SQUEEZING NOISES AS WEAK COUPLING LIMIT OF HAMILTONIAN SYSTEM

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We obtain a solution of quantum stochastic differential equation with squeezing noises (covariance is not a diagonal matrix) as weak coupling limit of a Hamiltonian system with a non gauge invariant quasi-free state.

§ 0. Introduction

The notion of squeezing noise has been introduced by Gardiner and Collet [9]. A squeezing noise is a quantum Brownian motion whose covariance has a nontrivial off-diagonal part. Physically this corresponds to the fact that the rate of creation (resp. of annihilation) of *pairs* of noise quanta in non zero. By taking Bogolubov transformations of standard quantum Brownian motions, it is easy to produce mathematical examples of squeezing noises. However this mathematical construction does not give insight into the physical meaning of the parameters defining the Bogolubov transformations. In [10] it was proved that the imaginary part of the off-diagonal terms of the covariance of the noise is related to a nonlinearity of the quantum Langevin equation and in the note [11] this fact was used to propose a simple experiment to distinguish between squeezing and non squeezing noise.

However, it is well known that, in the conditions of the weak coupling limit, the quantum Brownian motions are good approximations to the electromagnetic field and the quantum stochastic differential equations are limiting cases of the Schrödinger equation in interaction representation [1], [2], [3]. In view of this fact it would be desirable to express the coefficients of the noise, and in particular the squeezing coefficients, as functions of some quantities defining the original Hamiltonian model. The solution of this problem is the main goal of the present paper. Starting with a system coupled with the interaction (0.7) to an electromagnetic field in a non gauge invariant state, we deduce, in the weak coupling limit, a quantum stochastic differential equation driven by a squeezing Brownian motion whose coefficients are explicitly determined (cf. equation (6.6) below and the corresponding expressions (6.2), (6.3), (6.4), (6.5) for the coefficients).

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First of all let us formulate the problem precisely.

In [0], [1], [2], [3], and [4], a new approach to the weak coupling limit has been proposed, leading to the explicit determination not only of the limiting reduced dynamics but of the whole limiting process.

In the present paper, the notations and the model will be the same as in [1], but we shall start from a squeezing (i.e. non gauge invariant) Hamiltonian reservoir and we shall prove that the limiting process is driven by a squeezing noise.

Let H_0, H_1 be complex Hilbert spaces; Q a *real linear* operator on H_1; $W(H_1)$ the Weyl-algebra on H_1, φ_Q a mean zero quasi-free state on $W(H_1)$, with covariance **Q**, that is

$$\varphi_Q(W(f)) = \exp(-\tfrac{1}{2} Re\langle f, Qf\rangle),\ f\in H_1.$$

Let $\{H_Q, \pi_Q, \Phi_Q\}$ be the GNS triple of $(W(H_1), \varphi_Q)$; denote $\pi_Q(W(.))$ by $W_Q(.)$ and let $A_Q(f)$, $A_Q^+(f), f\in H_1$ be the associated annihilation and creation operators. Let H_R be the free Hamiltonian of the reservoir, H_S the free Hamiltonian of the system and

$$H^{(\lambda)} := H_S \otimes 1 + 1 \otimes H_R + \lambda V \tag{0.1}$$

with

$$V := -\frac{1}{i}(D\otimes A_Q^+(g) - D^+\otimes A_Q(g)),$$

$$H_R = d\Gamma(-H) \tag{0.2}$$

the total Hamiltonian of the composite system. H is a self-adjoint operator on H_1, $S_t^0 = e^{itH}$ is a unitary group, D is a bounded operator on H_0, and we assume that

$$Ade^{-itH_s}(D) = e^{-i\omega_0 t}D,\ \omega_0 \geqslant 0 \tag{0.3}$$

(rotating wave approximation). Put

$$U^{(\lambda)}(t) = e^{itH^{(o)}}e^{-itH^{(\lambda)}}. \tag{0.4}$$

One has

$$\frac{d}{dt}U^{(\lambda)}(t) = \frac{1}{i}\lambda V(t)\,U^{(\lambda)}(t), \tag{0.5}$$

where

$$H^{(o)} := H_S\otimes 1 + 1\otimes H_R, \tag{0.6}$$

$$\begin{aligned} V(t) := e^{-itH^{(o)}}Ve^{itH^{(o)}} &= -\frac{1}{i}(D\otimes e^{-i\omega_0 t}A_Q^+(S_t^0 g) - D^+\otimes e^{i\omega_0 t}A_Q(S_t^0 g)) \\ &= -\frac{1}{i}(D\otimes A_Q^+(S_t g) - D^+\otimes A_Q(S_t g)) \end{aligned} \tag{0.7}$$

and

$$S_t = e^{-i\omega_0 t} S_t^0, \ t \in \mathbf{R}.$$

Moreover, we assume that

$$QS_t = S_t Q. \tag{0.8}$$

As in [1] suppose that there exists a non-zero subspace $K \subseteq \mathrm{Dom}(Q)$ (in all the examples it will be a dense subspace) such that

$$\int_{\mathbf{R}} |\langle f_1, S_t Q f_2 \rangle| \, dt < \infty, \ \forall f_1, f_2 \in K. \tag{0.9}$$

This condition implies that the sesquilinear form

$$f_1, f_2 \colon K \to (f_1 | f_2) := \int_{\mathbf{R}} \langle f_1, S_t Q f_2 \rangle \, dt \tag{0.10}$$

defines a pre-scalar product on K. We shall also denote by K the associated Hilbert space, i.e. the completion of the quotient of K by the zero norm elements with respect to the scalar product (0.13). The scalar product on K will also be denoted $(.|.)$.

In [1] and [2] we have proved that, for $Q \geqslant 1$ *positive*, the limits

$$\lim_{\lambda \to 0} \langle u \otimes W_Q(\lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1 \, du) \Phi_Q, U^{(\lambda)}(t/\lambda^2) v \otimes W_Q(\lambda \int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_2 \, du) \Phi_Q \rangle \tag{0.11a}$$

and

$$\lim_{\lambda \to 0} \langle u \otimes W_Q(\lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1 \, du) \Phi_Q,$$

$$U^{(\lambda)}(t/\lambda^2)(X \otimes 1) U^{(\lambda)}(t/\lambda^2)^+ v \otimes W_Q(\lambda \int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u f_2 \, du) \Phi_Q \rangle \tag{0.11b}$$

exist and are equal to

$$\langle u \otimes W_{1 \otimes Q}(\chi_{[S_1, T_1]} \otimes f_1) \Psi_{1 \otimes Q}, U(t) v \otimes W_{1 \otimes Q}(\chi_{[S_2, T_2]} \otimes f_2) \Psi_{1 \otimes Q} \rangle, \tag{0.12a}$$

$$\langle u \otimes W_{1 \otimes Q}(\chi_{[S_1, T_1]} \otimes f_1) \Psi_{1 \otimes Q}, U(t)(X \otimes 1) U(t)^+ v \otimes W_{1 \otimes Q}(\chi_{[S_2, T_2]} \otimes f_2) \Psi_{1 \otimes Q} \rangle, \tag{0.12b}$$

respectively, where, $\{H_{1 \otimes Q}, \Psi_{1 \otimes Q}, W_{1 \otimes Q}(\chi_{[S,T]} \otimes f)\}$ is the Brownian motion on $L^2(\mathbf{R}, dt; K)$. Moreover, from [1] and [2], we know that $U(t)$ satisfies the quantum stochastic differential equation

$$U(t) = 1 + \int_0^t (D \otimes dA_{1 \otimes Q}^+(s, g) - D^+ \otimes dA_{1 \otimes Q}(s, g) -$$

$$- (Q^+ g | Q^+ g)_- D^+ D \otimes 1 \, ds - (Q^- g | Q^- g)_+ DD^+ \otimes 1 \, ds)) U(s). \tag{0.13}$$

In the present paper we shall study the limits (0.11a) for Q *real linear* and are given by the expression (0.12a), (0.12b) respectively but now $U(t)$ is the solution of the stochastic differential equation with squeezing noise.

In the sequel, we shall need the following results proved in [1].

LEMMA (0.1). *For each $g \in \mathrm{Dom}(Q)$ and for any $-\infty < S \leqslant T < +\infty$, the integral*

$$\int_S^T S_t g dt \tag{0.14}$$

is well defined and belongs to $\mathrm{Dom}(Q)$, *moreover,*

$$Q \int_S^T S_t g dt = \int_S^T Q S_t g dt. \tag{0.15}$$

LEMMA (0.2). *For each pair $f, g \in K$ satisfying* (0.11) *and for any $S_1, T_1, S_2, T_2 \in \mathbf{R}$ $(S_j \leqslant T_j, j = 1, 2)$, one has*

$$\lim_{\lambda \to o} \langle \lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f du, \lambda \int_{S_2/\lambda^2}^{T_2/\lambda^2} S_u g du \rangle = \langle \chi_{[S_1, T_1]}, \chi_{[S_2, T_2]} \rangle \int_{\mathbf{R}} \langle f, S_t g \rangle dt, \tag{0.16}$$

where, the scalar product of the characteristic functions is defined in $L^2(\mathbf{R})$ and the limit is uniform for S_1, T_1, S_2, T_2 in a bounded set of $\mathbf{R}$.

LEMMA (0.3). *For each $n \in \mathbf{N}, f_1, \ldots, f_n \in K, S_1, T_1, \ldots, S_n, T_n \in \mathbf{R}, x_1, \ldots, x_n \in \mathbf{R}$, the limit*

$$\lim_{\lambda \to o} \langle \Phi_Q, W_Q(x_1 \lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} S_u f_1 du) \ldots W_Q(x_n \lambda \int_{S_n/\lambda^2}^{T_n/\lambda^2} S_u f_n du) \Phi_Q \rangle \tag{0.17}$$

exists uniformly for $x_1, \ldots, x_n, S_1, \ldots, S_n, T_1, \ldots, T_n$ in a bounded set of $\mathbf{R}$.

In the paper, we would like to write

$$\begin{aligned} V_g(t) &:= i(D \otimes A_Q^+(S_t g) - D^+ \otimes A_Q(S_t g)) \\ &= \frac{i}{2}(D \otimes B_Q(S_t ig) + iD \otimes B_Q(S_t g) - D^+ \otimes B_Q(S_t ig) + iD^+ \otimes B_Q(S_t g)) \\ &= \frac{i}{2}((D - D^+) \otimes B_Q(S_t ig) + i(D + D^+) \otimes B_Q(S_t g)) \\ &=: \frac{i}{2} \sum_{\varepsilon \in \{0, 1\}} D_\varepsilon \otimes B_Q(S_t i^\varepsilon g), \end{aligned} \tag{0.18}$$

where

$$D_\varepsilon = \begin{cases} D - D^+, & \text{if } \varepsilon = 1; \\ i(D + D^+), & \text{if } \varepsilon = 0. \end{cases} \tag{0.19}$$

So, one can get the following

$$U^{(\lambda)}(t/\lambda^2) = 1 + \sum_{n=1}^{\infty} (-i)^n \lambda^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n$$

$$\sum_{\varepsilon\in\{0,1\}^n} \left(\frac{i}{2}\right)^n D_{\varepsilon(1)} \ldots D_{\varepsilon(n)} \otimes B_Q(i^{\varepsilon(1)} S_t g) \ldots B_Q(i^{\varepsilon(n)} S_t g). \quad (0.20)$$

For simplicity, in the following, we shall denote W_Q, B_Q, A_Q, A_Q^+ by W, B, A, A^+, respectively.

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§ 1. The collective terms and the negligible terms

As in [1], we begin to investigate the collective terms and the negligible terms.

LEMMA (1.1). *For each* $n \in \mathbf{N}$, $f, f', g_1, \ldots, g_n \in H$,

$$\langle W(f)\Phi_Q, B(g_1)\ldots B(g_n) W(f')\Phi_Q\rangle$$
$$= \langle W(f)\Phi_Q, W(f')\Phi_Q\rangle \cdot P_n(s_1, \ldots, s_n; t_{1,2}, \ldots, t_{n-1,n}), \quad (1.1)$$

where

$$s_j := s_j(f, f', g_j) := \frac{i}{2} Re(\langle g_j, Q(f'-f)\rangle + \langle f'-f, Qg_j\rangle) + Im\langle f+f', g_j\rangle, \quad (1.2)$$

$$t_{j,k} := t_{j,k}(g_j, g_k) := \tfrac{1}{2} Re(\langle g_j, Qg_k\rangle + \langle g_k, Qg_j\rangle) + iIm\langle g_j, g_k\rangle \quad (1.3)$$

and P_n *is a polynomial defined by:*

$$P_0 = 1, \quad (1.4)$$

$$P_n(s_1, \ldots, s_n; t_{1,2}, \ldots, t_{n-1,n}) = \sum_{m=0}^{[n/2]} \sum_{2 \leq q_1 < \ldots < q_m \leq n}$$

$$\sum_{\substack{1 \leq p_1, \ldots, p_m \leq n, |\{p_h\}_{h=1}^m| = m \\ \{p_h\}_{h=1}^m \cap \{q_h\}_{h=1}^m = \emptyset,\ p_h < q_h,\ h = 1, \ldots, m}} \prod_{h=1}^{m} t_{p_h, q_h} \cdot \prod_{\alpha \in \{1, \ldots, n\} \setminus \{p_h, q_h\}_{h=1}^m} s_\alpha. \quad (1.5)$$

Moreover, the polynomials P_n *satisfy the following recursion relation*:

$$P_{n+1}(s_1, \ldots, s_n, s_{n+1}; t_{1,2}, \ldots, t_{n,n+1}) = s_{n+1} P_n(s_1, \ldots, s_n; t_{1,2}, \ldots, t_{n-1,n}) +$$

$$+ \sum_{j=1}^{n} \frac{\partial}{\partial s_j} P_n(s_1, \ldots, s_n; t_{1,2}, \ldots, t_{n-1,n}) \cdot t_{j,n+1}. \quad (1.6)$$

Remark: Lemma (1.1) is an improvement of Lemma (3.2) of [6] in the sense that, while in [6] only the recursion relation (1.6) was proved, here we derive the explicit form of polynomials P_n.

Proof: The results are proved by induction on n. The lemma is obviously true for $n = 0, 1$. Suppose now that (1.6) holds for $n = 0, 1, \ldots, k$, then we have

$$\langle W(f)\Phi_Q, B(g_1)\ldots B(g_{k+1})W(f')\Phi_Q\rangle$$

$$= \frac{d}{id\lambda}[\langle W(f)\Phi_Q, B(g_1)\ldots B(g_k)W(\lambda g_{k+1}+f')\Phi_Q\rangle e^{-iIm\langle\lambda g_{k+1},f'\rangle}]_{\lambda=0}$$

$$= \frac{d}{id\lambda}[\langle W(f)\Phi_Q, W(\lambda g_{k+1}+f')\Phi_Q\rangle e^{-iIm\langle\lambda g_{k+1},f'\rangle}\cdot$$

$$\cdot P_k(s_1(\lambda), \ldots, s_k(\lambda); t_{1,2}, \ldots, t_{k-1,k})]_{\lambda=0}, \tag{1.7}$$

where $t_{j,k}$ is same as in (1.3) and

$$s_j(\lambda) := \frac{i}{2}Re(\langle g_j, Q(f'+\lambda g_{k+1}-f)\rangle + \langle f'+\lambda g_{k+1}-f, Qg_j\rangle) + Im\langle f+\lambda g_{k+1}+f', g_j\rangle. \tag{1.8}$$

Thus,

$$\frac{d}{id\lambda}s_j(\lambda) = t_{j,k+1}. \tag{1.9}$$

Moreover,

$$\frac{d}{id\lambda}[\langle W(f)\Phi_Q, W(\lambda g_{k+1}+f')\Phi_Q\rangle e^{-iIm\langle\lambda g_{k+1},f'\rangle}]_{\lambda=0}$$

$$= \frac{d}{id\lambda}\Big[\exp\Big(-\frac{1}{2}Re\langle\lambda g_{k+1}+f'-f, Q(\lambda g_{k+1}+f'-f)\rangle\Big)\cdot$$

$$\cdot\exp(-iIm\langle\lambda g_{k+1},f'\rangle - iIm\langle -f, \lambda g_{k+1}+f'\rangle)\Big]_{\lambda=0}$$

$$= \langle W(f)\Phi_Q, W(f')\Phi_Q\rangle\cdot\frac{1}{i}\Big[-\frac{1}{2}Re(\langle g_{k+1}, Q(f'-f)\rangle + \langle f'-f, Qg_{k+1}\rangle) +$$

$$+ i(Im\langle f', g_{k+1}\rangle + Im\langle f, g_{k+1}\rangle)\Big] = \langle W(f)\Phi_Q, W(f')\Phi_Q\rangle\cdot s_{k+1}. \tag{1.10}$$

Using (1.9), (1.10) to compute the derivative in (1.7), we get (1.6). Now, we prove (1.5).

Notice that (1.4), (1.6) define a unique polynomial P_n. In fact, if $P_n^{(1)}$, $P_n^{(2)}$ satisfy (1.4), (1.6), then, $P_n := P_n^{(1)} - P_n^{(2)}$ satisfies (1.6) with $P_0 = 0$, therefore, for each

$n \geqslant 1$, $P_n = 0$. Now we show that the right-hand side of (1.5) satisfies (1.4) and (1.6). In the following, we shall denote the right-hand side of (1.5) by Q_n. Clearly, $Q_0 = 1$, and for n even ($= 2k$),

$$\sum_{j=1}^{n} \frac{\partial}{\partial s_j} \sum_{m=0}^{[n/2]} \sum_{2 \leqslant q_1 < \ldots < q_m \leqslant n} \sum_{\substack{1 \leqslant p_1, \ldots, p_m \leqslant n,\, p_h < q_h \\ |\{p_h, q_h\}_{h=1}^m| = 2m}} \prod_{h=1}^{m} t_{p_h, q_h} \cdot \prod_{\alpha \in \{1, \ldots, n\} \setminus \{p_h, q_h\}_{h=1}^m} s_\alpha \cdot t_{j, n+1}$$

$$= \sum_{m=0}^{[n/2]-1} \sum_{2 \leqslant q_1 < \ldots < q_m \leqslant n} \sum_{\substack{1 \leqslant p_1, \ldots, p_m \leqslant n,\, p_h < q_h \\ |\{p_h, q_h\}_{h=1}^m| = 2m}} \sum_{j \in \{1, \ldots, n\} \setminus \{p_h, q_h\}_{h=1}^m} \prod_{h=1}^{m} t_{p_h, q_h} \cdot \prod_{\alpha \in \{1, \ldots, n\} \setminus (\{p_h, q_h\}_{h=1}^m \cup \{j\})} s_\alpha \cdot t_{j, n+1}, \tag{1.11}$$

where $0 \leqslant m \leqslant [n/2]-1$ in the right-hand side of (1.11) is due to the fact that if $m = [n/2]$ then the polynomial in the left-hand side of (1.11) has only variables $t_{i,j}$, so the derivative for s_j is zero. Put $q_{m+1} = n+1$, $p_{m+1} = j$, then the left-hand side of (1.11) is equal to

$$\sum_{m=0}^{[n/2]-1} \sum_{2 \leqslant q_1 < \ldots < q_m < q_{m+1} = n+1} \sum_{\substack{1 \leqslant p_1, \ldots, p_m, p_{m+1} \leqslant n+1,\, p_h < q_h \\ |\{p_h, q_h\}_{h=1}^{m+1}| = 2(m+1)}} \prod_{h=1}^{m+1} t_{p_h, q_h} \cdot \prod_{\alpha \in \{1, \ldots, n+1\} \setminus \{p_h, q_h\}_{h=1}^{m+1}} s_\alpha . \tag{1.12}$$

Put $m' = m+1$, then, $1 \leqslant m' \leqslant [n/2] = [2k/2] = k = [(2k+1)/2] = [(n+1)/2]$, thus the left-hand side of (1.11) is equal

$$\sum_{m'=1}^{[(n+1)/2]} \sum_{2 \leqslant q_1 < \ldots < q_{m'} = n+1} \sum_{\substack{1 \leqslant p_1, \ldots, p_{m'} \leqslant n+1,\, p_h < q_h \\ |\{p_h, q_h\}_{h=1}^{m'}| = 2m'}} \prod_{h=1}^{m'} t_{p_h, q_h} \cdot \prod_{\alpha \in \{1, \ldots, n+1\} \setminus \{p_h, q_h\}_{h=1}^{m'}} s_\alpha . \tag{1.13}$$

For n odd,

$$\sum_{j=1}^{n} \frac{\partial}{\partial s_j} \sum_{m=0}^{[n/2]} \sum_{2 \leqslant q_1 < \ldots < q_m \leqslant n} \sum_{\substack{1 \leqslant p_1, \ldots, p_m \leqslant n,\, p_h < q_h \\ |\{p_h, q_h\}_{h=1}^m| = 2m}} \prod_{h=1}^{m} t_{p_h, q_h} \cdot \prod_{\alpha \in \{1, \ldots, n\} \setminus \{p_h, q_h\}_{h=1}^m} s_\alpha \cdot t_{j, n+1}$$

$$= \sum_{m=0}^{[(n-1)/2]} \sum_{2 \leqslant q_1 < \ldots < q_m \leqslant n} \sum_{\substack{1 \leqslant p_1, \ldots, p_m \leqslant n,\, p_h < q_h \\ |\{p_h, q_h\}_{h=1}^m| = 2m}} \sum_{j \in \{1, \ldots, n\} \setminus \{p_h, q_h\}_{h=1}^m} \prod_{h=1}^{m} t_{p_h, q_h} \cdot \prod_{\alpha \in \{1, \ldots, n\} \setminus (\{p_h, q_h\}_{h=1}^m \cup \{j\})} s_\alpha \cdot t_{j, n+1} . \tag{1.14}$$

Put $q_{m+1} = n+1$, $p_{m+1} = j$, $m' = m+1$, then $1 \leqslant m' \leqslant [(n-1)/2]+1 = [(n+1)/2]$, so, the left-hand side of (1.14) is equal to

$$\sum_{m'=1}^{[(n+1)/2]} \sum_{2 \leqslant q_1 < \ldots < q_{m'} = n+1} \sum_{\substack{1 \leqslant p_1, \ldots, p_{m'} \leqslant n+1,\, p_h < q_h \\ |\{p_h, q_h\}_{h=1}^{m'}| = 2m'}} \prod_{h=1}^{m'} t_{p_h, q_h} \cdot \prod_{\alpha \in \{1, \ldots, n+1\} \setminus \{p_h, q_h\}_{h=1}^{m'}} s_\alpha . \quad (1.15)$$

Hence, for each $n \geqslant 1$, (1.13) is equal to the left-hand side of (1.11), i.e. for each $n \geqslant 1$,

$$Q_{n+1} - \sum_{j=1}^{n} \frac{\partial}{\partial s_j} Q_n \cdot t_{j,n+1}$$

$$= \sum_{m=0}^{[(n+1)/2]} \sum_{2 \leqslant q_1 < \ldots < q_m \leqslant n+1} \sum_{\substack{1 \leqslant p_1, \ldots, p_m \leqslant n+1,\, p_h < q_h \\ |\{p_h, q_h\}_{h=1}^{m}| = 2m}} \prod_{h=1}^{m} t_{p_h, q_h} \cdot \prod_{\alpha \in \{1, \ldots, n+1\} \setminus \{p_h, q_h\}_{h=1}^{m}} s_\alpha -$$

$$- \sum_{m=1}^{[(n+1)/2]} \sum_{2 \leqslant q_1 < \ldots < q_m = n+1} \sum_{\substack{1 \leqslant p_1, \ldots, p_m \leqslant n+1,\, p_h < q_h \\ |\{p_h, q_h\}_{h=1}^{m}| = 2m}} \prod_{h=1}^{m} t_{p_h, q_h} \cdot \prod_{\alpha \in \{1, \ldots, n+1\} \setminus \{p_h, q_h\}_{h=1}^{m}} s_\alpha . \quad (1.16)$$

Notice that in the first term, m starts at 0 and $q_m \leqslant n+1$, while in the second, m starts at 1 and $q_m = n+1$. Thus (1.16) is equal to

$$\prod_{\alpha \in \{1, \ldots, n+1\}} s_\alpha + \sum_{m=1}^{[(n+1)/2]} \sum_{2 \leqslant q_1 < \ldots < q_m \leqslant n} \sum_{\substack{1 \leqslant p_1, \ldots, p_m \leqslant n,\, p_h < q_h \\ |\{p_h, q_h\}_{h=1}^{m}| = 2m}} \prod_{h=1}^{m} t_{p_h, q_h} \cdot s_{n+1} \cdot \prod_{\alpha \in \{1, \ldots,\} \setminus \{p_h, q_h\}_{h=1}^{m}} s_\alpha . \quad (1.17)$$

If n is even, then $[(n+1)/2] = [n/2]$. If n is odd, then since $q_m \leqslant n$, $p_h < q_h$, therefore

$$|\{1, \ldots, n+1\} \setminus \{p_h, q_h\}_{h=1}^{m}| \geqslant 2 \quad (1.18)$$

and hence,

$$m \leqslant \tfrac{1}{2}(n+1-2) = \tfrac{1}{2}(n-1) = [n/2]. \quad (1.19)$$

Thus, in both cases, (1.17) is

$$s_{n+1} \cdot Q_n$$

and this ends the proof.

In the following, we shall rewrite $P_n(\{s_j\}_{j=1}^{n}, \{t_{j,k}\}_{1 \leqslant j < k \leqslant n})$ as the sum of two terms:

$$P_n(\{s_j\}_{j=1}^{n}, \{t_{j,k}\}_{1 \leqslant j < k \leqslant n}) = I_g(n) + II_g(n) \quad (1.20)$$

with

$$I_g(n) = \sum_{m=0}^{[n/2]} \sum_{\substack{2 \leqslant q_1 < \ldots < q_m \leqslant n \\ \{q_h - 1\}_{h=1}^{m} \cap \{q_h\}_{h=1}^{m} = \emptyset}} \prod_{h=1}^{m} t_{q_h - 1, q_h} \cdot \prod_{\alpha \in \{1, \ldots, n\} \setminus \{q_h, q_h - 1\}_{h=1}^{m}} s_\alpha \quad (1.21)$$

and

$$II_g(n) = \sum_{m=0}^{[n/2]} \sum_{2 \leqslant q_1 < \ldots < q_m \leqslant n} \sum_{(p_1, q_1, \ldots, p_m, q_m)}' \prod_{h=1}^{m} t_{p_h, q_h} \cdot \prod_{\alpha \in \{1, \ldots, n\} \setminus \{q_h, p_h\}_{h=1}^m} s_\alpha, \tag{1.22}$$

where, $\sum'_{(p_1, q_1, \ldots, p_m, q_m)}$ means the sum for all $1 \leqslant p_1, \ldots, p_m \leqslant n$, such that $p_h \neq p_k$ for $h \neq k$, $p_h < q_h$, $h = 1, \ldots, m$, and for some $h = 1, \ldots, m$, $q_h - p_h \geqslant 2$.

For each $n \in \mathbf{N}$, $\varepsilon = (\varepsilon(1), \ldots, \varepsilon(n)) \in \{0, 1\}^n$, $g \in K$, denote

$$i^{\varepsilon(k)} S_{\tau_k} g =: g_k$$

and replace f, f' in (1.2), respectively, with

$$\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f\, du, \quad \lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f'\, du. \tag{1.23}$$

So, for each j, k,

$$t_{j,k} := t^{\varepsilon}_{j,k}$$

$$= \tfrac{1}{2} Re(\langle S_{\tau_j} i^{\varepsilon(j)} g, S_{\tau_k} Q i^{\varepsilon(k)} g\rangle + \langle S_{\tau_j} i^{\varepsilon(j)} g, S_{\tau_k} Q i^{\varepsilon(k)} g\rangle) + i Im \langle S_{\tau_j} i^{\varepsilon(j)} g, S_{\tau_k} i^{\varepsilon(k)} g\rangle \tag{1.24}$$

and

$$s_j := s^{\varepsilon}_j(\lambda) = \frac{i}{2} Re(\langle S_{\tau_j} i^{\varepsilon(j)} g, \lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u Q f'\, du - \lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u Q f\, du\rangle +$$

$$+ \langle \lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f'\, du - \lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f\, du, S_{\tau_j} Q i^{\varepsilon(j)} g\rangle) +$$

$$+ Im \langle \lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f'\, du + \lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f\, du, S_{\tau_j} i^{\varepsilon(j)} g\rangle. \tag{1.25}$$

The expressions obtained from (1.21) and (1.22) with these substitutions will be denoted by $I^{n;\varepsilon}_{g,\lambda}$ and $II^{n;\varepsilon}_{g,\lambda}$, respectively. Finally, for each $n \in \mathbf{N}$, $\varepsilon \in \{0, 1\}^n$, we define

$$I^{\varepsilon}_g(n, \lambda) = \lambda^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n_1}} dt_n I^{n;\varepsilon}_{g,\lambda}, \tag{1.26}$$

$$II^{\varepsilon}_g(n, \lambda) = \lambda^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n_1}} dt_n II^{n;\varepsilon}_{g,\lambda}. \tag{1.27}$$

§ 2. The limit of the collective terms and of the negligible terms

In the section, we shall investigate the limit of $I^{\varepsilon}_g(n, \lambda)$ and of $II^{\varepsilon}_g(n, \lambda)$ as $\lambda \to 0$.

THEOREM (2.1). *For each $g \in K$, $n \in \mathbf{N}$, $\varepsilon \in \{0, 1\}^n$,*

$$\lim_{\lambda \to 0} II^{\varepsilon}_g(n, \lambda) = 0. \tag{2.1}$$

Proof: For each $m = 0, 1, \ldots, [n/2]$, $2 \leqslant q_1 < \ldots < q_m \leqslant n$, $1 \leqslant p_1, \ldots, p_m \leqslant n-1$, one has

$$|II_g^\varepsilon(n, \lambda)| \leqslant \sum_{m=0}^{[n/2]} \sum_{2 \leqslant q_1 < \ldots < q_m \leqslant n} \sum_{(p_1, q_1, \ldots, p_m, q_m)}' \int_0^t dt_1 \ldots \int_0^{t_{q_1-2}} dt_{q_1-1} \int_{-t_{p_1}/\lambda^2}^{(t_{q_1-1}-t_{p_1})/\lambda^2} dt_{q_1} \int_0^{t_{q_1}} dt_{q_1+1} \ldots$$

$$\int_0^{t_{q_m-2}} dt_{q_m-1} \ldots \int_{-t_{p_m}/\lambda^2}^{(t_{q_m-1}-t_{p_m})/\lambda^2} dt_{q_m} \int_0^{t_{q_m}} dt_{q_m+1} \ldots \int_0^{t_{n-1}} dt_n$$

$$\prod_{h=1}^m \left[\frac{1}{2}(|\langle i^{\varepsilon(p_h)} g, S_{t_{q_h}} Q i^{\varepsilon(q_h)} g\rangle| + |\langle Q i_{\varepsilon(p_h)} g, S_{t_{q_h}} i^{\varepsilon(q_h)} g\rangle|) + |\langle i_{\varepsilon(p_h)} g, S_{t_{q_h}} i^{\varepsilon(q_h)} g\rangle|\right]$$

$$\prod_{\alpha \in \{1, \ldots, n\} \setminus \{p_h, q_h\}_{h=1}^m} \left[\frac{1}{2} \int_{-\infty}^{\infty} du(|\langle g, S_u f\rangle| + |\langle g, S_u f'\rangle|) \int_{-\infty}^{\infty} du(|\langle g, S_u Q f'\rangle| + \right.$$

$$\left. + |\langle g, S_u Q f\rangle| + |\langle Q i^{\varepsilon(\alpha)} g, S_u f'\rangle| + |\langle Q i^{\varepsilon(\alpha)} g, S_u f\rangle|)\right]. \tag{2.2}$$

From the proof of Lemma (4.2) in [1] one knows that there exists a constant C_1, such that

$$|II_g^\varepsilon(n, \lambda)| \leqslant \sum_{m=0}^{[n/2]} \sum_{2 \leqslant q_1 < \ldots < q_m \leqslant n} \sum_{(p_1, q_1, \ldots, p_m, q_m)}' C_1^{n-2m}.$$

$$\cdot \int_0^t dt_1 \ldots \int_0^{t_{q_1-2}} dt_{q_1-1} \int_{-t_{p_1}/\lambda^2}^{(t_{q_1-1}-t_{p_1})/\lambda^2} dt_{q_1} \int_0^{t_{q_1}} dt_{q_1+1} \ldots \int_0^{t_{q_m-2}} dt_{q_m-1} \ldots$$

$$\int_{-t_{p_m}/\lambda^2}^{(t_{q_m-1}-t_{p_m})/\lambda^2} dt_{q_m} \int_0^{t_{q_m}} dt_{q_m+1} \ldots \int_0^{t_{n-1}} dt_n \prod_{h=1}^m \left[\tfrac{1}{2}(|\langle i^{\varepsilon(p_h)} g, S_{t_{q_h}} Q i^{\varepsilon(q_h)} g\rangle| + \right.$$

$$\left. + |\langle Q i_{\varepsilon(p_h)} g, S_{t_{q_h}} i^{\varepsilon(q_h)} g\rangle|) + |\langle i_{\varepsilon(p_h)} g, S_{t_{q_h}} i^{\varepsilon(q_h)} g\rangle|\right]. \tag{2.3}$$

By the definition of $\sum'_{(p_1, q_1, \ldots, p_m, q_m)}$, we know that there exists at least one $h = 1, \ldots, m$, such that $q_h - p_h \geqslant 2$. For this h one has $t_{q_h-1} - t_{p_h} < 0$ almost everywhere, so, applying Lemma (4.2) of [1], we deduce that

$$\lim_{\lambda \to 0} |II_g^\varepsilon(n, \lambda)| = 0.$$

THEOREM (2.2). *For each* $n \in \mathbf{N}$, $\varepsilon \in \{0, 1\}^n$,

$$\lim_{\lambda \to 0} I_g^\varepsilon(n, \lambda) = \sum_{m=0}^{[n/2]} \sum_{\substack{2 \leqslant q_1 < \ldots < q_m \leqslant n, \\ \{q_h - 1\}_{h=1}^m \cap \{q_h\}_{h=1}^m = \emptyset}}$$

$$\int_{0\leqslant t_n\leqslant\ldots\leqslant t_{q_m}\leqslant\ldots\leqslant t_{q_1}\leqslant\ldots\leqslant t_1\leqslant t} dt_1\ldots dt_{q_1}\ldots dt_{q_m}\ldots dt_n$$

$$\prod_{h=1}^{m}\left[\frac{1}{2}Re\big((i^{\varepsilon(q_h-1)}g|\,Qi^{\varepsilon(q_h)}g)_- + \overline{(Qi^{\varepsilon(q_h-1)}g\,|\,i^{\varepsilon(q_h)}g)_-}\big)+iIm\,(i^{\varepsilon(q_h-1)}g\,|\,i^{\varepsilon(q_h)}g)_-\right]\cdot$$

$$\prod_{\alpha\in\{1,\ldots,n\}\setminus\{q_h,\,q_h-1\}_{h=1}^m}\left[\frac{i}{2}Re\big((i^{\varepsilon(\alpha)}g|Qf')\cdot\chi_{[S',T']}(t_\alpha)-(i^{\varepsilon(\alpha)}g|Qf)\cdot\chi_{[S,T]}(t_\alpha)+\right.$$
$$+(f'|Qi^{\varepsilon(\alpha)}g)\cdot\chi_{[S',T']}(t_\alpha)-(f|Qi^{\varepsilon(\alpha)}g)\cdot\chi_{[S,T]}(t_\alpha)\big)-$$
$$\left.-Im\,(i^{\varepsilon(\alpha)}g|f)\cdot\chi_{[S,T]}(t_\alpha)-Im\,(i^{\varepsilon(\alpha)}g|f')\cdot\chi_{[S',T']}(t_\alpha)\right]. \tag{2.4}$$

Proof: Since these are the type I terms, it follows that for each $m = 0, 1, \ldots, [n/2]$ and for each $2 \leqslant q_1 < \ldots < q_m \leqslant n$, one has $q_h - 1 = p_h$. Therefore with the same change of variables as used in the proof of (5.10) in [1], one gets

$$I_g^\varepsilon(n,\lambda)=\sum_{m=0}^{[n/2]}\sum_{\substack{2\leqslant q_1<\ldots<q_m\leqslant n\\ \{q_h-1\}_{h=1}^m\cap\{q_h\}_{h=1}^m=\emptyset}}$$

$$\int_0^t dt_1\ldots\int_0^{t_{q_1-2}}dt_{q_1-1}\int_{-t_{q_1-1}/\lambda^2}^{0}dt_{q_1}\int_0^{\lambda^2t_{q_1}+t_{q_1-1}}dt_{q_1+1}$$

$$\ldots\int_0^{t_{q_m-2}}dt_{q_m-1}\int_{t_{q_m-1}/\lambda^2}^{0}dt_{q_m}\int_0^{\lambda^2t_{q_m}+t_{q_m-1}}dt_{q_m+1}\ldots\int_0^{t_{n-1}}dt_n$$

$$\prod_{h=1}^{m}\left[\frac{1}{2}Re\big(\langle i^{\varepsilon(q_h-1)}g,\,S_{t_{q_h}}Qi^{\varepsilon(q_h)}g\rangle+\overline{\langle Qi^{\varepsilon(q_h-1)}g,\,S_{t_{q_h}}i^{\varepsilon(q_h)}g\rangle}\big)+\right.$$
$$\left.+iIm\,\langle i^{\varepsilon(q_h-1)}g,\,S_{t_{q_h}}i^{\varepsilon(q_h)}g\rangle\right]\cdot$$

$$\prod_{\alpha\in\{1,\ldots,n\}\setminus\{q_h,\,q_h-1\}_{h=1}^m}\left[\frac{i}{2}Re\Big(\langle i^{\varepsilon(\alpha)}g,\int_{(S'-t_\alpha)\lambda^2}^{(T'-t_\alpha)/\lambda^2}S_uQf'\rangle du-\langle i^{\varepsilon(\alpha)}g,\int_{(S-t_\alpha)\lambda^2}^{(T-t_\alpha)/\lambda^2}S_uQf\rangle du+\right.$$
$$+\int_{(S'-t_\alpha)\lambda^2}^{(T'-t_\alpha)/\lambda^2}\langle S_uf',\,Qi^{\varepsilon(\alpha)}g\rangle du-\int_{(S-t_\alpha)\lambda^2}^{(T-t_\alpha)/\lambda^2}\langle S_uf,\,Qi^{\varepsilon(\alpha)}g\rangle du\Big)+$$
$$\left.+Im\Big(\int_{(S'-t_\alpha)\lambda^2}^{(T'-t_\alpha)/\lambda^2}\langle S_uf',\,i^{\varepsilon(\alpha)}g\rangle du-\int_{(S-t_\alpha)\lambda^2}^{(T-t_\alpha)/\lambda^2}\langle S_uf,\,i^{\varepsilon(\alpha)}g\rangle du\Big)\right]. \tag{2.5}$$

Letting $\lambda\to 0$ in (2.5), one gets (2.4).

§ 3. The uniform estimate

In the section, we shall obtain the uniform estimate of $I_g^\varepsilon(n, \lambda)$ and $II_g^\varepsilon(n, \lambda)$ by adapting our case to the corresponding arguments of [1].

THEOREM (3.1). *For each $g, f, f' \in K$, $S, T, S', T' \in \mathbf{R}$, $Q \geq 1$, there exists a constant C_1 such that for each $n \in \mathbf{N}$, $\varepsilon \in \{0, 1\}^n$ and $t \geq 0$,*

$$|I_g^\varepsilon(n, \lambda)| \leq C_1^n \frac{(t \vee 1)^n}{[n/2]!}. \tag{3.1}$$

Proof: Notice that in (2.5), for each $h = 1, \ldots, m$, since $t_{q_h} \in [-t_{q_h-1}/\lambda^2, 0]$, one has $0 \leq \lambda^2 t_{q_h} + t_{q_h-1} \leq t_{q_h-1}$. So, the following estimate will be obtained easily from (2.5)

$$|I_g^\varepsilon(n, \lambda)| \leq \sum_{m=0}^{[n/2]} \sum_{\substack{2 \leq q_1 < \ldots < q_m \leq n \\ \{q_h-1\}_{h=1}^m \cap \{q_h\}_{h=1}^m = \emptyset}}$$

$$\int_0^t dt_1 \ldots \int_0^{t_{q_1-2}} dt_{q_1-1} \int_0^{t_{q_1-1}} dt_{q_1+1} \ldots \int_0^{t_{q_m-2}} dt_{q_m-1} \int_0^{t_{q_m-1}} dt_{q_m+1}$$

$$\ldots \int_0^{t_{n-1}} dt_n \cdot \int_{-\infty}^0 dt_{q_1} \ldots \int_{-\infty}^0 dt_{q_m}$$

$$\prod_{h=1}^m \left[\tfrac{1}{2} Re\left(|\langle g, S_{t_{q_h}} Q i^{\varepsilon(q_h)} g\rangle| + |\langle Q i^{\varepsilon(q_h-1)} g, S_{t_{q_h}} g\rangle|\right) + |\langle g, S_{t_{q_h}} g\rangle|\right] \cdot C_{1,1}^{n-2m}, \tag{3.2}$$

where

$$C_{1,1} := \tfrac{1}{2} \int_{-\infty}^{\infty} (|\langle g, S_u Q f\rangle| + |\langle g, S_u Q f'\rangle| +$$

$$+ |\langle Qg, S_u f\rangle| + |\langle Qg, S_u f'\rangle| + |\langle Qig, S_u f\rangle| + |\langle Qig, S_u f'\rangle|) du +$$

$$+ \int_{-\infty}^{\infty} (|\langle g, S_u f\rangle| + |\langle g, S_u f'\rangle|) du. \tag{3.3}$$

Put

$$C_{1,2} := C_{1,1} \bigvee \max_{\varepsilon \in \{0,1\}^2} \int_{-\infty}^0 (|\langle g, S_u g\rangle| +$$

$$+ |\langle g, S_u Q i^{\varepsilon(1)} g\rangle| + |\langle Q i^{\varepsilon(2)} g, S_u g\rangle|) du \tag{3.4}$$

then one has

$$|I_g^\varepsilon(n, \lambda)| \leq \sum_{m=0}^{[n/2]} \sum_{\substack{2 \leq q_1 < \ldots < q_m \leq n \\ \{q_h-1\}_{h=1}^m \cap \{q_h\}_{h=1}^m = \emptyset}} C_{1,2}^n \cdot \frac{t^{n-m}}{(n-m)!}$$

$$\leqslant \sum_{m=0}^{[n/2]} C_{1,2}^n \frac{(t \vee 1)^n}{(n-m)!} \binom{n}{m} \leqslant C_{1,2}^n \cdot \frac{n}{2} \cdot 2^n \cdot \frac{(t \vee 1)^n}{[n/2]!}. \tag{3.5}$$

Taking

$$C_1 := 2C_{1,2} + 1 \tag{3.6}$$

(3.1) is proved.

THEOREM (3.2). *For each $g, f, f' \in K$, S, T, S', $T' \in \mathbf{R}$, $Q \geqslant 1$, there exists a constant C_2 such that for each $n \in \mathbf{N}$, $\varepsilon \in \{0, 1\}^n$ and $t \geqslant 0$,*

$$|II_g^\varepsilon(n, \lambda)| \leqslant C_2^n \frac{(t \vee 1)^n}{[n/3]!}. \tag{3.7}$$

Proof: Put

$$C_{1,2} := \int_{-\infty}^{\infty} du \big(|\langle g, S_u Q f' \rangle| + |\langle g, S_u Q f \rangle| + |\langle g, S_u f' \rangle| + |\langle g, S_u f \rangle| +$$
$$+ \sup_{\varepsilon \in \{1,0\}} (|\langle Q i^\varepsilon g, S_u f' \rangle| + |\langle Q i^\varepsilon g, S_u f \rangle|)\big). \tag{3.8}$$

Using (5.28) and (5.29) in [1], one obtains

$$|II_g^\varepsilon(n, \lambda)| \leqslant \sum_{m=0}^{[n/2]} \sum_{2 \leqslant q_1 < \ldots < q_m \leqslant n} \sum'_{(p_1, q_1, \ldots, p_m, q_m)} \sum_{\sigma \in S_m^0}$$
$$\lambda^{-2m} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \prod_{h=1}^m F\left(\frac{t_{q_h} - t_{q_h - 1}}{\lambda^2}\right) C_{2,1}^{n-2m}, \tag{3.9}$$

where

$$F(u) := \max_{\varepsilon \in \{0,1\}^2} |\langle i^{\varepsilon(1)} g, S_u Q i^{\varepsilon(2)} g \rangle| + |\langle g, S_u g \rangle|. \tag{3.10}$$

Thus

$$(II_g^\varepsilon(n, \lambda)| \leqslant \Big(\sum_{m=0}^{[n/3]} + \sum_{m=[n/3]+1}^{[n/2]}\Big) \sum_{2 \leqslant q_1 < \ldots < q_m \leqslant n} \sum_{1 \leqslant p_1 < \ldots < p_m \leqslant n-1} \sum_{\sigma \in S_m^0}$$
$$\lambda^{-2m} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \prod_{h=1}^m F\left(\frac{t_{q_h} - t_{q_h - 1}}{\lambda^2}\right) C_{2,1}^{n-2m} \leqslant \frac{1}{3} n \cdot \binom{n}{m} \cdot \frac{[n/3]!}{[2n/3]!} C_{2,2}^n +$$
$$+ \sum_{m=[n/3]+1}^{[n/2]} \sum_{2 \leqslant q_1 < \ldots < q_m \leqslant n} \sum_{1 \leqslant p_1 < \ldots < p_m \leqslant n-1} \sum_{\sigma \in S_m^0}$$
$$\lambda^{-2m} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \prod_{h=1}^m F\left(\frac{t_{q_h} - t_{q_h - 1}}{\lambda^2}\right) C_{2,1}^{n-2m}, \tag{3.11}$$

where

$$C_{2,2} := C_{2,1} \bigvee \sup_{\varepsilon\in\{0,1\}^2} \int_{-\infty}^{0} [|\langle g, S_u Q i^{\varepsilon(1)} g\rangle| + + |\langle g, S_u Q i^{\varepsilon(2)} g\rangle| + |\langle g, S_g g\rangle|]\, du. \tag{3.12}$$

Using the proof of the Lemma (5.3) in [1], one gets easily (3.7).

§4. The weak coupling limit

In the section we shall obtain the explicit form of the limit (0.11a).

THEOREM (4.1). *For each* $g, f, f' \in K$, $u, v \in H_0$, $D \in B(H_0)$ *and*

$$V_g = i\big(D \otimes A^+(g) - D^+ \otimes A(g)\big)$$

the limit

$$\lim_{\lambda\to 0} \langle u \otimes W\big(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f\, du\big)\Phi_Q,\ U^{(\lambda)}(t/\lambda^2) v \otimes W\big(\lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f'\, du\big)\Phi_Q\rangle \tag{4.1}$$

exists and is equal to

$$\sum_{n=0}^{\infty} \sum_{\varepsilon\in\{0,1\}^n} \frac{1}{2^n} \langle u, D_{\varepsilon(1)} \dots D_{\varepsilon(n)} v\rangle \sum_{m=0}^{[n/2]} \sum_{\substack{2\leqslant q_1 < \dots < q_m \leqslant n,\\ \{q_h - 1\}_{h=1}^m \cap \{q_h\}_{h=1}^m = \emptyset}}$$

$$\int_{0\leqslant t_n \leqslant \dots \leqslant t_{q_m} \leqslant \dots \leqslant t_{q_1} \leqslant \dots \leqslant t_1 \leqslant t} dt_1 \dots dt_{q_1} \dots dt_{q_m} \dots dt_n$$

$$\prod_{h=1}^{m} \left[\frac{1}{2} Re\Big((i^{\varepsilon(q_h-1)} g | Q i^{\varepsilon(q_h)} g)_- + \overline{(Q i^{\varepsilon(q_h-1)} g | i^{\varepsilon(q_h)} g)_-}\Big) + i Im (i^{\varepsilon(q_h-1)} g | i^{\varepsilon(q_h)} g)_-\right] \cdot$$

$$\cdot \prod_{\alpha\in\{1,\dots,n\}\setminus\{q_h, q_h-1\}_{h=1}^m} \Big[\frac{i}{2} Re\Big((i^{\varepsilon(\alpha)} g | Q f') \cdot \chi_{[S',T']}(t_\alpha) - (i^{\varepsilon(\alpha)} g | Q f) \cdot \chi_{[S,T]}(t_\alpha) +$$

$$+ (f' | Q i^{\varepsilon(\alpha)} g) \cdot \chi_{[S',T']}(t_\alpha) - (f | Q i^{\varepsilon(\alpha)} g) \cdot \chi_{[S,T]}(t_\alpha)\Big) -$$

$$- Im (i^{\varepsilon(\alpha)} g | f) \cdot \chi_{[S,T]}(t_\alpha) - Im (i^{\varepsilon(\alpha)} g | f') \cdot \chi_{[S',T']}(t_\alpha)\Big]$$

$$\langle W(\chi_{[S,T]} \otimes f) \Psi_{1\otimes Q},\ W(\chi_{[S',T']} \otimes f') \Psi_{1\otimes Q}\rangle. \tag{4.2}$$

Proof: Expanding $U^{(\lambda)}(t/\lambda^2)$ according to the iterated series (0.20) one finds

$$\lim_{\lambda\to 0} \langle u \otimes W\big(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f\, du\big)\Phi_Q,\ U^{(\lambda)}(t/\lambda^2) v \otimes W\big(\lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f' du\big)\Phi_Q\rangle$$

$$= \lim_{\lambda\to 0} \langle u \otimes W\big(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f\, du\big)\Phi_Q,\ \sum_{n=0}^{\infty} \sum_{\varepsilon\in\{0,1\}^n} \frac{1}{2^n} D_{\varepsilon(1)} \dots D_{\varepsilon(n)} \otimes$$

$$\lambda^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n B(i^{\varepsilon(1)} S_{t_1} g) \ldots B(i^{\varepsilon(n)} S_{t_n} g) v \otimes W(\lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f' du) \Phi_Q\rangle. \tag{4.3}$$

Because of the uniform estimate we can exchange the limit and the series, so we obtain

$$\sum_{n=0}^{\infty} \sum_{\varepsilon \in \{0,1\}^n} \frac{1}{2^n} \langle u, D_{\varepsilon(1)} \ldots D_{\varepsilon(n)} v\rangle$$

$$\lim_{\lambda \to 0} \lambda^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n (I_g^{\varepsilon}(n, \lambda) + II_g^{\varepsilon}(n, \lambda))$$

$$\langle W(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f du) \Phi_Q, W(\lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f' du) \Phi_Q\rangle. \tag{4.4}$$

Applying Lemma (2.1) and Lemma (2.2) to (4.4), we complete the proof.

§ 5. The properties of the weak coupling limit

Our goal is now to identify the limit (4.2), obtained in the previous section, with the matrix element (0.12a). To this end we shall deduce, in this section, an ordinary differential equation satisfied by this limit (cf. Theorem (5.1)). In the following section we shall identify this equation with the weak form of a quantum stochastic differential equation. By Theorems (4.1) and (3.1), one knows that (4.2) is continuous in $u, v \in H_0$, so, for each $u, v \in H_0$, there exists a $G(t) \in H_0$, such that

$$\langle u, G(t)\rangle = \sum_{n=0}^{\infty} \sum_{\varepsilon \in \{0,1\}^n} \frac{1}{2^n} \langle u, D_{\varepsilon(1)} \ldots D_{\varepsilon(n)} v\rangle \sum_{m=0}^{[n/2]} \sum_{\substack{2 \leqslant q_1 < \ldots < q_m \leqslant n, \\ \{q_h - 1\}_{h=1}^m \cap \{q_h\}_{h=1}^m = \emptyset}}$$

$$\int_{0 \leqslant t_n \leqslant \ldots \leqslant t_{q_m} \leqslant \ldots \leqslant t_{q_1} \leqslant \ldots \leqslant t_1 \leqslant t} dt_1 \ldots dt_{q_1} \ldots dt_{q_m} \ldots dt_n$$

$$\prod_{h=1}^{m} \left[\frac{1}{2} Re\left((i^{\varepsilon(q_h - 1)} g | Q i^{\varepsilon(q_h)} g)_- + \overline{(Q i^{\varepsilon(q_h - 1)} g | i^{\varepsilon(q_h)} g)_-}\right) + i Im (i^{\varepsilon(q_h - 1)} g | i^{\varepsilon(q_h)} g)_-\right] \cdot$$

$$\cdot \prod_{\alpha \in \{1, \ldots, n\} \setminus \{q_h, q_h - 1\}_{h=1}^m} \left[\frac{i}{2} Re\left((i^{\varepsilon(\alpha)} g | Q f') \cdot \chi_{[S', T']}(t_\alpha) - (i^{\varepsilon(\alpha)} g | Q f) \cdot \chi_{[S, T]}(t_\alpha) + \right.\right.$$

$$\left. + (f' | Q i^{\varepsilon(\alpha)} g) \cdot \chi_{[S', T']}(t_\alpha) - (f | Q i^{\varepsilon(\alpha)} g) \cdot \chi_{[S, T]}(t_\alpha)\right) -$$

$$\left. - Im (i^{\varepsilon(\alpha)} g | f) \cdot \chi_{[S, T]}(t_\alpha) - Im (i^{\varepsilon(\alpha)} g | f') \cdot \chi_{[S', T']}(t_\alpha)\right]$$

$$\langle W(\chi_{[S, T]} \otimes f) \Psi_{1 \otimes Q}, W(\chi_{[S', T']} \otimes f') \Psi_{1 \otimes Q}\rangle. \tag{5.1}$$

That is

$$\langle u, G(t)\rangle = \lim_{\lambda \to 0} \langle u \otimes W(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f du) \Phi_Q,$$

$$U^{(\lambda)}(t/\lambda^2)v \otimes W\big(\lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f'du\big)\Phi_Q\rangle. \tag{5.2}$$

Clearly,

$$\langle u, G(0)\rangle = \lim_{\lambda\to 0} \langle u \otimes W\big(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f\,du\big)\Phi_Q,\, v \otimes W\big(\lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f'du\big)\Phi_Q\rangle$$
$$= \langle u \otimes W(\chi_{[S,T]}\otimes f)\Psi_{1\otimes Q},\, v \otimes W(\chi_{[S',T']}\otimes f')\Psi_{1\otimes Q}\rangle. \tag{5.3}$$

Denote the right-hand side of (5.2) by

$$\sum_{n=0}^{\infty} G_n(t). \tag{5.4}$$

For each $n = 1, 2, \ldots$, split $G_n(t)$ into two parts according to $q_1 = 2$ and $q_1 > 2$ and denote them by $G_n(1, t)$ and $G_n(2, t)$, respectively. Thus for each $n \in \mathbf{N}$ we have

$$G_n(1, t) = \sum_{\varepsilon\in\{0,1\}^n} \frac{1}{2^n}\langle u, D_{\varepsilon(1)}\ldots D_{\varepsilon(n)}v\rangle \sum_{m=1}^{[n/2]} \sum_{\substack{2\leqslant q_1<q_2<\ldots<q_m\leqslant n\\ \{q_h-1\}_{h=1}^m\cap\{q_h\}_{h=1}^m=\emptyset}}$$
$$\int_{0\leqslant t_n\leqslant\ldots\leqslant t_{q_m}\leqslant\ldots\leqslant t_{q_1}\leqslant\ldots\leqslant t_1\leqslant t} dt_1\ldots dt_{q_1}\ldots dt_{q_m}\ldots dt_n$$
$$\prod_{h=1}^{m}\left[\frac{1}{2}Re\big((i^{\varepsilon(q_h-1)}g|Qi^{\varepsilon(q_h)}g)_- + \overline{(Qi^{\varepsilon(q_h-1)}g|i^{\varepsilon(q_h)}g)_-}\big) + iIm(i^{\varepsilon(q_h-1)}g|i^{\varepsilon(q_h)}g)_-\right]\cdot$$
$$\cdot \prod_{\alpha\in\{1,\ldots,n\}\setminus\{q_h,q_h-1\}_{h=1}^m}\Big[\frac{i}{2}Re\big((i^{\varepsilon(\alpha)}g|Qf')\cdot\chi_{[S',T']}(t_\alpha) - (i^{\varepsilon(\alpha)}g|Qf)\cdot\chi_{[S,T]}(t_\alpha) +$$
$$+ (f'|Qi^{\varepsilon(\alpha)}g)\cdot\chi_{[S',T']}(t_\alpha) - (f|Qi^{\varepsilon(\alpha)}g)\cdot\chi_{[S,T]}(t_\alpha)\big) -$$
$$- Im(i^{\varepsilon(\alpha)}g|f)\cdot\chi_{[S,T]}(t_\alpha) - Im(i^{\varepsilon(\alpha)}g|f')\cdot\chi_{[S',T']}(t_\alpha)\Big]$$
$$\langle W(\chi_{[S,T]}\otimes f)\Psi_{1\otimes Q},\, W(\chi_{[S',T']}\otimes f')\Psi_{1\otimes Q}\rangle \tag{5.5}$$

and

$$G_n(2, t) = \sum_{\varepsilon\in\{0,1\}^n} \frac{1}{2^n}\langle u, D_{\varepsilon(1)}\ldots D_{\varepsilon(n)}v\rangle \sum_{m=0}^{[n/2]} \sum_{\substack{2\leqslant q_1<\ldots<q_m\leqslant n,\\ \{q_h-1\}_{h=1}^m\cap\{q_h\}_{h=1}^m=\emptyset}}$$
$$\int_{0\leqslant t_n\leqslant\ldots t_{q_m}\leqslant\ldots t_{q_1}\leqslant\ldots t_2\leqslant t_1\leqslant t} dt_1 dt_2\ldots dt_{q_1}\ldots dt_{q_m}\ldots dt_n$$
$$\prod_{h=1}^{m}\left[\frac{1}{2}Re\big((i^{\varepsilon(q_h-1)}g|Qi^{\varepsilon(q_h)}g)_- + \overline{(Qi^{\varepsilon(q_h-1)}g|i^{\varepsilon(q_h)}g)_-}\big) + iIm(i^{\varepsilon(q_h-1)}g|i^{\varepsilon(q_h)}g)_-\right]\cdot$$

$$\cdot \prod_{\alpha\in\{1,\ldots,n\}\setminus\{q_h,\, q_h-1\}_{h=1}^m} \Big[\frac{i}{2} Re\big((i^{\varepsilon(\alpha)}g|Qf')\cdot\chi_{[S',T']}(t_\alpha) - (i^{\varepsilon(\alpha)}g|Qf)\cdot\chi_{[S,T]}(t_\alpha) +$$
$$+ (f'|Qi^{\varepsilon(\alpha)}g)\cdot\chi_{[S',T']}(t_\alpha) - (f|Qi^{\varepsilon(\alpha)}g)\cdot\chi_{[S,T]}(t_\alpha)\big) -$$
$$- Im(i^{\varepsilon(\alpha)}g|f)\cdot\chi_{[S,T]}(t_\alpha) - Im(i^{\varepsilon(\alpha)}g|f')\cdot\chi_{[S',T']}(t_\alpha)\Big]$$
$$\langle W(\chi_{[S,T]}\otimes f)\Psi_{1\otimes Q},\, W(\chi_{[S',T']}\otimes f')\Psi_{1\otimes Q}\rangle. \tag{5.6}$$

Moreover, for each $n\in\mathbf{N}$,

$$\frac{d}{dt}G_n(1,t) = \sum_{\varepsilon\in\{0,1\}^n} \frac{1}{2^n}\langle u, D_{\varepsilon(1)}\ldots D_{\varepsilon(n)}v\rangle \sum_{m=1}^{[n/2]} \sum_{\substack{2=q_1<q_2<\ldots<q_m\leq n,\\ \{q_h-1\}_{h=1}^m\cap\{q_h\}_{h=1}^m=\emptyset}}$$
$$\int_{0\leq t_n\leq\ldots\leq t_{q_m}\leq\ldots\leq t_{q_2}\leq\ldots\leq t_3\leq t} dt_3\ldots dt_{q_2}\ldots dt_{q_m}\ldots dt_n$$
$$\Big[\frac{1}{2}Re\big((i^{\varepsilon(1)}g|Qi^{\varepsilon(2)}g)_- + \overline{(Qi^{\varepsilon(1)}g|i^{\varepsilon(2)}g)_-}\big) + iIm(i^{\varepsilon(1)}g|i^{\varepsilon(2)}g)_-\Big]\cdot$$
$$\cdot\prod_{h=1}^m \Big[\frac{1}{2}Re\big((i^{\varepsilon(q_h-1)}g|Qi^{\varepsilon(q_h)}g)_- + \overline{(Qi^{\varepsilon(q_h-1)}g|i^{\varepsilon(q_h)}g)_-}\big) + iIm(i^{\varepsilon(q_h-1)}g|i^{\varepsilon(q_h)}g)_-\Big]\cdot$$
$$\cdot \prod_{\alpha\in\{1,\ldots,n\}\setminus\{q_h,\, q_h-1\}_{h=1}^m} \Big[\frac{i}{2} Re\big((i^{\varepsilon(\alpha)}g|Qf')\cdot\chi_{[S',T']}(t_\alpha) - (i^{\varepsilon(\alpha)}g|Qf)\cdot\chi_{[S,T]}(t_\alpha) +$$
$$+ (f'|Qi^{\varepsilon(\alpha)}g)\cdot\chi_{[S',T']}(t_\alpha) - (f|Qi^{\varepsilon(\alpha)}g)\cdot\chi_{[S,T]}(t_\alpha)\big) -$$
$$- Im(i^{\varepsilon(\alpha)}g|f)\cdot\chi_{[S,T]}(t_\alpha) - Im(i^{\varepsilon(\alpha)}g|f')\cdot\chi_{[S',T']}(t_\alpha)\Big]$$
$$\langle W(\chi_{[S,T]}\otimes f)\Psi_{1\otimes Q},\, W(\chi_{[S',T']}\otimes f')\Psi_{1\otimes Q}\rangle. \tag{5.7}$$

Denoting $n' = n-2$, $\varepsilon'\in\{0,1\}^{n'}$, $\varepsilon'(h) := \varepsilon(h+2)$, $m' = m-1$, $q'_h = q_{h+1}$, $h = 1,\ldots,m'$, and $t'_h = t_{h+2}$, $h = 1,\ldots,n'$, the vector $(t_3,\ldots,t_n)$ takes the form $(t'_1,\ldots,t'_{n'})$ and the sum

$$\sum_{\varepsilon\in\{0,1\}^n}$$

becomes the sum

$$\sum_{\varepsilon\in\{0,1\}^2}\sum_{\varepsilon'\in\{0,1\}^{n'}}.$$

Moreover, n' and n have the same parity and $[n/2]-1 = [n'/2]$, $0\leq m'\leq n'$. Thus

$$\frac{d}{dt}G_n(1,t) = \frac{1}{4}\sum_{\varepsilon\in\{0,1\}^2}\sum_{\varepsilon'\in\{0,1\}^{n'}}\frac{1}{2^{n'}}\sum_{m'=0}^{[n'/2]}\sum_{\substack{2\leq q'_1<\ldots<q'_{m'}\leq n',\\ \{q'_h-1\}_{h=1}^{m'}\cap\{q'_h\}_{h=1}^{m'}=\emptyset}}$$

$$\langle u, D_{\varepsilon(1)} D_{\varepsilon(2)} \cdot D_{\varepsilon'(1)} \dots D_{\varepsilon'(n')} v\rangle$$

$$\int_{0 \leqslant t'_{n'} \leqslant \dots \leqslant t'_{q'_{m'}} \leqslant \dots \leqslant t'_{q'_1} \leqslant \dots \leqslant t'_1 \leqslant t} dt'_1 \dots dt'_{q'_1} \dots dt'_{q'_{m'}} \dots dt'_{n'}$$

$$\left[\frac{1}{2} Re\left((i^{\varepsilon(1)} g | Q i^{\varepsilon(2)} g)_- + \overline{(Q i^{\varepsilon(1)} g | i^{\varepsilon(2)} g)_-}\right) + i Im (i^{\varepsilon(1)} g | i^{\varepsilon(2)} g)_-\right] \cdot$$

$$\cdot \prod_{h=1}^{m} \left[\frac{1}{2} Re\left((i^{\varepsilon'(q'_h - 1)} g | Q i^{\varepsilon'(q'_h)} g)_- + \overline{(Q i^{\varepsilon'(q'_h - 1)} g | i^{\varepsilon'(q'_h)} g)_-}\right) + i Im (i^{\varepsilon'(q'_h - 1)} g | i^{\varepsilon'(q'_h)} g)_-\right] \cdot$$

$$\cdot \prod_{\alpha \in \{1, \dots, n'\} \setminus \{q'_h, q'_h - 1\}_{h=1}^{m'}} \left[\frac{i}{2} Re\left((i^{\varepsilon'(\alpha)} g | \{f') \cdot \chi_{[S', T']}(t'_\alpha) - (i^{\varepsilon'(\alpha)} g | Q f) \cdot \chi_{[S, T]}(t'_\alpha) + \right.\right.$$

$$+ (f' | Q i^{\varepsilon'(\alpha)} g) \cdot \chi_{[S', T']}(t'_\alpha) - (f | Q i^{\varepsilon'(\alpha)} g) \cdot \chi_{[S, T]}(t'_\alpha)\Big) -$$

$$\left. - Im (i^{\varepsilon'(\alpha)} g | f) \cdot \chi_{[S, T]}(t'_\alpha) - Im (i^{\varepsilon'(\alpha)} g | f') \cdot \chi_{[S', T']}(t'_\alpha)\right]$$

$$\langle W(\chi_{[S, T]} \otimes f) \Psi_{1 \otimes Q}, W(\chi_{[S', T']} \otimes f') \Psi_{1 \otimes Q}\rangle. \tag{5.8}$$

Therefore, there exists a constant C_3, such that

$$\left|\frac{d}{dt} G_n(1, t)\right| \leqslant C_3^n \frac{(t \vee 1)^n}{[(n-2)/2]!} \tag{5.9}$$

and moreover,

$$4 \sum_{n=2} \frac{d}{dt} G_n(1, t) = \sum_{\varepsilon \in \{0, 1\}^2} \left[\frac{1}{2} Re\left((i^{\varepsilon(1)} g | Q i^{\varepsilon(2)} g)_- + \right.\right.$$

$$\left. + \overline{(Q i^{\varepsilon(1)} g | i^{\varepsilon(2)} g)_-}\Big) + i Im (i^{\varepsilon(1)} g | i^{\varepsilon(2)} g)_-\right] \langle D^+_{\varepsilon(2)} D^+_{\varepsilon(1)} u, G(t)\rangle$$

$$= \left[\frac{1}{2} Re\left((g | Q g)_- + \overline{(Q g | g)_-}\right) + i Im (g | g)_-\right] \cdot \langle -(D + D^+)(D + D^+) u, G(t)\rangle +$$

$$+ \left[\frac{1}{2} Re\left((g | Q i g)_- + \overline{i (Q g | g)_-}\right) + i Re (g | g)_-\right] \cdot i \langle (D^+ - D)(D + D^+) u, G(t)\rangle +$$

$$+ \left[\frac{1}{2} Re\left(- i (g | Q g)_- + \overline{(Q i g | g)_-}\right) - i Re (g | g)_-\right] \cdot i \langle (D + D^+)(D^+ - D) u, G(t)\rangle +$$

$$+ \left[\frac{1}{2} Re\left(- i (g | Q i g)_- + \overline{i (Q i g | g)_-}\right) + i Im (g | g)_-\right] \cdot \langle (D^+ - D)(D^+ - D) u, G(t)\rangle. \tag{5.10}$$

On the other hand, for each $n \in \mathbf{N}$,

$$G_n(2, t) = \sum_{\varepsilon \in \{0, 1\}^n} \frac{1}{2^n} \langle u, D_{\varepsilon(1)} \dots D_{\varepsilon(n)} v\rangle \sum_{m=0}^{[n/2]} \sum_{\substack{2 \leqslant q_1 < \dots < q_m \leqslant n, \\ \{q_h - 1\}_{h=1}^m \cap \{q_h\}_{h=1}^m = \emptyset}}$$

$$\int_0^t dt_1 \left[\frac{i}{2} Re\big((i^{\varepsilon(1)}g|Qf')\cdot\chi_{[S',T']}(t_1) - (i^{\varepsilon(1)}g|Qf)\cdot\chi_{[S,T]}(t_1) + \right.$$
$$+ (f'|Qi^{\varepsilon(1)}g)\cdot\chi_{[S',T']}(t_1) - (f|Qi^{\varepsilon(1)}g)\cdot\chi_{[S,T]}(t_1)\big) -$$
$$\left. - Im(i^{\varepsilon(1)}g|f)\cdot\chi_{[S,T]}(t_1) - Im(i^{\varepsilon(1)}g|f')\cdot\chi_{[S',T']}(t_1)\right]$$

$$\int_{0\leqslant t_n\leqslant\ldots\leqslant t_{q_m}\leqslant\ldots\leqslant t_{q_1}\leqslant\ldots\leqslant t_2\leqslant t_1} dt_2\ldots dt_{q_1}\ldots dt_{q_m}\ldots dt_n$$
$$\prod_{h=1}^{m}\left[\frac{1}{2}Re\big((i^{\varepsilon(q_h-1)}g|Qi^{\varepsilon(q_h)}g)_- + \overline{(Qi^{\varepsilon(q_h-1)}g|i^{\varepsilon(q_h)}g)_-}\big) + iIm(i^{\varepsilon(q_h-1)}g|i^{\varepsilon(q_h)}g)_-\right]\cdot$$
$$\prod_{\alpha\in\{2,\ldots,n\}\setminus\{q_h,q_h-1\}_{h=1}^m}\left[\frac{i}{2}Re\big((i^{\varepsilon(\alpha)}g|Qf')\cdot\chi_{[S',T']}(t_\alpha) - (i^{\varepsilon(\alpha)}g|Qf)\cdot\chi_{[S,T]}(t_\alpha) + \right.$$
$$+ (f'|Qi^{\varepsilon(\alpha)}g)\cdot\chi_{[S',T']}(t_\alpha) - (f|Qi^{\varepsilon(\alpha)}g)\cdot\chi_{[S,T]}(t_\alpha)\big) -$$
$$\left. - Im(i^{\varepsilon(\alpha)}g|f)\cdot\chi_{[S,T]}(t_\alpha) - Re(i^{\varepsilon(\alpha)}g|f')\cdot\chi_{[S',T']}(t_\alpha)\right]$$
$$\langle W(\chi_{[S,T]}\otimes f)\Psi_{1\otimes Q},\, W(\chi_{[S',T']}\otimes f')\Psi_{1\otimes Q}\rangle \tag{5.11}$$

Denoting again $n' = n-1$, $t'_h = t_{h+1}$, $h = 1,\ldots,n'$, $q'_h = q_h+1$, $h = 1,\ldots,m$, and $\varepsilon'\in\{0,1\}^{n'}$, $\varepsilon(h) := \varepsilon(h+1)$, $h = 1,\ldots,n'$, the vector $(t_2,\ldots,t_n)$ takes the form $(t'_1,\ldots,t_{n'})$; the sum $\sum_{2<q_1<\ldots<q_m\leqslant n}$ becomes the sum $\sum_{2\leqslant q'_1<\ldots<q'_m\leqslant n'}$ and sum $\sum_{\varepsilon\in\{0,1\}^n}$ becomes the sum $\sum_{\varepsilon\in\{0,1\}}\cdot\sum_{\varepsilon'\in\{0,1\}^{n'}}$. Moreover, for m odd, $[n/2] = [(n-1)/2] = [n'/2]$; for n even, since $2 < q_1$, so $m \leqslant [(n-1)/2]$, that is $m \leqslant [n'/2]$. Therefore, we obtain

$$G_n(2,t) = \frac{1}{2}\sum_{\varepsilon\{0,1\}}\sum_{\varepsilon'\in\{0,1\}^{n'}}\frac{1}{2^{n'}}\langle u, D\varepsilon\cdot D_{\varepsilon'(1)}\ldots D_{\varepsilon(n')}v\rangle\sum_{m=0}^{[n'/2]}\sum_{\substack{2\leqslant q'_1<\ldots<q'_m\leqslant n',\\ \{q'_h-1\}_{h=1}^m\cap\{q'_h\}_{h=1}^m=\emptyset}}$$
$$\int_0^t dt_1\left[\frac{i}{2}Re\big((i^\varepsilon g|Qf')\cdot\chi_{[S',T']}(t_1) - (i^\varepsilon g|Qf)\cdot\chi_{[S,T]}(t_1) + \right.$$
$$+ (f'|Qi^\varepsilon g)\cdot\chi_{[S',T']}(t_1) - (f|Qi^\varepsilon g)\cdot\chi_{[S,T]}(t_1)\big) -$$
$$\left. - Im(i^\varepsilon g|f)\cdot\chi_{[S,T]}(t_1) - Im(i^\varepsilon g|f')\cdot\chi_{[S',T']}(t_1)\right]$$
$$\int_{0\leqslant t'_{n'}\leqslant\ldots\leqslant t'_{q'_m}\leqslant\ldots\leqslant t'_{q'_1}\leqslant\ldots\leqslant t'_1\leqslant t_1} dt'_1\ldots dt'_{q'_1}\ldots dt'_{q'_m}\ldots dt'_{n'}$$
$$\prod_{h=1}^{m}\left[\frac{1}{2}Re\big((i^{\varepsilon'(q'_h-1)}g|Qi^{\varepsilon'(q'_h)}g)_- + \overline{(Qi^{\varepsilon'(q'_h-1}g|i^{\varepsilon(q'_h)}g)_-}\big) + iIm(i^{\varepsilon'(q'_h-1)}g|i^{\varepsilon'(q'_h)}g)_-\right]\cdot$$

$$\cdot \prod_{\alpha \in \{1, \ldots, n'\} \setminus \{q'_h, q'_h - 1\}_{h=1}^{m}} \left[\frac{i}{2} Re\big((i^{\varepsilon'(\alpha)} g | Qf') \cdot \chi_{[S', T']}(t'_\alpha) - (i^{\varepsilon'(\alpha)} g | Qf) \cdot \chi_{[S, T]}(t'_\alpha) + \right.$$

$$+ (f' | Q i^{\varepsilon'(\alpha)} g) \cdot \chi_{[S', T']}(t'_\alpha) - (f | Q i^{\varepsilon'(\alpha)} g) \cdot \chi_{[S, T]}(t'_\alpha)\big) -$$

$$\left. - Im(i^{\varepsilon'(\alpha)} g | f) \cdot \chi_{S, T]}(t'_\alpha) - Im(i^{\varepsilon'(\alpha)} g | f') \cdot \chi_{[S', T']}(t'_\alpha) \right]$$

$$\langle W(\chi_{[S, T]} \otimes f) \Psi_{1 \otimes Q}, W(\chi_{[S', T']} \otimes f') \Psi_{1 \otimes Q} \rangle. \tag{5.12}$$

Thus

$$2 \sum_{n=1}^{\infty} G_n(2, t) = \sum_{\varepsilon \in \{0, 1\}} \int_0^t dt_1 \left[\frac{i}{2} Re\big((i^\varepsilon g | Qf') \cdot \chi_{[S', T']}(t_1) - (i^\varepsilon g | Qf) \cdot \chi_{[S, T]}(t_1) + \right.$$

$$+ (f' | Q i^\varepsilon g) \cdot \chi_{[S', T']}(t_1) - (f | Q i^\varepsilon g) \cdot \chi_{[S, T]}(t_1)\big) -$$

$$\left. - Im(i^\varepsilon g | f) \cdot \chi_{[S, T]}(t_1) - Im(i^\varepsilon g | f') \cdot \chi_{[S', T']}(t_1) \right] \langle D_\varepsilon^+ u, G(t_1) \rangle$$

$$= \int_0^t dt_1 \left\{ \left[\frac{i}{2} Re\big((g | Qf') \cdot \chi_{[S', T']}(t_1) - (g | Qf) \cdot \chi_{[S, T]}(t_1) + \right. \right.$$

$$+ (f' | Qg) \cdot \chi_{[S', T']}(t_1) - (f | Qg) \cdot \chi_{[S, T]}(t_1)\big) -$$

$$\left. - Im(g | f) \cdot \chi_{[S, T]}(t_1) - Im(g | f') \cdot \chi_{[S', T']}(t_1) \right] \cdot i \langle (D^+ + D) u, G(t_1) \rangle +$$

$$+ \left[\frac{i}{2} Re\big(-i(g | Qf') \cdot \chi_{[S', T']}(t_1) + i(g | Qf) \cdot \chi_{[S, T]}(t_1) + \right.$$

$$+ (f' | Qig) \cdot \chi_{[S', T']}(t_1) - (f | Qig) \cdot \chi_{[S, T]}(t_1)\big) +$$

$$\left. \left. + Re(g | f) \cdot \chi_{[S, T]}(t_1) + Re(g | f') \cdot \chi_{[S', T']}(t_1) \right] \langle (D^+ - D) u, G(t_1) \rangle \right\}. \tag{5.14}$$

Moreover, we have the following

THEOREM (5.1). *For each* $u, v \in H_0$, $D \in B(H_0)$ *satisfying* (1.1), $g, f, f' \in K$, S, T, S', $T' \in \mathbf{R}$,

$$4 \langle u, G(t) \rangle =$$

$$= 4 \langle u, G(0) \rangle + \int_0^t dt_1 \left\{ \left[\frac{1}{2} Re(g | Qg) + i Im(g | g)_- \right] \cdot \langle -(D + D^+)(D + D^+) u, G(t_1) \rangle + \right.$$

$$+ \left[\frac{1}{2} Re\big((g | Qig)_- + \overline{i(Qg | g)_-}\big) + i Re(g | g)_- \right] \cdot i \langle D^+ - D)((D + D^+) u, G(t_1) \rangle +$$

$$+\left[\frac{1}{2}Re\left(-i(g|Qg)_- + \overline{(Qig|g)_-}\right) - iRe(g|g)_-\right]\cdot i\langle (D+D^+)(D^+-D)u,\ G(t_1)\rangle +$$

$$+\left[\frac{1}{2}Re(ig|Qig) + iIm(g|g)_-\right]\cdot\langle (D^+-D)(D^+-D)u,\ G(t_1)\rangle$$

$$+2\left[\frac{i}{2}Re\big((g|Qf')\cdot\chi_{[S',T']}(t_1) - (g|Qf)\cdot\chi_{[S,T]}(t_1) + \right.$$

$$+ (f'|Qg)\cdot\chi_{[S',T']}(t_1) - (f|Qg)\cdot\chi_{[S,T]}(t_1)\big) -$$

$$\left. -Im(g|f)\cdot\chi_{[S,T]}(t_1) - Im(g|f')\cdot\chi_{[S',T']}(t_1)\right]\cdot i\langle (D^+ + D)u,\ G(t_1)\rangle +$$

$$+2\left[\frac{i}{2}Re\big((ig|Qf')\cdot\chi_{[S',T']}(t_1) + i(g|Qf)\cdot\chi_{[S,T]}(t_1) + \right.$$

$$+ (f'|Qig)\cdot\chi_{[S',T']}(t_1) - (f|Qig)\cdot\chi_{[S,T]}(t_1)\big) +$$

$$\left. + Re(g|f)\cdot\chi_{[S,T]}(t_1) + Re(g|f')\cdot\chi_{[S',T']}(t_1)\right]\langle (D^+-D)u,\ G(t_1)\rangle\Big\}. \qquad (5.15)$$

Proof: It is clear that there exists a constant C such that for each $n\in\mathbf{N}$,

$$|G_n(1,t)| + |G_n(2,t)| + \left|\frac{d}{dt}G_n(1,t)\right| \leqslant C^n\frac{(t\vee 1)^n}{[(n-2)/2]!}. \qquad (5.16)$$

Combining (5.4), (5.5), (5.6), (5.10), (5.14) and (5.16), one gets (5.15) immediately.

§6. The quantum stochastic differential equation

In the section, we shall research the relation between the weak coupling limit and the solution of some quantum stochastic differential equation.

First of all, by expanding the operator products in (5.15), we write it in the form

$$\langle u,\ G(t)\rangle = \langle u,\ G(0)v\rangle +$$

$$+\tfrac{1}{2}\int_0^t dt_1\big(\langle Du,\ G(t_1)\rangle F_+(t_1) + \langle -D^+u,\ G(t_1)\rangle F_-(t_1)\big) +$$

$$+\tfrac{1}{4}\big(\langle -DDu,\ G(t_1)\rangle F_{-,-} + \langle -D^+Du,\ G(t_1)\rangle F_{+,-} +$$

$$+ \langle -DD^+u,\ G(t_1)\rangle F_{-,+} + \langle -D^+D^+u,\ G(t_1)\rangle F_{+,+}\big), \qquad (6.1)$$

where we introduced the notations

$$F_{-,-} := \frac{1}{2}Re(g|Qg) + iIm(g|g)_- + \frac{i}{2}Re\big((ig|Qg)_+ + (g|Qig)_-\big) - Re(g|g)_- +$$

$$+\frac{i}{2}Re\big((ig|Qg)_- + (g|Qig)_+\big) + Re(g|g)_- - \frac{1}{2}Re(ig|Qig) - iIm(g|g)_-$$

$$= \frac{1}{2}\big[Re\big((g|Qg) - (ig|Qig)\big) + iRe\big((ig|Qg) + (g|Qig)\big)\big], \tag{6.2}$$

$$F_{+,-} := \frac{1}{2}Re(g|Qg) + iIm(g|g)_- - \frac{i}{2}Re\big((ig|Qg)_+ + (g|Qig)_-\big) + Re(g|g)_- +$$

$$+\frac{i}{2}Re\big((ig|Qg)_- + (g|Qig)_+\big) + Re(g|g)_- + \frac{1}{2}Re(ig|Qig) + iIm(g|g)_-$$

$$= \frac{1}{2}\big[Re\big((g|Qg) + (ig|Qig)\big) + iRe\big((ig|Qg)_- + (g|Qig)_+ - (ig|Qg)_+ - (g|Qig)_-\big) + 4(g|g)_-\big]$$

$$= \frac{1}{2}\big[Re\big((g|Qg) + (ig|Qig)\big) + 2(g|g) +$$

$$+ iRe\big((ig|Qg)_- + (g|Qig)_+ + 4Im(g|g)_- - (ig|Qg)_+ - (g|Qig)_-\big)\big], \tag{6.3}$$

$$F_{-,+} := \frac{1}{2}Re(g|Qg) + iIm(g|g)_- + \frac{i}{2}Re\big((ig|Qg)_+ + (g|Qig)_-\big) - Re(g|g)_- -$$

$$-\frac{i}{2}Re\big((ig|Qg)_- + (g|Qig)_+\big) - Re(g|g)_- + \frac{1}{2}Re(ig|Qig) + iIm(g|g)_-$$

$$= \frac{1}{2}\big[Re\big((g|Qg) + (ig|Qig)\big) - iRe\big((ig|Qg)_- + (g|Qig)_+ - (ig|Qg)_+ - (g|Qig)_-\big) - 4(g|g)_+\big]$$

$$= \frac{1}{2}\big[Re\big((g|Qg) + (ig|Qig)\big) - 2(g|g) -$$

$$- iRe\big((ig|Qg)_- + (g|Qig)_+ + 4Im(g|g)_+ - (ig|Qg)_+ - (g|Qig)_-\big)\big], \tag{6.4}$$

$$F_{+,+} := \frac{1}{2}Re(g|Qg) + iIm(g|g)_- - \frac{i}{2}Re\big((ig|Qg)_+ + (g|Qig)_-\big) - Re(g|g)_- -$$

$$-\frac{i}{2}Re\big((ig|Qg)_- + (g|Qig)_+\big) - Re(g|g)_- - \frac{1}{2}Re(ig|Qig) - iIm(g|g)_-$$

$$= \frac{1}{2}\big[Re\big((g|Qg) - (ig|Qig)\big) - iRe\big((ig|Qg) + (g|Qig)\big)\big]. \tag{6.5}$$

Now, let us consider the quantum stochastic differential equation

$$U(t) = 1 + \int_0^t \big(D \otimes dA^+(s, g) - D^+ \otimes dA(s, g) -$$

$$-\tfrac{1}{4}(F_{-,+}DD^{+}\otimes 1+F_{+,-}D^{+}D\otimes 1+F_{-,-}D^{+}D^{+}\otimes 1+F_{+,+}DD\otimes 1)\,ds\big)U(s) \qquad (6.6)$$

on $H_0 \otimes \Gamma\big(L^2(\mathbf{R})\otimes(K;(\cdot|\cdot))\big)$ with

$$A(t,g):=A(\chi_{[0,t]}\otimes g).$$

From [5], we know that the q.s.d.e. (6.6) has a unique solution which is given by the iterated series, that is, if we put

$$U_0(t):=1, \qquad (6.7)$$

$$U_{n+1}(t):=\int_0^t\big(D\otimes dA^{+}(s,g)-D^{+}\otimes dA(s,g)-$$

$$-\tfrac{1}{4}(F_{-,+}DD^{+}\otimes 1+F_{+,-}D^{+}D\otimes 1+F_{-,-}D^{+}D^{+}\otimes 1+F_{+,+}DD\otimes 1)ds\big)\,Un(s) \qquad (6.8)$$

then

$$U(t)=\sum_{n=0}^{\infty}U_n(t),\ t\geqslant 0. \qquad (6.9)$$

LEMMA (6.1). *For each* $n\in\mathbf{N}$,

$$U_n(t)=\int_0^t\int_0^{t_1}\dots\int_0^{t_{n-1}}\sum_{\varepsilon\in\{0,1,2\}^n}D_{\varepsilon(1)}\dots D_{\varepsilon(n)}\otimes dA^{\varepsilon(1)}(t_1)\dots dA^{\varepsilon(n)}(t_n), \qquad (6.10)$$

where

$$D_0:=-D^{+},\qquad D_1:=D,$$

$$D_2:=-\tfrac{1}{4}(F_{-,+}DD^{+}\otimes 1+F_{+,-}D^{+}D\otimes 1+F_{-,-}D^{+}D^{+}\otimes 1+F_{+,+}DD\otimes 1), \qquad (6.11)$$

$$A^0(s):=A(s,g),\qquad A^1(s):=A^{+}(s,g),\qquad A^2(s):=s. \qquad (6.12)$$

The proof is the same as the proof of Lemma (4.1) in [3].

In order to prove the unitarity of the solution of the q.s.d.e. (6.6), we begin by establishing the Ito table for a non gauge invariant quantum Brownian motion.

LEMMA (6.2). *For each* $g\in K$,

$$dA(t,g)dA(t,g)=\tfrac{1}{4}\big[Re\big((ig|Qig)-(g|Qg)\big)-iRe\big((ig|Qg)+(g|Qig)\big)\big]\,dt \qquad (6.13)$$

and

$$dA^{+}(t,g)\,dA^{+}(t,g)=\tfrac{1}{4}\big[Re\big((ig|Qig)-(g|Qg)\big)+iRe\big((ig|Qg)+(g|Qig)\big)\big]\,dt. \qquad (6.14)$$

Proof: By (1.5), for each $g,f,f'\in K$, $\xi,\ \xi'\in L^2(\mathbf{R})$,

$$\langle W(\xi\otimes f)\Phi_{1\otimes Q},\,dA(t,g)\,dA(t,g)\,W(\xi'\otimes f')\Phi_{1\otimes Q}\rangle$$

$$=\tfrac{1}{4}\langle W(\xi\otimes f)\Phi_{1\otimes Q},\,d\big(B(t,ig)-iB(t,g)\big)\,d\big(B(t,ig)-iB(t,g)\big)\,W(\xi'\otimes f')\Phi_{1\otimes Q}\rangle$$

$$=\tfrac{1}{4}\langle W(\xi\otimes f)\Phi_{1\otimes Q},\,W(\xi'\otimes f')\Phi_{1\otimes Q}\rangle$$

$$\Big[Re\langle\chi_{[t,t+dt]}\otimes ig,(1\otimes Q)\chi_{[t,t+dt]}\otimes ig\rangle+iIm\langle\chi_{[t,t+dt]}\otimes ig,\chi_{[t,t+dt]}\otimes ig\rangle-$$

$$-\frac{i}{2}Re(\langle\chi_{[t,t+dt]}\otimes ig,(1\otimes Q)\chi_{[t,t+dt]}\otimes g\rangle+\langle\chi_{[t,t+dt]}\otimes g,(1\otimes Q)\chi_{[t,t+dt]}\otimes ig\rangle)+$$

$$+Im\langle\chi_{[t,t+dt]}\otimes ig,\chi_{[t,t+dt]}\otimes g\rangle-$$

$$-\frac{i}{2}Re(\langle\chi_{[t,t+dt]}\otimes ig,(1\otimes Q)\chi_{[t,t+dt]}\otimes g\rangle+\langle\chi_{[t,t+dt]}\otimes g,(1\otimes Q)\chi_{[t,t+dt]}\otimes ig\rangle)+$$

$$+Im\langle\chi_{[t,t+dt]}\otimes g,\chi_{[t,t+dt]}\otimes ig\rangle-$$

$$-Re\langle\chi_{[t,t+dt]}\otimes g,(1\otimes Q)\chi_{[t,t+dt]}\otimes g\rangle+iIm\langle\chi_{[t,t+dt]}\otimes g,\chi_{[t,t+dt]}\otimes g\rangle\Big]$$

$$=\tfrac{1}{4}\langle W(\xi\otimes f)\Phi_{1\otimes Q},W(\xi'\otimes f')\Phi_{1\otimes Q}\rangle$$

$$\big[Re\big((ig|Qig)-(g|Qg)\big)-iRe\big((ig|Qg)+(g|Qig)\big)\big]\,dt. \tag{6.15}$$

So, one gets (6.13). Since

$$dA(t,g)\,dA(t,g)=\overline{dA^{+}(t,g)\,dA^{+}(t,g)}$$

(6.14) follows from (6.13).

LEMMA (6.3). *For each* $g\in K$,

$$dA^{+}(t,g)\,dA(t,g)=\tfrac{1}{4}\big[Re\big((ig|Qig)+(g|Qg)\big)-2(g|g)\big]\,dt. \tag{6.16}$$

Proof: By (1.5), for each $g,f,f'\in K$, $\xi,\xi'\in L^2(\mathbf{R})$,

$$\langle W(\xi\otimes f)\Phi_{1\otimes Q},dA^{+}(t,g)\,dA(t,g)\,W(\xi'\otimes f')\Phi_{1\otimes Q}\rangle$$

$$=\tfrac{1}{4}\langle W(\xi\otimes f)\Phi_{1\otimes Q},d\big(B(t,ig)+iB(t,g)\big)d\big(B(t,ig)-iB(t,g)\big)W(\xi'\otimes f')\Phi_{1\otimes Q}\rangle$$

$$=\tfrac{1}{4}\langle W(\xi\otimes f)\Phi_{1\otimes Q},W(\xi'\otimes f')\Phi_{1\otimes Q}\rangle$$

$$\Big[Re\langle\chi_{[t,t+dt]}\otimes ig,(1\otimes Q)\chi_{[t,t+dt]}\otimes ig\rangle+iIm\langle\chi_{[t,t+dt]}\otimes ig,\chi_{[t,t+dt]}\otimes ig\rangle-$$

$$-\frac{i}{2}Re\big(\langle\chi_{[t,t+dt]}\otimes ig,(1\otimes Q)\chi_{[t,t+dt]}\otimes g\rangle+\langle\chi_{[t,t+dt]}\otimes g,(1\otimes Q)\chi_{[t,t+dt]}\otimes ig\rangle\big)+$$

$$+Im\langle\chi_{[t,t+dt]}\otimes ig,\chi_{[t,t+dt]}\otimes g\rangle+$$

$$+\frac{i}{2}Re\big(\langle\chi_{[t,t+dt]}\otimes ig,(1\otimes Q)\chi_{[t,t+dt]}\otimes g\rangle+\langle\chi_{[t,t+dt]}\otimes g,(1\otimes Q)\chi_{[t,t+dt]}\otimes ig\rangle\big)-$$

$$-Im\langle\chi_{[t,t+dt]}\otimes g,\chi_{[t,t+dt]}\otimes ig\rangle+$$

$$+Re\langle\chi_{[t,t+dt]}\otimes g,(1\otimes Q)\chi_{[t,t+dt]}\otimes g\rangle+iIm\langle\chi_{[t,t+dt]}\otimes g,\chi_{[t,t+dt]}\otimes g\rangle\Big]$$

$$=\tfrac{1}{4}\langle W(\xi\otimes f)\Phi_{1\otimes Q},W(\xi'\otimes f')\Phi_{1\otimes Q}\rangle$$

$$\big[Re\big((ig|Qig)+(g|Qg)\big)-2(g|g)\big]\,dt. \tag{6.17}$$

So, one gets (6.16).

LEMMA (6.4). *For each* $g \in K$,

$$dA(t, g)\, dA^+(t, g) = \tfrac{1}{4}\big[Re\big((ig|Qig) + (g|Qg)\big) + 2(g|g)\big]\, dt. \tag{6.18}$$

Proof: By (1.5), for each $g, f, f' \in K$, $\xi, \xi' \in L^2(\mathbf{R})$,

$$\langle W(\xi \otimes f)\Phi_{1\otimes Q}, dA(t, g)\, dA^+(t, g)\, W(\xi' \otimes f')\Phi_{1\otimes Q}\rangle$$

$$= \tfrac{1}{4}\langle W(\xi \otimes f)\Phi_{1\otimes Q}, d\big(B(t, ig) - iB(t, g)\big)\, d\big(B(t, ig) + iB(t, g)\big)\, W(\xi' \otimes f')\Phi_{1\otimes Q}\rangle$$

$$= \tfrac{1}{4}\langle W(\xi \otimes f)\Phi_{1\otimes Q}, W(\xi' \otimes f')\Phi_{1\otimes Q}\rangle$$

$$\Big[Re\,\langle \chi_{[t,t+dt]} \otimes ig, (1 \otimes Q)\,\chi_{[t,t+dt]} \otimes ig\rangle + iIm\,\langle \chi_{[t,t+dt]} \otimes ig, \chi_{[t,t+dt]} \otimes ig\rangle +$$

$$+ \frac{i}{2} Re\big(\langle \chi_{[t,t+dt]} \otimes ig, (1 \otimes Q)\,\chi_{[t,t+dt]} \otimes g\rangle + \langle \chi_{[t,t+dt]} \otimes g, (1 \otimes Q)\,\chi_{[t,t+dt]} \otimes ig\rangle\big) -$$

$$- Im\,\langle \chi_{[t,t+dt]} \otimes ig, \chi_{[t,t+dt]} \otimes g\rangle -$$

$$- \frac{i}{2} Re\big(\langle \chi_{[t,t+dt]} \otimes ig, (1 \otimes Q)\,\chi_{[t,t+dt]} \otimes g\rangle + \langle \chi_{[t,t+dt]} \otimes g, (1 \otimes Q)\,\chi_{[t,t+dt]} \otimes ig\rangle\big) +$$

$$+ Im\,\langle \chi_{[t,t+dt]} \otimes g, \chi_{[t,t+dt]} \otimes ig\rangle +$$

$$+ Re\,\langle \chi_{[t,t+dt]} \otimes g, (1 \otimes Q)\,\chi_{[t,t+dt]} \otimes g\rangle + iIm\,\langle \chi_{[t,t+dt]} \otimes g, \chi_{[t,t+dt]} \otimes g\rangle\Big]$$

$$= \tfrac{1}{4}\langle W(\xi \otimes f)\Phi_{1\otimes Q}, W(\xi' \otimes f')\Phi_{1\otimes Q}\rangle$$

$$\big[Re\big((ig|Qig) + (g|Qg)\big) + 2(g|g)\big]\, dt. \tag{6.19}$$

So, one gets (6.18).

Using the Lemma (6.2), Lemma (6.3) and Lemmas (6.4) to (6.2)–(6.5), one obtains

$$\tfrac{1}{4}F_{-,-}\,dt = \tfrac{1}{8}\big[Re\big((g|Qg) - (ig|Qig)\big) + iRe\big((ig|Qg) + (g|Qig)\big)\big]\, dt$$

$$= -\tfrac{1}{2}dA(t, g)\, dA(t, g), \tag{6.20}$$

$$\tfrac{1}{4}F_{+,-}\,dt = \tfrac{1}{8}\big[Re\big((g|Qg) + (ig|Qig)\big) + 2(g|g) +$$

$$+ iRe\big((ig|Qg)_- + (g|Qig)_+ + 4Im(g|g)_- - (ig|Qg)_+ - (g|Qig)_-\big)\big]\, dt$$

$$= \tfrac{1}{2}dA(t, g)dA^+(t, g) + i\tfrac{1}{8}Re\big((ig|Qg)_- + (g|Qig)_+ + 4Im(g|g)_+ - (ig|Qg)_+ - (g|Qig)_-\big)dt, \tag{6.21}$$

$$\tfrac{1}{4}F_{-,+}\,dt = \tfrac{1}{8}\big[Re\big((g|Qg) + (ig|Qig)\big) - 2(g|g) -$$

$$- iRe\big((ig|Qg)_- + (g|Qig)_+ + 4Im(g|g)_- - (ig|Qg)_+ - (g|Qig)_-\big)\big]\, dt$$

$$= \tfrac{1}{2}dA^+(t,g)dA(t,g) - i\tfrac{1}{8}Re\big((ig|Qg)_- + (g|Qig)_+ + 4Im(g|g)_+ - (ig|Qg)_+ - (g|Qig)_-\big)dt, \tag{6.22}$$

$$\tfrac{1}{4}F_{+,+}\,dt = \tfrac{1}{8}\big[Re\big((g|Qg) - (ig|Qig)\big) - iRe\big((ig|Qg) + (g|Qig)\big)\big]dt = -\tfrac{1}{2}dA^+(t, h)dA^+(t, g). \tag{6.23}$$

Now let us denote

$$L := \tfrac{1}{8} Re\big((ig|Qg)_- + (g|Qig)_+ + 4Im(g|g)_- - (ig|Qg)_+ - (g|Qig)_-\big) D^+D -$$
$$- \tfrac{1}{8} Re\big((ig|Qg)_- + (g|Qig)_+ + 4Im(g|g)_+ - (ig|Qg)_+ - (g|Qig)_-\big) DD^+, \tag{6.24}$$

$$\sigma_{1,1} := \tfrac{1}{4}\big[Re\big((g|Qg) + (ig|Qig)\big) + 2(g|g)\big], \tag{6.25}$$

$$\sigma_{2,2} := \tfrac{1}{4}\big[Re\big((g|Qg) + (ig|Qig)\big) - 2(g|g)\big], \tag{6.26}$$

$$\sigma_{2,1} := \tfrac{1}{4}\big[Re\big((g|Qg) - (ig|Qig)\big) - iRe\big((ig|Qg) + (g|Qig)\big)\big], \tag{6.27}$$

$$\sigma_{1,2} := \tfrac{1}{4}\big[Re\big((g|Qg) - (ig|Qig)\big) + iRe\big((ig|Qg) + (g|Qig)\big)\big], \tag{6.28}$$

then, L is a self-adjoint operator and the q.s.d.e. (6.6) can be written in the following form:

$$U(t) = 1 + \int_0^t \Bigg\{ (1 \otimes dA^+(s,g), 1 \otimes dA(s,g)) \begin{pmatrix} D \otimes 1 \\ -D^+ \otimes 1 \end{pmatrix} +$$
$$+ \left[\frac{1}{2} \begin{pmatrix} D \otimes 1 \\ -D^+ \otimes 1 \end{pmatrix}^+ \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{pmatrix} \begin{pmatrix} D \otimes 1 \\ -D^+ \otimes 1 \end{pmatrix} + iL \right] ds \Bigg\} U(s) \tag{6.29}$$

with L given by (6.24) and $\sigma_{j,k}$ $(j, k = 1, 2)$ by (6.25)–(6.28). Moreover,

$$\begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{pmatrix} dt = \begin{pmatrix} dA(t,g)\,dA^+(t,g) & dA(t,g)\,dA(t,g) \\ dA^+(t,g)\,dA^+(t,g) & dA^+(t,g)\,dA(t,g) \end{pmatrix}. \tag{6.30}$$

From this one easily deduces that

LEMMA (6.5). *The solution of q.s.d.e.* (6.6) *is a unitary process.*

THEOREM (6.6). *For each* $g, f, f' \in K$, $u, v \in H_0$, $D \in B(H_0)$ *satisfying* (0.3), S, T, S', $T' \in \mathbf{R}$, $t \geqslant 0$ *and* Q *a real linear operator on* H_1, *the weak coupling limit*

$$\lim_{\lambda \to 0} \langle u \otimes W(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f\,du)\Phi_Q,\; U^{(\lambda)}(t/\lambda^2) \cdot v \otimes W(\lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f'\,du)\Phi_Q \rangle$$

exists and is equal to

$$\langle u \otimes W(\chi_{[S,T]} \otimes f)\Psi_{1 \otimes Q},\; U(t) v \otimes W(\chi_{[S',T']} \otimes f')\Psi_{1 \otimes Q} \rangle, \tag{6.31}$$

where $U(t)$ *is the unique solution of q.s.d.e.* (6.6) *on* $H_0 \otimes \Gamma\big(L^2(\mathbf{R}) \otimes (K; (\cdot|\cdot))\big)$.

Proof: For each $u, v \in H_0, f, f' \in K$, S, T, S', $T' \in \mathbf{R}$, put

$$\langle u, F(t) \rangle := \langle u, W(\chi_{[S,T]} \otimes f)\Psi_{1 \otimes Q},\; U(t) v \otimes W(\chi_{[S',T']} \otimes f')\Psi_{1 \otimes Q} \rangle, \tag{6.32}$$

then

$$\langle u, F(0) \rangle := \langle u \otimes W(\chi_{[S,T]} \otimes f)\Psi_{1 \otimes Q},\; v \otimes W(\chi_{[S',T']} \otimes f')\Psi_{1 \otimes Q} \rangle$$
$$= \lim_{\lambda \to 0} \langle u \otimes W(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f\,du)\Psi_Q,\; v \otimes W(\lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f'\,du)\Phi_Q \rangle = \langle u, G(0) \rangle \tag{6.33}$$

and

$$\langle u, F(t)\rangle = \langle u, F(0)\rangle + \tfrac{1}{4}\int_0^t dt_1 \, [F_{+,-}\langle -D^+Du, F(t_1)\rangle +$$

$$+ F_{-,+}\langle -DD^+u, F(t_1)\rangle + F_{-,-}\langle -DDu, F(t_1)\rangle + F_{+,+}\langle -D^+D^+u, F(t_1)\rangle] +$$

$$+ \langle Du \otimes W(\chi_{[S,T]} \otimes f)\Psi_{1\otimes Q}, \int_0^t 1 \otimes dA^+(s, g)\, U(s)\cdot v \otimes W(\chi_{[S',T']} \otimes f')\Psi_{1\otimes Q}\rangle +$$

$$+ \langle -D^+u \otimes W(\chi_{[S,T]} \otimes f)\Psi_{1\otimes Q}, \int_0^t 1 \otimes dA(s, g)\, U(s)\cdot v \otimes W(\chi_{[S',T']} \otimes f')\Psi_{1\otimes Q}\rangle. \tag{6.34}$$

Put

$$F_1(t) := \langle Du \otimes W(\chi_{[S,T]} \otimes f)\Psi_{1\otimes Q}, \int_0^t 1 \otimes dA^+(s, g) U(s)\cdot v \otimes W(\chi_{[S',T']} \otimes f')\Psi_{1\otimes Q}\rangle \tag{6.35}$$

then one has

$$F_1(t) = \sum_{n=0}^{\infty} \langle Du \otimes W(\chi_{[S,T]} \otimes f)\Psi_{1\otimes Q},$$

$$\int_0^t 1 \otimes dA^+(s, g)\, U_n(s)\cdot v \otimes W(\chi_{[S',T']} \otimes f')\Psi_{1\otimes Q}\rangle, \tag{6.36}$$

where $U_n(t)$, $t \geqslant 0$ is defined by (6.7) and (6.8). Thus by Lemma (6.1) one has

$$F_1(t) = \sum_{n=0}^{\infty} \sum_{\varepsilon\in\{0,1,2\}^n} \int_0^t \int_0^s \int_0^{s_1} \dots \int_0^{s_{n-1}}$$

$$\langle Du \otimes W(\chi_{[S,T]} \otimes f)\Psi_{1\otimes Q}, D_{\varepsilon(1)} \dots D_{\varepsilon(n)} \otimes$$

$$dA^{\varepsilon(1)}(s_1, g) \dots dA^{\varepsilon(n)}(s_n, g)\, dA^+(s, g)\cdot v \otimes W(\chi_{[S',T']} \otimes f')\Psi_{1\otimes Q}\rangle$$

$$= \sum_{n=0}^{\infty} \sum_{\varepsilon\in\{0,1,2\}^n} \langle Du, D_{\varepsilon(1)} \dots D_{\varepsilon(n)} v\rangle$$

$$\int_0^t \int_0^s \int_0^{s_1} \dots \int_0^{s_{n-1}} \langle W(\chi_{[S,T]} \otimes f)\Psi_{1\otimes Q},$$

$$dA^{\varepsilon(1)}(s_1, g) \dots dA^{\varepsilon(n)}(s_n, g)\, dA^+(s, g)\cdot W(\chi_{[S',T']} \otimes f')\Psi_{1\otimes Q}\rangle. \tag{6.37}$$

Now, in order to apply Lemma (1.1), we express the creation and annihilation operators $A^{\varepsilon}(s, g)$ in terms of the field operators, denoted by $B(s, g)$. Thus (6.37) becomes

$$\sum_{n=0}^{\infty} \sum_{\varepsilon\in\{0,1,2\}^n} \langle Du, D_{\varepsilon(1)}\dots D_{\varepsilon(n)} v\rangle$$

$$\int_0^t\int_0^s\int_0^{s_1}\dots\int_0^{s_{n-1}} \langle W\chi_{[S,T]}\otimes f)\Psi_{1\otimes Q},$$

$$PL_n(\varepsilon, dB)\tfrac{1}{2}\big(dB(s, ig)+idB(s, g)\big)\cdot W(\chi_{[S',T']}\otimes f')\Psi_{1\otimes Q}\rangle, \tag{6.38}$$

where

$$PL_n(\varepsilon, dB) := dA^{\varepsilon(1)}(s_1, g)\dots dA^{\varepsilon(n)}(s_n, g)$$

is a polynomial in dA, dA^+ and so in dB.

Now notice that the expression $PL_n(\varepsilon, dB)\cdot\big(dB(s, ig)+idB(s, g)\big)$ is a sum of products of the form

$$dB(s_1, i^{\varepsilon(1)}g)\dots dB(s_n, i^{\varepsilon(n)}g)\,dB(s, i^{\varepsilon(n+1)}g) \tag{6.39}$$

with the intervals ds_j, ds pairwise disjoint. Thus if we apply to the matrix element of each of these terms the recurrence relation (1.6), the term with the $\frac{d}{ds_j}$-derivative will disappear because it will be a sum of polynomials each of which is multiplied by some variable of the form

$$t_{j,n+1} = \langle \chi_{[s_j, s_j+ds_j]}\otimes g, \chi_{[s,s+ds]}\otimes g\rangle = \langle \chi_{[s_j,s_j+ds_j]}, \chi_{[s,s+ds]}\rangle\cdot\|g\|^2 = 0. \tag{6.40}$$

This means that, in our case, we can apply, instead of (1.6) the much simpler relation

$$P_{n+1} = s_{n+1} P_n \tag{6.41}$$

with s_{n+1} being either

$$\frac{i}{2}Re\big((ig|Qf')\chi_{[S',T']}(s)-(ig|Qf)\chi_{[S,T]}(s)+(f'|Qig)\chi_{[S',T']}(s)-(f|Qig)\chi_{[S,T]}(s)\big)+$$
$$+Im(f'|ig)\chi_{[S',T']}(s)+Im(f|ig)\chi_{[S,T]}(s) \tag{6.42a}$$

or

$$\frac{i}{2}Re\big((g|Qf')\chi_{[S',T']}(s)-(g|Qf)\chi_{[S,T]}(s)+(f'|Qg)\chi_{[S',T']}(s)-(f|Qg)\chi_{[S,T]}(s)\big)+$$
$$+Im(f'|g)\chi_{[S',T']}(s)+Im(f|g)\chi_{[S,T]}(s). \tag{6.42b}$$

Resumming P_n, corresponding to the different products in the form (6.39), we obtain the same matrix element of $PL_n(\varepsilon, dB)$ (without the additional factor $dB(s, ig)+$ $+idB(s, g)$). Summing up, we have proved the identity

$$\int_0^t\int_0^s\int_0^{s_1}\dots\int_0^{s_{n-1}} \langle W(\chi_{[S,T]}\otimes f)\Psi_{1\otimes Q}, PL_n(\varepsilon, dB)$$

$$\tfrac{1}{2}(dB(s, ig)+idB(s, g))\cdot W(\chi_{[S', T']}\otimes f')\Psi_{1\otimes Q}\rangle$$

$$= \tfrac{1}{2}\int_0^t ds \int_0^s \int_0^{s_1} \ldots \int_0^{s_{n-1}} \langle W(\chi_{[S,T]}\otimes f)\Psi_{1\otimes Q}, PL^n(\varepsilon, dB)\cdot W(\chi_{[S',T']}\otimes f')\Psi_{1\otimes Q}\rangle\cdot$$

$$\cdot\left\{\left[\frac{i}{2}Re\big((ig|Qf')\chi_{[S',T']}(s)-(ig|Qf)\chi_{[S,T]}(s)+(f'|Qig)\chi_{[S',T']}(s)-(f|Qig)\chi_{[S,T]}(s)\big)+\right.\right.$$

$$\left.+Im(f'|ig)\chi_{[S',T']}(s)+Im(f|ig)\chi_{[S,T]}(s)\right]+$$

$$+i\left[\frac{i}{2}Re\big((g|Qf')\chi_{[S',T']}(s)-(g|Qf)\chi_{[S,T]}(s)+(f'|Qg)\chi_{[S',T']}(s)-(f|Qg)\chi_{[S,T]}(s)\big)+\right.$$

$$\left.\left.+Im(f'|g)\chi_{[S',T']}(s)+Im(f|g)\chi_{[S,T]}(s)\right]\right\}. \qquad (6.43)$$

Using this identity in (6.38), we find

$$F_1(t) = \tfrac{1}{2}\int_0^t ds\,\langle Du\otimes W(\chi_{[S,T]}\otimes f)\Psi_{1\otimes Q}, U(s)\cdot v\otimes W(\chi_{[S',T']}\otimes f')\Psi_{1\otimes Q}\rangle\cdot$$

$$\cdot\left\{\left[\frac{i}{2}Re\big((ig|Qf')\chi_{[S',T']}(s)-(ig|Qf)\chi_{[S,T]}(s)+(f'|Qig)\chi_{[S',T']}(s)-(f|Qig)\chi_{[S,T]}(s)\big)+\right.\right.$$

$$\left.+Im(f'|ig)\chi_{[S',T']}(s)+Im(f|ig)\chi_{[S,T]}(s)\right]+$$

$$+i\left[\frac{i}{2}Re\big((g|Qf')\chi_{[S',T']}(s)-(g|Qf)\chi_{[S,T]}(s)+(f'|Qg)\chi_{[S',T']}(s)-(f|Qg)\chi_{[S,T]}(s)\big)+\right.$$

$$\left.\left.+Im(f'|g)\chi_{[S',T']}(s)+Im(f|g)\chi_{[S,T]}(s)\right]\right\}$$

$$= \tfrac{1}{2}\int_0^t ds\,\langle Du, F(s)\rangle\cdot$$

$$\cdot\left\{\left[\frac{i}{2}Re\big((ig|Qf')\chi_{[S',T']}(s)-(ig|Qf)\chi_{[S,T]}(s)+(f'|Qig)\chi_{[S',T']}(s)-(f|Qig)\chi_{[S,T]}(s)\big)+\right.\right.$$

$$\left.+Im(f'|ig)\chi_{[S',T']}(s)+Im(f|ig)\chi_{[S,T]}(s)\right]+$$

$$+i\left[\frac{i}{2}Re\big((g|Qf')\chi_{[S',T']}(s)-(g|Qf)\chi_{[S,T]}(s)+(f'|Qg)\chi_{[S',T']}(s)-(f|Qg)\chi_{[S,T]}(s)\big)+\right.$$

$$\left.\left.+Im(f'|g)\chi_{[S',T']}(s)+Im(f|g)\chi_{[S,T]}(s)\right]\right\} = \tfrac{1}{2}\int_0^t ds\,\langle Du, F(s)\rangle\cdot F_+(s). \qquad (6.44)$$

Similarly, one can get

$$\langle -D^{+}u\otimes W(\chi_{[S,T]}\otimes f)\Psi_{1\otimes Q}, \int_0^t 1\otimes dA(s,g)\,U(s)\cdot v\otimes W(\chi_{[S',T']}\otimes f')\Psi_{1\otimes Q}\rangle$$
$$= \tfrac{1}{2}\int_0^t ds\,\langle -D^{+}u, F(s)\rangle\cdot F_{-}(s). \qquad (6.45)$$

Hence, (6.34) becomes

$$\langle u, F(t)\rangle = \langle u, F(0)\rangle + \tfrac{1}{4}\int_0^t ds\,[F_{+,-}\langle -D^{+}Du, F(s)\rangle +$$
$$+F_{-,+}\langle -DD^{+}u, F(s)\rangle + F_{-,-}\langle -DDu, F(s)\rangle + F_{+,+}\langle -D^{+}D^{+}u, F(s)\rangle] +$$
$$+\tfrac{1}{2}\int_0^t ds\,[\langle Du, F(s)\rangle F_{+}(s) + \langle -D^{+}u, F(s)\rangle F_{-}(s)]. \qquad (6.46)$$

Therefore, $\langle u, F(t)\rangle = \langle u, G(t)\rangle$, $t \geqslant 0$ and this ends the proof.

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