QUANTUM STOP TIMES

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Abstract

The notion of stop-time can be naturally translated in a quantum probabilistic framework and this problem has been studied by several authors [1], [2], [3], [4], [5]. Recently Parthasarathy and Sinha [4] have established a factorization property of the L^2-space over the Wiener space (regarded as the Fock space over $L^2(\mathbf{R}_+)$) based on the notion of quantum stop time which is a quantum probabilistic analogue of the strong Markov property. In this note we prove a stronger result which has no classical analogue namely that the algebra generated by the stopped Weyl operators in the sense of [4] (i.e.the past algebra with respect to a stop time S), is the algebra of all the bounded operators on L^2 of the Wiener space.

1 INTRODUCTION

Let us recall some notations from [4]. Let $\mathcal{H} = L^2(\mathbf{R}_+)$ denote the Fock space over $L^2(\mathbf{R}_+)$; Φ be the vacuum vector in $\mathcal{H}$. For each $t \geq 0$ one has

$$\mathcal{H} \cong \Gamma(L^2(0,t)) \otimes \Gamma(L^2(t,\infty)) \tag{1}$$

and we use the notations:

$$\mathcal{H}_{[t} = \Phi_{t]} \otimes \Gamma(L^2(t,\infty)) \qquad ; \qquad \mathcal{H}_{t]} = \Gamma(L^2(0,t)) \otimes \Phi_{[t} \tag{2}$$

We denote

$$\mathcal{B} = \mathcal{B}(\mathcal{H}) \tag{3}$$

the algebra of all bounded operators on $\mathcal{H}$ and

$$\mathcal{B}_{t]} = \mathcal{B}(\mathcal{H}_{t]}) \otimes 1_{[t} \qquad ; \qquad \mathcal{H}_{[t} = 1_{t]} \otimes \mathcal{B}(\mathcal{H}_{[t}) \tag{4}$$

denote the past and future filtrations. One has

$$\mathcal{B} \cong \mathcal{B}_{t]} \otimes \mathcal{B}_{[t} \tag{5}$$

we shall also use the notation

$$\mathcal{B} \cong \mathcal{B}_{\infty]} \tag{6}$$

We denote $\chi_{[0,t]}$ the characteristic function of the interval $[0,t]$ and, for each $f \in L^2(\mathbf{R}_+)$

$$f_{t]} = \chi_{[0,t]} f \qquad ; \qquad f_{[t} = \chi_{[t,\infty]} f \tag{7}$$

The shift on $L^2(\mathbf{R}_+)$ will be denoted θ_s. By definition:

$$\theta_s f(t) = f(t-s) \qquad ; \qquad f(u) = 0 \text{ if } u < 0$$

so that

$$\theta_s \theta_s^+ = \text{multiplication by } \chi_{[0,s]}$$

The Weyl operators on $L^2(\mathbf{R}_+)$ are denoted W(f) ($f \in L^2(\mathbf{R}_+)$). They are unitary operators on $\mathcal{H}$ satisfying the canonical commutation relations:

$$W(f)W(g) = e^{iIm<f,g>}W(f+g) \tag{8}$$

let $E_{t]} : \mathcal{B} \longrightarrow \mathcal{B}_{t]}$ denote the Fock conditional expectation characterized by the propery:

$$E_{t]}(a_{t]} \otimes a_{[t)}) = a_{t]} < \Phi, a_{[t}\Phi > \tag{9}$$

for every $a_{t]} \in \mathcal{B}_{t]}$, $a_{[t} \in \mathcal{B}_{[t}$. Denoting $P_{t]} : \mathcal{H} \longrightarrow \mathcal{H}_{t]}$ the orthogonal projection one has:

$$E_{t]}(a) = P_{t]} \cdot a \cdot P_{t]} \otimes 1_{[t} \tag{10}$$

Definition 1 *A* **stop time** *is a spectral measure on S from the Borel subsets of $\mathbf{R}_+$ with values in the projections of $\mathcal{B}$ such that for each $t \geq 0$:*

$$S_{[0,t]} \in \mathcal{B}_{t]} \tag{11}$$

Equivalently a stop time can be defined by an increasing right continuous family S(o, t) of projections satisfying (11).

Example (1.) The classical stop time. Let $(\Omega, \mathcal{F}, P)$ be a probability space , $\mathcal{F}_{t]}$ a past filtration and $\tau : \Omega \longrightarrow \bar{\mathbf{R}}_+ = \bar{\mathbf{R}}_+ \cup \{\infty\}$ a random variable. Then the family

$$S_{[0,t]} = \chi_{[0,t]}(\tau) \tag{12}$$

is a stop time in the sense of definition (1) above and clearly the family (13) uniquely determines the random variable τ. **Example (2.)** Let $(\Omega, \mathcal{F}, P)$ denote the Wiener space , $\mathcal{F}_{t]}$ the corresponding past filtration and w_t a $\mathbf{R}^n$ -valued Brownian motion. Fix $A \subseteq \mathbf{R}^\mathbf{n}$ and define

$$S_A(o,t) = \chi_{\bigcap_{s \leq t}[w_t \in A]} \tag{13}$$

$$S_A(+\infty) = 1 - \lim_{t\to\infty} S_A(o,t) \tag{14}$$

Then the family $S_A(o,t)$ is a stop time in the sense of definition (1). Moreover it has the property:

$$S_A(o,t) = S_A(o,s) \otimes S_A(s,t) \qquad ; \qquad o < s < t \tag{15}$$

Example (3.) Let e(.) be a spectral measure in $L^2(\mathbf{R}_+)$ with the property that for each interval $I \subseteq \mathbf{R}_+$ one has

$$e(I) \leq \chi_I \tag{16}$$

Then

$$S(\ .\ ) = \Gamma(e(\ .\ )) \tag{17}$$

is a stop time. In the following we will assume that

$$S(+\infty) = 0 \tag{18}$$

We will also use the notation:

$$S_{[t} = S(t,\infty) \tag{19}$$

Lemma 1 *For each $f \in L^2(\mathbf{R}_+)$ the integral*

$$\int_0^\infty S(ds)W(f_{s]}) = S \circ W(f) \tag{20}$$

is well defined on the exponential vectors and defines a unique unitary operator on $\mathcal{H}$.

Proof. The Riemann sums corresponding to the integral (20) are well defined operators. Applying them to an exponential vector $\psi(g)$, we can use Lemma **??** of [4] to prove the existence of the limit and the isometry property.

Definition 2 *The (left) S–past algebra is the von Neumann algebra $\mathcal{A}_{S]}$ generated by the operators*

$$\int_a^b S(ds)W(f_{s]}) = S \circ W(f) \tag{21}$$

for all $0 \le a < b \le \infty$ and $f \in L^2(\mathbf{R}_+)$. Note that, since

$$\int_a^b S(ds)W(f_{s]}) = S(a,b) \cdot \int_0^\infty S(ds)W(f_{s]})$$

the operators (21) are contractions. The operators (21) are called the left

S–stopped Weyl operators or simply, since in this note we shall consider this type of operators for a fixed S, the stopped Weyl operators.

Lemma 2 *For each $f \in L^2(\mathbf{R}_+)$ the integral*

$$\int_0^\infty S(ds)W(\theta_s f) \tag{22}$$

is well defined on the exponential vectors and defines a unique unitary operator on $\mathcal{H}$. In analogy with the terminology adopted by Parthasarathy and

Sinha, the operators (22) will be called the **shifted Weyl operators**.

Definition 3 *The S–future algebra is the von Neumann algebra $\mathcal{A}_{[S}$ generated by the operators (22) for $f \in L^2(\mathbf{R}_+)$.*

We shall denote S" the von Neumann algebra generated by the spectral projections S(a,b) with $a, b \in \mathbf{R}_+$ and S' its commutant.

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2 The S–past algebra

Theorem 1 *If for all* $0 < b < \infty$

$$S_{[b} \neq 0 \tag{23}$$

then

$$\mathcal{A}_{S]} = \mathcal{B}(\mathcal{H}) \tag{24}$$

Proof. First notice that the von Neumann algebra generated by the set

$$\{S(a,b) \cdot W(f_{b]}) \;\; 0 \leq a \leq b \leq \infty \;\; f \in L^2(\mathbf{R}_+)\} \tag{25}$$

is contained in $\mathcal{A}_{S]}$ and that

$$S'' \subseteq \mathcal{A}_{S]} \tag{26}$$

In fact choosing $f = f_{a]}$ in (21) we obtain

$$\int_a^b S(ds)W(f_{s]}) = S(a,b) \cdot W(f_{a]}) \tag{27}$$

which proves that $\mathcal{A}_{S]}$ contains all the operators (25). (26) follows from this inclusion by putting $f = 0$ identically in (27).
Let now X be an orthogonal projection in $\mathcal{A}'_{S]}$. We shall prove that $X = 0$, 1 and this will imply our thesis.
By (27) with $b = \infty$ one has that for any $0 \leq a \leq \infty$, $f \in L^2(\mathbf{R}_+)$

$$(S(a,\infty) \cdot W(-f_{a]}))^* = W(f_{a]}) \cdot S(a,\infty) \in \mathcal{A}_{[S} \tag{28}$$

hence for all $0 \leq a \leq \infty$ *and* $f \in L^2(\mathbf{R}_+)$ one has

$$X \cdot W(f_{a]}) \cdot S(a,\infty) = W(f_{a]}) \cdot S(a,\infty) \cdot X \tag{29}$$

Taking $E_{a]}$-expectations of both sides of (29) we obtain

$$E_{a]}(X) \cdot W(f_{a]}) \cdot S(a,\infty) = W(f_{a]}) \cdot S(a,\infty) \cdot E_{a]}(X) \tag{30}$$

Therefore for each $b_{a]} \in \mathcal{B}_{a]}$

$$E_{a]}(X) \cdot b_{a]} \cdot S(a,\infty) = b_{a]} \cdot S(a,\infty) \cdot E_{a]}(X)$$

Now we distinguish two possibilities:

i) There exists a sequence (a_n) in $\mathbf{R}_+$ such that $a_n \uparrow \infty$ and

$$S_{[a_n} \cdot E_{a_n]}(X) \neq 0 \qquad \forall n$$

ii) There exists $\bar{a} > 0$ such that, for each $a > \bar{a}$

$$S_{[}a \cdot E_{a]}(X) = 0$$

In case i), for each integer n there exists a vector ξ_n in $\mathcal{H}_{a_n]}$ such that $\eta_n :=$

$S_{[a_n} \cdot E_{a_n]}(X)\xi_n \neq 0$. In fact otherwise for each pair of vectors ξ , $\eta \in \mathcal{H}_{a_n]}$ one would have:

$$< \xi, S_{[a_n} \cdot E_{a_n]}(X)\eta >= 0$$

which implies $S_{[a_n} \cdot E_{a_n]}(X) = 0$, since $S_{[a_n} \cdot E_{a_n]}(X) \in \mathcal{B}_{a_n]}$. If $\xi_n \in \mathcal{H}_{a_n]}$ is a vector as described above then for every $b_{a_n]} \in \mathcal{B}_{a_n]}$ one has

$$E_{a_n]}(X) \cdot b_{a_n]} \cdot S_{[a_n}\xi_n = b_{a_n]} \cdot S_{[a_n} \cdot E_{a_n]}(X)\xi_n = \eta_n$$

and since both η_n and $S_{[a_n}\xi_n$ are $\neq 0$ (in fact $S_{[a_n}$ commutes with $E_{a_n]}(X)$) and $b_{a_n]} \in \mathcal{B}_{a_n]}$ is arbitrary, it follows that

$$E_{a_n]}(X) \cdot \mathcal{H} \supseteq E_{a_n]}(X) \cdot \mathcal{H}_{a_n]} \supseteq \mathcal{H}_{a_n]}$$

in particular

$$P_{a_n]} \cdot X \cdot \mathcal{H} \supseteq P_{a_n]} \cdot X \cdot \mathcal{H}_{a_n]} = E_{a_n]}(X)\mathcal{H}_{a_n]} \supseteq \mathcal{H}_{a_n]}$$

and therefore, for each n

$$X \cdot \mathcal{H} = P_{a_n]} \cdot X \cdot \mathcal{H} \oplus P^{\perp}_{a_n]} \cdot X \cdot \mathcal{H} \supseteq P_{a_n]} \cdot X \cdot \mathcal{H}_{a_n]} \supseteq \mathcal{H}_{a_n]}$$

Letting $n \to \infty$ we obtain that the range of X is $\mathcal{H}$, i.e. X is the identity. In case ii) let a be any number such that $a > \bar{a}$. Then from (29) one has:

$$X \cdot W(f_{a]}) \cdot S(a, \infty) = W(f_{a]}) \cdot S(a, \infty) \cdot X$$

which, taking $E_{a]}$- expectations of both sides, implies

$$E_{a]}(X) \cdot W(f_{a]}) \cdot S(a, \infty) = W(f_{a]}) \cdot S(a, \infty) \cdot E_{a]}(X) = 0 \qquad (31)$$

since $a > \bar{a}$. Now, since $S_{[a}$ is localized on $\mathcal{H}_{a]}$ it follows that $S_{[a} \neq 0$ if and only if $S_{[a}$ restricted to $\mathcal{H}_{a]}$ is $\neq 0$. Therefore the space generated by the vectors $W(f_{a]}) \cdot S(a,\infty) \cdot \mathcal{H}_{a]}$ with $f \in L^2(\mathbf{R}_+)$ is dense in $\mathcal{H}_{a]}$ and therefore (31) implies that for each pair of vectors ξ , $\eta \in \mathcal{H}_{a]}$ one has

$$< \xi, E_{a]}(X)\eta >=< \xi P_{a]} \cdot X \cdot P_{a]}\eta >= 0$$

or equivalently

$$P_{a]} \cdot X \cdot P_{a]} = 0$$

and since in our assumptions $S_{[a}$ is $\neq 0$ for every $a > 0$ it follows that

$$X = \lim_{a \to \infty} P_{a]} \cdot X \cdot P_{a]} = 0$$

and this ends the proof.

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