

**MARTINGALE CONVERGENCE OF GENERALIZED
CONDITIONAL EXPECTATIONS**

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Abstract

We prove that the martingale convergence theorem for generalized conditional expectations in von Neumann algebras holds in the weak topology without restrictions. The situation is therefore different from the strong topology case, where there are restrictive conditions which distinguish between increasing and decreasing sequences of von Neumann algebras. Moreover known counterexamples show that in the decreasing case the strong martingale convergence theorem might not hold.

1 Introduction

Motivated by the Doob–Moy [19] operator characterization of classical conditional expectations and from some results of Segal for finite W^* -algebras [24], Umegaki [27], [28] initiated a systematic study of the notion of conditional expectation in C^* - and W^* -algebras. In particular he proved the first martingale convergence theorems in this framework [29]. Several types of martingale convergence theorems have subsequently been proved by Cuculescu [6], (see also [1] where an asymptotic abelianess condition is used), Tsukada [26] completed the analysis of the strong martingale convergence, Lance [15], Dang–Ngoc [7] and Goldstein [11] investigated the almost sure convergence case, and Petz [20], [21] unified the technique of proof of martingale and ergodic almost sure convergence theorems.

All the above results were obtained for conditional expectations in Umegaki's sense, i.e. norm one projections. However, for the construction of quantum Markov chains, the identification of norm one projections with conditional expectations is too restrictive and a more general notion, called here *generalized conditional expectation* is needed (cf. [23] for details on this topic). This more general notion was introduced in [2] and soon after, the first martingale convergence theorems, corresponding to this notion of conditional expectation, were proved by Petz [21], Hiai and Tsukada [12], Cecchini and Petz [20]. In particular, [12] contains necessary and sufficient conditions for the strong martingale convergence of conditional expectations in the sense of [2] corresponding to increasing or decreasing nets of von Neumann algebras.

These results however left open a natural problem: in classical probability theory the convergence of martingales corresponding to increasing or decreasing nets of W^* -algebras is a universal phenomenon, i.e. independent

of specific properties of the nets of algebras considered; on the other hand, the convergence results of Hiai and Tsukada [12] involves restrictive conditions on the nets of algebras involved. Moreover these conditions are different in the decreasing and increasing case.

A martingale convergence theorem for generalized conditional expectations is, in its essence, a theorem on the convergence of the modular structures associated to a decreasing (resp. increasing) sequence of von Neumann algebras, to the corresponding modular structure associated to the intersection (closure of the union) of the given sequence.

In the increasing case, approximation theorems of this kind were first obtained by Hugenholz and Wieringa [14] and by Araki [3] and generalized in [16], [9] where a general continuity property of the polar decomposition for graph increasing sequences of linear operators was proved. The decreasing case is relevant for problems arising in operator algebras [17], [18] where the canonical endomorphism in the sense of [18] (which is the generalized conditional expectation of [2] in the case of a common cyclic and separating vector) plays a relevant role. In the present paper we show how to modify the basic ideas, developed in [16], [9] for the increasing case, so to adapt them to the decreasing case,

This adaptation goes through some nontrivial steps, the main difference with the martingale convergence theorem contained in [18], being that the usual convergence in the resolvent sense has to be replaced, in our case, by the convergence of continuous functions of the operators vanishing at infinity. The origins of this difference are explained in Remark (2.7) below.

As a corollary of this result we prove the universality of the martingale convergence theorems in the weak operator topology, both in the increasing and the decreasing case. Our result is optimal, in the sense that in the decreasing case there are known counterexamples to the strong martingale convergence (cf. Cecchini [5], Hiai and Tsukada [12]).

We also explain the reason for the apparent asymmetry between the increasing and the decreasing for the case (cf. the remark at the end of Section (2.)). For notational of simplicity we shall formulate our results for *sequences* of von Neumann algebras, however our arguments work without changes in the case of *nets*.

2 Notations

Let \mathcal{H} be a Hilbert space and $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ a linear operator where $D(A)$, the *domain* of A , is a not necessarily dense linear subspace of \mathcal{H} . If A, B are linear operators on \mathcal{H} , we write $A \subset B$ if

$$D(A) \subset D(B) \quad \text{and} \quad A\xi = B\xi, \quad \forall \xi \in D(A) \quad (1)$$

namely the graph of A is contained in the graph of B . A sequence (A_j) of linear operators is called *decreasing* (resp. *increasing*) if for each j

$$A_{j+1} \subset A_j; \quad (\text{resp } A_{j+1} \supset A_j) \quad (2)$$

If (A_j) is a decreasing sequence of closed operators, the intersection of their graphs is still the graph of a closed linear an operator denoted $A_\infty : D(A_\infty) \subset \mathcal{H} \rightarrow \mathcal{H}$ and we shall also use the notation $A_\infty = \bigcap_j A_j$; if (A_j) is an increasing sequence of linear operator, the union of the graphs of A_j is the graph of a linear operator $A_\infty^0 = \bigcup_j A_j$; if A_∞^0 is closable, we denote by A_∞ its closure.

If A is a closed linear operator, one can define its polar decomposition

$$A = VH \quad (3)$$

by regarding A as a linear operator from $\overline{D(A)}$, the closure of $D(A)$, to \mathcal{H} ; H is a selfadjoint operator on $\overline{D(A)}$ and V a partial isometry from $\overline{D(A)}$ to \mathcal{H} .

In the following we shall often identify V and H with VP_A and HP_A , where $P_A : \mathcal{H} \rightarrow \overline{D(A)}$ is the orthogonal projection. So both V and H will be considered as acting on \mathcal{H} .

3 Continuity of the polar decomposition: decreasing sequences

Theorem 1 *Let \mathcal{H} be a Hilbert space and (A_j) a decreasing sequence of closed linear (or antilinear) operators on \mathcal{H} with intersection*

$$A_\infty := \bigcap A_n \quad (4)$$

Denote, for $n \in \mathbf{N} \cup \{\infty\}$, by \mathcal{H}_n the closure of $D(A_n)$ in \mathcal{H} , by $P_n : \mathcal{H} \rightarrow \mathcal{H}_n$ the orthogonal projection and by $A_n = V_n H_n$ the polar decomposition of A_n .

Then for every continuous function $f : \mathbf{R} \rightarrow \mathbf{R}$ vanishing at infinity ($\lim_{t \rightarrow +\infty} f(t) = 0$)

$$s - \lim_{n \rightarrow \infty} f(H_n)P_n = f(H_\infty)P_\infty \quad (5)$$

$$w - \lim_{n \rightarrow \infty} V_n P_n = V_\infty P_\infty \quad (6)$$

The same result is valid if the sequence (A_n) is increasing, provided $\bigcup A_n$ is closable and A_∞ denotes its closure as above [9].

Remark (2.2) Notice that the limit

$$s - \lim_{n \rightarrow \infty} P_n = P \quad (7)$$

always exists and

$$P \geq P_\infty \quad (8)$$

but in general $P \neq P_\infty$. A corollary of this fact is that, if f does not vanish at infinity, then (5) is not true in general (otherwise, taking $f(t) = 1$ ($t \in \mathbf{R}$) one would have $P_n \downarrow P_\infty$).

Remark (2.3) If $f : \mathbf{R} \rightarrow \mathbf{R}$ is continuous and $\lim_{t \rightarrow +\infty} f(t) = \lambda$, then (5) implies that

$$s - \lim_{n \rightarrow \infty} f(H_n)P_n = f(H_\infty)P_\infty - \lambda(P - P_\infty) \quad (9)$$

in particular, because of (5), for any such f :

$$s - \lim_{n \rightarrow \infty} f(H_n)P_\infty = f(H_\infty)P_\infty \quad (10)$$

and therefore (10) will hold for any continuous bounded function f by an elementary 3ε arguments, see [30, proof of theorems VIII.20]. The same is true if the continuity of f holds only on a open set of spectral measure 1 for all the operators H_n, H_∞ (in this form it will be used to establish the relation (37)). Moreover

$$s - \lim_{n \rightarrow \infty} V_n P_\infty = V_\infty P_\infty \quad (11)$$

because the weak topology coincides with the strong topology on the set of

isometries. **Remark (2.4)** Theorem 4 could obviously be stated for operators between different Hilbert spaces \mathcal{H} and \mathcal{K} ; by replacing \mathcal{H} with $\ker(A_1)^\perp$ (the orthogonal complement of the null space of A_1) we may thus assume that all the A_j are nonsingular (in the decreasing case).

Proof of Theorem 4 We begin by considering the decreasing case. For $n \in \mathbf{N} \cup \{\infty\}$, let \mathcal{H}'_n denote the Hilbert space $D(A_n)$ equipped with the graph norm $\|\xi\|_{A_n}^2 = \|\xi\|^2 + \|A_n\xi\|^2$ and

$$T_n : \mathcal{H}'_n \rightarrow \mathcal{H}_n \subset \mathcal{H} \quad (12)$$

the identification operator; clearly $\|T_n\| \leq 1$.

Notice that \mathcal{H}'_{n+1} is a closed subspace of \mathcal{H}'_n and that the restriction of T_n on \mathcal{H}'_{n+1} is T_{n+1} . We denote $T := T_1$, $\mathcal{H}' := \mathcal{H}'_1$.

Since T is one-to-one, for any $\xi, \eta \in D(A_n)$, $T^{-1}\xi, T^{-1}\eta$ are well defined and one has:

$$\langle T^{-1}\xi, T^{-1}\eta \rangle_{\mathcal{H}'_n} = \langle \xi, \eta \rangle + \langle A_n\xi, A_n\eta \rangle$$

Thus, if $\xi, \eta \in D(H_n^2)$ which is a core for H ,

$$\langle T^{-1}\xi, T^{-1}\eta \rangle_{\mathcal{H}'_n} = \langle \xi, \eta \rangle + \langle A_n^* A_n \xi, \eta \rangle = \quad (13)$$

$$= \langle (1 + H_n^2)\xi, \eta \rangle \quad (14)$$

$$= \langle (1 + H_n^2)^{1/2}\xi, (1 + H_n^2)^{1/2}\eta \rangle \quad (15)$$

Therefore the map

$$T^{-1}(1 + H_n^2)^{-1/2} : D(H_n^2) \rightarrow \mathcal{H}' \quad (16)$$

extends to an isometry $W_n : \mathcal{H}_n \rightarrow \mathcal{H}'_n$. We still denote by W_n the partial isometry $W_n P_n : \mathcal{H} \rightarrow \mathcal{H}'$.

The sequence of projections $W_n W_n^* : \mathcal{H}' \rightarrow \mathcal{H}'$ is decreasing to $W_\infty W_\infty^*$, hence the sequence $(T W_n W_n^* T^*)$ is strongly convergent to $T W_\infty W_\infty^* T^*$.

It follows, using (16) that:

$$T W_n W_n^* T^* = (1 + H_n^2)^{-1/2} P_n (1 + H_n^2)^{-1/2} P_n = (1 + H_n^2)^{-1} P_n \quad (17)$$

converges strongly to $(1 + H_\infty^2)^{-1} P_\infty$. Therefore, since the sequence (17) is uniformly bounded in norm, for any polynomial p in one variable with $p(0) = 0$,

$$s - \lim_{n \rightarrow \infty} p \left(\frac{1}{1 + H_n^2} \right) P_n = p \left(\frac{1}{1 + H_\infty^2} \right) P_\infty \quad (18)$$

and since, by the Stone–Weierstrass theorem, the corresponding functions $p\left(\frac{1}{1+t^2}\right)$ ($t \in \mathbf{R}_+$) are norm dense among the continuous functions on \mathbf{R}_+ vanishing at $+\infty$, (5) holds.

Now we study the convergence of the sequence (W_n) . To this end, let $\xi \in D((TT^*)^{-1})$, $\eta \in \mathcal{H}$, and notice that, using (2.2) with $f(t) = (1+t^2)^{-1/2}$, the quantity

$$\begin{aligned} & \langle T^{-1}\xi, (W_n - W_\infty)\eta \rangle_{\mathcal{H}'} = \\ & = \langle T^{-1}\xi, T^{-1} [(1 + H_n^2)^{-1/2}P_n - (1 + H_\infty^2)^{-1/2}P_\infty] \eta \rangle_{\mathcal{H}'} \\ & = \langle (TT^*)^{-1}\xi, [(1 + H_n^2)^{-1/2}P_n - (1 + H_\infty^2)^{-1/2}P_\infty] \eta \rangle \end{aligned}$$

tends to zero as $n \rightarrow \infty$. Since the range of $(TT^*)^{-1}$ is dense (because from (13) it follows that on a dense set this operator coincides with $(1 + H^2)$), and $D((TT^*)^{-1})$ is a core for T^{-1} , it follows that $T^{-1}D((TT^*)^{-1})$ is dense in \mathcal{H}' and therefore (W_n) converges weakly to W_∞ .

Now notice that

$$w - \lim A_n T W_n = A_\infty T W_\infty \quad (19)$$

as follows from the weak convergence of W_n to W_∞ and the identity

$$A_n T W_n - A_\infty T W_\infty = A_1 T (W_n - W_\infty) \quad (20)$$

because $A_1 T : \mathcal{H}'_1 \rightarrow \mathcal{H}$ is bounded.

But (19) implies that

$$A_n T W_n = V_n H_n (1 + H_n^2)^{-1/2} P_n$$

hence (19) is equivalent to

$$w - \lim V_n H_n (1 + H_n^2)^{-1/2} P_n = V_\infty H_\infty (1 + H_\infty^2)^{-1/2} P_\infty \quad (21)$$

Now notice that

$$(V_n - V_\infty) H_n (1 + H_n^2)^{-1/2} P_n = \quad (22)$$

$$= [V_n H_n (1 + H_n^2)^{-1/2} P_n - V_\infty H_\infty (1 + H_\infty^2)^{-1/2} P_\infty] \quad (23)$$

$$+ [V_\infty H_\infty (1 + H_\infty^2)^{-1/2} P_\infty - V_\infty H_n (1 + H_n^2)^{-1/2} P_n] \quad (24)$$

The first term in square brackets in (22) tends to zero weakly because of (21). To the second one, we apply Remark (2.3) with the function

$$f(t) = t(1 + t^2)^{-1/2} \rightarrow 1, \quad \text{as } t \rightarrow +\infty \quad (25)$$

and conclude that its strong limit, as $n \rightarrow \infty$, exists and is equal to

$$-V_\infty(P - P_\infty) \quad (26)$$

where P is defined in (7). But $V_\infty = V_\infty P_\infty$ (cf. the remark after (3)) and $P \geq P_\infty$, hence the quantity (26) is zero and (22) weakly converges to zero.

Now we show that

$$B_n := V_n - V_\infty = V_n P_n - V_\infty P_\infty \quad (27)$$

converges weakly to zero by showing that any of its weak limit points is zero. The sequence (27) has weak limit points since it is bounded, let B be one of these and let (B_{n_k}) be a subnet weakly converging to B . Because of (25) it follows by Remark (2.3) as before that $H_n(1 + H_n^2)^{-1/2}P_n$ strongly converges to $H_\infty(1 + H_\infty^2)^{-1/2}P + (P - P_\infty)$, hence

$$w - \lim_{k \rightarrow \infty} B_{n_k} H_{n_k} (1 + H_{n_k}^2)^{-1/2} P_{n_k} = B [H_\infty (1 + H_\infty^2)^{-1/2} P_\infty + (P - P_\infty)] \quad (28)$$

But from the discussion after (22) we know that the limit of the left hand side of (28) is zero.

Since by Remark (2.4) we may suppose that each H_n is nonsingular, it follows that the range of $H_\infty(1 + H_\infty^2)^{-1/2}P_\infty$ is dense in $P_\infty \mathcal{H}$. Hence the vanishing of the left hand side of (28) is equivalent to $BP = 0$. But, since $P_n \geq P$

$$V_n(1 - P) = V_n P_n(1 - P) = V_n(P_n - P)$$

which tends to zero strongly for $n \rightarrow \infty$. Therefore $V_\infty(1 - P) = V_\infty P_\infty(1 - P) = 0$.

Thus $w - \lim(V_n - V_\infty)(1 - P) = 0$ and therefore also $B(1 - P) = 0$. Thus $B = 0$ and, since B is an arbitrary weak limit point of $V_n - V_\infty$, (6) follows.

We conclude this proof by mentioning that the increasing case, already dealt in [10], can be treated by the same argument as above, with the simplification that $P = P_\infty$ automatically. The above discussion remains valid with the change of notations:

$$T := T_\infty ; \quad \mathcal{H}' := \mathcal{H}_\infty$$

(the sequence of projection $(W_n W_n^*)$ is here increasing, but still converges strongly to $W_\infty W_\infty^*$). See also [16], [10]. **Corollary (2.6)** The limit in (6)

holds in the strong topology if and only if $P = P_\infty$, i.e. if and only if P_n converges strongly to P_∞ . **Proof.** Both statements follow from (11) and the

identity

$$V_n P_n = V_n P_\infty + V_n (P_n - P_\infty) \quad (29)$$

In fact, if $(V_n P_n)$ converges strongly to $V_\infty P_\infty$, then by (16), $V_n (P_n - P_\infty)$ tends to 0 strongly and therefore also $(P_n - P_\infty)$.

Conversely, if (P_n) converges strongly to P_∞ , then again by (11) and (29), it follows that $(V_n P_n)$ converges strongly to $V_\infty P_\infty$. **Remark (2.7)** In the

increasing case $P_n \uparrow P_\infty$ automatically therefore, in contrast to the decreasing case, H_n converges to H_∞ in the strong resolvent sense. This explains the apparent discrepancy between the increasing and decreasing case which was met in the previous literature on this problem [7], [12], [13], [22].

4 Convergence of modular structures

In this section (\mathcal{A}_n) shall denote a sequence of von Neumann algebras acting on a Hilbert space \mathcal{H} with a separating vector Φ . For each $n \in \mathbf{N}$ we denote

$$\mathcal{H}_n = [\mathcal{A}_n \Phi] \quad (30)$$

the closed cyclic space of \mathcal{A}_n with respect to Φ , and S_n the Tomita involution on \mathcal{H}_n , i.e. the closure of

$$a_n \Phi \mapsto a_n^* \Phi ; a_n \in \mathcal{A}_n$$

The polar decomposition of S_n is denoted

$$S_n = J_n \Delta_n^{1/2}$$

with the convention (1.4), if $P_n : \mathcal{H} \rightarrow \mathcal{H}_n$ denotes the orthogonal projection, one has

$$J_n = J_n P_n ; \Delta_n = \Delta_n P_n \quad (31)$$

Proposition 1 *Let (\mathcal{A}_n) be a decreasing sequence of von Neumann algebras and let*

$$\mathcal{A}_\infty = \bigcap_n \mathcal{A}_n \quad (32)$$

with S, J, Δ the corresponding Tomita operators and Q the projection on the space $[\mathcal{A}_\infty\Phi]$. Then (with the notations (3)):

$$S = \bigcap_n S_n \quad (33)$$

Proof. Let $S_\infty := \bigcap_n S_n$ and let $S_\infty = J_\infty\Delta_\infty$ denote its polar decomposition. The first step of the proof shows that S and S_∞ act on the same Hilbert space (i.e. $[\mathcal{A}_\infty\Phi]$), see also [17].

Let P_∞ be the orthogonal projection on the closure of the domain of S_∞ and $x \in \mathcal{A}'_1$. Then $x \in \mathcal{A}'_n$ and $xP_n \in \mathcal{A}'_n$ for each $n \in \mathbf{N}$. Therefore $J_n x J_n \in \mathcal{A}_n P_n$ so there exists $y_n \in \mathcal{A}_n$ satisfying

$$y_n P_n = J_n x J_n P_n ; \quad \|y_n\| = \|x\| \quad (34)$$

If the subnet (y_{n_k}) converges weakly to y , then

$$y_{n_k} \Phi = J_{n_k} x \Phi \quad (35)$$

and, because of Theorem 4, the right hand side of (35) converges weakly to $J_\infty x \Phi$. Thus

$$y \Phi = J_\infty x \Phi = P_\infty J_\infty x \Phi$$

On the other hand $y \in \mathcal{A}_\infty$ so $Qy\Phi = y\Phi$, therefore, since $x \in \mathcal{A}'_1$ is arbitrary:

$$QP_\infty J_\infty = P_\infty J_\infty$$

so that $Q \geq P_\infty$. Since $S \subset S_\infty$, we have $Q \leq P_\infty$ and therefore

$$Q = P_\infty \quad (36)$$

Remark (2.3) and (3.7) imply

$$s - \lim_{n \rightarrow \infty} \Delta_n^{\text{it}} Q = \Delta_\infty^{\text{it}} Q ; \quad t \in \mathbf{R} \quad (37)$$

Moreover, for each $x \in \mathcal{A}_\infty^+$ and for each $n \in \mathbf{N}$

$$\Delta_n^{\text{it}} x \Delta_n^{-\text{it}} Q = \sigma_t^{(n)}(x) Q \in \mathcal{A}_n Q$$

in particular $\Delta_n^{\text{it}} x \Phi \in \mathcal{A}_n^+ \Phi$ and therefore there exists $y_n \in \mathcal{A}_n^+$ satisfying

$$\Delta_n^{\text{it}} x \Phi = y_n \Phi ; \quad \|y_n\| = \|x\| \quad (38)$$

Taking a weakly convergent subnet of y_n in (38) we conclude that

$$\Delta_\infty^{\text{it}} x\Phi \in \mathcal{A}_\infty^+ \Phi ; \quad t \in \mathbf{R} \quad (39)$$

Since \mathcal{A}_∞ is spanned by its positive elements, there exists a positive linear map σ_t^∞ of \mathcal{A}_∞ in \mathcal{A}_∞ determined by

$$\sigma_t^\infty(x)\Phi = \Delta_\infty^{\text{it}} x\Phi ; \quad x \in \mathcal{A}_\infty \quad (40)$$

Since σ_{-t}^∞ is the inverse of σ_t^∞ , the maps σ_t^∞ form a one-parameter group of isometries (by positivity) of \mathcal{A}^∞ , hence a one-parameter group of automorphisms of \mathcal{A}^∞ [31]. As is well-known the canonicity of the modular operator (cf. [4], proof of theorem (3.2.18)) then implies that $\Delta_\infty^{\text{it}}$ and Δ^{is} commute for all $t, s \in \mathbf{R}$, hence $\Delta_\infty^{1/2}$ and $\Delta^{1/2}$ have a common core and therefore also S and S_∞ have a common core. But then $S_\infty = S$ because $S \subset S_\infty$ and both are closed operators.

Corollary (3.2) With the above notations $J_n P_n \rightarrow JP$ weakly and $f(\Delta_n)P_n \rightarrow f(\Delta)P$ strongly for every continuous $f : \mathbf{R}_+ \rightarrow \mathbf{R}$ vanishing at $+\infty$. Proof.

Immediate by Theorem 4 and Proposition 1.

5 Martingale convergence

Let now $\mathcal{B} \subseteq \mathcal{A}$ be an inclusion of von Neumann algebras and φ a faithful normal state of \mathcal{A} ; by considering the GNS representation we may assume that φ is determined by a cyclic and separating vector Φ for \mathcal{A} : $\varphi(x) = (x\Phi, \Phi)$, $x \in \mathcal{A}$. Let $E \in \mathcal{B}'$ be the orthogonal projection onto the closure $\overline{\mathcal{B}\Phi}$ of $\mathcal{B}\Phi$ and $J_\mathcal{A}$ and $J_\mathcal{B}$ the corresponding modular involutions (where $J_\mathcal{B}$ acts on $\overline{\mathcal{B}\Phi}$). The generalized expectation $\varepsilon : \mathcal{A} \rightarrow \mathcal{B}$ associated with φ is the completely positive map ε of \mathcal{A} into \mathcal{B} given by

$$\varepsilon(x)E = J_\mathcal{B} E J_\mathcal{A} x J_\mathcal{A} E J_\mathcal{B}$$

note in fact that

$$J_\mathcal{B} E J_\mathcal{A} \mathcal{A} J_\mathcal{A} E J_\mathcal{B} = J_\mathcal{B} E \mathcal{A}' E J_\mathcal{B} \subset J_\mathcal{B} E \mathcal{B}' E J_\mathcal{B} = \mathcal{B} E$$

by a double use of Tomita's theorem. Notice that if \mathcal{B} is globally invariant under the modular group of \mathcal{A} associated with φ , then ε is the usual Takesaki conditional expectation [25] while if Φ is cyclic for \mathcal{A} then ε is the canonical endomorphism of \mathcal{A} into \mathcal{B} [18], [19].

Theorem 2 *In the notations and assumptions of Proposition 1, let $\varepsilon_\infty : \mathcal{A}_1 \rightarrow \mathcal{A}_\infty$ denote the generalized conditional expectation associated with φ . Then (ε_n) converges pointwise weakly to ε_∞ .*

Proof. Let $x \in \mathcal{A}_1$. Since $(\varepsilon_n(x))$ is bounded, it will be sufficient to prove weak convergence on the dense subspace $\mathcal{A}'_1\Phi$. But, for each $n \in \mathbf{N} \cup \{\infty\}$

$$\varepsilon_n(x)\Phi = J_n P_n J_1 x \Phi$$

We can apply Corollary (3.2) to the sequence $(J_n P_n)$ and conclude that

$$w - \lim \varepsilon_n(x)\Phi = \varepsilon_\infty(x)\Phi$$

The statement easily follows from the cyclicity of Φ for \mathcal{A}'_1 .

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