The Stochastic Limit of the Fröhlich Hamiltonian: Relations With the Quantum Hall Effect

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We propose a model of an approximatively two-dimensional electron gas in a uniform electric and magnetic field and interacting with a positive background through the Fröhlich Hamiltonian. We consider the stochastic limit of this model and we find the quantum Langevin equation and the generator of the master equation. This allows us to calculate the explicit form of the conductivity and the resistivity tensors and to deduce a *fine tuning condition* (FTC) between the electric and the magnetic fields. This condition shows that the *x*-component of the current is zero unless a certain quotient, involving the physical parameters, takes values in a finite set of physically meaningful rational numbers. We argue that this behavior is quite similar to that observed in the quantum Hall effect. We also show that, under some conditions on the form factors entering in the definition of the model, also the plateaux and the "almost" linear behavior of the Hall resistivity can be recovered. Our FTC does not distinguish between fractional and integer values.

KEY WORDS: stochastic limit; Fröhlich Hamiltonian.

1. INTRODUCTION

The Hamiltonian for the quantum Hall effect (QHE) is, see for instance Bagarello *et al.* (1993),

$$H^{(N)} = H_0^{(N)} + \lambda\big(H_c^{(N)} + H_B^{(N)}\big) \tag{1}$$

where $H_0^{(N)}$ is the Hamiltonian for the free N electrons, $H_c^{(N)}$ is the Coulomb interaction:

$$H_c^{(N)} = \frac{1}{2}\sum_{i\neq j}^{N}\frac{e^2}{|\underline{r}_i - \underline{r}_j|} \tag{2}$$

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and $H_B^{(N)}$ is the interaction of the charges with the positive uniform background.

In the present paper we consider a model defined by an Hamiltonian

$$H = H_{0,e} + H_{0,R} + \lambda H_{eb} \qquad (3)$$

which is obtained from the Hamiltonian (1) by introducing the following approximations (for a more precise description of H see the next section):

- the Coulomb background–background interaction is replaced by the free bosons Hamiltonian $H_{O,R,}$ (12);
- the Coulomb electron–electron and electron–background interaction is replaced by the Fröhlich Hamiltonian H_{eb} (16) which is only quadratic rather than quartic in the fermionic operators.

These are certainly strong approximations. However since, as explained in Strocchi (xxxx), from the Fröhlich Hamiltonian it is possible, with a canonical transformation, to recover a quartic interaction, one can say that the Fröhlich Hamiltonian describes an effective electron–electron interaction which may mimic at least some aspects of the original Coulomb interaction. From this point of view it seems natural to conjecture that some dynamical phenomena deduced from this Hamiltonian might have implications in the study of the real QHE. There exists a huge bibliography concerning QHE. Here we refer only to Chakraborty and Pietiläinen (1988) and, for a more recent review, to Girvin (1999).

This conjecture is supported by our main result, given by formulae (83) and (84) where we deduce, directly from the dynamics, and not from phenomenological arguments, an obstruction to the presence of a nonzero x-component of the current, which is quantized according to the values of a finite set of rational numbers. This result is, to our knowledge, new and the fact that such a mechanism can arise even in relatively idealized models of electrical conductivity, seems to be at least worth of some attention. More precisely we prove that the x-component of the mean value of the density current operator is necessarily zero unless a certain quotient ($\frac{2\pi e E}{m \omega^2 L_x}$, cf. (9) and (11) for the definition of these parameters), involving the magnitudes of the physical quantities defining the model, takes a rational value. This is what we call a *fine tuning condition* (FTC).

The rational numbers that appear in the FTC are quotients of the Bohr frequencies of the free single-electron Hamiltonian. It is quite reasonable to expect that, in a concrete physical situation, only a small number of these frequencies will play a relevant role for the scale of phenomena involved. In this approximation we can say that the x-component of the mean value of the density current operator is nonzero only in correspondence of a finite number of rational values of the fine tuning parameter. This feature will be discussed in Section 6.

The fine tuning condition strongly reminds the rational values of the filling factor for which the plateaux are observed in the real QHE. Again, in Section 6 we will relate these two facts.

The specification of the values of these physically relevant rational numbers and the comparison with those rational numbers which are experimentally measured in the QHE, requires a detailed analysis which will be done elsewhere.

We use the technique of the stochastic limit of quantum theory and we refer to the paper Accardi *et al.* (1996) for a synthetic description, to Accardi *et al.* (1999) for more recent results, to Accardi *et al.* (1990) and Accardi and Lu (1960) for mathematical details, and to Accardi *et al.* (2001) for a systematic exposition.

2. THE SINGLE-ELECTRON PROBLEM

In these notes we discuss a model of $N < \infty$ charged interacting particles concentrated around a two-dimensional layer contained in the (x, y)-plane and subjected to a uniform electric field $\underline{E} = E\hat{j}$, along y, and to an uniform magnetic field $\underline{B} = B\hat{k}$ along z.

The Hamiltonian for the free N electrons $H_0^{(N)}$, is the sum of N contributions:

$$H_0^{(N)} = \sum_{i=1}^{N} H_0(i) \tag{4}$$

where $H_0(i)$describes the minimal coupling of the ith electrons with the field:

$$H_0(i) = \frac{1}{2m} \left(\underline{p} + \frac{e}{c}\underline{A}(r_i) \right)^2 + e\underline{E} \cdot \underline{r}_i \tag{5}$$

To $H_0^{(N)}$ we still have to add the interaction with the background and, then, the free Hamiltonian for the background itself. This will be made in the following section.

We fix the Landau gauge $\underline{A} = -B(y, 0, 0)$. In this gauge the Hamiltonian becomes

$$H_0(i) = \frac{1}{2m} \left[\left(p_x - \frac{eB}{c}y \right)^2 + p_y^2 \right] + eEy \tag{6}$$

which, obeys the commutation rule $[p_x, H_0] = 0$. The solutions of the eigenvalue equation for the single-charge Hamiltonian (6)

$$H_0\psi_{np}(\underline{r}) = \varepsilon_{np}\psi_{np}(\underline{r}), \quad n \in \mathbf{N}, \, p \in \mathbf{Z} \tag{7}$$

(where the double index is due to the fact that, two quantum numbers are necessary to fix the eigenstate) are known, (Accardi *et al.*, 1990; Accardi and Lu, 1960), to be of the form: $\psi(\underline{r}) = Ce^{ikx}\varphi(y)$, where C is a normalization constant fixed by the geometry of the system. Using this factorization, the time-independent Schrödinger equation (7) can be rewritten as an harmonic oscillator equation

$$\left(\frac{1}{2m} p_y^2 + \frac{1}{2} m\omega^2(y - y_0)^2 \right) \varphi(y) = \varepsilon'\varphi(y) \tag{8}$$

depending on the parameters

$$\omega = \frac{eB}{mc}; \quad \varepsilon' = \varepsilon - \frac{\hbar^2 k^2}{2m} + \frac{1}{2}m\omega^2 y_0^2;$$

$$y_0 = \frac{1}{m\omega^2}(\hbar^2 k\omega - eE); \quad \varepsilon = \frac{k^2}{2m} - \frac{1}{2}m\omega^2 y_0^2 \tag{9}$$

where k is the momentum along the x-axis. If we require periodic boundary condition on x, $\psi(-L_x/2, y) = \psi(L_x/2, y)$, for almost all y, we also conclude that the momentum k along x, cannot take arbitrary values but must be quantized. In particular, if the system is infinitely extended along y, then all the possible, values of k are

$$k = \frac{2\pi}{L_x}p, \quad p \in \mathbf{Z} \tag{10}$$

Normalizing the wave functions in the strip $[-L_x/2, L_x/2] \times \mathbf{R}$, we finally get

$$\psi_{np}(\underline{r}) = \frac{e^{i\frac{2\pi px}{L_x}}}{\sqrt{L_x}}\varphi_n\left(y - y_0^{(p)}\right)$$

$$\varepsilon_{np} = \hbar\omega(n + 1/2) - \frac{eE}{2m\omega^2}\left(eE - \frac{h\hbar\omega\pi p}{L_x}\right) \tag{11}$$

where φ_n is the nth eigenstate of the one-dimensional harmonic oscillator, ω and y_0 are given by (9), and p is fixed by (10).

Equation (11) shows that the wave function $\psi_{np}(\underline{r})$ factorizes in a x-dependent part, which is labeled by the quantum number p, and a part, only depending on y, which is labeled by both n **and** p due to the presence of $y_0^{(p)}$ in the argument of the function φ_n.

It may be interesting to remark that when $E = 0$ the model collapses to the one of a simple harmonic oscillator, see Girvin (1999) and Bagarello *et al.*, (1993) for instance, and an infinite degeneracy in p of each Landau level (n fixed) appears. Following the usual terminology we will call *lowest Landau level* (LLL) the energy level corresponding to $n = 0$.

3. THE SECOND QUANTIZED MODEL

The Hamiltonian $H_0^{(N)}$ contains the interaction of the electrons with the electric and the magnetic field. In this paper we add the Fröhlich interaction of the electrons with a background bosonic field. The free Hamiltonian of the background boson field is

$$H_{0,R} = \int \omega(\underline{k})b^+(\underline{k})b(\underline{k})\,dk \tag{12}$$

where $\omega(\underline{k})$ is the dispersion for the free background. Its analytical form will be kept general in this paper.

The electron–background interaction is given here by the Fröhlich Hamiltonian (Strocchi, xxxx)

$$H_{eb} = \int \psi^\dagger(\underline{r})\psi(\underline{r})\tilde{F}(\underline{r} - \underline{r}')\phi(\underline{r}')\,dr\,dr' \tag{13}$$

where $\psi(\underline{r})$ and $\phi(\underline{r}')$ are respectively the electron and the bosonic fields, while $\tilde{F}$ is a form factor. Expanding $\phi(\underline{r}')$ in plane waves, $\psi(\underline{r})$ in terms of the eigenstates $\psi_\alpha(\underline{r})$, see (11), introducing the form factors

$$g_{\alpha\beta}(\underline{k}) := \frac{1}{\sqrt{(2\pi)^2}}\frac{\hat{V}_{\alpha\beta(\underline{k})}}{\sqrt{2\omega(\underline{k})}} \tag{14}$$

where

$$\hat{V}_{\alpha\beta'}(\underline{k}) := \int \overline{\psi_\alpha(\underline{r})}e^{i\underline{k}\cdot\underline{r}}\psi\beta'(\underline{r})\,dr \tag{15}$$

and taking $\tilde{F}(\underline{r}) = e^2\delta(\underline{r})$ (Strocchi, xxxx), we can write

$$H_{eb} = e^2\sum_{\alpha\beta} a_\alpha^+ a_\beta\big(b(g_{\alpha\beta}) + b^+(\overline{g_{\beta\alpha}})\big) \tag{16}$$

which is quadratic in the fermionic operators a_α, a_α^+ are fermionic operators satisfying

$$\{a_\alpha, a_\beta\} = \{a_\alpha^+, a_\beta^+\} = 0 \quad \{a_\alpha, a_\beta^+\} = \delta_{\alpha\beta} \tag{17}$$

The boson operators $b(\underline{k})$ satisfy the canonical commutation relations:

$$[b(\underline{k}), b^+(\underline{k}')] = \delta(\underline{k} - \underline{k}') \quad [b(\underline{k}), b(\underline{k}')] = [b^+(\underline{k}), b^+(\underline{k}')] = 0 \tag{18}$$

The form factors $g_{\alpha\beta}$ depend on the level indices (α, β). Notice that we have adopted here and in the following the simplifying notation for the quantum numbers $\alpha = (n_\alpha, p_\alpha)$ and that we have introduced the smeared operators

$$b(g_{\beta\alpha}) = \int dk\, b(\underline{k})g_{\beta\alpha}(\underline{k}) \tag{19}$$

In terms of the fermion operators, the free electron Hamiltonian (4) becomes

$$H_{0,e} = \sum_\alpha \varepsilon_\alpha a_\alpha^+ a_\alpha \tag{20}$$

where ε_α the are the single-electron energies, labeled by the pairs $\alpha = (n, p)$ as explained in formula (11).

Therefore, the total Hamiltonian is

$$H = H_{0,e} + H_{0,R} + \lambda H_{eb} = H_0 + \lambda H_{eb} \tag{21}$$

4. THE STOCHASTIC LIMIT OF THE MODEL

In this section we briefly outline how to apply the stochastic limit procedure to the model introduced above. The stochastic limit describes the dominating contribution to the dynamics in time scales of the order t/λ^2, where λ is the coupling constant. The *stochastic golden rule* is a prescription which, given a usual Hamiltonian equation, allows to write, with a few simple calculations, the Langevin and the master equation (Accardi *et al.*, 1990, 1996, 2001; Accardi and Lu, 1960). In this paper we will be mainly concerned with the master equation.

The starting point is the Hamiltonian (21) together with the commutation relations (17) and (18). Of course, the Fermi and the Bose operators commute. The interaction Hamiltonian H_{eb} for this model is given by (16) and the free Hamiltonian H_0 is given by (20), (12), and (21).

The time evolution of H_{eb}, in the interaction picture is then

$$
\begin{aligned}
H_{eb}(t) &= e^{iH_0t} H_{eb} e^{-iH_0t} \\
&= e^2 \sum_{\alpha\beta} a_\alpha^+ a_\beta \big(b \big(g_{\alpha\beta} e^{-it(\omega-\varepsilon_\alpha\beta)} \big) + b^+ \big(\bar{g}_{\beta\alpha} e^{it(\omega-\varepsilon_{\beta\alpha})} \big) \big)
\end{aligned}
\tag{22}
$$

where

$$
\varepsilon_{\alpha\beta} = \varepsilon_\alpha - \varepsilon_\beta
\tag{23}
$$

Therefore, the Schrödinger equation in interaction representation is

$$
\partial_t U_t^{(\lambda)} = -i\lambda H_{eb}(t) U_t^{(\lambda)}
\tag{24}
$$

After the time rescaling $t \to t/\lambda^2$, Eq. (24) becomes

$$
\partial_t U_{t/\lambda^2}^{(\lambda)} = -\frac{i}{\lambda} H_{eb}(t/\lambda^2) U_{t/\lambda^2}^{(\lambda)}
\tag{25}
$$

whose integral form is

$$
U_{t/\lambda^2}^{(\lambda)} = \mathbb{I} - \frac{i}{\lambda} \int_0^t H_{eb}(t'/\lambda^2) U_{t'/\lambda^2}^{(\lambda)} \, dt'
\tag{26}
$$

We see that the rescaled Hamiltonian

$$
\frac{1}{\lambda} H_{eb}(t/\lambda^2) = e^2 \sum_{\alpha\beta} a_\alpha^\dagger \alpha\beta \frac{1}{\lambda} b \big(e^{-i\frac{t}{\lambda^2}(\omega-\varepsilon_{\alpha\beta})} g_{\alpha\beta} \big) + \text{h.c.}
\tag{27}
$$

depends on the rescaled fields

$$
b_{\alpha\beta,\lambda}(t) = \frac{1}{\lambda} b \big(e^{-i\frac{t}{\lambda^2}(\omega-\varepsilon_{\alpha\beta})} g_{\alpha\beta} \big)
\tag{28}
$$

The first statement of the stochastic golden rule see Accardi *et al.* (2001) is that the rescaled fields converge (in the sense of correlators) to a quantum white noise

$$b_{\alpha\beta}(t) = \lim_{\lambda \to 0} \frac{1}{\lambda} b\big(g_{\alpha\beta} e^{-i\frac{t}{\lambda^2}(\omega - \varepsilon_{\alpha\beta})}\big) \tag{29}$$

characterized by the following commutation relations

$$[b_{\alpha\beta}(t), b_{\alpha'\beta'}(t')] = [b_{\alpha\beta}^+(t), b_{\alpha'\beta'}^+(t')] = 0 \tag{30}$$

$$[b_{\alpha\beta}(t), b_{\alpha'\beta'}^+(t')] = \delta_{\varepsilon_{\alpha\beta}, \varepsilon_{\alpha'\beta'}} \delta(t - t') G^{\alpha\beta\alpha'\beta'} \tag{31}$$

where the constants $G^{\alpha\beta\alpha'\beta'}$ are given by

$$G^{\alpha\beta\alpha'\beta'} = \int_{-\infty}^{\infty} d\tau \int dk g_{\alpha\beta}(\underline{k}) \overline{g_{\alpha'\beta'}(\underline{k})} e^{i\tau(\omega(\underline{k}) - \varepsilon_{\alpha\beta})}$$

$$= 2\pi \int dk g_{\alpha\beta}(k) g_{\alpha\beta}(k) \delta(\omega(\underline{k}) - \varepsilon_{\alpha\beta}) \tag{32}$$

will be denoted by η_0. The vacuum of the master fields $b_{\alpha\beta}(t)$

$$b_{\alpha\beta}(t)\eta_0 = 0 \quad \forall\alpha\beta, \forall t \tag{33}$$

Moreover, the appearance of $\delta_{\varepsilon_{\alpha\beta}, \varepsilon_{\alpha'\beta'}}$ in the commutator (31) and of the δ-function in (32) is a first indication of the relevance of the integer numbers for this model. This point will be better clarified in the following and will be relevant in the computation of the conductivity tensor.

The limit Hamiltonian is, then, see Accardi *et al.* (2001).

$$H_{eb}^{(sl)}(t) = e^2 \sum_{\alpha\beta} (a_\alpha^+ a_\beta b_{\alpha\beta}(t) + \text{h.c.}) \tag{34}$$

In this sense we say that $H_{eb}^{(sl)}(t)$ is the "stochastic limit" of $H_{eb}(t)$ in (22). Moreover, the stochastic limit of the equation of motion is (25)

$$\partial_t U_t = -i H_{eb}^{(sl)}(t) U_t \tag{35}$$

or, in integral form,

$$U_t = \mathbb{I} - i \int_0^t H_{eb}^{(sl)}(t') U_{t'} \, dt' \tag{36}$$

Finally, the stochastic limit of the (Heisenberg) time evolution of any observable X of the system is

$$j_t(X) = U_t^+ X U_t = U_t^+ (X \otimes 1_R) U_t \tag{37}$$

Since the $b_{\alpha\beta}(t)$ are quantum white noises, Eq. (35), and the corresponding differential equation for $j_t(\tilde{X})$, are singular equations and to give them a meaning we

bring them in normal form. This normally ordered evolution equation is called *the quantum Langevin equation*. Its explicit form is

$$\partial_t j_t(X) = e^2 \sum_{\alpha\beta} \{ j_t([a_\alpha^+ a_\beta, X]\Gamma_-^{\alpha\beta} - \Gamma_-^{\alpha\beta}[a_\beta^+ a_\alpha, X]) \}$$

$$+ie^2 \sum_{\alpha\beta} \{ b_{\alpha\beta}^+(t) j_t([a_\beta^+ a_\alpha, X]) + j_t([a_\alpha^+ a_\beta, X]) b_{\alpha\beta}(t) \} \qquad (38)$$

where

$$\Gamma_-^{\alpha\beta} := \sum_{\alpha'\beta'} \delta_{\varepsilon_{\alpha\beta}, \varepsilon_{\alpha'\beta'}} a_{\beta'}^+ a_{\alpha'} G_-^{\alpha\beta\alpha'\beta'} \qquad (39)$$

$$G_-^{\alpha\beta\alpha'\beta'} = \int_{-\infty}^0 d\tau \int dk g_{\alpha\beta}(\underline{k}) \overline{g_{\alpha'\beta'}(\underline{k})} e^{i\tau(\omega(\underline{k}) - \varepsilon_{\alpha\beta})}$$

$$= \frac{1}{2} G^{\alpha\beta\alpha'\beta'} - i \, \text{P.P.} \quad \int g_{\alpha\beta}^{(k)} \overline{g_{\alpha'\beta'}^{(k)}} \frac{1}{\omega_k - \varepsilon_{\alpha\beta}} \qquad (40)$$

The master equation is obtained by taking the mean value of (38) in the state $\eta_0^{(\xi)} = \eta \otimes \xi$, ξ being a generic vector of the system. This gives

$$\langle \partial_t j_t(X) \rangle_{\eta_0^{(\xi)}} = e^2 \sum_{\alpha\beta} \langle j_t([a_\alpha^+ a_\beta, X]\Gamma_-^{\alpha\beta} - \Gamma_-^{\alpha\beta+}[a_\beta^+, a_\alpha, X]) \rangle_{\eta_0^{(\xi)}} \qquad (41)$$

and from this we find for the generator

$$L(X) = e \sum_{\alpha\beta\alpha'\beta'} \delta_{\varepsilon_{\alpha\beta}, \varepsilon_{\alpha'\beta'}} \{ [a_\alpha^+ a_\beta, X] a_{\beta'}^+ a_{\alpha'} G_-^{\alpha\beta\alpha'\beta'}$$

$$-a_{\alpha'}^+ a_{\beta'} [a_\beta^+ a_\alpha, X] \overline{G_-^{\alpha\beta\alpha'\beta'}} \} \qquad (42)$$

The expressions for $L(X)$ obtained above will be the starting point for our successive analysis.

5. THE CURRENT OPERATOR IN SECOND QUANTIZATION

The current is proportional to the sum of the velocities of the electrons:

$$\vec{J}_\Lambda(t) = \alpha_c \sum_{i=1}^N \frac{d}{dt} \vec{R}_i(t) \qquad (43)$$

Here Λ is the two-dimensional region corresponding to the physical layer, α_c is a proportionality constant which takes into account the electron charge, the area of the surface of the physical device and other physical quantities, and $\vec{R}_i(t)$ is the position operator for the ith electron. Moreover, N is the number of electrons

contained in Λ. Defining

$$\vec{X}_\Lambda(t) = \sum_{i=1}^{N} \vec{R}_i(t) \qquad (44)$$

we conclude that

$$\vec{J}_\Lambda(t) = \alpha_c \dot{\vec{X}}_\Lambda(t) \qquad (45)$$

Since $\vec{X}_\Lambda$ is a sum of single-electron operators its expression in second quantization is given by

$$\vec{X}_\Lambda = \sum_{\gamma\mu} \vec{X}_{\gamma\mu} a_\gamma^+ a_\mu \qquad (46)$$

where

$$\vec{X}_{\gamma\mu} = \langle \psi_\gamma, \vec{X}_\Lambda \psi_\mu \rangle = \int \psi_\gamma(\underline{r})\underline{r}\psi_\mu(\underline{r})\,dr \qquad (47)$$

Recall that the $\psi_\gamma(\underline{r})$ are the single-electron wave functions given by (11) and a_α and a_α^+ satisfy the anticommutation relations (17).

The next step consists in computing the matrix elements (47). This can be done exactly, due to the known expression for $\psi_\gamma(\underline{r})$, even without restricting the analysis to the LLL. In fact the two components of $\vec{X}_{\gamma\mu}$ in (47) have the form:

$$X_{\gamma\mu}^{(1)} = \frac{1}{L_x} \int_{-L_x/2}^{L_x/2} x\,e^{2\pi i(p_\mu - p_\gamma)x/L_x}dx$$
$$\cdot \int_{-\infty}^{+\infty} \overline{\varphi_{n_\gamma}\left(y - y_0^{(p\gamma)}\right)}\varphi_{n_\mu}\left(y - y_0^{(p_\mu)}\right)dy \qquad (48)$$

$$X_{\gamma\mu}^{(2)} = \frac{1}{L_x} \int_{-L_x/2}^{L_x/2} e^{2\pi i(p_\mu - p_\gamma)x/L_x}dx$$
$$\cdot \int_{-\infty}^{+\infty} \overline{y\varphi_{n_\gamma}\left(y - y_0^{(p\gamma)}\right)}\varphi_{n_\mu}\left(y - y_0^{(p_\mu)}\right)dy \qquad (49)$$

and these integrations can be easily performed by making use of the following formulas (cf. Cohen-Tannoudji *et al.*, 1977; Gradshteyn and Ryzhik, 1980):

$$\int_{-\infty}^{+\infty} dx\,e^{-x^2}H_m(x+y)H_n(x+z) = 2^n\sqrt{\pi}m!z^{n-m}\cdot L_m^{n-m}(-2yz) \qquad (50)$$

if $m \leq n$, and

$$\int_{-\infty}^{+\infty} \overline{\varphi_n(y)}y\varphi_m(y)\,dy = \sqrt{\frac{\hbar}{2mn}}[\sqrt{m+1}\delta_{n,m+1} + \sqrt{m}\delta_{n,m-1}] \qquad (51)$$

where H_m and L_m^{n-m} are respectively Hermite and Laguerre polynomials. With these ingredients we get

$$X_{\gamma\mu}^{(1)} = (1 - \delta_{p_\mu p_\gamma})(-1)^{p_\mu - p_\gamma} \frac{L_x e^{-y_{p_\mu p_\gamma}^2}}{2\pi i (p_\mu - p_\gamma)} \mathcal{L}_{\gamma\mu} \tag{52}$$

$$X_{\gamma\mu}^{(2)} = \delta_{p_\mu p_\gamma} \left\{ y_0^{(p_\gamma)} \delta_{n_\gamma n_\mu} + \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n_\mu + 1} \delta_{n_\gamma, n_\mu + 1} + \sqrt{n_\mu} \delta_{n_\gamma n_\gamma - 1} \right) \right\} \tag{53}$$

where

$$\mathcal{L}_{\gamma\mu} := \begin{cases} \sqrt{\frac{2^n \gamma n_\mu!}{2^n \mu n_\gamma!}} y_{p_\mu p_\gamma}^{n_\gamma - n_\mu} L_{n_\mu}^{n_\gamma - n_\mu}(2y_{p_\mu p_\gamma}^2) & \text{if } n_\mu \le n_\gamma \\ \sqrt{\frac{2^n \mu n_\gamma!}{2^n \gamma n_\mu!}} \left(- y_{p_\mu p_\gamma} \right)^{n_\mu - n_\gamma} L_{n_\gamma}^{n_\mu - n_\gamma}(2y_{p_\mu p_\gamma}^2) & \text{if } n_\gamma \le n_\mu \end{cases} \tag{54}$$

$$y_{p_\mu p_\gamma} := \sqrt{\frac{m\omega}{4\hbar}} \left(y_0^{(p_\mu)} - y_0^{(p_\gamma)} \right) = \frac{\pi}{L_x} \sqrt{\frac{\hbar}{m\omega}} (p_\mu - p_\gamma) \tag{55}$$

Notice that, whenever $p_\mu = p_\gamma$, formula (52) must be interpreted simply as $X_{\gamma\mu}^{(1)} = 0$.

These results are simpler if we restrict to the LLL. In this case we have $n_\gamma = n_\mu = 0$ and therefore, since $L_0^a(x) = 1$, we simply get

$$X_{\gamma\mu}^{(1)} = (1 - \delta_{p_\mu p_\gamma})(-1)^{p_\mu - p_\gamma} L_x \frac{e^{-y_{p_\mu p_\gamma}^2}}{2\pi i (p_\mu - p_\gamma)} \tag{56}$$

$$X_{\gamma\mu}^{(2)} = y_0^{(p_\gamma)} \delta_{o_\mu p_\gamma} \tag{57}$$

To show how these results can be useful in the computation of the electron current we start noticing that, if ϱ is a state of the electron system, then

$$\langle \vec{J}_\Lambda(t) \rangle_\varrho = \alpha_c \left\langle \frac{d}{dt} \vec{X}_\Lambda(t) \right\rangle_\varrho = \alpha_c \langle L(\vec{X}_\Lambda(t)) \rangle_\varrho = \alpha_c \text{Tr}(\varrho L(\vec{X}_\Lambda(t))) \tag{58}$$

The vector $\langle \vec{J}_\Lambda(t) \rangle$ will be computed in the next section for a particular class of states ϱ, and we will use this result to get the expressions for the conductivity tensor and for its inverse, the resistivity matrix.

6. THE FINE TUNING CONDITION AND THE RESISTIVITY TENSOR

In this section we will use formula (58) above in order to obtain the conductivity and the resistivity tensors. To do this we begin computing the electric current. We first need to find $L(\vec{X}_\Lambda)$, L being the generator given in (42). Since

$\vec{X}_\Lambda = \vec{X}_\Lambda^\dagger$, we have

$$L(\vec{X}_\Lambda) = L_1(\vec{X}_\Lambda) + \text{h.c.}$$

where, as we find after a few computations,

$$L_1(\vec{X}_\Lambda) = e^2 \sum_{\alpha\beta\alpha'\beta',\gamma} \delta_{\epsilon_{\alpha\beta},\epsilon_{\alpha'\beta'}} G_-^{\alpha\beta\alpha'\beta'} \left(\vec{X}_{\beta_\gamma} a_\alpha^+ a_\gamma a_{\beta'}^+ a_{\alpha'} - \vec{X}_{\gamma\alpha} a_\gamma^+ a_\beta a_{\beta'}^+ a_{\alpha'} \right) \quad (59)$$

In the present paper we consider a situation of zero temperature and we compute the mean value of $L_1(\vec{X}_\Lambda)$ on a Fock N–particle state ψ_I:

$$\psi_I = a_{i_1}^+ \ldots a_{i_{N_I}}^+ \psi_0, \quad i_k \neq i_l, \forall k \neq l \quad (60)$$

where I is a set of possible quantum numbers ($I \subset (\mathbf{N}_0, \mathbf{Z})$), N_I is the number of elements in I and ψ_0 is the vacuum vector of the fermionic operators, $a_\alpha \psi_0 = 0$ for all α. The order of the elements of I is important to fix uniquely the phase of ψ_I. Equation (58) gives now

$$\langle \psi_I, \vec{J}_\Lambda(t)\psi_I \rangle|_{t=0} = \alpha_c \langle \psi_I, L(\vec{X}_\Lambda)\psi_I \rangle \quad (61)$$

Introducing now the characteristic function of the set I,

$$\chi_I(\alpha) = \begin{cases} 1 & \text{if} \quad \alpha \in I \\ 0 & \text{if} \quad \alpha \notin I \end{cases} \quad (62)$$

we get

$$\langle a_\gamma^\dagger a_\alpha \psi_I, a_{\beta'}^\dagger a_{\alpha'} \psi_I \rangle = \delta_{\alpha\gamma} \delta_{\alpha'\beta'} \chi_I(\alpha) \chi_I(\alpha') + \delta_{\alpha\alpha'} \delta_{\gamma\beta'} \chi_I(\alpha)(1 - \chi_I(\gamma)). \quad (63)$$

Using this equality, together with

$$\delta_{\varepsilon_{\alpha\beta},\varepsilon_{\alpha'\alpha'}} = \delta_{\varepsilon_\alpha,\varepsilon_\beta} \quad \delta_{\varepsilon_{\alpha\beta},\varepsilon_{\alpha\beta'}} = \delta_{\varepsilon_\beta,\varepsilon_{\beta'}} \quad (64)$$

we find that the average current is proportional to

$$\langle L(\vec{X}_\Lambda) \rangle_{\psi_I} = \mathcal{L}_1(\vec{X}_\Lambda) + \mathcal{L}_2(\vec{X}_\Lambda) \quad (65)$$

where we isolate two contributions of different structure:

$$\mathcal{L}_1(\vec{X}_\Lambda) = e^2 \sum_{\alpha\beta\alpha'} \delta_{\varepsilon_\alpha,\varepsilon_\beta} \{\chi_I(\alpha) - \chi_I(\beta)\} \cdot \chi_I(\alpha') (\vec{X}_{\alpha\beta} \overline{G_-^{\alpha\beta\alpha'\alpha'}} + \vec{x}_{\beta\alpha} G_-^{\beta\alpha\alpha'\alpha'}) \quad (66)$$

$$\mathcal{L}_2(\vec{X}_\Lambda) = e^2 \sum_{\alpha\beta\beta'} \delta_{\varepsilon_\beta,\varepsilon_{\beta'}} \{\vec{X}_{\beta\beta'} [G_-^{\alpha\beta\alpha\beta'} \chi_I(\alpha)(1 - \chi_I(\beta'))$$

$$-\overline{G_-^{\beta\alpha\beta'\alpha}} \chi_I(\beta')(1 - \chi_I(\alpha))] - \vec{X}_{\beta'\beta} [G_-^{\beta\alpha\beta'\alpha} \chi_I(\beta')(1 - \chi_I(\alpha))$$

$$-\overline{G_-^{\alpha\beta\alpha\beta'}} \chi_I(\alpha)(1 - \chi_I(\beta'))]\} \quad (67)$$

Remark 6.1. It is interesting to notice that if we replace $\delta_{\varepsilon_\alpha, \varepsilon_\beta}$ by $\delta_{\alpha,\beta}$ and $\delta_{\varepsilon_\beta, \varepsilon'_\beta}$ by $\delta_{\beta,\beta'}$, then we easily obtain $\langle L(X_\Lambda^{(1)}) \rangle = 0$, which would imply that no current transportation is compatible with this constraint. This means that this approximation (taking $\alpha = \beta$ and $\beta = \beta'$ means to consider only one among the many contributions in the sums in (66), (67)!) is too strong and must be avoided in order not to get trivial results.

Using Eqs. (52) and (53) for $X_{\gamma\mu}^{(i)}$ we are able to obtain $\mathcal{L}_1(X_\Lambda^{(i)})$ and $\mathcal{L}_2(X_\Lambda^{(i)})$, $i = 1, 2$. First of all we can show that, even if $\mathcal{L}_1(X_\Lambda^{(1)})$ is not zero, nevertheless it does not depend on the electric field. Therefore

$$\frac{\partial}{\partial E} \mathcal{L}_1(X_\Lambda^{(1)}) = 0 \tag{68}$$

Secondly, the computation of $\mathcal{L}_2(X_\Lambda^{(1)})$ gives rise to an interesting phenomenon: due to the definition of $X_{\gamma\mu}^{(1)}$, the sum in (67) is different from zero only if $p_\beta \neq p_{\beta'}$. Moreover, we also must have $\varepsilon_\beta = \varepsilon_{\beta'}$, that is

$$n_\beta - n_{\beta'} = \frac{2\pi eE}{m\omega^2 L_x}(p_{\beta'} - p_\beta) \tag{69}$$

This equality can be satisfied in two different ways: let us denote $\mathcal{R}$ the set of all possible quotients of the form $(n_\beta - n_{\beta'})/(p_{\beta'} - p_\beta)$. This set, in principle, coincides with the set of the rational numbers. Therefore, $0 \in R$. Then

(1) if $\frac{2\pi eE}{m\omega^2 L_x}$ is not in $\mathcal{R}$, (69) can be satisfied only if $\beta = \beta'$. But this condition implies in particular that $p_\beta = p_{\beta'}$, and we know already that whenever this condition holds, then $X_{\beta\beta'}^{(1)} = 0$, so that $\mathcal{L}_2(X_\Lambda^{(1)} = 0)$.

(2) If $\frac{2\pi eE}{m\omega^2 L_x}$ is in $\mathcal{R}$, then we have two possibilities: the first one is again

$$\beta = \beta'$$

which, as we have just shown, does not contribute to $\mathcal{L}_2(X_\Lambda^{(1)}) = 0$. The second is

$$\frac{n_\beta - n_{\beta'}}{p_{\beta'} - p_\beta} = \frac{2\pi eE}{m\omega^2 L_x} \tag{70}$$

which gives a nontrivial contribution to the current.

Therefore, we can state the following

Proposition 1. *In the context of Model (21) there exists a set of rational numbers $\mathcal{R}$ with the following property: if the electric and the magnetic fields are such that if the quotient*

$$\frac{2\pi eE}{m\omega^2 L_x}$$

does not belong to $\mathcal{R}$ then

$$\langle J_\Lambda^{(1)}(t) \rangle_{\psi_I} = 0$$

On the other hand, if condition (70) is satisfied, we can conclude that the sum $\Sigma_{\alpha\beta\beta'}\delta_{\varepsilon_\beta,\varepsilon_{\beta'}}(\ldots)$ in (67) can be replaced by

$$\sum_{\alpha\beta\beta'}\delta_{\varepsilon_\beta,\varepsilon_{\beta'}}(\ldots) = \sum_{\alpha}\sum_{\beta\beta'}{}'(\ldots) \tag{71}$$

where $\Sigma_\alpha \Sigma'_{\beta\beta}$ means that the sum is extended to all the α and to those β and β' with $p_\beta \neq p_{\beta'}$ satisfying (70) (which automatically implies that $\varepsilon_\beta = \varepsilon_{\beta'}$).

Since, as it is easily seen, $g_{\alpha\beta}(k)\overline{g_{\alpha'\beta'}(k)}$ does not depend on $\vec{E}$, we find that

$$\frac{\partial}{\partial E}G_{-}^{\alpha\beta\alpha'\beta'} = -i\frac{he}{m\omega L_x}(p_\alpha - p_\beta)\Lambda_{-}^{\alpha\beta\alpha'\beta'} \tag{72}$$

where

$$\Lambda_{-}^{\alpha\beta\alpha'\beta'} = \int_{-\infty}^{0}dt\int dk g_{\alpha\beta}(\underline{k})\overline{g_{\alpha'\beta'}(\underline{k})}e^{i\tau(\omega(\underline{k})-\varepsilon_{\alpha\beta})} \tag{73}$$

so that, using also (71), we get

$$\frac{\partial}{\partial E}\mathcal{L}_2\big(X_\Lambda^{(1)}\big) = \frac{he}{m\omega L_x}\Theta_x \tag{74}$$

where

$$\Theta_x := \sum_{a}\sum_{\beta\beta'}{}'(p_\beta - p_\alpha)\tilde{x}_{\beta\beta'}^{(1)}\{\chi_I(\alpha)(1-\chi_I(\beta'))\cdot(\Lambda_{-}^{\alpha\beta\alpha\beta'} + \overline{\Lambda_{-}^{\alpha\beta\alpha\beta'}})$$

$$-\chi_I(\beta')(1-\chi_I(\alpha))(\Lambda_{-}^{\beta\alpha\beta'\alpha} + \overline{\Lambda_{-}^{\beta\alpha\beta'\alpha}})\} \tag{75}$$

and

$$\tilde{x}_{\beta\beta'}^{(1)} = iX_{\beta\beta'}^{(1)} \quad (\in \mathbf{R}) \tag{76}$$

Therefore, we conclude that

$$\frac{\partial}{\partial E}\big\langle J_\Lambda^{(1)}(t)\big\rangle_{\psi_I} = \frac{\alpha_c h e^3}{m\omega L_x}\Theta_x \tag{77}$$

Let us now compute the second component of the average current: $\langle\psi_I, L(X_\Lambda^{(2)})\psi_0\rangle = \mathcal{L}_1(X_\Lambda^{(2)}) + \mathcal{L}_2(X_\Lambda^{(2)})$.

The first contribution is easily shown, from (66) and (53), to be identically zero, since

$$\delta_{\varepsilon_\alpha,\varepsilon_\beta}\delta_{p_\alpha p_\beta} = \delta_{\alpha\beta} \tag{78}$$

On the contrary the second term, $\mathcal{L}_2(X_\Lambda^{(2)})$, is different from zero and it has an interesting expression: in fact due to the factor δ_{p_μ,p_γ}, the only nontrivial contributions in the sum $\Sigma_{\beta\beta'}\delta_{\varepsilon_\beta,\varepsilon_\beta}$, in (67), are exactly those with $\beta = \beta'$. Taking all

this into account, we find that

$$\mathcal{L}_2\left(X_\Lambda^{(2)}\right) = e^2 \sum_{\alpha\beta} \left(y_0^{(p_\beta)} - y_0^{(p_\alpha)}\right)\chi_I(\alpha)(1 - \chi_I(\beta))(G_-^{\alpha\beta\alpha\beta} + G_-^{\overline{\alpha\beta\alpha\beta}}) \quad (79)$$

which is different from zero. Furthermore, using (72), we get

$$\frac{\partial}{\partial E}\mathcal{L}_2(X_\Lambda^{(2)}) = -2e^3\left(\frac{h}{m\omega L_x}\right)^2 \Theta_y$$

were we have defined

$$\Theta_y = \sum_{\alpha,\beta}(p_\alpha - p_\beta)^2\chi_I(\alpha)(1 - \chi_I(\beta))\mathfrak{J}(\Lambda_-^{\alpha\beta\alpha\beta}) \quad (80)$$

and $\Lambda_-^{\alpha\beta\alpha\beta}$ is given by (73). If we call now

$$j_{x,E} = \frac{\partial\langle J_\Lambda^{(1)}(t)\rangle_{\psi_I}}{\partial E}|_{t=0} = \alpha_c\frac{\partial\langle L(X_\Lambda^{(1)})\rangle_{\psi_I}}{\partial E}$$

$$j_{y,E} = \frac{\partial\langle J_\Lambda^{(2)}(t)\rangle_{\psi_I}}{\partial E}|_{t=0} = \alpha_c\frac{\partial\langle L(X_\Lambda^{(2)})\rangle_{\psi_I}}{\partial E}$$

we obtain the conductivity tensor (see Chakraborty and Pietiläinen, 1988)

$$\sigma_{xx} = \sigma_{yy} = j_{y,E} \quad \sigma_{xy} = -\sigma_{yx} = j_{x,E} \quad (81)$$

and the resistivity tensor

$$\rho_{xx} = \rho_{yy} = \frac{\sigma_{yy}}{\sigma_{yy}^2 + \sigma_{xy}^2}, \quad \rho_{xy} = -\sigma_{yx} = \frac{\sigma_{xy}}{\sigma_{yy}^2 + \sigma_{xy}^2} \quad (82)$$

After minor computations we conclude that

$$\rho_{xy} = \begin{cases} 0 & \text{if } \frac{2\pi eE}{m\omega^2 L_x} \notin \mathcal{R} \\ \frac{m\omega L_x}{2e^3 h\alpha_c}\frac{\Theta_x}{\left[\Theta_x^2 + \left(\frac{h}{m\omega L_x}\right)^2\Theta_y^2\right]} & \text{if } \frac{2\pi eE}{m\omega^2 L_x} \in \mathcal{R} \end{cases} \quad (83)$$

$$\rho_{xx} = \begin{cases} -\left(\frac{m\omega L_x}{h}\right)^2\frac{1}{2\alpha_c e^3\Theta_y} & \text{if } \frac{2\pi eE}{m\omega^2 L_x} \notin \mathcal{R} \\ -\frac{1}{2e^3\alpha_c}\frac{\Theta_y}{\left[\Theta_x^2 + \left(\frac{h}{m\omega L_x}\right)^2\Theta_y^2\right]} & \text{if } \frac{2\pi eE}{m\omega^2 L_x} \in \mathcal{R} \end{cases} \quad (84)$$

We want to relate these results with the experimental graphs concerning the components of the resistivity tensor, see Girvin (1999). To avoid confusions, let us remark that our choice for the direction of the electric field, the *y*-axis, is not the usual one, the *x*-axis, see Girvin (1999). Therefore, in our notation, the Hall resistivity is really ρ_{xx}, while our ρ_{xy} corresponds to the *xx* component of ρ as given in Girvin (1999).

Let us now comment these results that are consequences of the basic relation (70). As it is evident from the formula above, the fact that the *fine tuning condition* (FTC) ($\frac{2\pi eE}{m\omega^2 L_x} \in \mathcal{R}$) is satisfied implies that $\rho_{xy} \neq 0$, so that the resistivity tensor is nondiagonal. Vice versa, if the FTC is not satisfied, then $\rho = \rho_{xx}\mathbb{I}$, $\mathbb{I}$, being the 2×2 identity matrix. This implies that, whenever the FTC holds, then the x-component of the mean value of the density current operator is in general different from zero, while it is necessarily zero if the FTC is not satisfied.

If the physical system is prepared in such a way that $\frac{2\pi eE}{m\omega^2 L_x} \in \mathcal{R}$, then an experimental device should be able to measure a nonzero current along the x-axis. Otherwise, this current should be zero whenever $\frac{2\pi eE}{m\omega^2 L_x} \notin \mathcal{R}$. A crucial point is now the determination of the set $\mathcal{R}$, of rational numbers. From a mathematical point of view, all the natural integers n_α and all the relative integer p_α are allowed. However, physics restricts the experimentally relevant values to a rather small set. In fact eigenstates corresponding to high values of n_α and p_α are energetically not favored because the associated eigenenergies $\epsilon_{n_\alpha p_\alpha}$, in (11) increases and the probabilities of finding an electron in the corresponding eigenstate decrease (this is a generalization of the standard argument that restrict the analysis of the fractional QHE to the first few Landau levels). Moreover, high positive values of $-p_\alpha$ are not compatible with the fact that H_0 must be bounded from below, to be a "honest" Hamiltonian.

Therefore, in formula (70) not all the rational numbers are physically allowed but only those compatible with the above constraints. For this reason it is quite reasonable to expect that the set $\mathcal{R}$ consists only of a *finite set* of rational values. The determination of this set strongly depends on the physics of the experimental setting and we shall discuss it in a future paper.

We end this section, and the paper, with the following two remarks.

The sharp values of the magnetic field involved in the FTC may be a consequence of the approximation intrinsic in the stochastic limit procedure, which consists in taking $\lambda \to 0$ and $t \to \infty$. In intermediate regions ($\lambda > 0$ and $t < \infty$), it is not hard to imagine that the δ-function giving rise to the FTC becomes a smoother function.

Under special assumptions on the B-dependence of Θ_x and Θ_y, together with some reasonable physical constraint on the value of the magnetic field, it is not difficult to check that ρ_{xx} has plateaux, corresponding to the zeros of ρ_{xy} and that, outside of these plateaux, it grows linearly with B.

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