage faible d'un système interagissant avec un réservoir de bosons sans supposer l'approximation d'onde tournante, c'est-à-dire que nous permettons à l'hamiltonien du système d'avoir plusieurs fréquences caractéristiques au lieu d'une seule [*voir* équation (1.4)]. Notre résultat principal est une preuve de l'existence de la limite de couplage faible des éléments de matrice par rapport à certains vecteurs collectifs de la solution de l'équation de Schrödinger dans la représentation interaction (c'est-à-dire de l'opérateur

(*) On leave of absence from Beijing Normal University.

d'onde à l'instant t). De plus cette limite est solution d'une équation différentielle stochastique contenant une famille de mouvements browniens quantiques indépendants, un par fréquence caractéristique de l'hamiltonien.

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1. INTRODUCTION

The attempt to produce a satisfactory quantum description of irreversible phenomena has motivated many investigations. In the last 30 years these investigations have produced a number of interesting models, but the theoretical status of these models has remained, for a long time, uncertain and even (in some cases) contradictory [14]. The standard approach to the problem is the following: one starts from a quantum system coupled to a reservoir (or heat bath, or noise – according to the interpretations). The reservoir is physically distinguished by the system because its time correlations decay much faster and one expects that, by the effect of the interaction, and in some limiting situations some energy will flow irreversibly from the system to the reservoir. One of the best known, among these limiting situations, is the so-called van Hove (or weak coupling) limit, in which the strength of the coupling system-reservoir is given by a constant λ and one studies the average values of the observables of the system, evolved up to time t with the coupled Heisenberg evolution of the system + reservoir, in the limit $\lambda \to 0$, $\lambda^2 t \to \tau$ as $\lambda \to 0$. This limiting procedure singles out the **long time cumulative behaviour** of the observables and the weakness of the coupling implies, in the limit, an effect of lost of memory (Markovian approximation).

In a first stage of development of this approach one only considered averages with respect to the Fock vacuum or to a fixed thermal state. This limitation has the effect of sweeping away, in the limit, all the terms of the iterated series except those which, in the Wick ordering procedure, correspond to the scalar terms. The resulting **reduced evolution** was a quantum Markovian semigroup and the corresponding equation – a quantum master equation (*cf.* [6], [7], [8], [11]).

Independently of these developments, and on a less rigorous mathematical level, the notion of **quantum Brownian motion** emerged from investigations of quantum optics, especially in connection with laser theory and, with the work of Hudson an Parthasarathy this notion was brought ot its full power with the construction of the quantum stochastic calculus for the Fock Brownian motion [13].

The new idea of quantum stochastic calculus is that one does not limit oneself to the reduced evolution (*i.e.* the master equation) but one considers the noise (quantum Brownian motion) as an idealized reservoir and one studies the coupled evolution of the system coupled to the noise . This evolution is not a standard quantum mechanical one, because it is not described by the usual Heisenberg equation, but by a quantum stochastic differential equation, called the **quantum Langevin equation**.

The physical importance of this more complete description has been pointed out by many authors and, from an intuitive point of view is quite clear: if the noise is looked at as an approximate description of the reservoir field (or gas), then in order to extract experimental information on the coupled system, one can choose to measure **either** the system **or** the reservoir, while in the previous approach one could only predict the behavior of the observables of the system ([9], [10]).

In order to put to effective use this new connection with the experimental evidence, a last step had to be accomplished: **to prove the internal coherence of this picture**. This means essentially two things:

(i) to explain precisely in which sense the quantum Brownian motions are approxiamtions of the quantum fields (or of Boson or Fermion gases at a given temperature);

(ii) to explain precisely in which sense a quantum stochastic differential equation is an approximation of a ordinary Hamiltonian equation.

Once answered the questions (i) and (ii), all the previous results on the master equation follow easily applying a (by now standard) quantum probabilistic technique – the quantum Feynman-Kac formula (*cf.* [0]).

In a series of papers [1], ..., [5] (*cf.* also [12], for more recent results), we have solved the problems (i) and (ii) in a variety of models which include the most frequently used models in quantum optics.

The present paper points out two qualitatively new phenomena which arise when there are several characteristic frequencies in the original system:

(i) In the limit, not a single quantum Brownian motion arises, but several independent and mutually non isomorphic ones: one for each frequency of the system.

(ii) The **collective vectors** suitable to evidentiate the weak coupling behavior, in this case **are not the same** as in previous papers. This suggests the appearence of a interesting new phenomenon, namely: **that the chioce of the correct collective vectors** needed to evidentiate the weak coupling limit behaviour **should depend** on the form of the interaction. We expect this new phenomenon to play an important part in the study of **macroscopic**, *i.e.* collective, quantum effects.

A natural extrapolation of our result to the case of continuous spectrum, leads to conjecture that in this case one should have a continuous tensor product of quantum Brownian motions labeled by the energy spectrum of the original system. However it is not clear to us how to adapt the techniques developed so far to the case of a system with continuous spectrum.

Let us now fix the notations and state our basic assumptions.

We consider a quantum "System + Reservoir" model. Let H_0 be the system Hilbert space; H_1 the one particle reservoir Hilbert space; H_S the system Hamiltinian; $H_R = d\Gamma(-\Delta)$ where Δ is a negative self-adjoint operator – the one particle reservoir Hamiltonian – and $S_t^0 : = e^{-it\Delta}$. The total space is

$$H_0 \otimes \Gamma(H_1)$$

where $\Gamma(H_1)$ is the Fock space over H_1 and the total Hamiltonian is

$$H^{(\lambda)} = H_S \otimes 1 + 1 \otimes H_R + \lambda V \tag{1.1}$$

where,

$$V = V_g : = -\frac{1}{i}(D \otimes A^+(g) - D^+ \otimes A(g)) \tag{1.2}$$

D is a bounded operator on H_0 and $g \in H_1$.

The rotating wave approximation corresponds to the two following assumptions:

(i) in the interaction there are no terms of the form $D \otimes A(g)$ or $D^+ \otimes A^+(g)$.

(ii) D is an eigenvector of the free evolution of the system, *i.e.*

$$D(t) : = e^{-itH_S} D e^{itH_S} = e^{-it\omega_0} D \tag{1.3}$$

In [6], condition (1.3) is replaced by the following weaker condition: H_S has pure non-degenerate discrete spectrum and the system Hilbert space is finite dimensional. This assumption implies that

$$D(t) : = e^{-itH_S} D e^{itH_S} = \sum_{d=1}^{N} D_d e^{-it\omega_d} \tag{1.4}$$

where the D_d $(d = 1, \ldots, N)$ are bounded operators on H_0 and $\omega_d \neq \omega_{d'}$ for $d \neq d'$. In the present paper, we shall prove that if condition (1.3) is replaced by condition (1.4), then all the results of [2] are still valid with the only difference that the resulting quantum stochastic defferential equation is driven not by a single quantum Brownian motion, but by N-independent ones [in the sense of Definition (1.1)]. The reason of the appearance of these N-independent Brownian motions is explained in Section 2.

Now let $H^{(0)} = H^{(\lambda)} - \lambda V = H_S \otimes 1 + 1 \otimes H_R$, then the operator

$$U^{(\lambda)}(t) = e^{-itH^{(0)}} e^{itH_\lambda} \tag{1.5}$$

satisfies the equation

$$\frac{d}{dt} U^{(\lambda)}(t) = \frac{1}{i} V_g(t) U^{(\lambda)}(t) \tag{1.6}$$

where

$$V_g(t) := -\frac{1}{i} (D(t) \otimes A^+(S_t^0 g) - D^+(t) \otimes A(S_t^0 g)) \tag{1.7}$$

In the following, we shall use the notation

$$V_g(t) = -\frac{1}{i} \sum_{d=1}^{N} (D_d \otimes A^+(S_t^d g) - D_d^+ \otimes A(S_t^d g))$$
$$= i \sum_{d=1}^{N} \sum_{\varepsilon \in \{0, 1\}} D_d^\varepsilon \otimes A^\varepsilon (S_t^d g) \tag{1.8}$$

where,

$$D_d^0 := -D_d^+; \qquad D_d^1 := D_d \tag{1.9}$$
$$A^0 := A; \qquad A^1 := A^+ \tag{1.10}$$

and

$$S_t^d := e^{-i\omega_d t} S_t^0 \tag{1.11}$$

In our assumptions, the iterated series

$$U^{(\lambda)}(t) = \sum_{n=0}^{\infty} (-i)^n \lambda^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n_1}} dt_n V_g(t_1) \dots V_g(t_n) \tag{1.12}$$

converges weakly on the domain of vectors of the form $u \otimes \psi$, where $u \in H_0$ and ψ is a coherent vector in $\Gamma(H_1)$.

In order to formulate our results we have to introduce the notion of N-independent Boson Brownian motions.

DEFINITION (1.1). – Let $K_1, \dots, K_N$ be Hilbert spaces and let, for each $d = 1, \dots, N$, be given a self-adjoint operator $Q_d \geqq 1$ on K_d. The process of N-independent Boson Brownian motions, respectively with values in $K_1, \dots, K_N$ and covariances $Q_1, \dots, Q_N$ is the process obtained by taking the tensor product of the Q_d-quantum Brownian motions on $L^2(\mathbf{R}, dt; K_d)$

$$\{ \mathscr{H}_{Q_d}, \mathscr{D}_{Q_d}, W_{Q_d}(\chi_{(s_d, t_d]} \otimes b_d); (s_d, t_d] \subset \mathbf{R}; b_d \in K_d \} \tag{1.13}$$

(*cf.* Def. (2.3) of [2]) for $d = 1, \dots, N$. In other terms this is the cyclic quasi-free representation W of the CCR on

$$\bigoplus_{d=1}^{N} L^2(\mathbf{R}, dt; K_d)$$

with respect to the state φ characterized by

$$\varphi\left(W\left(\bigoplus_{d=1}^{n} \chi_d \otimes f_d\right)\right) = \exp\left(-\frac{1}{2}\sum_{d=1}^{N} \|\chi_d\|^2 \langle f_d, Q_d f_d \rangle\right), \qquad (1.14)$$

$$\chi_d \in L^2(\mathbf{R}); \qquad d=1, \ldots, N$$

If

$$Q_d = 1; \qquad d=1, \ldots, N$$

then we speak of N-independent Boson Fock Brownian motions.

The Hilbert space where this representation lives is the tensor product of the spaces $\mathscr{H}_{Q_d}(d=1, \ldots, N)$. Denoting

$$A_d(\chi_{(s,t]} \otimes f_d), \qquad A_d^+(\chi_{(s,t]} \otimes f_d) \qquad ((s,t] \subseteq \mathbf{R}, f_d \in K_d)$$

the annihilation and creation operators acting on the d-th factor of the tensor product, one often uses the unbounded form of the quantum Brownian motion process given by the pair of operators

$$\left.\begin{array}{c} A_d(t, f_d) = A_d(\chi_{[0,t]} \otimes f_d) \\ A_d^+(t, f_d) = A_d^+(\chi_{[0,t]} \otimes f_d), \\ t \in \mathbf{R}_+, \quad f_d \in K_d, \quad d=1, \ldots, N \end{array}\right\} \qquad (1.15)$$

All these operators are defined on the domain of the vectors

$$W\left(\bigoplus_{d=1}^{N} \chi_d \otimes f_d\right) \cdot \left(\bigotimes_{d=1}^{N} \Psi_d\right) = \bigotimes_{d=1}^{N} [W_{Q_d}(\chi_d \otimes f_d) \Psi_d] =: \bigotimes_{d=1}^{N} \Psi_d(\chi_d \otimes f_d) \qquad (1.16)$$

Where $\chi_d \in L^2(\mathbf{R})$, $f_d \in K_d$ and Ψ_d is the cyclic vector of the representation W_{Q_d}.

As in [2], we shall suppose that there exists a non-zero subspace $K \subset H_1$, such that

$$\int_{\mathbf{R}} |\langle f, S_t^0 f' \rangle| \, dt < \infty, \quad \text{for each } f, f' \in K \qquad (1.17)$$

This condition implies that, for each $d=1, \ldots, N$, the sesquilinear form

$$f, f': \quad K \to (f | f')_d := \int_{\mathbf{R}} \langle f, S_t^d f' \rangle \, dt \qquad (1.18)$$

[where S_t^d is defined by (1.11)] defines a pre-scalar product on K. In the following, for each $d=1, \ldots, N$, K_d shall denote the Hilbert space completion of the quotient space of K for $(.|.)_d$-null space.

With these notations we can state our main result:

THEOREM (1.2). – *Under the condition* (1.4) *and the notation* (1.15), (1.16), *for each* $N \in \mathbf{N}$, g, $\{f_d\}_{d=1}^N$, $\{f'_d\}_{d=1}^N \subset K$, $\{S_d, T_d, S'_d, T'_d\}_{d=1}^N \subset \mathbf{R}$,

$t \geqq 0$, u, $v \in H_0$, *the limit as* $\lambda \to 0$ *of the matrix element*

$$\left\langle u \otimes W\left(\lambda \sum_{d=1}^{N} \int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d \, du\right)\Phi, \right.$$

$$\left. U^{(\lambda)}(t/\lambda^2)\, v \otimes W\left(\lambda \sum_{d=1}^{N} \int_{S'_{d'}/\lambda^2}^{T'_{d'}/\lambda^2} S_u^d f'_d \, du\right)\Phi \right\rangle \quad (1.19)$$

exists and, in the notation (1.16), *is equal to*

$$\left\langle u \otimes \bigotimes_{d=1}^{N} \Psi_d(\chi_{[S_d,\, T_d]} \otimes f_d),\ U(t)\, v \otimes \bigotimes_{d=1}^{N} \Psi_d(\chi_{[S'_d,\, T'_d]} f'_d) \right\rangle \quad (1.20)$$

where, $U(t)$ *is the solution of the quantum stochastic differential equation*

$$U(t) = 1 + \int_0^t \sum_{d=1}^{N} (D_d \otimes d\tilde{A}_d^+(s, g)$$

$$- D_d^+ \otimes d\tilde{A}_d(s, g) - D_d^+ D_d \otimes \mathbf{1}^{\otimes N} . (g \,|\, g)_d, - ds)\, U(s) \quad (1.21)$$

on $H_0 \otimes \bigotimes_{d=1}^{N} \Gamma(L^2(\mathbf{R}) \otimes (K_d, (.\,|\,.)_d)$ *and where, by definition for each* $d = 1, \ldots, N$,

$$\tilde{A}_d(s, g) := \mathbf{1}^{\otimes (d-1)} \otimes A_d(s, g) \otimes \mathbf{1}^{\otimes (N-d)} \quad (1.22)$$

and $A_d(s, g)$ *is given by* (1.15).

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2. THE CONVERGENCE OF THE RESERVOIR PROCESS

In this section, we prove the convergence of the reservoir process to a N-independent quantum Brownian motion. This corresponds to the convergence of the 0-th order term of (1.13) in the series expansion (1.12).

The following result generalizes Lemma (3.2) of [2]:

LEMMA (2.1). — *In the notation* (1.18), *for each* $N \in \mathbf{N}$, $f_1, \ldots, f_N$, $g_1, \ldots, g_N \in K$ *and for each* $S_1, \ldots, S_N$, $T_1, \ldots, T_N$, $S'_1, \ldots, S'_N$, $T'_1, \ldots, T'_N$ *in* $\mathbf{R}$,

$$\lim_{\lambda\to 0}\left\langle \lambda\sum_{d=1}^{N}\int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d\,du,\ \lambda\sum_{d'=1}^{N}\int_{S'_{d'}/\lambda^2}^{T'_{d'}/\lambda^2} S_u^d g_d\,du\right\rangle$$

$$= \sum_{d=1}^{N}\langle \chi_{[S_d,T_d]}, \chi_{[S'_d,T'_d]}\rangle_{L^2(\mathbf{R})}(f_d\,|\,g_d)_d$$

$$=\left(\bigoplus_{d=1}^{N}(\chi_{[S_d,T_d]}\otimes f_d)\,\middle|\,\bigoplus_{d=1}^{N}(\chi_{[S'_d,T'_d]}\otimes g_d)\right) \quad (2.1)$$

Proof. — For each $f_1,\dots,f_N$, $g_1,\dots,g_N\in K$ and for each $S_1,\dots,S_N$, $T_1,\dots,T_N$, $S'_1,\dots,S'_N$, $T'_1,\dots,T'_N$,

$$\left\langle \lambda\sum_{d=1}^{N}\int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d\,du,\ \lambda\sum_{d=1}^{N}\int_{S'_d/\lambda^2}^{T'_d/\lambda^2} S_u^d g_d\,du\right\rangle$$
$$=\sum_{d=1}^{N}\sum_{d'=1}^{N}\left\langle \lambda\int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d\,du,\ \int_{S'_{d'}/\lambda^2}^{T'_{d'}/\lambda^2} S_{u'}^{d'} g_{d'}\,du'\right\rangle$$
$$=\sum_{d=1}^{N}\sum_{d'=1}^{N}\int_{S_d}^{T_d}du\int_{(S'_{d'}-u)/\lambda^2}^{(T'_{d'}-u)/\lambda^2}du'\,\langle f_d, S_{u'}^d g_{d'}\rangle\, e^{i(\omega_d-\omega_{d'})u/\lambda^2} \quad (2.2)$$

By the Riemann-Lebesgue Lemma, one gets

$$\lim_{\lambda\to 0}\int_{S_d}^{T_d}du\int_{(S'_{d'}-u)/\lambda^2}^{(T'_{d'}-u)/\lambda^2}du'\,\langle f_d, S_{u'}^d, g_{d'}\,du'\rangle\, e^{i(\omega_d-\omega_{d'})u/\lambda^2}$$
$$=\left\{\begin{matrix} 0,\ \text{if}\quad d\neq d' \\ \langle \chi_{[S_d,T_d]}, \chi_{[S'_d,T'_d]}\rangle_{L^2(\mathbf{R})}(f_d\,|\,g_d)_d,\quad \text{if}\quad d=d'\end{matrix}\right\} \quad (2.3)$$

and this ends the proof.

The following lemma shows in which sence the "collective Hamiltonian process" converges to N-independent quantum Brownian motion.

LEMMA (2.2). — *For each* $n\in\mathbf{N}$, $\{f_d^{(k)}\}_{1\leq d\leq N,\,1\leq k\leq n}\subset K$, $\{x_d^{(k)}\}_{1\leq d\leq N,\,1\leq k\leq n}\subset\mathbf{R}$. $\{S_d^{(k)}, T_d^{(k)}\}_{1\leq d\leq N,\,1\leq k\leq n}\subset\mathbf{R}$ *and* $S_d^{(k)}\leq T_d^{(k)}$, *the limit*

$$\lim_{\lambda\to 0}\Big\langle \Phi, W\left(\sum_{d=1}^{N}x_d^{(1)}\lambda\int_{S_d^{(1)}/\lambda^2}^{T_d^{(1)}/\lambda^2}S_u^d f_d^{(1)}\,du\right)\times\dots$$
$$\times W\left(\sum_{d=1}^{N}x_d^{(n)}\lambda\int_{S_d^{(n)}/\lambda^2}^{T_d^{(n)}/\lambda^2}S_u^d f_d^{(n)}\,du\right)\Phi\Big\rangle \quad (2.4)$$

exists uniformly for $\{x_d^{(k)}\}_{1\leqq d\leqq N,\,1\leqq k\leqq n}$, $\{S_d^{(k)}, T_d^{(k)}\}_{1\leqq d\leqq N,\,1\leqq k\leqq n}$ *in a bounded set of* $\mathbf{R}$ *and is equal to*

$$\prod_{d=1}^{N}\langle \psi_d, W(x_d^{(1)}\chi_{[S_d^{(1)}, T_d^{(1)}]}\otimes f_d^{(1)})\times\ldots \times W(x_d^{(n)}\chi_{[S_d^{(n)}, T_d^{(n)}]}\otimes f_d^{(n)})\Psi_d\rangle$$

$$= \left\langle \Psi_1\otimes\ldots\otimes\Psi_N, \bigotimes_{d=1}^{N} W(x_d^{(1)}\chi_{[S_d^{(1)}, T_d^{(1)}]}\otimes f_d^{(1)})\times\ldots \times\bigotimes_{d=1}^{N} W(x_d^{(n)}\chi_{[S_d^{(n)}, T_d^{(n)}]}\otimes f_d^{(n)})\Psi_1\otimes\ldots\otimes\Psi_N\right\rangle$$

$$= \left\langle \Psi_1\otimes\ldots\otimes\Psi_N, W\left(\bigoplus_{d=1}^{N} x_d^{(1)}\chi_{[S_d^{(1)}, T_d^{(1)}]}\otimes f_d^{(1)}\right)\times\ldots \times W\left(\bigoplus_{d=1}^{N} x_d^{(n)}\chi_{[S_d^{(n)}, T_d^{(n)}]}\otimes f_d^{(n)}\right)\Psi_1\otimes\ldots\otimes\Psi_N\right\rangle \quad (2.5)$$

where, Ψ_d *is the vaccum of* $\Gamma(L^2(\mathbf{R})\otimes(K_d, (.|.)_d)$.

Proof. — For each $n\in N$, $\{f_d^{(k)}\}_{1\leqq d\leqq N,\,1\leqq k\leqq n}\subset K$, $\{x_d^{(k)}\}_{1\leqq d\leqq N,\,1\leqq k\leqq n}\subset\mathbf{R}$, $\{S_d^{(k)}, T_d^{(k)}\}_{1\leqq d\leqq N,\,1\leqq k\leqq n}\subset\mathbf{R}$ and $S_d^{(k)}\leqq T_d^{(k)}$, $T_d^{(k)}\}_{1\leqq d\leqq N,\,1\leqq k\leqq n}\subset\mathbf{R}$ and $S_d^{(k)}\leqq T_d^{(k)}$, by the Riemann-Lebesgue Lemma,

$$\left\langle \Phi, W\left(\sum_{d=1}^{N} x_d^{(1)}\lambda\int_{S_d^{(1)}/\lambda^2}^{T_d^{(1)}/\lambda^2} S_u^d f_d^{(1)}\,du\right)\times\ldots \times W\left(\sum_{d=1}^{N} x_d^{(n)}\lambda\int_{S_d^{(n)}/\lambda^2}^{T_d^{(n)}/\lambda^2} S_u^d f_d^{(n)}\,du\right)\Phi\right\rangle$$

$$\exp\left(-\frac{1}{2}\sum_{1\leqq i,j\leqq n}\sum_{d,d'=1}^{N} x_d^{(i)}x_{d'}^{(j)}\lambda^2 \times\left\langle\int_{S_d^{(j)}/\lambda^2}^{T_d^{(j)}/\lambda^2} S_u^d f_d^{(i)}\,du, \int_{S_d^{(j)}/\lambda^2}^{T_d^{(j)}/\lambda^2} S_u^{d'} f_{d'}^{(i)}\,du\right\rangle\right)$$

$$\times\exp\left(-i\,\mathrm{Im}\sum_{1\leqq i,j\leqq n}\sum_{d,d'=1}^{N} x_d^{(i)}x_{d'}^{(j)}\lambda^2 \times\left\langle\int_{S_d^{(j)}/\lambda^2}^{T_d^{(j)}/\lambda^2} S_u^d f_d^{(i)}\,du, \int_{S_d^{(j)}/\lambda^2}^{T_d^{(j)}/\lambda^2} S_u^{d'} f_{d'}^{(i)}\,du\right\rangle\right)$$

$$\to\exp\left(-\frac{1}{2}\sum_{1\leqq i,j\leqq n}\sum_{d=1}^{N} x_d^{(i)}x_d^{(j)} \times\langle\chi_{[S_d^{(j)}, T_d^{(j)}]}\otimes f_d^{(i)}, \chi_{[S_d^{(j)}, T_d^{(j)}]}\otimes f_d^{(j)}\rangle\right)$$

$$\exp\left(-i\,\mathrm{Im}\sum_{1\leqq i,j\leqq n}\sum_{d=1}^{N} x_d^{(i)}x_d^{(j)}\right.$$

$$\times \left\langle \chi_{[S_d^{(i)}, T_d^{(i)}]} \otimes f_d^{(i)}, \chi_{[S_d^{(j)}, T_d^{(j)}]} \otimes f_d^{(j)} \right\rangle \Big)$$

$$= \exp\left(-\frac{1}{2} \sum_{1 \leq i, j \leq n} \left\langle \bigoplus_{d=1}^{N} x_d^{(i)} \chi_{[S_d^{(i)}, T_d^{(i)}]} \otimes f_d^{(i)}, \bigoplus_{d=1}^{N} x_d^{(j)} \chi_{[S_d^{(j)}, T_d^{(j)}]} \otimes f_d^{(j)} \right\rangle \right)$$

$$\exp\left(-i \operatorname{Im} \sum_{1 \leq i, j \leq n} \left\langle \bigoplus_{d=1}^{N} x_d^{(i)} \chi_{[S_d^{(i)}, T_d^{(i)}]} \otimes f_d^{(i)}, \bigoplus_{d=1}^{N} x_d^{(j)} \chi_{[S_d^{(j)}, T_d^{(j)}]} \otimes f_d^{(j)} \right\rangle \right) \quad (2.6)$$

and obviously, the convergence is uniform for $\{ x_d^{(k)}, S_d^{(k)}, T_d^{(k)} \}_{1 \leq d \leq N, 1 \leq k \leq n}$ in a bounded subset of $\mathbf{R}$.

3. THE WEAK COUPLING LIMIT

The strategy of the proofs in the section is similar to that applied in [2], therefore we shall only outline the proofs, giving the full details only of those statements which are qualitatively different from the corresponding ones in [2].

First of all, in analogy with Lemma (4.1) of [2], we have the following

LEMMA (3.1). — *For each $n \in \mathbf{N}$, the product*

$$(-i)^n V_g(t_1) \ldots V_g(t_n)$$

can be written as a sum of two types of terms (called terms of type I *and of type* II):

$$\sum_{\varepsilon \in \{0, 1\}^n} (I^{\varepsilon}_{g, D}(n, t) + II^{\varepsilon}_{g, D}(n, t)) \quad (3.1)$$

with

$$I^{\varepsilon}_{g, D}(n, t) := (-D^+(t_1)) \ldots D(t_{j_1}) \ldots D(t_{j_k}) \ldots (-D^+(t_n))$$

$$\otimes \sum_{m=0}^{k \wedge (n-k)} \sum_{\substack{1 \leq r_1 < \ldots < r_k \leq k \\ \{j_{r_h} - 1\}_{h=1}^{m} \cap \{j_h\}_{h=1}^{k} = \varnothing}} \prod_{h=1}^{m} \langle S^0_{t_{j_{r_h}} - 1} g, S^0_{t_{j_{r_h}}} g \rangle$$

$$\times \prod_{\alpha \in \{j_h\}_{h=1}^{k} \setminus \{j_{r_h}\}_{h=1}^{m}} A^+(S^0_{t_\alpha} g)$$

$$\prod_{\alpha \in \{1, \ldots, n\} \setminus (\{j_h\}_{h=1}^{k} \cup \{j_{r_h} - 1\}_{h=1}^{m})} A(S^0_{t_\alpha} g)$$

and

$$\mathrm{II}^{\varepsilon}_{g,\mathrm{D}}(n,t):=(-\mathrm{D}^{+}(t_1))\ldots\mathrm{D}(t_{j_1})\ldots\mathrm{D}(t_{j_k})\ldots(-\mathrm{D}^{+}(t_n))$$

$$\otimes\sum_{m=0}^{k\wedge(n-k)}\sum_{\substack{1\leqq q_1<\ldots<q_m\leqq n\\ \{q_h\}_{h=1}^{m}\subset\{j_h\}_{h=1}^{k}}}\sum'_{(p_1,\ldots,p_m,\{q_h\}_{h=1}^{m})}\prod_{h=1}^{m}\langle \mathrm{S}^0_{t_{p_h}}g,\mathrm{S}^0_{t_{q_h}}g\rangle$$

$$\times\prod_{\alpha\in\{j_h\}_{h=1}^{k}\setminus\{q_h\}_{h=1}^{m}}\mathrm{A}^{+}(\mathrm{S}^0_{t_\alpha}g)\prod_{\alpha\in\{1,\ldots,n\}\setminus(\{j_h\}_{h=1}^{k}\cup\{p_h\}_{h=1}^{m})}\mathrm{A}(\mathrm{S}^0_{t_\alpha}g)\quad(3.2\,b)$$

As in [2], $j_1<\ldots<j_k$ *are the indices of the creators;* q_h *(resp.* p_h*) are the indices of those creators (resp. annihilators) which have given rise to a scalar product. Moreover, we recall from* [2] *that* $\sum'_{(p_1,\ldots,p_m,\{q_h\}_{h=1}^{m})}$ *means that the sum which runs over all* $1\leqq p_1,\ldots,p_m\leqq m$ *satisfying*

$$|\{p_h\}_{h=1}^{m}|=m\quad(3.3\,a)$$

$$\{p_h\}_{h=1}^{m}\subset\{1,\ldots,n\}\setminus\{j_h\}_{h=1}^{k}\quad(3.3\,b)$$

$$p_h<q_h,\qquad h=1,\ldots,m\quad(3.3\,c)$$

$$q_h-p_h\geqq 2\quad\text{for some } h=1,\ldots,m\quad(3.3\,d)$$

The proof is the same as in [2] except for the fact that now the D depend on t.

LEMMA (3.2). – *For each* $g\in\mathrm{K}$, *and* $\mathrm{D}\in\mathrm{B}(\mathrm{H}_0)$, *there exists a constant* C, *such that for each* $u, v\in\mathrm{H}_0$, $n\in\mathbf{N}$, *and* $t\geqq 0$,

$$\Bigg|\Bigg\langle u\otimes \mathrm{W}\left(\lambda\sum_{d=1}^{\mathrm{N}}\int_{\mathrm{S}_d/\lambda^2}^{\mathrm{T}_d/\lambda^2}\mathrm{S}^d_u f_d\,du\right)\Phi,\ \lambda^n\int_0^t dt_1\int_0^{t_1}dt_2\ldots\int_0^{t_{n-1}}dt_n$$
$$\mathrm{V}_g(t_1)\ldots\mathrm{V}_g(t_n)v\otimes\mathrm{W}\left(\lambda\sum_{d=1}^{\mathrm{N}}\int_{\mathrm{S}'_{d'}/\lambda^2}^{\mathrm{T}'_{d'}/\lambda^2}\mathrm{S}^d_u f'_d\,du\right)\Phi\Bigg\rangle\Bigg|$$
$$\leqq\|u\|.\|v\|.\mathrm{C}^n\frac{1}{([(1/3)n])!}(t\vee 1)^n\quad(3.4)$$

Proof. – The difference between (3.4) and the uniform estimate of Lemmata (5.2), (5.3) of [2] consists only in the following two points:

(i) replacing D by $\mathrm{D}(t):=e^{-it\mathrm{H}_\mathrm{S}}\mathrm{D}e^{it\mathrm{H}_\mathrm{S}}$;

(ii) replacing the scalar product

$$\left|\left\langle\int_{\mathrm{S}/\lambda^2}^{\mathrm{T}/\lambda^2}\mathrm{S}_u f\,du,\ \mathrm{S}_{\tau/\lambda^2}g\right\rangle\right|\quad(3.5\,a)$$

by

$$\left| \left\langle \sum_{d=1}^{N} \int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d \, du, \; S_{\tau/\lambda^2}^0 g \right\rangle \right| \tag{3.5b}$$

Since $\| D(t) \| = \| D \|$, the difference (i) can't influence the uniform estimate. But also the difference (ii) is not important since (3.5*b*) is majorized by

$$\sum_{d=1}^{N} \int_{S_d/\lambda^2}^{T_d/\lambda^2} | \langle S_{u-\tau/\lambda^2}^d f_d, g \rangle e^{-i\omega_d \tau/\lambda^2} | \, du \leqq \sum_{d=1}^{N} \int_{-\infty}^{\infty} | \langle S_u^0 f_d, g \rangle | \, du \tag{3.6}$$

So, we get the uniform estimate with the same arguments as in the above mentioned Lemmata of [2].

The following lemma states the irrelevance, in the weak coupling limit, of the terms of type II in the decomposition (3.1).

LEMMA (3.3). — *For each* $g \in K$, *and* $D \in B(H_0)$, *there exists a constant* C, *such that for each* $u, v \in H_0$, $n \in N$, *and* $t \geqq 0$,

$$\lim_{\lambda \to 0} \Bigg| \Bigg\langle u \otimes W\left(\lambda \sum_{d=1}^{N} \int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d \, du \right) \Phi, \\ \lambda^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \, II_{g,D}^{\varepsilon}(n, t) \\ \times v \otimes W\left(\lambda \sum_{d=1}^{N} \int_{S'_d/\lambda^2}^{T'_d/\lambda^2} S_u^d f'_d \, du \right) \Phi \Bigg\rangle \Bigg| = 0 \tag{3.7}$$

Proof. — With the same arguments as in the proof of the Lemma (4.2) in [2] we obtain the inequality

$$\Bigg| \Bigg\langle u \otimes W\left(\lambda \sum_{d=1}^{N} \int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d \, du \right) \Phi, \\ \lambda^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \, II_{g,D}^{\varepsilon}(n, t) \\ \times v \otimes W\left(\lambda \sum_{d=1}^{N} \int_{S'_d/\lambda^2}^{T'_d/\lambda^2} S_u^d f'_d \, du \right) \Phi \Bigg\rangle \Bigg| \\ \leqq C_1^n \lambda^{-2m} \int_0^t dt_1 \dots \int_0^{t_{n-1}} dt_n \\ \times \sum_{m=0}^{k \wedge (n-k)} \sum_{(p_1, q_1, \dots, p_m, q_m)}' \prod_{h=1}^{m} | \langle S_{t_{p_h}/\lambda^2}^0 g, S_{t_{q_h}/\lambda^2}^0 g \rangle | \tag{3.8}$$

Replacing S_t by S_t^0, we see that the right hand side of (3.8) has the same form as (4.21) of [2]. So we can conclude the proof with the same arguments as in Lemma (4.2) of [2].

The following lemma gives the explicit from of the limit of the terms of type I.

THEOREM (3.4). — *For each* $n \in \mathbf{N}$, *the limit*

$$\lim_{\lambda \to 0} \left\langle u \otimes W\left(\lambda \sum_{d=1}^{N} \int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d \, du \right) \Phi, \right. \\ \lambda^n \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \, I_{g,D}^{\varepsilon}(n, t) \\ \left. v \otimes W\left(\lambda \sum_{d=1}^{N} \int_{S'_d/\lambda^2}^{T'_d/\lambda^2} S_u^d f'_d \, du \right) \Phi \right\rangle \quad (3.9)$$

exists and is equal to

$$\sum_{m=0}^{k \wedge (n-k)} \sum_{\substack{1 \leqq r_1 < \ldots < r_m \leqq k \\ \{ j_{r_h} - 1 \}_{h=1}^{m} \cap \{ j_h \}_{h=1}^{k} = \varnothing}} \sum_{1 \leqq d_1, \ldots, d_{j_{r_1}} - 1 = d_{j_{r_1}}, \ldots, d_{j_{r_m}} - 1 = d_{j_{r_m}}, \ldots, d_n \leqq N} \\ \times \int_{0 \leqq t_n \leqq \ldots \leqq t_{j_{r_m}} \leqq \ldots t_{j_{r_1}} \leqq \ldots \leqq t_1 \leqq t} dt_1 \ldots dt_{j_{r_1}} \ldots dt_{j_{r_m}} \ldots dt_n \\ \langle u, D_n(j_1, \ldots, j_k, \{ d_h \}_{h=1}^{n}) v \rangle \prod_{h=1}^{m} (g | g)_{d_{j_{r_h}}, -} \\ \times \prod_{\alpha \in \{ j_h \}_{h=1}^{k} \setminus \{ j_{r_j} \}_{h=1}^{m}} \chi_{[S_{d_\alpha}, T_{d_\alpha}]}(t_\alpha) (f_{d_\alpha} | g)_{d_\alpha} \\ \times \prod_{\alpha \in \{ 1, \ldots, n \} \setminus (\{ j_h \}_{h=1}^{k} \cup \{ j_{r_h} - 1 \}_{h=1}^{m})} \chi_{[S'_d, T'_d]}(t_\alpha) (g | f'_{d_\alpha})_{d_\alpha} \\ \left\langle W\left(\bigoplus_{d=1}^{N} \chi_{[S_d, T_d]} \otimes f_d \right) \Psi_1 \otimes \ldots \otimes \Psi_N, \right. \\ \left. W\left(\bigoplus_{d=1}^{N} \chi_{[S'_d, T'_d]} \otimes f'_d \right) \Psi_1 \otimes \ldots \otimes \Psi_N \right\rangle \quad (3.10)$$

where,

$$D_n(j_1, \ldots, j_k, \{ d_h \}_{h=1}^{n}) := (-D_{d_1}^{+}) \ldots D_{d_{j_1}} \ldots D_{d_{j_k}} \ldots (-D_{d_n}^{+}) \quad (3.11)$$

and

$$(g | g)_{d,-} := \int_{\infty}^{o} \langle g, S_t^d g \rangle \, dt \quad (3.12)$$

Proof. — Using the expression (3.2 *a*) of the type I terms, one has for each $\varepsilon \in \{0, 1\}^n$, $\{j_1, \ldots, j_k\} = \{j, \varepsilon(j) = 1\}$,

$$
\begin{aligned}
&\Big\langle u \otimes W\Big(\lambda \sum_{d=1}^{N} \int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d\, du\Big)\Phi,\\
&\qquad \lambda^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n\, I^{\varepsilon}_{g, D}(n, t)\\
&\qquad \times v \otimes W\Big(\lambda \sum_{d=1}^{N} \int_{S'_d/\lambda^2}^{T'_d/\lambda^2} S_u^d f'_d\, du\Big)\Phi\Big\rangle\\
&= \sum_{m=0}^{k \wedge (n-k)} \sum_{\substack{1 \leq r_1 < \ldots < r_m \leq k\\ \{j_{r_h} - 1\}_{h=1}^m \cap \{j_h\}_{h=1}^k = \varnothing}} \lambda^{-2m} \int_0^t dt_1 \ldots\\
&\qquad \times \int_0^{t_{n-1}} dt_n \prod_{h=1}^{m} \langle g, S^0_{(t_{j_{r_h}} - t_{j_{r_h} - 1})/\lambda^2} g\rangle\\
&\qquad \times \langle u, D_{\varepsilon(1)}(t_1/\lambda^2) \ldots D_{\varepsilon(n)}(t_n/\lambda^2) v\rangle\\
&\qquad \times \prod_{\alpha \in \{j_h\}_{h=1}^k \setminus \{j_{r_h}\}_{h=1}^m} \sum_{d=1}^{N} \int_{S_d/\lambda^2}^{T_d/\lambda^2} \langle S_u^d f_d, S^0_{t_\alpha/\lambda^2} g\rangle\, du\\
&\qquad \times \prod_{\alpha \varepsilon \{1, \ldots, n\} \setminus (\{j_h\}_{h=1}^k \cup \{j_{r_h} - 1\}_{h=1}^m)} \sum_{d=1}^{N} \int_{S'_d/\lambda^2}^{T'_d/\lambda^2} \langle S^0_{t_\alpha/\lambda^2} g, S_u^d f'_d\rangle\, du\\
&\qquad \times \Big\langle W\Big(\lambda \sum_{d=1}^{N} \int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d\, du\Big)\Phi,\\
&\qquad\qquad W\Big(\lambda \sum_{d=1}^{N} \int_{S'_d/\lambda^2}^{T'_d/\lambda^2} S_u^d f'_d\, du\Big)\Phi\Big\rangle
\end{aligned} \tag{3.13}
$$

From this, using (1.4), we know that (3.13) is equal to

$$
\begin{aligned}
&\sum_{m=0}^{k \wedge (n-k)} \sum_{\substack{1 \leq r_1 < \ldots < r_m \leq k\\ \{j_{r_h} - 1\}_{h=1}^m \cap \{j_h\}_{h=1}^k = \varnothing}} \lambda^{-2m} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n\\
&\qquad \times \sum_{1 \leq d_1, \ldots, d_n \leq N} \langle u, D_{d_1}^{\varepsilon(1)} \ldots D_{d_n}^{\varepsilon(n)} v\rangle\\
&\qquad \prod_{h=1}^{m} \langle g, S^0_{(t_{j_{r_h}} - t_{j_{r_h} - 1})/\lambda^2} g\rangle\, e^{i(\omega_{d_{j_{r_h}-1}} t_{j_{r_h}-1}/\lambda^2 - \omega_{d_{j_{r_h}}} t_{j_{r_h}}/\lambda^2)}\\
&\times \prod_{\alpha \in \{j_h\}_{h=1}^k \setminus \{j_{r_h}\}_{h=1}^m} \sum_{d=1}^{N} \int_{S_d/\lambda^2}^{T_d/\lambda^2} \langle S_u^d f, S^0_{t_\alpha/\lambda^2} g\rangle\, 2^{-i\omega_{d_\alpha} t_\alpha/\lambda^2}\, du
\end{aligned}
$$

$$\times \prod_{\alpha \in \{1, \ldots, n\} \setminus (\{j_h\}_{h=1}^{k} \cup \{j_{r_h}-1\}_{h=1}^{m})} \sum_{d=1}^{N} \int_{S'_d/\lambda^2}^{T'_d/\lambda^2} \langle S^0_{t_\alpha/\lambda^2} g, S^d_u f'_d \rangle$$

$$\times e^{i\omega_{d_\alpha} t_\alpha/\lambda^2} du \left\langle W\left(\lambda \sum_{d=1}^{N} \int_{S_d/\lambda^2}^{T_d/\lambda^2} S^d_u f_d \, du\right)\Phi, \right.$$

$$\left. W\left(\lambda \sum_{d=1}^{N} \int_{S'_d/\lambda^2}^{T'_d/\lambda^2} S^d_u f'_d \, du\right)\Phi \right\rangle \quad (3.14)$$

Now, multiplying and dividing by a factor $e^{-i\omega_d t}$ and using (1.4), we can reduce the $S^d - S^0$ - scalar products to $S^d - S^d$ - scalar products. Thus the expression (3.14) becomes

$$\sum_{m=0}^{k \wedge (n-k)} \sum_{\substack{1 \leq r_1 < \ldots < r_m \leq k \\ \{j_{r_h}-1\}_{h=1}^{m} \cap \{j_h\}_{h=1}^{k} = \varnothing}} \lambda^{-2m} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n$$

$$\times \sum_{1 \leq d_1, \ldots, d_n \leq N} \langle u, D^{\varepsilon(1)}_{d_1} \ldots D^{\varepsilon(n)}_{d_n} v \rangle$$

$$\times \prod_{h=1}^{m} \langle g, S^{d_{j_{r_h}}}_{(t_{j_{r_h}} - t_{j_{r_h}-1})/\lambda^2} g \rangle e^{i(\omega_{d_{j_{r_h}1}} - \omega_{d_{j_{r_h}}}) t_{j_{r_h}-1}/\lambda^2}$$

$$\times \prod_{\varepsilon \in \{j_h\}_{h=1}^{k} \setminus \{j_{r_h}\}_{h=1}^{m}} \sum_{d=1}^{N} \int_{S_d/\lambda^2}^{T_d/\lambda^2} \langle S^d_u f_d, S^d_{t_\alpha/\lambda^2} g \rangle e^{-(i\omega_{d_\alpha} - \omega_d) t_\alpha/\lambda^2} du$$

$$\times \prod_{\alpha \in \{1, \ldots, n\} \setminus (\{j_h\}_{h=1}^{k} \cup \{j_{r_h}-1\}_{h=1}^{m})} \sum_{d=1}^{N} \int_{S'_d/\lambda^2}^{T'_d/\lambda^2} \langle S^d_{t_\alpha/\lambda^2} g, S^d_u f'_d \rangle$$

$$\times e^{i(\omega_{d_\alpha} - \omega_d) t_\alpha/\lambda^2} du \left\langle W\left(\lambda \sum_{d=1}^{N} \int_{S_d/\lambda^2}^{T_d/\lambda^2} S^d_u f_d \, du\right)\Phi, \right.$$

$$\left. W\left(\lambda \sum_{d=1}^{N} \int_{S'_d/\lambda^2}^{T'_d/\lambda^2} S^d_u f'_d \, du\right)\Phi \right\rangle \quad (3.15)$$

With the same changes of variables as those used in Theorem (5.1) of [2] we reduce (3.15) to the form

$$\sum_{m=0}^{k \wedge (n-k)} \sum_{\substack{1 \leq r_1 < \ldots < r_m \leq k \\ \{j_{r_h}-1\}_{h=1}^{m} \cap \{j_h\}_{h=1}^{k} = \varnothing}} \lambda^{-2m} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n$$

$$\times \sum_{1 \leq d_1, \ldots, d_n \leq N} \langle u, D^{\varepsilon(1)}_{d_1} \ldots D^{\varepsilon(n)}_{d_n} v \rangle$$

$$\times \int_0^t dt_1 \ldots \int_{-t_{j_{r_1}-1}/\lambda^2}^{0} dt_{j_{r_1}} \int^{t_{j_{r_1}-1} + \lambda^2 t_{j_{r_1}}} dt_{j_{r_1}+1} \times \ldots$$

$$\times \int_{-t_{j_{r_m}-1}/\lambda^2}^{0} dt_{j_{r_m}} \int^{t_{j_{r_m}-1}+\lambda^2 t_{j_{r_m}}} dt_{j_{r_m}+1} \dots \int_0^{t_{n-1}} dt_n$$

$$\times \prod_{h=1}^{m} \langle g, S^{d_{j_{r_h}}}_{t_{j_{r_h}}/\lambda^2} g \rangle e^{i(\omega_{d_{j_{r_h}-1}} - \omega_{d_{j_{r_h}}}) t_{j_{r_h}-1}/\lambda^2}$$

$$\times \prod_{\alpha \in \{j_h\}_{h=1}^{k} \setminus \{j_{r_h}\}_{h=1}^{m}} \sum_{d=1}^{N} \int_{(S_d - t_\alpha)/\lambda^2}^{(T_d - t_\alpha)/\lambda^2} \langle S^d_u f_d, g \rangle e^{-(i\omega_{d_\alpha} - \omega_d) t_\alpha/\lambda^2} du$$

$$\times \prod_{\alpha \in \{1, \dots, n\} \setminus (\{j_h\}_{h=1}^{k} \cup \{j_{r_h} - 1\}_{h=1}^{m})} \sum_{d=1}^{N} \int_{(S'_d - t_\alpha)/\lambda^2}^{(T'_d - t_\alpha)/\lambda^2} \langle g, S^d_u \rangle$$

$$\times e^{i(\omega_{d_\alpha} - \omega_d) t_\alpha/\lambda^2} du \left\langle W\left(\lambda \sum_{d=1}^{N} \int_{S_d/\lambda^2}^{T_d/\lambda^2} S^d_u f_d \, du\right)\Phi, \right.$$

$$\left. W\left(\lambda \sum_{d=1}^{N} \int_{S'_d/\lambda^2}^{T'_d/\lambda^2} S^d_u f'_d \, du\right)\Phi \right\rangle \quad (3.16)$$

By the Riemann-Lebesgue lemma, in the limit only those terms will survive for which $\omega_{d_\alpha} = \omega_d$. Therefore the expression (3.15), in the limit $\lambda \to 0$, is equal to the limit, as $\lambda \to 0$, of the expression

$$\sum_{m=0}^{k \wedge (n-k)} \sum_{\substack{1 \leq r_1 < \dots < r_m \leq k \\ \{j_{r_h} - 1\}_{h=1}^{m} \cap \{j_h\}_{h=1}^{k} = \varnothing}} \lambda^{-2m} \int_0^t dt_1 \dots \int_0^{t_{n-1}} dt_n$$

$$\times \sum_{1 \leq d_1, \dots, d_n \leq N} \langle u, D^{\varepsilon(1)}_{d_1} \dots D^{\varepsilon(n)}_{d_n} v \rangle$$

$$\times \int_0^t dt_1 \dots \int_{-t_{j_{r_1}-1}/\lambda^2}^{0} dt_{j_{r_1}} \int_0^{t_{j_{r_1}-1}+\lambda^2 t_{j_{r_1}}} dt_{j_{r_1}+1} \times \dots$$

$$\times \int_{-t_{j_{r_m}-1}/\lambda^2}^{0} dt_{j_{r_m}} \int^{t_{j_{r_m}-1}+\lambda^2 t_{j_{r_m}}} dt_{j_{r_m}+1} \dots \int_0^{t_{n-1}} dt_n$$

$$\times \prod_{h=1}^{m} \langle g, S^{d_{j_{r_h}}}_{t_{j_{r_h}}/\lambda^2} g \rangle e_{i(\omega_{d_{j_{r_h}-1}}} - \omega_{d_{j_{r_h}}}) t_{j_{r_h}-1}/\lambda^2$$

$$\times \prod_{\alpha \in \{j_h\}_{h=1}^{k} \setminus \{j_{r_h}\}_{h=1}^{m}} \int_{(S_{d_\alpha} - t_\alpha)/\lambda^2}^{(T_{d_\alpha} - t_\alpha)/\lambda^2} \langle S^{d_\alpha}_u f_{d_\alpha}, g \rangle du$$

$$\times \prod_{\alpha \in \{1, \dots, n\} \setminus (\{j_h\}_{h=1}^{k} \cup \{j_{r_h} - 1\}_{h=1}^{m})} \int_{(S'_{d_\alpha} - t_\alpha)/\lambda^2}^{(T'_{d_\alpha} - t_\alpha)/\lambda^2} \langle g, S^{d_\alpha}_u f'_{d_\alpha} \rangle du$$

$$\times \left\langle W\left(\lambda \sum_{d=1}^{N} \int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d \, du\right)\Phi, \right.$$
$$\left. W\left(\lambda \sum_{d=1}^{N} \int_{S'_d/\lambda^2}^{T'_d/\lambda^2} S_u^d f'_d \, du\right)\Phi\right\rangle \quad (3.17)$$

Notice that,

$$\lim_{\lambda \to 0} \int_{(S-t)/\lambda^2}^{(T-t)/\lambda^2} \langle g, S_u^d f\rangle \, du = \chi_{[S, T]}(g \,|f)_d \quad (3.18)$$

and

$$\left| \int_{(S-t)/\lambda^2}^{(T-t)/\lambda^2} \langle g, S_u^d f\rangle \, du \right| \leqq \int_{-\infty}^{\infty} |\langle g, S_u^0 f\rangle| \, du < \infty \quad (3.19)$$

for each f, $g \in K$, $d=1, \ldots, N$ and $S, T \in \mathbf{R}$, $t \geqq 0$. So we can apply again the Riemann-Lebesgue Lemma and conclude that, in the limit for $\lambda \to 0$ of the expression (3.17), only the terms with

$$d_{j_{r_h}-1} = d_{j_{r_h}}; \qquad h=1, \ldots, m$$

will survive. This implies that apart from the (irrelevant) sum over $d_1, \ldots, d_n$, the terms non vanishing in the limit of the expression (3.17) are exactly of the same form as the expression (5.10) of Theorem (5.1) of [2]. Applying this Theorem, (3.10) follows easily, and this ends the proof.

From the above we obtain the explicit form of the limit (1.11).

THEOREM (3.5). — *For each* $N \in \mathbf{N}$, g, $\{f_d\}_{d=1}^N$, $\{f'_d\}_{d=1}^N \subset K$, $\{S_d, T^d, S'_d, T'_d\}_{d=1}^N \subset \mathbf{R}$, $t \geqq 0$, u, $v \in H_0$, *the limit*

$$\lim_{\lambda \to 0} \left\langle u \otimes W\left(\lambda \sum_{d=1}^{N} \int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d \, du\right)\Phi, \right.$$
$$\left. U^{(\lambda)}(t/\lambda^2)\, v \otimes W\left(\lambda \sum_{d=1}^{N} \int_{S'_d/\lambda^2}^{T'_d/\lambda^2} S_u^d f'_d \, du\right)\Phi\right\rangle \quad (3.20)$$

exists and is equal to

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{1 \leqq j_1 < \ldots < j_k \leqq n} \sum_{m=0}^{k \wedge (n-k)} \sum_{\substack{1 \leqq r_1 < \ldots < r_m \leqq k \\ \{j_{r_h}-1\}_{h=1}^m \cap \{j_h\}_{h=1}^k = \varnothing}}$$
$$\times \sum_{1 \leqq d_1, \ldots, d_{j_{r_1}-1} = d_{j_{r_1}}, \ldots, d_{j_{r_m}-1} = d_{j_{r_m}}, \ldots, d_n \leqq N}$$
$$\times \int_{0 \leqq t_n \leqq \ldots \leqq t_{j_{r_m}} \leqq \ldots t_{j_{r_1}} \leqq \ldots \leqq t_1 \leqq t} dt_1 \ldots dt_{j_{r_1}} \ldots dt_{j_{r_m}} \ldots dt_n$$

$$\times \langle u, D_n(j_1, \ldots, j_k, \{d_h\}_{h=1}^n) v \rangle \prod_{h=1}^m (g|g)_{d_{j_{r_h}}}, -$$

$$\times \prod_{\alpha \in \{j_h\}_{h=1}^k \setminus \{j_{r_h}\}_{h=1}^m} \chi_{[S_{d_\alpha}, T_{d_\alpha}]}(t_\alpha)(f_{d_\alpha}|g)_{d_\alpha}$$

$$\times \prod_{\alpha \in \{1, \ldots, n\} \setminus (\{j_h\}_{h=1}^k \cup \{j_{r_h}-1\}_{h=1}^m)} \chi_{[S'_{d_\alpha}, T'_{d_\alpha}]}(t_\alpha)(g|f'_{d_\alpha})_{d_\alpha}$$

$$\times \left\langle W\left(\bigoplus_{d=1}^N \chi_{[S_d, T_d]} \otimes f_d\right) \Psi_1 \otimes \ldots \otimes \Psi_N, \right.$$

$$\left. W\left(\bigoplus_{d=1}^N \chi_{[S'_d, T'_d]} \otimes f'_d\right) \Psi_1 \otimes \ldots \otimes \Psi_N \right\rangle \quad (3.21)$$

Proof. – Expand $U^{(\lambda)}(t/\lambda^2)$ using the iterated series and use Lemmas (3.1), (3.2), (3.3) and Theorem (3.4).

4. THE QUANTUM STOCHASTIC DIFFERENTIAL EQUATION

In the section, we shall prove that the limit (3.20) is the solution of a quantum stochastic differential equation (q. s. d. e.) whose explicit form we are going to determine.

We shall first determine an equation satisfied by the limit (3.20) and then we shall identify this equation with a q. s. d. e.

From Lamma (3.2) we know that for each $u \in H_0$, $t \geqq 0$, there exists a $G(t) \in H_0$, such that (3.21) can be written to

$$\langle u, G(t) \rangle \quad (4.1)$$

Denote for each $\lambda > 0$,

$$\langle u, G_\lambda(t) \rangle := \left\langle u \otimes W\left(\lambda \sum_{d=1}^N \int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d \, du\right) \Phi, \right.$$

$$\left. U^{(\lambda)}(t/\lambda^2) v \otimes W\left(\lambda \sum_{d=1}^N \int_{S'_d/\lambda^2}^{T'_d/\lambda^2} S_u^d f'_d \, du\right) \Phi \right\rangle \quad (4.2)$$

then Theorem (3.5) shows that

$$\lim_{\lambda \to 0} \langle u, G_\lambda(t) \rangle = \langle u, G(t) \rangle \quad (4.2a)$$

Moreover for each $n \geqq 1$,

$$\frac{d}{dt} \left\langle u \otimes W\left(\lambda \sum_{d=1}^N \int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d \, du\right) \Phi, \right.$$

$$\lambda^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n V_g(t_1) \ldots V_g(t_n)$$

$$\times v \otimes \mathrm{W}\left(\lambda \sum_{d=1}^{\mathrm{N}} \int_{\mathrm{S}'_d/\lambda^2}^{\mathrm{T}'_d/\lambda^2} \mathrm{S}_u^d f'_d\, du\right)\Phi\Big\rangle$$

$$= \Big\langle u \otimes \mathrm{W}\left(\lambda \sum_{d=1}^{\mathrm{N}} \int_{\mathrm{S}_d/\lambda^2}^{\mathrm{T}_d/\lambda^2} \mathrm{S}_u^d f_d\, du\right)\Phi,$$

$$\times \lambda^{n-2} \int_0^{t/\lambda^2} dt_2 \int_0^{t_2} dt_3 \dots \int_0^{t_{n-1}} dt_n \mathrm{V}(t/\lambda^2)\mathrm{V}_g(t_2)\dots\mathrm{V}_g(t_n)$$

$$\times v \otimes \mathrm{W}\left(\lambda \sum_{d=1}^{\mathrm{N}} \int_{\mathrm{S}'_d/\lambda^2}^{\mathrm{T}'_d/\lambda^2} \mathrm{S}_u^d f'_d\, du\right)\Phi\Big\rangle \quad (4.3)$$

Notice that $\|\mathrm{D}(t)\| = \|\mathrm{D}\|$ so, using the proof of Theorem (6.4) in [2], we know that there exists a constant C_2 such that

$$\Bigg|\frac{d}{dt}\Big\langle u \otimes \mathrm{W}\left(\lambda \sum_{d=1}^{\mathrm{N}} \int_{\mathrm{S}_d/\lambda^2}^{\mathrm{T}_d/\lambda^2} \mathrm{S}_u^d f_d\, du\right)\Phi,$$

$$\lambda^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \mathrm{V}_g(t_1)\dots\mathrm{V}_g(t_n)$$

$$\times v \otimes \mathrm{W}\left(\lambda \sum_{d=1}^{\mathrm{N}} \int_{\mathrm{S}'_d/\lambda^2}^{\mathrm{T}'_d/\lambda^2} \mathrm{S}_u^d f'_d\, du\right)\Phi\Big\rangle\Bigg| \leqq \mathrm{C}_2^n \frac{(t \vee 1)^n}{[n/3]!} \quad (4.4)$$

Therefore the function $t \to \langle u, \mathrm{G}_\lambda(t)\rangle$ is differentiable and its derivative is equal to

$$\frac{d}{dt}\Big\langle u \otimes \mathrm{W}\left(\lambda \sum_{d=1}^{\mathrm{N}} \int_{\mathrm{S}_d/\lambda^2}^{\mathrm{T}_d/\lambda^2} \mathrm{S}_u^d f_d\, du\right)\Phi,$$

$$\mathrm{U}^{(\lambda)}(t/\lambda^2)\, v \otimes \mathrm{W}\left(\lambda \sum_{d=1}^{\mathrm{N}} \int_{\mathrm{S}'_d/\lambda^2}^{\mathrm{T}'_d/\lambda^2} \mathrm{S}_u^d f'_d\, du\right)\Phi\Big\rangle$$

$$= \sum_{n=1}^{\infty} \frac{d}{dt}\Big\langle u \otimes \mathrm{W}\left(\lambda \sum_{d=1}^{\mathrm{N}} \int_{\mathrm{S}_d/\lambda^2}^{\mathrm{T}_d/\lambda^2} \mathrm{S}_u^d f_d\, du\right)\Phi,$$

$$\times (-i)^n \lambda^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n\, \mathrm{V}_g(t_1)\dots\mathrm{V}_g(t_n)$$

$$\times v \otimes \mathrm{W}\left(\lambda \sum_{d=1}^{\mathrm{N}} \int_{\mathrm{S}'_d/\lambda^2}^{\mathrm{T}'_d/\lambda^2} \mathrm{S}_u^d f'_d\, du\right)\Phi\Big\rangle \quad (4.5)$$

From (4.3) and (4.5), one gets

$$\frac{d}{dt}\Big\langle u \otimes \mathrm{W}\left(\lambda \sum_{d=1}^{\mathrm{N}} \int_{\mathrm{S}_d/\lambda^2}^{\mathrm{T}_d/\lambda^2} \mathrm{S}_u^d f_d\, du\right)\Phi,$$

$$\mathrm{U}^{(\lambda)}(t/\lambda^2)\, v \otimes \mathrm{W}\left(\lambda \sum_{d=1}^{\mathrm{N}} \int_{\mathrm{S}'_d/\lambda^2}^{\mathrm{T}'_d/\lambda^2} \mathrm{S}_u^d f'_d\, du\right)\Phi\Big\rangle$$

$$=(-i)\frac{1}{\lambda}\left\langle u\otimes W\left(\sum_{d=1}^{N}\int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d\,du\right)\Phi,\right.$$

$$\left.V_g(t/\lambda^2)\,U^{(\lambda)}(t/\lambda^2)\,v\otimes W\left(\lambda\sum_{d=1}^{N}\int_{S'_d/\lambda^2}^{T'_d/\lambda^2} S_u^d f'_d\,du\right)\Phi\right\rangle$$

$$=\frac{1}{\lambda}\left\langle u\otimes W\left(\sum_{d=1}^{N}\int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d\,du\right)\Phi,\right.$$

$$(D(t/\lambda^2)\otimes A^+(S^0_{t/\lambda^2}g)-D^+(t/\lambda^2)\otimes A(S^0_{t/\lambda^2}g))\,U^{(\lambda)}(t/\lambda^2)$$

$$\left.\times v\otimes W\left(\lambda\sum_{d=1}^{N}\int_{S'_d/\lambda^2}^{T'_d/\lambda^2} S_u^d f'_d\,du\right)\Phi\right\rangle$$

$$=\frac{1}{\lambda}\left\langle u\otimes W\left(\lambda\sum_{d=1}^{N}\int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d\,du\right)\Phi,\right.$$

$$\times\left(\sum_{d=1}^{N} D_d\otimes A^+(S^d_{t/\lambda^2}g)-\sum_{d=1}^{N} D_d^+\otimes A(S^d_{t/\lambda^2}g)\right)$$

$$\left.\times U^{(\lambda)}(t/\lambda^2)\,v\otimes W\left(\lambda\sum_{d=1}^{N}\int_{S'_d/\lambda^2}^{T'_d/\lambda^2} S_u^d f'_d\,du\right)\Phi\right\rangle \quad (4.6)$$

If we call $I(t, \lambda)$ [resp. $II(t, \lambda)$] the piece of the scalar product in (4.6) contaning the terms $D_d\otimes A^+$ (resp. $-D_d^+\otimes A$), then (4.6) can be written as

$$\frac{1}{\lambda}I(t,\lambda)+\frac{1}{\lambda}II(t,\lambda) \quad (4.6a)$$

Notice that

$$\lim_{\lambda\to 0}\frac{1}{\lambda}I(t,\lambda)$$

$$=\lim_{\lambda\to 0}\sum_{d,d'=1}^{N}\left\langle D_d^+\,u\otimes W\left(\lambda\sum_{d=1}^{N}\int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d\,du\right)\Phi,\right.$$

$$\left.U^{(\lambda)}(t/\lambda^2)\,v\otimes W\left(\lambda\sum_{d=1}^{N}\int_{S'_d/\lambda^2}^{T'_d/\lambda^2} S_u^d f'_d\,du\right)\Phi\right\rangle$$

$$\times\int_{S_d/\lambda^2}^{T_d/\lambda^2}\langle S_u^d f_d, S^{d'}_{t/\lambda^2}g\rangle\,du$$

$$=\lim_{\lambda\to 0}\sum_{d,d'=1}^{N}\left\langle D_d^+\,u\otimes W\left(\lambda\sum_{d=1}^{N}\int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d\,du\right)\Phi,\right.$$

$$\left.U^{(\lambda)}(t/\lambda^2)\,v\otimes W\left(\lambda\sum_{d=1}^{N}\int_{S'_d/\lambda^2}^{T'_d/\lambda^2} S_u^d f'_d\,du\right)\Phi\right\rangle$$

$$\times\int_{(S_d-t)/\lambda^2}^{(T_d-t)/\lambda^2}\langle S_u^d f_d, g\rangle\,e^{i(\omega_{d'}-\omega_d)t/\lambda^2}\,du \quad (4.7)$$

Therefore, by the Riemann-Lebesgue Lemma, Theorem (3.5), the definitions (4.1), (4.2) and dominated convergence, one has

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \int_0^t ds\, \mathrm{I}(s, \lambda) = \sum_{d=1}^{\mathrm{N}} \int_0^t ds\, \chi_{[\mathrm{S}, \mathrm{T}]}(s) \cdot (f_d \mid g)_d \cdot \langle \mathrm{D}_d^+ u, \mathrm{G}(s) \rangle \quad (4.8)$$

The $\frac{1}{\lambda}\mathrm{II}(t, \lambda)$ term is dealt with the same strategy for the term II_λ in [2] (*cf.* formula (6.14) of [2]). That is:

- one brings $-\mathrm{D}_d^+ \otimes 1$ on the left hand side of the scalar product.
- one writes

$$\begin{aligned} 1 \otimes \mathrm{A}(\mathrm{S}^d_{t/\lambda^2} g)\, \mathrm{U}^{(\lambda)}(t/\lambda^2) \\ = \mathrm{U}^{(\lambda)}(t/\lambda^2)\, 1 \otimes \mathrm{A}(\mathrm{S}^d_{t/\lambda^2} g) + [1 \otimes \mathrm{A}(\mathrm{S}^d_{t/\lambda^2} g), \mathrm{U}^{(\lambda)}(t/\lambda^2)] \quad (4.9) \end{aligned}$$

- the first term on the right hand side of (4.9) acts on the coherent vector giving rise to an expression similar to (4.7) which, in the limit $\lambda \to 0$ convergences a. e. to

$$\int_0^t ds \sum_{d=1}^{\mathrm{N}} \chi_{[\mathrm{S}, \mathrm{T}]}(s) \cdot (f_d \mid g)_d \cdot \langle \mathrm{D}_d u, \mathrm{G}(s) \rangle \quad (4.10)$$

- the commutator term is the sum, for $d=1, \ldots, \mathrm{N}$, of

$$\begin{aligned} \Big\langle -\mathrm{D}_d u \otimes \mathrm{W}\Big(\lambda \sum_{d=1}^{\mathrm{N}} \int_{\mathrm{S}_d/\lambda^2}^{\mathrm{T}_d/\lambda^2} \mathrm{S}_u^d f_d\, du\Big)\Phi, \\ [1 \otimes \mathrm{A}(\mathrm{S}^d_{t/\lambda^2} g), \mathrm{U}^{(\lambda)}(t/\lambda^2)] \\ \times v \otimes \mathrm{W}\Big(\lambda \sum_{d=1}^{\mathrm{N}} \int_{\mathrm{S}'_d/\lambda^2}^{\mathrm{T}'_d/\lambda^2} \mathrm{S}_u^d f'_d\, du\Big)\Phi \Big\rangle \quad (4.11) \end{aligned}$$

Expressing of $\mathrm{U}^{(\lambda)}(t/\lambda^2)$ in terms of the itrated series, the n-th order term in λ is

$$\begin{aligned} (-i)^n \lambda^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \\ \times \Big\langle -\mathrm{D}_d u \otimes \mathrm{W}\Big(\lambda \sum_{d=1}^{\mathrm{N}} \int_{\mathrm{S}_d/\lambda^2}^{\mathrm{T}_d/\lambda^2} \mathrm{S}_u^d f_d\, du\Big)\Phi, \\ [1 \otimes \mathrm{A}(\mathrm{S}^d_{t/\lambda^2} g), \mathrm{V}_g(t_1) \ldots \mathrm{V}_g(t_n)] \\ \times v \otimes \mathrm{W}\Big(\lambda \sum_{d=1}^{\mathrm{N}} \int_{\mathrm{S}'_d/\lambda^2}^{\mathrm{T}'_d/\lambda^2} \mathrm{S}_u^d f'_d\, du\Big)\Phi \Big\rangle \quad (4.12) \end{aligned}$$

But the limit, as $\lambda \to 0$, of (4.12) is the same as the limit, as $\lambda \to 0$, of

$$(-i)^n \lambda^{n-2} \int_0^t dt_1 \int_0^{t_1/\lambda^2} dt_2 \dots \int_0^{t_{n-1}} dt_n \times \Big\langle -D_d u \otimes W\Big(\lambda \sum_{d=1}^{N} \int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d \, du\Big)\Phi, [1 \otimes A(S^d_{t/\lambda^2} g), V_g(t_1/\lambda^2)] V_g(t_2) \dots V_g(t_n) \times v \otimes W\Big(\lambda \sum_{d=1}^{N} \int_{S'_d/\lambda^2}^{T'_d/\lambda^2} S_u^d f'_d \, du\Big)\Phi\Big\rangle \quad (4.13)$$

because all the other terms will contain scalar products of the form

$$\langle S^d_{t/\lambda^2} g, S^{d'}_{t_j/\lambda^2} g \rangle$$

with $j \geqq 2$ which are terms of type II in the sense of Lemma (3.3), and therefore vanishing in the limit $\lambda \to 0$. Moreover the commutator in the expression (4.13) is equal to

$$\sum_{d'=1}^{N} \langle S^d_{t/\lambda^2} g, S^{d'}_{t_1/\lambda^2} g \rangle \quad (4.14)$$

and since

$$\langle S^d_{t/\lambda^2} g, S^{d'}_{t_1/\lambda^2} g \rangle = \langle S^d_{t/\lambda^2} g, S^{d}_{t_1/\lambda^2} g \rangle . e^{i(\omega_d - \omega_{d'}) t_1/\lambda^2}$$

the Riemann-Lebesgue Lemma implies that only the term with $d = d'$ survives. So the limit, as $\lambda \to 0$, of integral of (4.11) is equal to the limit, as $\lambda \to 0$, of

$$(-i)^n \lambda^{n-2} \int_0^t dt_1 \int_0^{t_1/\lambda^2} dt_2 \dots \int_0^{t_{n-1}} dt_n \times \Big\langle -D_d u \otimes W\Big(\lambda \sum_{d=1}^{N} \int_{S_d/\lambda^2}^{T_d/\lambda^2} S_u^d f_d \, du\Big)\Phi, \langle S^d_{t/\lambda^2} g, S^{d}_{t_1/\lambda^2} g \rangle . e^{i(\omega_d - \omega_{d'} t_1)/\lambda^2} V_g(t_2) \dots V_g(t_n) \times v \otimes W\Big(\lambda \sum_{d=1}^{N} \int_{S'_d/\lambda^2}^{T'_d/\lambda^2} S_u^d f'_d \, du\Big)\Phi\Big\rangle \quad (4.15)$$

But this limit is exactly of the same type as that considered in (4.7) and therefore it exists. The estimate (4.4) guarantes that, in the expansion of (4.11) with the iterated series, one can go to the limit term by term. Resumming the limit (4.15) in n and using Theorem (3.5), one gets that the limit of (4.15) is equal to

$$\sum_{d=1}^{N} \int_0^t ds \, (\chi_{[S'_d, T'_d]}(s) (g \,|\, f'_d)_d . \langle -D_d u, G(s) \rangle + (g \,|\, g)_{d,-} . \langle -D_d^+ D_d u, G(s) \rangle) \quad (4.16)$$

Summing up:

THEOREM (4.1). — *For each* $u\in H_0$, *the map* $t\mapsto\langle u, G(t)\rangle$, *defined by* (4.1), *is differentiable almost everywhere and satisfies the integral equation*

$$\begin{aligned}\langle u, G(t)\rangle = \langle u, v\rangle \Big\langle & W\Big(\bigoplus_{d=1}^{N}\chi_{[S_d, T_d]}\otimes f_d\Big)\Psi_1\otimes\ldots\otimes\Psi_N,\\ & W\Big(\bigoplus_{d=1}^{N}\chi_{[S'_d, T'_d]}\otimes f'_d\Big)\Psi_1\otimes\ldots\otimes\Psi_N\Big\rangle\\ & +\sum_{d=1}^{N}\int_0^t ds\,[\chi_{[S_d, T_d]}(s)(f_d|g)_d.\langle D_d^+ u, G(s)\rangle\\ & +\chi_{[S'_d, T'_d]}(s)(g|f'_d)_d.\langle -D_d u, G(s)\rangle\\ & +(g|g)_{d,-}.\langle -D_d^+ D_d u, G(s)\rangle]\end{aligned}\qquad(4.17)$$

Proof. — Putting together (4.5), (4.6), (4.6*a*), (4.8), (4.9), (4.10), (4.16) we obtain:

$$\begin{aligned}\langle u, G(t)\rangle &= \lim_{\lambda\to 0}\langle u, G_\lambda(t)\rangle\\ &= \lim_{\lambda\to 0}\Big(\langle u, G_\lambda(0)\rangle + \int_0^t ds\frac{d}{ds}\langle u, G_\lambda(s)\rangle\Big)\\ &= \langle u, v\rangle\Big\langle W\Big(\bigoplus_{d=1}^{N}\chi_{[S_d, T_d]}\otimes f_d\Big)\Psi_1\otimes\ldots\otimes\Psi_N,\\ &\qquad W\Big(\bigoplus_{d=1}^{N}\chi_{[S'_d, T'_d]}\otimes f'_d\Big)\Psi_1\otimes\ldots\otimes\Psi_N\Big\rangle\\ &\qquad+\sum_{d=1}^{N}\int_0^t ds\,[\chi_{[S_d, T_d]}(s)(f_d|g)_d.\langle D_d^+ u, G(s)\rangle\\ &\qquad+\chi_{[S'_d, T'_d]}(s)(g|f'_d)_d.\langle -D_d u, G(s)\rangle\\ &\qquad+(g|g)_{d,-}.\langle -D_d^+ D_d u, G(s)\rangle]\end{aligned}\qquad(4.18)$$

From the above we can easily deduce the proof of our main result

Proof of Theorem (1.2). — For each $N\in\mathbf{N}$, g, $\{f_d\}_{d=1}^N$, $\{f'_d\}_{d=1}^N\subset K$, $\{S_d, T_d, S'_d, T'_d\}_{d=1}^N\subset\mathbf{R}$, $t\geqq 0$, u, $v\in H_0$, denote

$$\begin{aligned}F(u, t) := \Big\langle & u\otimes\bigotimes_{d=1}^{N} W(\chi_{[S_d, T_d]}\otimes f_d)\Psi_1\otimes\ldots\otimes\Psi_N,\\ & U(t)v\otimes\bigotimes_{d=1}^{N} W(\chi_{[S'_d, T'_d]}\otimes f'_d)\Psi_1\otimes\ldots\otimes\Psi_N\Big\rangle\end{aligned}\qquad(4.19)$$

It is clear that $F(u, t)$ satisfies (4.17) the equation with the same initial condition. Hence by the uniqueness of the solution of the q. s. d. e. (4.17), we have

$$F(u, t) = \langle u, G(t)\rangle,\qquad t\geqq 0\qquad(4.20)$$

which, because of (4.19) and (4.2 a), implies the thesis.

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