**Unitarity conditions for the**

**renormalized square of white noise**

by

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Abstract

The formal unitarity conditions for stochastic equations driven by the renormalized square of white noise are shown to hold rigorously in the framework of sesquilinear forms on the Fock space.

# 1 Introduction

Let $L^2_{\text{sym}}(\mathbb{R}^n)$ denote the space of square integrable functions on $\mathbb{R}^n$ which are symmetric under permutation of their arguments, and let $F := \bigoplus_{n=0}^{\infty} L^2_{\text{sym}}(\mathbb{R}^n)$ where: if $\psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in F$, then $\psi^{(0)} \in \mathbb{C}$, $\psi^{(n)} \in L^2_{\text{sym}}(\mathbb{R}^n)$, and

$$\|\psi\|^2 := |\psi(0)|^2 + \sum_{n=1}^{\infty} \int_{\mathbb{R}^n} |\psi^{(n)}(s_1, \ldots, s_n)|^2 ds_1 \ldots ds_n < +\infty$$

Denote by $S \subset L^2(\mathbb{R}^n)$ the Schwartz space of smooth functions decreasing at infinity faster than any polynomial and let

$$D := \left\{ \psi \in F/\psi^{(n)} \in S, \sum_{n=1}^{\infty} n \|\psi^{(n)}\|^2 < \infty \right\}$$

Following [5] for each $t \in \mathbb{R}$ define the linear operator valued distribution $b^+(t)$ by

$$(b^+(t)\psi)^{(n)} := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \delta(t - s_i)\psi^{(n-1)}(s_1, \ldots, \hat{s}_i, \ldots, s_n)$$

Then $B(t) := \int_0^t b(s)ds$, $B^+(t) := \int_0^t b^+(s)ds$ and

$$N(t) := \int_0^t b^+(s)b(s)ds$$

are, for each $t$, operators acting on $D$ which correspond to the "annihilation", "creation", and "number" operators respectively of linear quantum stochastic calculus (cf. [10]).

The "white noise functionals" $b$ and $b^+$ satisfy the commutation relation $[b(t), b^+(s)] = \delta(t - s)$, or more general $[b(t), b^+(s)] = \gamma \cdot \delta(t - s)$, where $\gamma \in (0, +\infty)$ is the variance of the quantum Brownian motion defined by $B$ and $B^+$ and $\delta$ is the delta function (cf. [5], [6]).

Linear quantum stochastic calculus dealt with evolutions driven by the stochastic differentials $dB(t) := b(t)dt$, $dB^+(t) := b^+(t)dt$, and $dN(t) := b^+(t)b(t)dt$. However, related to a problem arising from quantum optics, a need appeared for the inclusion of differentials with respect to the square of the white noise functionals $b(t)$ and $b^+(t)$, namely $dB_2(t) := b(t)^2 dt$, and $dB_2^+(t) = b^+(t)^2 dt$ (cf. [7]).

The "renormalized Itô table" corresponding to $dt$, $dB$, $dB^+$, $dB_2$, $dB_2^+$ and $dN$ was shown in [6] to be

| | $dt$ | $dB$ | $dB^+$ | $dB_2$ | $dB_2^+$ | $dN$ |
|---|---|---|---|---|---|---|
| $dt$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $dB$ | 0 | 0 | $\gamma dt$ | 0 | $2\gamma dB^+$ | $\gamma dB$ |
| $dB^+$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $dB_2$ | 0 | 0 | $2\gamma dB$ | 0 | $4\gamma dN$ | $2\gamma dB_2$ |
| $dB_2^+$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $dN$ | 0 | 0 | $\gamma dB^+$ | 0 | $2\gamma dB_2^+$ | $\gamma dN$ |

(1)

Since $L^2_{\text{sym}}(\mathbb{R}^n) = L^2_{\text{syn}}(R)^{\otimes n}$ we can identify $F$ with the symmetric (Boson) Fock space over $S$.

If $\psi = \{(n!)^{-1/2} f^{\otimes n}\}$ we denote $\psi$ by $\psi(f)$.

If the elements of $S$ are defined on $[0, +\infty)$ we denote the Fock space by $\Gamma(S_+)$. As usual (cf. [6]) we couple $\Gamma(S_+)$ with an "initial" Hilbert space $H_0$ and we define $H := H_0 \otimes \Gamma(S_+)$.

Let $B(H_0)$ denote the space of bounded linear operators from $H_0$ to itself and let $B(H)$ be defined similarly for $H$. Early attempts were made in [3] and [4] to study and give meaning to stochastic evolutions of the form

$$dU(t) = [A_1 dt + A_2 dB(t) + A_3 dB^+(t) + A_4 dB_2(t) + A_5 dB_2^+(t) + A_6 dN(t)]U(t) \tag{2}$$

$$U(0) = I \ , \quad 0 \leq t \leq T < +\infty$$

where $A_i \in B(H)$ for $i = 1, 2, \ldots, 6$ and $A_i = \hat{A}_i \otimes Id$ where $\hat{A}_i \in B(H_0)$, $Id$ is the identity on $\Gamma(S_+)$ and $I$ is the identity on $H$. Formal unitarity conditions for the "solution" $U = \{U(t)/0 \leq t \leq T\}$ of (2) were also obtained in [3].

In this paper we show that the framework of sesquilinear forms on Hilbert space enables us to give a rigorous meaning to (2) and to prove the unitarity of the solution (in the sense of sesquilinear forms).

In what follows we will use the notations:

$$N_1(t) = t \cdot 1 \tag{3}$$

$$N_2(t) = B(t)$$

$$N_3(t) = B^+(t)$$

$$N_4(t) = B_2(t)$$

$$N_5(t) = B_2^+(t$$
$$N_6(t) = N(t)$$

and

$$N_1^* = N_1 \ , \quad N_2^* = N_3 \ , \quad N_4^* = N_5 \ , \quad N_6^* = N_6 \qquad (4)$$

# 2 Stochastic integrals with respect to second order white noises

Let $t \in [0, T]$ with $T < +\infty$. For a sesquilinear form $U(t)$ defined on the linear span $E$ of the vectors of the form

$$u \otimes \psi(g) \ : \ u \in H_0 \ , \ g \in S_+$$

we will use the symbolic notation

$$U(t)(u \otimes \psi(f), v \otimes \psi(g)) =: \langle u \otimes \psi(f), \ U(t)v \otimes \psi(g)\rangle$$

In the following all the forms will be considered on this domain. The *identity form*, $U(t) = 1$, is defined by the condition, for all vectors in the form domain:

$$\langle u \otimes \psi(f), \ U(t)v \otimes \psi(g)\rangle = \langle u \otimes \psi(f), v \otimes \psi(g)\rangle_H$$

where the right hand side denotes the scalar product in $H$. If there exists a sesquilinear form $U^*(t)$ satisfying

$$\overline{U^*(t)[u \otimes \psi(f), \quad v \otimes \psi(g)]} = U(t)(u \otimes \psi(f), v \otimes \psi(g))$$

where $\overline{x}$ is the complex conjugate of $x$, we say that $U^*(t)$ is the adjoint of $U(t)$ and we use the notation

$$\langle u \otimes \psi(f), \ U(t)v \otimes \psi(g)\rangle = \langle U^*(t)u \otimes \psi(f), v \otimes \psi(g)\rangle$$

**Definition 1** *Let be given, for each $t \in [0,T]$, a sesquilinear form $U(t)$ and, for $i = 1, 2, \ldots, 6$, let $A_i \in \mathcal{B}(H_o)$. The stochastic integral*

$$\sum_{i=1}^{6} \int_0^t U(s) A_i(s) dN_i(s)$$

*is the sesquilinear form, defined on $E$ by:*

$$\sum_{i=1}^{6} \int_0^t U(s) A_i dN_i(s)(u \otimes \psi(f)\ , \quad v \otimes \psi(g)) := \sum_{i=1}^{6} \int_0^t \rho_i(s) ds U(s)(u \otimes \psi(f)$$

$$A_i v \otimes \psi(g)) := \sum_{i=1}^{6} \int_0^t \rho_i(s) ds \langle u \otimes \psi(f), U(s) A_i v \otimes \psi(g) \rangle \tag{5}$$

*where*

$$\rho_i = \begin{cases} 1 & i{=}1 \\ \underline{g} & i{=}2 \\ \overline{f} & i{=}3 \\ \underline{g}^2 & i{=}4 \\ \overline{f}^2 & i{=}5 \\ \overline{f} g & i{=}6 \end{cases} \tag{6}$$

*The adjoint stochastic integral of $\sum_{i=1}^{6} \int_0^t U(s) A_i dN_i(s)$ is the sesquilinear form*

$$\sum_{i=1}^{6} \int_0^t A_i^* U^*(s) dN_i^*(s)$$

*defined on $E$ by:*

$$\sum_{i=1}^{6} \int_0^t A_i^* U^*(s) dN_i^*(s)(u \otimes \psi(f), v \otimes \psi(g)) := \sum_{i=1}^{6} \int_0^t \sigma_i(s) ds U^*(s)(A_i u \otimes \psi(f) \tag{7}$$

$$v \otimes \psi(g)) := \sum_{i=1}^{6} \int_0^t \sigma_i(s) ds \langle A_i u \otimes \psi(f), U^*(s) v \otimes \psi(g) \rangle$$

*where* $A_i^*$ *is the adjoint operator of* $A_i$ *and*

$$\sigma_i = \begin{cases} 1 & i=1 \\ \overline{f} & i=2 \\ g & i=3 \\ \overline{f}^2 & i=4 \\ g^2 & i=5 \\ \overline{f}g & i=6 \end{cases} \tag{8}$$

*In order for (5) and (7) to make sense we assume that*

$$\int_0^T |U(t)(u \otimes \psi(f), v \otimes \psi(g))|^2 dt = \int_0^T |\langle u \otimes \psi(f), U(t) v \otimes \psi(g)\rangle|^2 dt < +\infty \tag{9}$$

*and*

$$\int_0^T |U^*(t)(u \otimes \psi(f), v \otimes \psi(g))|^2 dt = \int_0^T |\langle u \otimes \psi(f), U^*(t) v \otimes \psi(g)\rangle|^2 dt < +\infty \tag{10}$$

*for all* $u$*,* $v \in H_0$ *and* $f$*,* $g \in S_+$ *such that*

$$\int_0^T |\rho_i(t)|^2 dt < +\infty \ , \quad \int_0^T |\sigma_i(t)|^2 dt < +\infty \tag{11}$$

*for* $i = 1, 2, \dots, 6$.

Condition (11) is surely satistied if

$$|f(s)| \leq 1 \qquad ; \qquad |g(s)| \leq 1$$

for $s \in [0, T]$ and in the following we shall assume that all the test functions of the exponential vectors satisfy this condition.

**Definition (2.2)** For each $t \in [0, T]$ define a sequence $U_n(t)$, $n = -1, 0, \dots, +\infty$, of sesquilinear forms on $E$ by

$$U_{-1}(t) := 0 \tag{12}$$

$$U_n(t) := 1 \ + \ \sum_{i=1}^{6} \int_0^t U_{n-1}(s) A_i dN_i(s) \qquad ; \ n = 0, \dots, +\infty \tag{13}$$

Then we define the adjoint sequence $U_n^*(t)$ by

$$U_{-1}^*(t) := 0 \tag{14}$$

$$U_n^*(t) := 1 \ + \ \sum_{i=1}^{6} \int_0^t A_i^* U_{n-1}^*(s) dN_i^*(s) \qquad ; \ n = 0, \ldots, +\infty \tag{15}$$

**Proposition (2.3)** The complex valued sequences of functions

$$t \mapsto U_n(t)(u \otimes \psi(f) \ , \quad v \otimes \psi(g)) =: \langle u \otimes \psi(f), \ U_n(t) v \otimes \psi(g) \rangle \tag{16}$$

$$t \mapsto U_n^*(t)(u \otimes \psi(f) \ , \quad v \otimes \psi(g)) =: \langle u \otimes \psi(f), \ U_n^*(t) v \otimes \psi(g) \rangle \tag{17}$$

converge uniformly on $[0, T]$, as $n \to +\infty$ for all $u, v \in H_0$ and $f, g \in S_+$. **Proof**. By (13) and (5), for $n \geq 1$

$$U_n(t)(u \otimes \psi(f) \ , \quad v \otimes \psi(g)) =: \langle u \otimes \psi(f), \ U_n(t) v \otimes \psi(g) \rangle =$$

$$\langle u \otimes \psi(f), \ v \otimes \psi(g) \rangle + \sum_{k=1}^{n} \sum_{i_1, \ldots, i_k = 1}^{6} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} dt_1 \ldots dt_k \rho_{i_1}(t_1) \ldots \rho_{i_k}(t_k)$$
$$\langle u \otimes \psi(f), A_{i_k} \ldots A_{i_1} v \otimes \psi(g) \rangle \tag{18}$$

where $t_o = t$. Thus

$$| \ \langle u \otimes \psi(f), \ U_n(t) v \otimes \psi(g) \rangle - \langle u \otimes \psi(f), \ U_{n-1}(t) v \otimes \psi(g) \rangle \ | =$$

$$= | \sum_{i_1, \ldots, i_n = 1}^{6} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_1 \ldots dt_n \rho_{i_1}(t_1) \ldots \rho_{i_n}(t_n) \langle u \otimes \psi(f), A_{i_n} \ldots A_{i_1} v \otimes \psi(g) \rangle \ |$$

$$\leq \sum_{i_1, \ldots, i_n = 1}^{6} \frac{t^n}{n!} \parallel A_{i_1} \parallel \ldots \parallel A_{i_n} \parallel \parallel u \otimes \psi(f) \parallel \ \parallel \ v \otimes \psi(g) \parallel$$

$$\leq \frac{(6t)^n}{n!} (\max_{1 \leq \lambda \leq 6} \parallel A_{i_\lambda} \parallel)^n \parallel u \otimes \psi(f) \parallel \ \parallel \ v \otimes \psi(g) \parallel \tag{19}$$

so that

$$\sum_{n=1}^{\infty} | \ \langle u \otimes \psi(f), \ U_n(t) v \otimes \psi(g) \rangle - \langle u \otimes \psi(f), \ U_{n-1}(t) v \otimes \psi(g) \rangle \ | \leq$$

$$\exp 6t(max_{1\leq\lambda\leq 6} \parallel A_{i_\lambda} \parallel) \parallel u\otimes\psi(f) \parallel \parallel \ v\otimes\psi(g) \parallel \tag{20}$$

and from this the uniform convergence on $[0,T]$, as $n\to+\infty$, of the sequence $\langle u\otimes\psi(f),\ U_n(t)v\otimes\psi(g)\rangle$ follows. A similar argument holds for $U_n^*(t)$.

Let, for each $t\in[0,T]$, the sesquilinear forms $U(t)$ and $U^*(t)$ be defined on $E$ by

$$U(t)(u\otimes\psi(f)\ ,\quad v\otimes\psi(g)) := \lim_{n\to\infty} U_n(t)(u\otimes\psi(f)\ ,\quad v\otimes\psi(g)) \tag{21}$$

$$U^*(t)(u\otimes\psi(f)\ ,\quad v\otimes\psi(g)) := \lim_{n\to\infty} U_n^*(t)(u\otimes\psi(f)\ ,\quad v\otimes\psi(g)) \tag{22}$$

Then, in the sense of equality of sesquilinear forms

$$U(t) = 1\ +\ \sum_{i=1}^{6}\int_0^t U(s)A_i dN_i(s) \tag{23}$$

$$U^*(t) = 1\ +\ \sum_{i=1}^{6}\int_0^t A_i^* U^*(s) dN_i^*(s) \tag{24}$$

where the right hand sides of (23) and (24) are defined by (5) and (7) respectively. **Proof**.

$$\begin{aligned} &\mid \left[U(t) - \left(1\ +\ \sum_{i=1}^{6}\int_0^t U(s)A_i dN_i(s)\right)\right] \\ &(u\otimes\psi(f)\ ,\ v\otimes\psi(g)) \leq\mid [U(t)-U_n(t)]\,(u\otimes\psi(f)\ ,\ v\otimes\psi(g)) \mid + \\ &+\mid \left[\left(1\ +\ \sum_{i=1}^{6}\int_0^t U(s)A_i dN_i(s)\right)\right] - \left[\left(1\ +\ \sum_{i=1}^{6}\int_0^t U_{n-1}(s)A_i dN_i(s)\right)\right] \\ &(u\otimes\psi(f)\ ,\ v\otimes\psi(g)) \mid\leq\mid [U(t)-U_n(t)]\,(u\otimes\psi(f)\ ,\ v\otimes\psi(g)) \mid + \\ &+\sum_{i=1}^{6}\int_0^t ds\rho_i(s) \mid [U(s)\ -\ U_{n-1}(s)]\,(u\otimes\psi(f)\ ,\ A_i v\otimes\psi(g)) \mid \end{aligned} \tag{25}$$

and this tends to zero as $n\to\infty$ in (25), because the uniform convergence in $t$ allows to pass to the limit under the integral sign. The proof of (24) is similar.

# 3 The renormalized product of stochastic integrals

**Definition 3.1**. For each $t \in [0,T]$ let $U_n(t)$ and $U_m(t)$ be defined as in (12) and (13). For $n,\, m = 0,1,2,\dots$ we define the renormalized product of $U_n(t)$ and $U_m(t)$ to be the sesquilinear form $\langle U_n(t), U_m(t)\rangle$ defined on $E$ by the recursion

$$\langle U_n(t), xU_m(t)\rangle (u\otimes\psi(f), v\otimes\psi(g)) :=$$

$$\langle U_0(t), xU_0(t)\rangle (u\otimes\psi(f), v\otimes\psi(g)) + \int_0^t \Big[\sum_{i=1}^{6} \sigma_i(s)\cdot \langle U_{n-1}(s), xU_m(s)\rangle (A_i u\otimes\psi(f), v\otimes(g)) +$$

$$+\sum_{j=1}^{6} \rho_j(s)\cdot \langle U_n(s), xU_{m-1}(s)\rangle (u\otimes\psi(f), A_j v\otimes\psi(g)) +$$

$$+\sum_{i,j=1}^{6} \omega_{ij}^{\gamma}(s) \langle U_{n-1}(s), xU_{m-1}(s)\rangle (A_i u\otimes\psi(f), A_j v\otimes\psi(g))\Big] ds \qquad (26)$$

where $x$ is any bounded operator in the initial space (which we identify with $x\otimes 1$) and

$$\langle U_{-1}(t), xU_n(t)\rangle = \langle U_n(t), xU_{-1}(t)\rangle := 0 \quad n,m = -1,0,1,\dots \qquad (27)$$

$$\langle U_0(t), xU_0(t)\rangle := \langle I, xI\rangle := x \qquad (28)$$

$$\langle U_n(t), xU_0(t)\rangle := \langle U_n(t), xI\rangle := U_n^*(t)x \qquad n = 1,2,\dots \qquad (29)$$

$$\langle U_0(t), xU_m(t)\rangle := \langle I, xU_m(t)\rangle := xU_m(t) \quad m = 1,2,\dots \qquad (30)$$

$\sigma_i$, $\rho_j$ are as in (6) and (8) and

$$(\omega_{ij}^{\gamma}(s))_{i,j=1}^{6} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & 2\gamma\overline{f}(s) & \gamma g(s) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\gamma g(s) & 0 & 4\gamma\overline{f}(s)g(s) & 2\gamma g(s)^2 \\ 0 & 0 & \gamma\overline{f}(s) & 0 & 2\gamma\overline{f}(s)^2 & \gamma\overline{f}(s)g(s) \end{bmatrix} \qquad (31)$$

**Remarks**. Using (13), (26) can be written as

$$\langle\sum_{i=1}^{6}\int_0^t U_n(s)A_i dN_i(s), x\sum_{j=1}^{6}\int_0^t U_m(s)A_j dN_j(s)\rangle(u\otimes\psi(f), v\otimes\psi(g)) :=$$

$$:= \int_0^t \Big[\sum_{i=1}^{6}\sigma_i(s)\langle U_n(s), x\sum_{j=1}^{6}\int_0^s U_m(z)A_j dN_j(z)\rangle(A_i u\otimes\psi(f), v\otimes\psi(g))+$$

$$+\sum_{j=1}^{6}\rho_j(s)\langle\sum_{j=1}^{6}\int_0^s U_n(z)A_i dN_i(z), xU_m(s)\rangle(u\otimes\psi(f), A_j v\otimes\psi(g))+$$

$$+\sum_{i,j=1}^{6}\omega_{ij}^{\gamma}(s)\langle U_n(s), xU_m(s)\rangle(A_i u\otimes\psi(f), A_j v\otimes\psi(g))\Big]ds \tag{32}$$

which can be thought of as the definition of the renormalized product of the two stochastic integrals. Frequently, in the following we shall omit the symbol $x$ from the notation of the quadratic form $\langle U(t), xU(t)\rangle$.

The recursive scheme underlying (26) is

$$\begin{bmatrix} \langle U_0, U_0\rangle & \langle U_0, U_1\rangle \ldots \langle U_0, U_m\rangle \ldots \\ \langle U_1, U_0\rangle & \langle U_1, U_1\rangle \ldots \langle U_1, U_m\rangle \ldots \\ \vdots & \\ \langle U_{n-1}, U_0\rangle & \ldots \langle U_{n-1}, U_{m-1}\rangle \langle U_{n-1}, U_m\rangle \ldots \\ \langle U_n, U_0\rangle & \ldots \langle U_n, U_{m-1}\rangle \langle U_n, U_m\rangle \ldots \end{bmatrix} \tag{33}$$

where the first two rows and the first column are computed with the use of (27), (28), (29) and (30), and for $n \geq 2$ and $m \geq 1$ the entry $\langle U_n, U_m\rangle$ depends on $\langle U_{n-1}, U_m\rangle$, $\langle U_n, U_{m-1}\rangle$ and $\langle U_{n-1}, U_{m-1}\rangle$ only.

**Definition 3.2**. If $F$ is a sesquilinear form on $H \times H$ and $K \in B(H_0)$ then $KF$ and $FK$ are the sesquilinear forms defined on $E$ by

$$KF(u\otimes\psi(f), v\otimes\psi(g)) := F(K^* u\otimes\psi(f), v\otimes\psi(g)) \tag{34}$$

$$FK(u\otimes\psi(f), v\otimes\psi(g)) := F(u\otimes\psi(f), Kv\otimes\psi(g)) \tag{35}$$

**Definition 3.3**. Let $K$, $L \in B(H_0)$ and $t \in [0, T]$. We define the *renormalized product* of the sesquilinear forms $KU_n^*(t)$ and $LU_n^*(t)$ to be the sesquilinear form defined for $n$, $m = 0, 1, 2, \ldots$ by the recursion

$$\langle KU_n^*(t), LU_m^*(t)\rangle(u\otimes\psi(f), v\otimes\psi(g)) := \langle K, L\rangle(u\otimes\psi(f), v\otimes\psi(g))+$$

$$\int_0^t \Big[\sum_{i=1}^{6} \rho_i(s)\langle KA_i^* U_{n-1}^*(s), LU_m^*(s)\rangle(u\otimes\psi(f), v\otimes\psi(g))+$$

$$+\sum_{j=1}^{6} \sigma_j(s)\langle KU_n^*(s), LA_j^* U_{m-1}^*(s)\rangle[u\otimes\psi(f), v\otimes\psi(g))+$$

$$+\sum_{i,j=1}^{6} \omega_{ij}^{\gamma}(s)\langle KA_i^* U_{n-1}^*(s), LA_j^* U_{m-1}^*(s)\rangle(u\otimes\psi(f), v\otimes\psi(g))\Big] ds \qquad (36)$$

where

$$\langle KU_{\rho-1}^*(t), LU_m^*(t)\rangle = \langle KU_n^*(t), LU_{-1}^*(t)\rangle = 0 \quad n,m=-1,0,1,\dots \qquad (37)$$

$$\langle KU_n^*(t), L\rangle := U_n(t)K^*L \qquad n=0,1,2,\dots \qquad (38)$$

$$\langle K, LU_m^*(t)\rangle := K^*LU_m^*(t) \qquad n=0,1,2,\dots \qquad (39)$$

$\sigma_j$, $p_j$ and $\omega_{ij}^{\gamma}$ are as in Definition 3.1.

**Remarks**. For $K=L=I$ (36) can be written as

$$\langle\sum_{i=1}^{6}\int_0^t A_i^* U_i^*(s) dN_i^*(s), \sum_{j=1}^{6}\int_0^t A_j^* U_m^*(s) dN_j^*(s)\rangle(u\otimes\psi(f), v\otimes\psi(g)) :=$$

$$\Big\{\int_0^t \Big[\sum_{i=1}^{6} \rho_i(s)\langle A_i^* U_n^*(s), \sum_{j=1}^{6}\int_0^t A_j^* U_m^*(s) dN_j^*(s)\rangle+$$

$$+\sum_{j=1}^{6}\sigma_j(s)\langle\sum_{i=1}^{6}\int_0^t A_i^* U_n^*(s) dN_i^*(s), A_j^* U_m^*(s)\rangle+$$

$$+\sum_{i,j=1}^{6}\omega_{ij}^{\gamma}(s)\langle A_i^* U_n^*(s), A_j^* U_m^*(s)\rangle\Big] ds\Big\}(u\otimes\psi(f), v\otimes\psi(g)) \qquad (40)$$

We remark also that Definitions 3.1 and 3.3 have the renormalized Itô table (1) built in.

In fact one arrives at (26), (40) and then also to (26) and (36) by assuming that everything works as in the linear noise case and then use the renormalized Itô table (1) to compute the matrix element of the product of two stochastic integrals.

**Proposition 3.4**. Let $t \in [0,T]$, $u, v \in H_0$, $f$, $g \in S_+$ with $|f|$, $|g| \le 1$. For $n$, $m = 0, 1, \ldots$ define

$$\mathcal{E}_{n,m}(t) := |\langle U_n(t) - U_{n-1}(t), U_m(t)\rangle| \tag{41}$$

Then

$$\mathcal{E}_{n,m}(t)(u\otimes\psi(f), v\otimes\psi(g)) \le \left[\frac{W^{n+m}t^{\frac{n+m}{2}}}{\sqrt{n!m!}} + \frac{L^{n+m}t^n}{n!}\right] \|u\otimes\psi(f)\|\,\|v\otimes\psi(g)\| \tag{42}$$

where

$$W := 1 + 24 \max_{1\le i\le 6}\|A_i\|T^{1/2} + 4\sqrt{2}\gamma\left(6\max_{1\le i\le 6}\|A_i\|\right)^2 \tag{43}$$

and

$$L := (1+T)\left(1 + 6\max_{1\le i\le 6}\|A_i\|\right) + 4\gamma\left(6\max_{1\le i\le 6}\|A_i\|\right)^2 \tag{44}$$

**Proof**. We will prove (42) by using double induction on $n$, $m$ as follows: consider the matrix

$$\begin{bmatrix} \mathcal{E}_{0,0} & \mathcal{E}_{0,1} & \ldots & & \mathcal{E}_{0,m} \\ \mathcal{E}_{1,0} & \mathcal{E}_{1,1} & \ldots & & \mathcal{E}_{1,n} \\ \mathcal{E}_{2,0} & \mathcal{E}_{2,1} & \ldots & & \mathcal{E}_{2,m} \\ \vdots & \vdots & & & \mathcal{E}_{2,m} \\ \mathcal{E}_{n-1,0} & \mathcal{E}_{n-1,1} & \ldots & \mathcal{E}_{n-1,m-1} & \mathcal{E}_{n-1,m} \\ \mathcal{E}_{n,0} & \mathcal{E}_{n,1} & \ldots & \mathcal{E}_{n,m-1} & \mathcal{E}_{n,m} \end{bmatrix} \tag{45}$$

We first show that (42) holds for the first row and the first column. Then we assume that (42) holds for $\mathcal{E}_{n-1,m-1}$, $\mathcal{E}_{n-1,m}$, $\mathcal{E}_{n,m-1}$ and we show that it is also valid for $\mathcal{E}_{n,m}$.

The idea behind this inductive scheme lies in the inequality

$$\mathcal{E}_{n,m}(t)(u\otimes\psi(f), v\otimes\psi(g)) \le \int_0^t \Big[\sum_{i=1}^{6}\mathcal{E}_{n-1,m}(s)(A_iu\otimes\psi(f), v\otimes\psi(g)) +$$

$$+\sum_{j=1}^{6}\mathcal{E}_{n,m-1}(s)(u\otimes\psi(f), A_jv\otimes\psi(g)) + 4\gamma\sum_{i,j=1}^{6}\mathcal{E}_{n-1,m-1}(s)(A_iu\otimes\psi(f), A_jv\otimes\psi(g))\Big]ds \tag{46}$$

which is a direct consequence of (26) for $n,\ m = 1, 2, \ldots$. For $m = 0$ and $n = 0, 1, \ldots$ by (19), (27), (28) and (29)

$$\begin{aligned}
\mathcal{E}_{n,0}(t)(u \otimes \psi(f), v \otimes \psi(g)) &= |\langle U_n(t) - U_{n-1}(t), I\rangle (u \otimes \psi(f), v \otimes \psi(g))| \quad (47)\\
&\leq \frac{(6 \max_{1 \leq i \leq 6} \|A_i\|)^n t^n}{n!} \|u \otimes \psi(f)\| \, \|v \otimes \psi(g)\| \quad (48)\\
&\leq \frac{L^n t^n}{n!} \|u \otimes \psi(f)\| \, \|v \otimes \psi(g)\| \quad (49)
\end{aligned}$$

since $6 \max_{1 \leq i \leq 6} \|A_i\| < L$, so (42) holds. For $n = 0$ (42) reduces to

$$\mathcal{E}_{0,m}(t)(u \otimes \psi(f), v \otimes \psi(g)) \leq \left[\frac{W^m t^{m/2}}{\sqrt{m!}} + L^n\right] \|u \otimes \psi(f)\| \, \|v \otimes \psi(g)\| \quad (50)$$

which we will prove by induction on $m = 0, 1, 2, \ldots$.

For $m = 0$,

$$\mathcal{E}_{0,0}(t)(u \otimes \psi(f), v \otimes \psi(g)) = |\langle U_0(t) - U_{-1}(t), U_0(t)\rangle (u \otimes \psi(f), v \otimes \psi(g))|$$

$$= |\langle I - 0, I\rangle (u \otimes \psi(f), v \otimes \psi(g))| = |\langle u \otimes \psi(f), v \otimes \psi(g)\rangle_H| \leq \|u \otimes \psi(f)\| \, \|v \otimes \psi(g)\|$$

$$\leq \left[\frac{W^0}{0!} t^0 + L^0\right] \|u \otimes \psi(f)\| \, \|v \otimes \psi(g)\|$$

so (47) is valid. Suppose that (51) holds for $m - 1$. We will show that it also holds for $m$. By (5) and (13) we have

$$\mathcal{E}_{0,m}(u \otimes \psi(f), v \otimes \psi(g)) = |\langle U_0(t) - U_{-1}(t), U_m(t)\rangle (u \otimes \psi(f), v \otimes \psi(g))|$$

$$= |\langle I, U_m(t)\rangle (u \otimes \psi(f), v \otimes \psi(g))| = |U_m(t)(u \otimes \psi(f), v \otimes \psi(g))|$$

$$= \left| \left[ I + \sum_{i=1}^{6} \int_0^t U_{m-1}(s) A_i dN_i(s) \right] (u \otimes \psi(f), v \otimes \psi(g)) \right|$$

$$= \left| \langle u \otimes \psi(f), v \otimes \psi(g)\rangle_H + \sum_{i=1}^{6} \int_0^t U_{m-1} A_i dN_i(s)(u \otimes \psi(f), v \otimes \psi(g)) \right|$$

$$= \left| \langle u \otimes \psi(f), v \otimes \psi(g)\rangle_H + \sum_{i=1}^{6} \int_0^t \rho_i(s) U_{m-1}(s)(u \otimes \psi(f), A_i v \otimes \psi(g)) ds \right|$$

$$\leq \|u \otimes \psi(f)\| \, \|v \otimes \psi(g)\| + \sum_{i=1}^{6} \int_0^t |U_{m-1}(s)(u \otimes \psi(f), A_i v \otimes \psi(g))| ds$$

$$= \|u \otimes \psi(f)\| \; \|v \otimes \psi(g)\| + \sum_{i=1}^{6} \int_0^t \mathcal{E}_{0,m-1}(s)(u \otimes \psi(f), A_i v \otimes \psi(g) ds$$

$$\leq \|u\otimes\psi(f)\| \; \|v\otimes\psi(g)\| + \sum_{i=1}^{6} \int_0^t \left[ \frac{W^{n-1}}{\sqrt{(m-1)!}} S^{\frac{m-1}{2}} + L^{m-1} \right] ds \|u\otimes\psi(f)\| \; \|A_i v\otimes\psi(g)\|$$

$$\leq \|u\otimes\psi(f)\| \; \|v\otimes\psi(g)\| + 6 \max_{1\leq i\leq 6} \|A_i\| \left( \frac{W^{m-1}}{\sqrt{(m-1)!}} \frac{t^{m+\frac{1}{2}}}{\frac{m+1}{2}} + L^{m-1} t \right) \|u\otimes\psi(f)\| \; \|v\otimes\psi(g)\|$$

$$\leq \left[ 6 \max_{1\leq i\leq 6} \|A_i\| \frac{W^{m-1}}{\sqrt{m!}} t^m 2T^{1/2} + 1 + TL^{m-1} \right] \|u \otimes \psi(f)\| \; \|u \otimes \psi(g)\|$$

$$\leq \left[ 2T^{1/2} 6 \max_{1\leq i\leq 6} \|A_i\| \frac{W^{m-1}}{\sqrt{n!}} t^m + (1+T) L^{m-1} \right] \|u \otimes \psi(f)\| \; \|v \otimes \psi(g)\|$$

$$\leq \left[ \frac{W^M}{\sqrt{m!}} t^m + L^m \right] \|u \otimes \psi(f)\| \|v \otimes \psi(g)\|$$

thus proving (51). We have thus proved (42) to hold true for the first row and column of matrix (45). We will now show that if it holds true for $\mathcal{E}_{n-1,m-1}$, $\mathcal{E}_{n-1,m}$, and $\mathcal{E}_{n,m-1}$ then it does for $\mathcal{E}_{n,m}$ as well, thus completing our induction. In view of (46) we have

$$\mathcal{E}_{n,m}(t)(u\otimes\psi(f), v\otimes\psi(g)) \leq \int_0^t \Big[ \sum_{i=1}^{6} \left( \frac{W^{n+m-1} S^{\frac{n+m-1}{2}}}{\sqrt{(n-1)!n!}} + \frac{L^{n+m-1} s^{n-1}}{(n-1)!} \right) \|A_i\|$$

$$+ \sum_{i=1}^{6} \left( \frac{W^{n+m-1} s^{\frac{n+m-1}{2}}}{\sqrt{n!(m-1)!}} + \frac{L^{n+m-1} s^n}{n!} \right) \|A_j\| + 4\gamma \sum_{i,j=1}^{6} \Big( \frac{W^{n+m-2} s^{\frac{n+m-2}{2}}}{\sqrt{(n-1)!(m-1)!}} +$$

$$\frac{L^{n+m-2} s^{n-1}}{(n-1)!} \Big) \|A_i\| \; \|A_j\| \Big] ds \|u \otimes \psi(f)\| v \otimes \psi(g)\|$$

$$\leq \Big\{ \int_0^t \Big[ 6 \max_{1\leq i\leq 6} \left( \frac{W^{n+m-1} s^{\frac{n+m-1}{2}}}{\sqrt{(n-1)!m!}} + \frac{W^{n+m-1} s^{\frac{n+m-1}{2}}}{\sqrt{n!(m-1)!}} \right) +$$

$$4\gamma \left( 6 \max_{1\leq i\leq 6} \|A_i\| \right)^2 \frac{W^{n+m-2} s^{\frac{n+m-2}{2}}}{\sqrt{(n-1)!(m-1)!}} \Big] ds +$$

$$\int_0^t \Big[6 \max_{1\le i\le 6} \|A_i\| \left(\frac{L^{n+m-1}s^{n-1}}{(n-1)!} + \frac{L^{n+m-1}s^n}{n!}\right) + 4\gamma \left(6 \max_{1\le i\le 6} \|A_i\|\right)^2$$

$$\cdot \frac{L^{n+m-2}s^{n-1}}{(n-1)!}\Big] ds \Big\} \|u\otimes\psi(f)\|\ \|v\otimes\psi(g)\|$$

$$\le \Big\{ \Big[6 \max_{1\le i\le 6} \|A_i\| \left( \frac{W^{n+m-1}t^{\frac{n+m-1}{2}}}{\sqrt{(n-1)!m!}(n+m+1)/2} + \frac{W^{n+m-1}t^{\frac{n+m+1}{2}}}{\sqrt{n!(m-1)!}(n+m+1)/2} \right)$$

$$+4\gamma \left(6 \max_{1\le i\le 6} \|A_i\|\right)^2 \frac{W^{n+m-2}t^{\frac{n+m}{2}}}{\sqrt{(n-1)!(m-1)!}(n+m)/2}\Big] + \Big[6 \max_{1\le i\le 6} \|A_i\| \Big( \frac{L^{n+m-1}t^n}{n!} +$$

$$\frac{L^{n+m-1}t^nT}{n!}\Big) + 4\gamma \left(6 \max_{1\le i\le 6} \|A_i\|\right)^2 \frac{L^{n+m-2}t^n}{n!}\Big]\Big\} \|u\otimes\psi(f)\|\ \|v\otimes\psi(g)\|$$

$$\le \Big\{ \Big[6 \max_{1\le i\le 6} \|A_i\| \left( \frac{W^{n+m-1}t^{\frac{n+m}{2}}2T^{1/2}}{\sqrt{n!m!}} + \frac{W^{n+m-1}t^{\frac{n+m}{2}}2T^{1/2}}{\sqrt{n!m!}} \right)$$

$$+4\gamma \left(6 \max_{1\le i\le 6} \|A_i\|\right)^2 \frac{W^{n+m-1}t^{\frac{n+m}{2}}\sqrt{2}}{\sqrt{n!m!}}\Big] + \Big[6 \max_{1\le i\le 6} \|A_i\| \Big( \frac{L^{n+m-1}t^n}{n!} +$$

$$\frac{TL^{n+m-1}t^n}{n!}\Big) + 4\gamma \left(6 \max_{1\le i\le 6} \|A_i\|\right)^2 \frac{L^{n+m-1}t^n}{n!}\Big]\Big\} \|u\otimes\psi(f)\|v\otimes\psi(g)\|$$

(where we have used the inequality $\frac{2}{n+m} \le \frac{\sqrt{2}}{\sqrt{nm}}$)

$$\le \Big\{ \left[24 \max_{1\le i\le 6} \|A_i\|T^{1/2} + 4\sqrt{2}\gamma \left(6 \max_{1\le i\le 6} \|A_i\|\right)^2\right] \frac{W^{n+m-1}t^{\frac{n+m}{2}}}{\sqrt{n!m!}} +$$

$$\left[6 \max_{1\le i\le 6} \|A_i\|(1+T) + 4\gamma \left(6 \max_{1\le i\le 6} \|A_i\|\right)^2\right] \frac{L^{n+m-1}t^n}{n!}\Big\} \|u\otimes\psi(f)\|\|v\otimes\psi(g)\|$$

$$\le \left[\frac{W^{n+m}t^{\frac{n+m}{2}}}{\sqrt{n!m!}} + \frac{L^{n+m}t^n}{n!}\right] \|u\otimes\psi(f)\|\ \|v\otimes\psi(g)\|$$

**Remarks**. Since $\langle U_n(t), U_m(t) - U_{m-1}(t)\rangle(u\otimes\psi(f), v\otimes\psi(g))$ is the complex–conjugate of $\langle U_m(t) - U_{m-1}(t), U_n(t)\rangle(v\otimes\psi(g), u\otimes\psi(f))$ it follows that $J_{n,m}(t) := |\langle U_n(t), U_m(t) - U_{m-1}(t)\rangle|$ satisfies

$$J_{n,m}(t)(u\otimes\psi(f), v\otimes\psi(g)) \leq \left[\frac{W^{n+m}t^{\frac{n+m}{2}}}{\sqrt{n!m!}} + \frac{L^{n+m}t^n}{n!}\right] \|u\otimes\psi(f)\| \, \|v\otimes\psi(g)\| \tag{51}$$

where $W$, $L$ are as in (43) and (44).

Moreover, a method similar to that used in the proof of Proposition 3.4 can be employed to show that if $k$, $\Lambda \in B(H_0)$ then $\mathcal{E}^*_{n,m}(t) := |\langle k(U^*_n(t) - U^*_{n-1}(t)), \Lambda U^*_m(t)\rangle|$ and $J^*_{n,m}(t) := |\langle kU^*_n(t), \Lambda(U^*_m(t) - U^*_{m-1}(t))\rangle|$ satisfy the inequalities

$$\mathcal{E}^*_{n,m}(t)(u\otimes\psi(f), v\otimes\psi(g)) \leq \left[\frac{M^{n+m}t^{\frac{n+m}{2}}}{\sqrt{n!m!}} + \frac{C^{n+m}t^n}{n!}\right] \|u\otimes\psi(f)\| \, \|v\otimes\psi(g)\| \tag{52}$$

and

$$J^*_{n,m}(t)(u\otimes\psi(f), v\otimes\psi(g)) \leq \left[\frac{M^{n+m}t^{\frac{n+m}{2}}}{\sqrt{n!m!}} + \frac{C^{n+m}t^n}{n!}\right] \|u\otimes\psi(f)\| \, \|v\otimes\psi(g)\| \tag{53}$$

for suitable constants $M$, $C > 0$ which are independent of $t$, $n$, $m$.

**Proposition 3.5**. For each $k = 0, 1, 2, \ldots$ and $t \in [0, T]$ the sequences $\{a_{n,k}(t)\}_{n=k}^{+\infty}$ and $\{b_{n,k}(t)\}_{n=k}^{+\infty}$ of sesquilinear forms defined pointwise on $E$ by

$$a_{n,k}(t)(u \otimes \psi(f), v \otimes \psi(g)) := \langle U_{n-k}(t), U_n(t)\rangle(u \otimes \psi(f), v \otimes \psi(g)) \tag{54}$$

and

$$b_{n,k}(t)(u \otimes \psi(f), v \otimes \psi(g)) := \langle U_n(t), U_{n-k}(t)\rangle(u \otimes \psi(f), v \otimes \psi(g)) \tag{55}$$

converge as $n \to +\infty$, uniformly with respect to $t \in [0, T]$ and $k$.

Moreover, if

$$a_{\infty,k}(t) := \lim_n a_{n,k}(t) \tag{56}$$

and

$$b_{\infty,k}(t) := \lim_n b_{n,k}(t) \tag{57}$$

then

$$a_{\infty,k_1}(t) = a_{\infty,k_2}(t) := a_\infty(t) \tag{58}$$

and

$$b_{\infty,k_1}(t) = b_{\infty,k_2}(t) := b_\infty(t) \tag{59}$$

for all $k_1$, $k_2 = 0, 1, 2, \ldots$ and $t \in [0, T]$ and also

$$a_\infty(t) = b_\infty(t) := \lambda(t) \tag{60}$$

**Proof**.

$$|a_{n,k}(t)(u \otimes \psi(f), v \otimes \psi(g)) - a_{n-1,k}(t)(u \otimes \psi(f), v \otimes \psi(g))| =$$

$$|\langle U_{n-k}(t), U_n(t)\rangle(u\otimes\psi(f), v\otimes\psi(g)) - \langle U_{n-1-k}(t), U_{n-1}(t)\rangle(u\otimes\psi(f), v\otimes\psi(g))|$$

$$\leq |\langle U_{n-k}(t), U_n(t)\rangle(u\otimes\psi(f), v\otimes\psi(g)) - \langle U_{n-1-k}(t)U_n(t)\rangle(u\otimes\psi(f), v\otimes\psi(g))| +$$

$$|\langle U_{n-1-k}(t), U_n(t)\rangle(u\otimes\psi(f), v\otimes\psi(g)) - \langle U_{n-1-k}(t), U_{n-1}(t)\rangle(u\otimes\psi(f), v\otimes\psi(g))|$$

$$= |\langle U_{n-k}(t) - U_{n-k-1}(t), U_n(t)\rangle(u \otimes \psi(f), v \otimes \psi(g))| +$$

$$|\langle U_{n-k-1}(t), U_n(t) - U_{n-1}(t)\rangle(u \otimes \psi(f), v \otimes \psi(g))| \tag{61}$$

By (42) for $m = n$

$$|\langle U_{n-k}(t) - U_{n-k-1}(t), U_n(t)\rangle(u \otimes \psi(f), v \otimes \psi(g))| \leq$$

$$\left[\frac{W^{2n-k} t^{\frac{2n-k}{2}}}{\sqrt{(n-k)!n!}} + \frac{L^{2n-k} t^n}{n!}\right] \|u \otimes \psi(f)\| \; \|v \otimes \psi(g)\| \tag{62}$$

Distinguishing the cases $t \in [0, 1]$ and $t \in (1, T]$ we obtain

$$|\langle U_{n-k}(t) - U_{n-k-1}(t), U_n(t)\rangle(u \otimes \psi(f), v \otimes \psi(g))| \leq$$

$$\leq \left[\frac{W^{2n}(\max(1,T))^n}{\sqrt{n!}} + \frac{L^{2n}(\max(1,T))^n}{n!}\right] \|u \otimes \psi(f)\| \; \|v \otimes \psi(g)\| \tag{63}$$

and so

$$|\langle U_{n-k}(t) - U_{n-k-1}(t), U_n(t)\rangle(u \otimes \psi(f), v \otimes \psi(g))| \to 0 \tag{64}$$

as $n \to +\infty$ uniformly with respect to $t$ and $k$.

Similarly, from (51) with $m = n - k - 1$, we obtain that

$$|\langle U_{n-k-1}(t), U_n(t) - U_{n-1}(t)\rangle(u \otimes \psi(f), v \otimes \psi(g))| \to 0$$

as $n \to +\infty$ uniformly with respect to $t$ and $k$. Thus, by (61), $\{a_{n,k}(t)(u \otimes \psi(f), v \otimes \psi(g))\}_{n=k}^{+\infty}$ converges as $n \to +\infty$, uniformly with respect to $t$ and $k$. Similarly, $\{b_{n,k}(t)(u \otimes \psi(f), v \otimes \psi(g))\}_{n=k}^{+\infty}$ converges as $n \to +\infty$, uniformly with respect to $t$ and $k$. For $k = 0, 1, 2, \ldots$

$$|a_{\infty,k+1}(t)(u \otimes \psi(f), v \otimes \psi(g)) - a_{\infty,k}(t)(u \otimes \psi(f), v \otimes \psi(g))|$$

$$\leq |a_{\infty,k+1}(t)(u \otimes \psi(f), v \otimes \psi(g)) - a_{n,k+1}(t)(u \otimes \psi(f), v \otimes \psi(g))| +$$
$$|a_{n,k+1}(t)(u \otimes \psi(f), v \otimes \psi(g)) - a_{n,k}(t)(u \otimes \psi(f), v \otimes \psi(g))| +$$
$$|a_{n,k}(t)(u \otimes \psi(f), v \otimes \psi(g)) - a_{\infty,k}(t)(u \otimes \psi(f), v \otimes \psi(g))|$$

$\to 0$ as $n \to +\infty$ by (55) and the fact that

$$|a_{n,k+1}(t)(u \otimes \psi(f), v \otimes \psi(g)) - a_{n,k}(t)(u \otimes \psi(f), v \otimes \psi(g))|$$
$$= |\langle U_{n-k}(t) - U_{n-k-1}(t), U_n(t)\rangle(u \otimes \psi(f), v \otimes \psi(g))|$$

$\to 0$ as $n \to +\infty$ by (64). Thus

$$a_{\infty,k+1}(t) = a_{\infty,k}(t) \tag{65}$$

for all $k = 0, 1, 2, \ldots$ and $t \in [0, T]$ which implies (58). The proof of (59) is similar.

Since $a_{n,0}(t) = b_{n,0}(t)$ and by (56), (57) $a_\infty(t) = \lim_n a_{n,0}(t)$ and $b_\infty(t) = \lim_n b_{n,0}(t)$ it follows that $a_\infty(t) = b_\infty(t)$.

**Remarks**. For $k = 0$ the sequences defined in (54) and (55) correspond to the main diagonal of (33). For $k = 1, 2, 3, \ldots$ they correspond to diagonals above and below the main diagonal of (33) respectively. The analogue of (33) for the renormalized products $\langle KU_n^*(t), LU_m^*(t)\rangle$ where $K$, $L \in B(H_0)$ and $n$, $m = 0, 1, 2, \ldots$ is the matrix

$$\begin{bmatrix} \langle KU_0^*, LU_0^*\rangle \langle KU_0^*, LU_1^*\rangle & \ldots & \langle KU_0^*, LU_m^*\rangle \ldots \\ \langle KU_1^*, LU_0^*\rangle \langle KU_1^*, LU_1^*\rangle & \ldots & \langle KU_1^*, LU_m^*\rangle \ldots \\ \vdots & & \\ \langle KU_{n-1}^*, LU_0^*\rangle & \ldots & \langle KU_{n-1}^*, LU_m^*\rangle \ldots \\ \langle KU_n^*, LU_0^*\rangle & \ldots & \langle KU_n^*, LU_m^*\rangle \ldots \end{bmatrix} \tag{66}$$

In a way similar to that used in the proof of Proposition 3.5 we can show that all left–to–right diagonal sequences $\{\langle KU_{n-k}^*(t), LU_n^*(t)\rangle(u \otimes \psi(f), v \otimes \psi(g))\}_{n=k}^{+\infty}$ and

$$\{\langle KU_n^*(t), LU_{n-k}^*(t)\rangle(u \otimes \psi(f), v \otimes \psi(g))\}_{n=k}^{+\infty}\ , \quad k = 0, 1, 2, \ldots$$

converge as $n \to \infty$ to the same limit and the convergence is uniform with respect to $k$ and $t \in [0, T]$.

**Theorem 3.6**. For all $u,\ v \in H_0,\ f, g \in S_+$ with $|f| \leq 1$ and $|g| \leq 1$, $t \in [0, T]$, and $K,\ L \in B(H_0)$ the double sequences of sesquilinear forms

$$\{\langle U_n(t), U_m(t)\rangle(u \otimes \psi(f), v \otimes \psi(g))\}_{n,m=0}^{+\infty} \tag{67}$$

and

$$\{\langle KU_n^*(t), LU_m^*(t)\rangle(u \otimes \psi(f), v \otimes \psi(g))\}_{n,m=0}^{+\infty} \tag{68}$$

converge as $n,\ m \to +\infty$ and the convergence is uniform with respect to $t \in [0, T]$.

**Proof**. We will show that the sequences defined by (67) and (68) converge to the common limit of the diagonal sequences of the infinite matrices (33) and (66) respectively.

Let $\lambda(t)(u \otimes \psi(f), v \otimes \psi(g))$ be the common limit, uniform in $t$ and $k$, of the left–to–right diagonal sequences of (33). By the uniformity of the limit, with respect to $k$, in Proposition 3.5, every neighborhood of $\lambda(t)\ (u \otimes \psi(f), v \otimes \psi(g))$ will eventually contain all terms of all diagonal sequences of (33). Therefore it will eventually contain all terms of the double sequence (67) which therefore converges to $\lambda(t)\ (u \otimes \psi(f), v \otimes \psi(g))$ and the convergence is, in view of Proposition 3.5, uniform with respect to $t$. The rest of the proof is similar.

**Definition 3.7**. Let $K,\ L \in B(H_0)$ and let $U_n(t)$ and $U_m(t)$ be as in Definition 2.2.

The sesquilinear form $\langle U_n(t)K, U_n(t)L\rangle$ is defined for $n,\ m = 0, 1, \ldots$ on $E$ by

$$\langle U_n(t)K, U_m(t)L\rangle(u \otimes \psi(f), v \otimes \psi(g)) := \langle U_n(t), U_m(t)\rangle(Ku \otimes \psi(f), Lv \otimes \psi(g)) \tag{69}$$

**Definition 3.8**. In view of Theorem 3.6 and Definition 3.7, for $t \in [0, T]$ and $K,\ L \in B(H_0)$ we define sesquilinear forms $\langle U(t)K, U(t)L\rangle$ and $\langle KU^*(t), LU^*(t)\rangle$ on $E$ by

$$\langle U(t)K, U(t)L\rangle(u \otimes \psi(f), v \otimes \psi(g)) := \lim_{n,m} \langle U_n(t)K, U_m(t)L\rangle(u \otimes \psi(f), v \otimes \psi(g))$$

$$:= \lim_{n,m} \langle U_n(t), U_m(t)\rangle(Ku \otimes \psi(f), Lv \otimes \psi(g)) = \tag{70}$$

and

$$\langle KU^*(t), LU^*(t)\rangle(u\otimes\psi(f), v\otimes\psi(g)) := \lim_{n,m}\langle KU_n^*(t), LU_m^*(t)\rangle(u\otimes\psi(f), v\otimes\psi(g)) \tag{71}$$

# 4 Unitarity in the sense of sesquilinear forms

**Theorem 4.1**. If the coefficients $A_i \in B(H_0)$, $i = 1, \ldots, 6$ satisfy the conditions

$$\begin{aligned}
A_1 + A_1^* + \gamma A_2 A_2^* &= 0\\
A_1^* + A_1 + \gamma A_3^* A_3 &= 0\\
A_2 + A_3^* + 2\gamma A_4 A_2^* + \gamma A_2 A_6^* &= 0\\
A_3^* + A_2 + 2\gamma A_5^* A_3 + \gamma A_3^* A_6 &= 0\\
A_4 + A_5^* + 2\gamma A_4 A_6^* &= 0\\
A_5^* + A_4 + 2\gamma A_5^* A_6 &= 0\\
A_6 + A_6^* + 4\gamma A_4 A_4^* + \gamma A_6 A_6^* &= 0\\
A_6^* + A_6 + 4\gamma A_5^* A_5 + \gamma A_6^* A_6 &= 0
\end{aligned} \tag{72}$$

then for all $t \in [0,T]$, $u$, $v \in H_0$, and $f$, $g \in S_+$ with $|f|$, $|g| \leq 1$

$$\langle U(t), U(t)\rangle(u\otimes\psi(f), v\otimes\psi(g)) = \langle I, I\rangle(u\otimes\psi(f), v\otimes\psi(g)) \tag{73}$$

and

$$\langle U^*(t), U^*(t)\rangle(u\otimes\psi(f), v\otimes\psi(g)) = \langle I, I\rangle(u\otimes\psi(f), v\otimes\psi(g)) \tag{74}$$

where the left hand sides of (73) and (74) are in the sense of Definition 3.8 and the right hand sides equal $\langle u\otimes\psi(f), v\otimes\psi(g)\rangle_H$, the inner product of $u\otimes\psi(f)$ and $v\otimes\psi(g)$.

**Proof**. To prove (74) we notice that using (71) and, i.e. the recurrent definition of $U_n^*$:

$$[\langle U^*(t), U^*(t)\rangle - \langle I, I\rangle](u\otimes\psi(f), v\otimes\psi(g)) =$$

$$\lim_n [\langle U_n^*(t), U_n^*(t)\rangle - \langle I, I\rangle](u\otimes\psi(f), v\otimes\psi(g)) =$$

$$= \lim_n \Big[\langle I, \sum_{i=1}^6 \int_0^t A_i^* U_{n-1}^*(s) dN_i^*(s)\rangle + \langle \sum_{i=1}^6 \int_0^t A_i^* U_{n-1}^*(s) dN_i^*(s), I\rangle$$

$$+ \langle \sum_{i=1}^6 \int_0^t A_i^* U_{n-1}^*(s) dN_i^*(s), \sum_{j=1}^6 \int_0^t A_j^* U_{n-1}^*(s) dN_j^*(s)\rangle\Big](u\otimes\psi(f), v\otimes\psi(g))$$

(by (15))

$$= \lim_n \Big\{\int_0^t \sum_{i=1}^6 \sigma_i(s)\langle I, A_i^* U_{n-1}^*(s)\rangle ds + \int_0^t \sum_{i=1}^6 \rho_i(s)\langle A_i^* U_{n-1}^*(s), I\rangle ds$$

$$+ \int_0^t \Big[\sum_{i=1}^6 \rho_i(s)(\langle A_i^* U_{n-1}^*(s), U_n^*(s)\rangle - \langle A_i^* U_{n-1}^*(s), I\rangle) +$$

$$\sum_{j=1}^6 \sigma_j(s)(\langle U_n^*(s), A_j^* U_{n-1}^*(s)\rangle - \langle I, A_j^* U_{n-1}^*(s)\rangle) +$$

$$\sum_{i,j=1}^6 \omega_{ij}^\gamma(s)\langle A_i^* U_{n-1}^*(s), A_j^* U_{n-1}^*(s)\rangle\Big] ds\Big\}(u\otimes\psi(f), v\otimes\psi(g))$$

(by (40) and (15))

$$= \lim_n \int_0^t \Big[\sum_{i=1}^6 \rho_i(s)\langle A_i^* U_{n-1}^*(s), U_n^*(s)\rangle + \sum_{i=1}^6 \sigma_i(s)\langle U_n^*(s), A_i^* U_{n-1}^*(s)\rangle$$

$$+ \sum_{i,j=1}^6 \omega_{ij}^\gamma(s)\langle A_i^* U_{n-1}^*(s), A_j^* U_{n-1}^*(s)\rangle\Big] ds\ (u\otimes\psi(f), v\otimes\psi(g))$$

$$= \int_0^t \Big[\sum_{i=1}^6 \rho_i(s)\langle A_i^* U^*(s), U^*(s)\rangle + \sum_{i=1}^6 \sigma_i(s)\langle U^*(s), A_i^* U^*(s)\rangle$$

$$+ \sum_{i,j=1}^6 \omega_{ij}^\gamma(s)\langle A_i^* U^*(s), A_j^* U^*(s)\rangle\Big] ds\ (u\otimes\psi(f), v\otimes\psi(g))$$

by dominated convergence, Theorem 3.6, and Definition 3.8 this is

$$= \int_0^t \langle U^*(s), \left[ \sum_{i=1}^{6} \rho_i(s) A_i + \sum_{i=1}^{6} \sigma_i(s) A_i^* + \sum_{i,j=1}^{6} \omega_{ij}^{\gamma}(s) A_i A_j^* \right] U^*(s) \rangle ds \, (u \otimes \psi(f), v \otimes \psi(g))$$

Let $C$ denote the quantity in brackets. We will show that $C = 0$. By the definition of $\sigma_i$, $\rho_i$ and $\omega_{ij}^{\gamma}$, after collecting terms we have

$$C = [A_1^* + A_1 + \gamma A_3^* A_3] + \overline{f}[A_2^* + A_3 + 2\gamma A_3^* A_5 + \gamma A_6^* A_3] +$$

$$g[A_3^* + A_2 + \gamma A_3^* A_6 + 2\gamma A_5^* A_3] + \overline{f}^2 [A_4^* + A_5 + 2\gamma A_6^* A_5] +$$

$$g^2 [A_5^* + A_4 + 2\gamma A_5^* A_6] + \overline{f} g [A_6^* + A_6 + 4\gamma A_5^* A_5 + \gamma A_6^* A_6] = 0$$

by (72), thus proving (74). To prove (73) we notice that, for $n \geq 1$ and $t \in [0, T]$, using (13) we have

$$\langle U_n(t) u \otimes \psi(f), U_n(t) v \otimes \psi(g) \rangle = \langle U_n(t), U_n(t) \rangle (u \otimes \psi(f), v \otimes \psi(g)) =$$

$$= \Big[ \langle I, I \rangle + \int_0^t \Big( \sum_{i=1}^{6} \sigma_i(s) \langle U_{n-1}(s) A_i, U_n(s) \rangle + \sum_{i=1}^{6} \rho_i(s) \langle U_n(s), U_{n-1}(s) A_i \rangle$$

$$+ \sum_{i,j=1}^{6} \omega_{ij}^{\gamma}(s) \langle U_{n-1}(s) A_i, U_{n-1}(s) A_j \rangle \Big) ds \Big] (u \otimes \psi(f), v \otimes \psi(g)) \qquad (75)$$

By (75)

$$\frac{d}{dt} \langle U_n(t) u \otimes \psi(f), U_n(t) v \otimes \psi(g) \rangle =$$

$$\Big[ \sum_{i=1}^{6} \sigma_i(t) \langle U_{n-1}(t) A_i, U_n(t) \rangle + \sum_{i=1}^{6} \rho_i(t) \langle U_n(t), U_{n-1}(t) A_i \rangle +$$

$$+ \sum_{i,j=1}^{6} \omega_{ij}^{\gamma}(t) \langle U_{n-1}(t) A_i, U_{n-1}(t) A_j \rangle \Big] (u \otimes \psi(f), v \otimes \psi(g)) \qquad (76)$$

By Theorem 3.6 the right hand side of (76) converges uniformly in $t$ as $n \to +\infty$, so the sequence $\{\langle U_n(t) u \otimes \psi(f), U_n(t) v \otimes \psi(g) \rangle\}_{n=0}^{\infty}$ and its time derivative converge uniformly with respect to $t$ as $n \to +\infty$. Thus we may let $n \to +\infty$ in (4.5) and interchange the limit and differentiation operations to obtain

$$\frac{d}{dt} \langle U(t), U(t) \rangle (u \otimes \psi(f), v \otimes \psi(g)) =$$

$$= \Big[\sum_{i=1}^{6} \sigma_i(t)\langle U(t)A_i, U(t)\rangle + \sum_{i=1}^{6} \rho_i(t)\langle U(t), U(t)A_i\rangle +$$

$$+ \sum_{i,j=1}^{6} \omega_{ij}^{\gamma}(t)\langle U(t)A_i, U(t)A_j\rangle\Big](u\otimes\psi(f), v\otimes\psi(g)) \qquad (77)$$

For $t \in [0,T]$ let $F(t) : H_0 \times H_0 \to \mathbb{C}$ be the sesquilinear form defined, for fixed $f$, $g \in S_+$ with $|f| \leq 1$ and $|g| \leq 1$, by

$$F(t)(u,v) := \langle U(t), U(t)\rangle(u\otimes\psi(f), v\otimes\psi(g)) = \langle U(t)u\otimes\psi(f), U(t)v\otimes\psi(g)\rangle \qquad (78)$$

Since

$$|F(t)(u,v)| = |\langle U(t), U(t)\rangle(u\otimes\psi(f), v\otimes\psi(g))| = \lim_n |\langle U_n(t), U_n(t)\rangle(u\otimes\psi(f), v\otimes\psi(g))|$$

$$= \lim_n \left|\left[\langle U_0(t), U_0(t)\rangle + \sum_{k=1}^{n}(\langle U_k(t), U_k(t)\rangle - \langle U_{k-1}(t), U_{k-1}(t)\rangle)\right](u\otimes\psi(f), v\otimes\psi(g))\right|$$

$$\leq |\langle u\otimes\psi(f), v\otimes\psi(g)\rangle_H| + \lim_n \sum_{k=1}^{n} |(\langle U_k(t), U_k(t)\rangle - \langle U_{k-1}(t), U_{k-1}(t)\rangle)(u\otimes\psi(f), v\otimes\psi(g))|$$

$$\leq \|u\otimes\psi(f)\|\|v\otimes\psi(g)\| + \lim_n \sum_{k=1}^{n}(|\langle U_k(t) - U_{k-1}(t), U_k(t)\rangle(u\otimes\psi(f), v\otimes\psi(g))| +$$

$$|\langle U_{k-1}(t), U_k(t) - U_{k-1}(t)\rangle(u\otimes\psi(f), v\otimes\psi(g))|)$$

$$\leq \|u\otimes\psi(f)\|\;\|v\otimes\psi(g)\| + \lim_n \sum_{k=1}^{n}\left[\left(\frac{W^{2k}t^k}{k!} + \frac{L^{2k}t^k}{k!}\right) + \left(\frac{W^{2k-1}t^{k-\frac{1}{2}}}{\sqrt{(k-1)!k!}} + \frac{L^{2k-1}t^k}{k!}\right)\right]$$

$$\cdot\|u\otimes\psi(f)\|\;\|v\otimes\psi(g)\|$$

(by (42) and (51))

$$\leq \left[1 + \lim_n \sum_{k=1}^{n}\left(\left(\frac{(W^2)^k t^k}{k!} + \frac{(L^2)^k t^k}{k!}\right) + \left(\frac{WT(W^2)^{k-1}t^{k-1}}{(k-1)!} + \frac{1}{L}\,\frac{(L^2)^k t^k}{k!}\right)\right)\right]$$

$$\cdot\|u\otimes\psi(f)\|\;\|v\otimes\psi(g)\|$$

$$\leq \left[1 + e^{W^2T} + e^{L^2T} + WTe^{W^2T} + \frac{1}{L}\,e^{L^2T}\right]\|\psi(f)\|\;\|\psi(g)\|\;\|u\|\cdot\|v\|$$

$$\leq \left[ \left( 1 + (1+WT)e^{W^2T} + \left(1+\frac{1}{L}\right) e^{L^2T} \right) e^{2T} \right] \|u\| \, \|v\| \qquad (79)$$

it follows that $F(t)$ is, for each $t \in [0,T]$ and $f$, $g \in S^+$ with $|f| \leq 1$ and $|g| \leq 1$, a bounded sesquilinear form on $H_0 \times H_0$. Thus there exists an operator $F(t) \in B(H_0)$, depending on $f$, $g$, such that for all $u$, $v \in H_0$

$$\langle u, F(t)v \rangle = F(t)(u,v) = \langle U(t), U(t) \rangle (u \otimes \psi(f), v \otimes \psi(g)) \qquad (80)$$

By (77) $F(t)$ satisfies the ordinary Banach space differential equation

$$\frac{dF}{dt} = \sum_{i=1}^{6} \sigma_i(t) A_i^* F(t) + \sum_{i=1}^{6} \rho_i(t) F(t) A_i + \sum_{i,j=1}^{6} \omega_{ij}^{\gamma}(t) A_i^* F(t) A_j \qquad (81)$$

with initial condition

$$F(0) = \langle \psi(f), \psi(g) \rangle I \qquad (82)$$

where $I$ is the identity operator in $B(H_0)$, since by (80)

$$\begin{aligned} \langle u, F(0)v \rangle &= \langle U(0), U(0) \rangle (u \otimes \psi(f), v \otimes \psi(g)) && (83) \\ &= \lim_n \langle U_n(0), U_n(0) \rangle (u \otimes \psi(f), v \otimes \psi(g)) = \langle I, I \rangle (u \otimes \psi(f), v \otimes \psi(g)) && (84) \\ &= \langle u \otimes \psi(f), v \otimes \psi(g) \rangle = \langle u, v \rangle \langle \psi(f), \psi(g) \rangle = \langle u, \langle \psi(f), \psi(g) \rangle I v \rangle && (85) \end{aligned}$$

We will show that

$$F(t) = \langle \psi(f), \psi(g) \rangle I \qquad (86)$$

is the unique solution of (81), (82). Clearly (82) is satisfied. Moreover

$$\frac{dF}{dt} = \frac{d}{dt} \left( \langle \psi(f), \psi(g) \rangle I \right) = 0 \qquad (87)$$

and

$$\sum_{i=1}^{6} \sigma_i(t) A_i^* F(t) + \sum_{i=1}^{6} \rho_i(t) F(t) A_i + \sum_{i,j=1}^{6} \omega_{ij}^{\gamma}(t) A_i^* F(t) A_j =$$

$$= \left[ \sum_{i=1}^{6} \sigma_i(t) A_i^* + \sum_{i=1}^{6} \rho_i(t) A_i + \sum_{i,j=1}^{6} \omega_{ij}^{\gamma}(t) A_i^* A_j \right] \langle \psi(f), \psi(g) \rangle$$

Let $D$ denote the quantity in brackets. As before, by (72), $D = 0$. Thus

$$\sum_{i=1}^{6} \sigma_i(t) A_i^* F(t) + \sum_{i=1}^{6} \rho_i(t) F(t) A_i + \sum_{i,j=1}^{6} \omega_{ij}^{\gamma}(t) A_i^* F(t) A_j = 0 \qquad (88)$$

By (87) and (88), (86) defines a solution of (81), (82). It is a well known result of classical theory that such a solution is unique. Thus, by (78), (80), and (86)

$$\langle U(t), U(t)\rangle(u \otimes \psi(f), v \otimes \psi(g)) = \langle u \otimes \psi(f), v \otimes \psi(g)\rangle_H \tag{89}$$

proving (73).

**Remark (1)**. It was shown in [1] that if

$$A_1 = iH - \frac{\gamma}{2} L^*L,\ A_2 = -L^*W,\ A_3 = L$$

$$A_4 = -\left(\frac{I - \text{ Re } W}{8\gamma^2}\right)^{1/2} MW,\ A_5 = M^*\left(\frac{I - \text{ Re } W}{8\gamma^2}\right)^{1/2}$$

and

$$A_6 = \frac{W - I}{2\gamma}$$

where $L$, $H$, $W$, $M \in B(H_0)$ with $H$ self–adjoint and $W$, $M$ unitary operators satisfying

$$L^*(I - W) + \sqrt{2}(I - \text{ Re } W)^{1/2} ML = 0 \tag{90}$$

then $A_1$, $A_2, \ldots, A_6$ satisfy the unitarity conditions (72).

**Remark (2)**. Our theory can be immediately extended to include coefficients of the form

$$\hat{A}_i(t) = \alpha_i(t) \cdot A_i \tag{91}$$

where $\alpha_i$ is a scalar function on $[0, T]$ and $A_i \in \mathcal{B}(H_0)$.

**Remark (3)**. Concerning the quantum stochastic differential equation

$$dU(t) = -i \cdot c(t)U(t)dt - ig(t)U(t)dB_2^+(t) - ig(t)U(t)dB_2(t) - i\omega(t)U(t)dN(t)$$

$$U(0) = I \tag{92}$$

where $c$, $g$, $w$ are complex–valued functions of $t$, which was derived in [7], we remark that unitarity conditions (72) are satisfied if

$$\begin{gathered} \text{Im } c = 0 \\ 2 \text{ Im } \omega + |g|^2 4\gamma + |\omega|^2\gamma = 0 \\ \text{Im } g + \gamma\bar{g}\omega = 0 \end{gathered} \tag{93}$$

# References


[1] Accardi L., *Quantum Stochastic Calculus*, In "Proceedings IV Vilnius Conference on Probability and Mathematical Statistics", VNU Science, 1986.

[2] Accardi L., *A mathematical theory of quantum noise*, in "Proceedings 1st World Congress of the Bernoulli Society" (Y. Prohorov and V. Sazonov, Eds.), Vol 1, pp. 427-444, VNU Science, 1987.

[3] Accardi, L., Boukas A., *Unitarity conditions for stochastic differential equations driven by non-linear quantum noise*.

[4] Accardi, L., Boukas A., *Stochastic evolutions driven by non-linear quantum noise*.

[5] Accardi L., Fagnola F., Quagebeur J.,*Representation free quantum stochastic calculus*, *J. Func. Anal.*, 104 (1992), 140-197.

[6] Accardi L., Lu Y.G., Obata N.,*Towards a non-linear extension of stochastic calculus,* Publications of the Research Institute for Mathematical Sciences, Kyoto, RIMS Kokyuroku 957, Obata N. (ed.), (1996), 1-15.

[7] Accardi L., Lu Y.G., Volovich I.:*White noise approach to stochastic calculus and nonlinear ItC tables,* submitted to: Nagoya Journal of Mathematics.

[8] Accardi L., Volovich I., *Quantum white noise with singular non linear interaction,* submitted.

[9] Fagnola F., *On quantum stochastic differential equations with unbounded coefficients,* Probab. Th. Rel. Fields 86, 501-516 (1990).

[10] Obata N., *Quantum stochastic differential equations in terms of quantum white noise,* Nonlinear Analysis, Theory, Methods and Applications, vol. 30, No. 1, pp. 279–290, 1997.

[11] Parthasarathy, K.R., *An Introduction to Quantum Stochastic calculus*, Birkhauser Boston Inc., 1992.

[12]