Control of elementary quantum flows

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Abstract

We consider the problem of controlling the size of an elementary quantum stochastic flow generated by a unitary stochastic evolution driven by first order white noise.

The quantum stochastic analogue of the problem of minimizing a quadratic performance criterion associated with a classical stochastic differential equation was solved with the use of the representation free quantum stochastic calculus of [1] in [6]. Simpler versions, corresponding to quantum stochastic differential equations (qsde) driven by first order white noise, can be found in [4, 5] where a simple quantum analogue of the Bucy-Kalman filter was obtained (see [2] for a general version). In a simple form, related to what follows, the result reads as follows: Let $U = \{U_t \,/\, t \geq 0\}$ be an adapted process satisfying the qsde (see [1, 7] for proofs of existence and uniqueness theorems and definitions of related concepts)

$$dU_t = (F_t U_t + u_t)\, dt + \Psi_t \, U_t \, dA_t + \Phi_t \, U_t \, dA_t^\dagger, \tag{1}$$

$$U_0 = I,\, t \in [0, T]$$

where $T > 0$ is a fixed finite horizon, $dA_t^\dagger$ and dA_t are the differentials of the creation and annihilation processes of [7], and the coefficient processes are adapted, bounded, strongly continuous and square integrable processes living on the exponential vectors domain $\mathcal{E} = \text{span}\,\{h = h_0 \otimes \psi(f)\}$ of the tensor product $H_0 \otimes \Gamma$ of a system (separable Hilbert) space H_0 and the Boson (noise) Fock space Γ on $L^2([0, T], \mathbf{C})$. Treating $u = \{u_t \,/\, t \geq 0\}$ as a control process, we can show that the quadratic performance functional

$$J_{h,T}(u) = \int_0^T [< U_t h, X^* X \, U_t h > + < u_t h, u_t h >]\, dt$$

$$+ < U_T h, M \, U_T h > \tag{2}$$

where X, M are bounded operator on H_0, identified with their ampliations $X \otimes I$, $M \otimes I$ to $H_0 \otimes \Gamma$, with $M \geq 0$, is minimized by the feedback control $u_t =$

$-P_t U_t$, where the bounded, positive, self-adjoint process $\{P_t\,/\,t \in [0,T]\}$, with $P_T = M$, is the solution of the quantum stochastic Riccati equation

$$dP_t + (P_t F_t + F_t^* P_t + \Phi_t^* P_t \Phi_t - P_t^2 + X^* X)\,dt + \tag{3}$$

$$(P_t \Psi_t + \Phi_t^* P_t)\,dA_t + (P_t \Phi_t + \Psi_t^* P_t)\,dA_t^\dagger = 0$$

and the minimum value is $< h, P_0 h >$.

Turning to quantum flows, if U_t is for each t a unitary operator, in which case (1) takes the form

$$dU_t = -((iH + \frac{1}{2}\,L^* L)\,dt + L^* dA_t - L\,dA_t^\dagger)\,U_t, \tag{4}$$

$$U_0 = I,\, t \in [0,T]$$

with adjoint

$$dU_t^* = -U_t^*((-iH + \frac{1}{2}\,L^* L)\,dt - L^* dA_t + L\,dA_t^\dagger), \tag{5}$$

$$U_0^* = I,\, t \in [0,T]$$

where H, L are bounded operators on H_0 with H self-adjoint, then the family $\{j_t(X)/\,t \in [0,T]\}$ of bounded linear operators on $H_0 \otimes \Gamma$ defined by $j_t(X) = U_t^*\,X\,U_t$ (see [3] for a general theory) is called an *elementary quantum flow* (EQF). Using quantum Itô's formula for first order white noise, namely $dA_t\,dA_t^\dagger = dt$ and all other products of differentials are equal to zero, we can show that $\{j_t(X)/\,t \in [0,T]\}$ satisfies the qsde

$$\begin{aligned} dj_t(X) &= j_t(i[H,X] - \\ &\quad \frac{1}{2}(L^* L X + X L^* L - 2L^* X L))\,dt \\ &+ j_t([L^*,X])\,dA_t + j_t([X,L])\,dA_t^\dagger \\ j_0(X) &= X,\, t \in [0,T] \end{aligned} \tag{6}$$

In the case when X is self-adjoint and $U_t = e^{-itH}$, $\{j_t(X)/\,t \in [0,T]\}$ describes the time evolution of the quantum mechanical observable X and (6) is the *Heisenberg equation* for observables. In the general case, (6) is interpreted as the *Heisenberg picture of the Schrödinger equation in the presence of noise* or as a quantum probabilistic analogue of the *Langevin equation.* Moreover a *-unital homomorphism solution of (0.6) can always be written as an EQF. Looking at (4) as (1) with $u_t = -\frac{1}{2}L^*LU_t$ and taking $M = \frac{1}{2}L^*L$, (2) becomes

$$J_{h,T}(L) = \int_0^T [\, \|j_t(X)h\|^2 + \frac{1}{4}\|j_t(L^*L)h\|^2 \,]\, dt + \tag{7}$$

$$\frac{1}{2}\|j_T(L)h\|^2$$

Thinking of L as a control, we interpret the first term of (7) as a measure of the size of the flow over $[0,T]$, the second as a measure of the control effort over $[0,T]$ and the third as a "penalty" for allowing the evolution to go on for a long time. In order for L to be optimal it must satisfy $\frac{1}{2}L^*L = P_t$ where P_t is the solution of (3) for $F_t = -iH$, $\Phi_t = L$ and $\Psi_t = -L^*$. For these choices (3) reduces, by the time independence of P_t and the linear independence of dt, dA_t and $dA_t^\dagger$, to the equations

$$[L, L^*] = 0 \text{ (i.e } L \text{ is normal)} \tag{8}$$

and

$$\frac{i}{2}[H, P_\infty] + \frac{1}{4}P_\infty^2 + X^*X = 0 \tag{9}$$

where $P_\infty = \frac{1}{2}L^*L$. We recognize (9) as a special case of the algebraic Riccati equation (ARE) (see [8]). It is known that if there exists a bounded linear operator K on H_0 such that $\frac{i}{2}H + KX^*$ is the generator of an asymptotically stable semigroup (i.e if the pair $(\frac{i}{2}H,\ X^*)$ is stabilizable) then (9) has a positive self-adjoint solution P_∞. We may summarize as follows: *Let $h \in \mathcal{E}$, $0 < T < +\infty$, and let H, L, X be bounded linear operators on H_0 such that H is self-adjoint and the pair $(\frac{i}{2}H,\ X^*)$ is stabilizable. The quadratic performance criterion $J_{h,T}(L)$ of (7) associated with the EQF $\{j_t(X) = U_t^*\, X\, U_t \,/\, t \geq 0\}$, where $U = \{U_t \,/\, t \geq 0\}$ is the solution of (4), is minimized by*

$$L = \sqrt{2}\, P_\infty^{1/2}\, W \;\; (\textit{polar decomposition of } L) \tag{10}$$

where P_∞ is a positive self-adjoint solution of the ARE (9) and W is any bounded unitary linear operator on H_0 commuting with P_∞. Moreover $\min_L\, J_{h,T}(L) = < h, P_\infty h >$ *independent of T.*

Riferimenti bibliografici

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