Commutators Associated With The Renormalized Powers Of Quantum White Noise

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Abstract. Let $\delta(t)$ denote the Dirac delta function. We show how, when the renormalization constant $c > 0$ in $\delta^2(t) = c\,\delta(t)$ is large or approaches $+\infty$, the commutation relations for the Renormalized Powers of Quantum White Noise (RPQWN) can be truncated to yield either the Heisenberg Canonical Commutation Relations (CCR) or the Renormalized Square of White Noise (RSWN) commutation relations of [18], parametrized by the order of the white noise functionals. The, still open, problem of choosing a renormalization of the powers of the delta function that will lead to a Fock representation of the RPQWN commutation relations is described.

1. Introduction

The standard boson white noise Lie algebra is defined by its generators, $b_t, b_s^\dagger, 1$ (central element) satisfying the (first order white noise) commutation relations

$$[b_t, b_s^\dagger] = \delta(t-s)\cdot 1$$

and

$$[b_t^\dagger, b_s^\dagger] = [b_t, b_s] = 0$$

where $t, s \geq 0$ and $\delta(t)$ is the Dirac delta function. The, so called, Hida white noise functionals b_t and $b_t^\dagger$ can be rigorously defined as follows: Let $L^2_{sym}(\mathbf{R}^n)$ denote the space of square integrable functions on $\mathbf{R}^n$ symmetric under permutation of their arguments, and let $F := \bigoplus_{n=0}^\infty L^2_{sym}(\mathbf{R}^n)$ where if $\psi := \{\psi^{(n)}\}_{n=0}^\infty \in F$, then $\psi^{(0)} \in \mathbf{C}$, $\psi^{(n)} \in L^2_{sym}(\mathbf{R}^n)$ and

$$\|\psi\|^2 = \|\psi(0)\|^2 + \sum_{n=1}^\infty \int_{\mathbf{R}^n} |\psi^{(n)}(s_1,\ldots,s_n)|^2 ds_1\ldots ds_n$$

The subspace of vectors $\psi = \{\psi^{(n)}\}_{n=0}^\infty \in F$ with $\psi^{(n)} = 0$ for almost all n will be denoted by D_0. Denote by $S \subset L^2(\mathbf{R}^n)$ the Schwartz space of smooth functions decreasing at infinity faster than any polynomial and let $D := \{\psi \in F | \psi^{(n)} \in S, \sum_{n=1}^\infty n|\psi^{(n)}|^2 < \infty\}$. For each $t \in \mathbf{R}$ define the linear operator $b_t : D \to F$ by

$$(b_t\psi)^{(n)}(s_1,\ldots,s_n) := \sqrt{n+1}\,\psi^{(n+1)}(t, s_1,\ldots,s_n)$$

and the operator valued distribution (cf. [18] for details) b_t^+ by

$$(b_t^+\psi)^{(n)}(s_1,\ldots,s_n) := \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\delta(t-s_i)\psi^{(n-1)}(s_1,\ldots,\hat{s}_i,\ldots,s_n)$$

where ˆ denotes omission of the corresponding variable. The possibility of giving a meaning to the higher powers of white noise, i.e. to the symbolic expressions $b_t^n, {b_s^\dagger}^k$, where $n,k\in\{0,1,2,....\}$ is an old problem of quantum field theory which has been the subject of recent research activity (see e.g [3],[4], [14], [16]). For $n,k\in\{0,1,2,...\}$ we will introduce the notation $\epsilon_{n,k} := 1-\delta_{n,k}$, where $\delta_{n,k}$ is Kronecker's delta, and use "falling" factorial powers $x^{(y)}$ defined by $x^{(y)} = x(x-1)\cdots(x-y+1)$ with $x^{(0)} = 1$.

Lemma 1. *For $l\in\mathbb{N}$ let $\delta^l(t-s)$ denote the formal l–th power of the δ–function ($\delta^0 := 1$). For all $t,s\in\mathbb{R}_+$ and $n,k\geq 0$,*

$$[b_t^n, {b_s^\dagger}^k] = \epsilon_{n,0}\epsilon_{k,0}\textstyle\sum_{l\geq 1}\binom{n}{l}\,k^{(l)}\,{b_s^\dagger}^{k-l}\,b_t^{n-l}\,\delta^l(t-s) \tag{1.1}$$

Proof. We will let k be arbitrary and use induction on n. The cases $n=0$ and/or $k=0$ are obvious. For $n=1$ and $k>0$ we have

$$
\begin{aligned}
[b_t, b_s^{\dagger k}] &= b_t b_s^{\dagger k} - b_s^{\dagger k} b_t \\
&= b_t b_s^{\dagger} b_s^{\dagger k-1} - b_s^{\dagger k} b_t \\
&= \left(b_s^{\dagger} b_t + \delta(t-s)\right) b_s^{\dagger k-1} - b_s^{\dagger k} b_t \\
&= b_s^{\dagger} b_t b_s^{\dagger k-1} + \delta(t-s) b_s^{\dagger k-1} - b_s^{\dagger k} b_t \\
&= b_s^{\dagger} b_t b_s^{\dagger} b_s^{\dagger k-2} + \delta(t-s) b_s^{\dagger k-1} - b_s^{\dagger k} b_t \\
&= b_s^{\dagger} \left(b_s^{\dagger} b_t + \delta(t-s)\right) b_s^{\dagger k-2} + \delta(t-s) b_s^{\dagger k-1} - b_s^{\dagger k} b_t \\
&= b_s^{\dagger 2} b_t b_s^{\dagger k-2} + \delta(t-s) b_s^{\dagger k-1} + \delta(t-s) b_s^{\dagger k-1} - b_s^{\dagger k} b_t \\
&= b_s^{\dagger 2} b_t b_s^{\dagger k-2} + 2\delta(t-s) b_s^{\dagger k-1} - b_s^{\dagger k} b_t \\
&= b_s^{\dagger 2} b_t b_s^{\dagger} b_s^{\dagger k-3} + 2\delta(t-s) b_s^{\dagger k-1} - b_s^{\dagger k} b_t \\
&= b_s^{\dagger 2} \left(b_s^{\dagger} b_t + \delta(t-s)\right) b_s^{\dagger k-3} + 2\delta(t-s) b_s^{\dagger k-1} - b_s^{\dagger k} b_t \\
&= b_s^{\dagger 3} b_t b_s^{\dagger k-3} + 3\delta(t-s) b_s^{\dagger k-1} - b_s^{\dagger k} b_t \\
&\ \ \vdots \\
&= b_s^{\dagger k} b_t + k\delta(t-s) b_s^{\dagger k-1} - b_s^{\dagger k} b_t \\
&= k\delta(t-s) b_s^{\dagger k-1} \\
&= \sum_{l \geq 1} \binom{1}{l} k^{(l)}\, b_s^{\dagger k-l}\, b_t^{1-l}\, \delta^l(t-s).
\end{aligned}
$$

Thus (1.1) is true for $n = 1$. Suppose that it is true for $n = m$. We will show that it is true for $n = m + 1$. We have

$$
\begin{aligned}
b_t^{m+1} b_s^{\dagger k} &= b_t b_t^m b_s^{\dagger k} \\
&= b_t \left(b_s^{\dagger k} b_t^m + \sum_{l\geq 1} \binom{m}{l} k^{(l)}\, b_s^{\dagger k-l}\, b_t^{m-l}\, \delta^l(t-s) \right) \\
&= b_s^{\dagger k} b_t^{m+1} + k b_s^{\dagger k-1} b_t^m \delta(t-s) + \sum_{l\geq 1} \binom{m}{l} k^{(l)}\, b_t\, b_s^{\dagger k-l}\, b_t^{m-l}\, \delta^l(t-s) \\
&= b_s^{\dagger k} b_t^{m+1} + k b_s^{\dagger k-1} b_t^m \delta(t-s) + \sum_{l\geq 1} \binom{m}{l} k^{(l)}\, (b_s^{\dagger k-l}\, b_t \\
&+ (k-l) b_s^{\dagger k-l-1} \delta(t-s)) b_t^{m-l} \delta^l(t-s) \\
&= b_s^{\dagger k} b_t^{m+1} + k b_s^{\dagger k-1} b_t^m \delta(t-s) + \sum_{l\geq 1} \binom{m}{l} k^{(l)}\, b_s^{\dagger k-l}\, b_t^{m-l+1} \delta^l(t-s) \\
&+ \sum_{l\geq 1} \binom{m}{l} k^{(l)} (k-l)\, b_s^{\dagger k-l-1}\, b_t^{m-l} \delta^{l+1}(t-s)
\end{aligned}
$$

which, upon letting $L = l+1$ in the last sum, becomes

$$
\begin{aligned}
&= b_s^{\dagger k} b_t^{m+1} + k b_s^{\dagger k-1} b_t^m \delta(t-s) + \sum_{l\geq 1} \binom{m}{l} k^{(l)}\, b_s^{\dagger k-l}\, b_t^{m-l+1} \delta^l(t-s) \\
&+ \sum_{L\geq 2} \binom{m}{L-1} k^{(L-1)} (k-L+1)\, b_s^{\dagger k-L}\, b_t^{m-L+1} \delta^L(t-s) \\
&= b_s^{\dagger k} b_t^{m+1} + k b_s^{\dagger k-1} b_t^m \delta(t-s) + m\, k b_s^{\dagger k-1} b_t^m \delta(t-s) \\
&+ \sum_{l=2}^{m} \left(\binom{m}{l} + \binom{m}{l-1} \right) k^{(l)}\, b_s^{\dagger k-l}\, b_t^{m-l+1} \delta^l(t-s) \\
&+ k^{(m)} (k-m) b_s^{\dagger k-m-1} \delta^m(t-s) \\
&= b_s^{\dagger k} b_t^{m+1} + (m+k) b_s^{\dagger k-1} b_t^m \delta(t-s) \\
&+ \sum_{l=2}^{m} \left(\binom{m}{l} + \binom{m}{l-1} \right) k^{(l)}\, b_s^{\dagger k-l}\, b_t^{m-l+1} \delta^l(t-s) \\
&+ k^{(m+1)} b_s^{\dagger k-m-1} \delta^m(t-s)
\end{aligned}
$$

Using $\binom{m}{l} + \binom{m}{l-1} = \binom{m+1}{l}$ this becomes

$$
\begin{aligned}
&= b_s^{\dagger k} b_t^{m+1} + (m+1)k b_s^{\dagger k-1} b_t^m \delta(t-s) \\
&+ \sum_{l=2}^{m} \binom{m+1}{l} k^{(l)}\, b_s^{\dagger k-l}\, b_t^{m-l+1} \delta^l(t-s) \\
&+ k^{(m+1)} b_s^{\dagger k-m-1} \delta^m(t-s) \\
&= b_s^{\dagger k} b_t^{m+1} + \sum_{l\geq 1} \binom{m+1}{l} k^{(l)}\, b_s^{\dagger k-l}\, b_t^{m-l+1} \delta^l(t-s).
\end{aligned}
$$

□

Lemma 2. *For all $t, s \in \mathbb{R}_+$ and $n, k, N, K \geq 0$,*

$$
b_t^{\dagger n}\, b_t^k\, b_s^{\dagger N}\, b_s^K = b_t^{\dagger n}\, b_s^{\dagger N}\, b_t^k\, b_s^K + \epsilon_{k,0}\epsilon_{N,0} \textstyle\sum_{l\geq 1} \binom{k}{l} N^{(l)}\, b_t^{\dagger n}\, b_s^{\dagger N-l}\, b_t^{k-l}\, b_s^K\, \delta^l(t-s) \tag{1.2}
$$

Proof.

$$
\begin{aligned}
b_t^{\dagger n} b_t^k b_s^{\dagger N} b_s^K &= b_t^{\dagger n} (b_t^k b_s^{\dagger N}) b_s^K \\
&= b_t^{\dagger n} \left([b_t^k, b_s^{\dagger N}] + b_s^{\dagger N} b_t^k \right) b_s^K \\
&= b_t^{\dagger n} \left(\epsilon_{k,0}\epsilon_{N,0} \sum_{l\geq 1} \binom{k}{l} N^{(l)}\, b_s^{\dagger N-l} b_t^{k-l} \delta^l(t-s) + b_s^{\dagger N} b_t^k \right) b_s^K \\
&= \epsilon_{k,0}\epsilon_{N,0} \sum_{l\geq 1} \binom{k}{l} N^{(l)}\, b_t^n b_s^{\dagger N-l} b_t^{k-l} b_s^K \delta^l(t-s) + b_t^n b_s^{\dagger N} b_t^k b_s^K \\
&= \epsilon_{k,0}\epsilon_{N,0} \sum_{l\geq 1} \binom{k}{l} N^{(l)}\, b_t^n b_s^{\dagger N-l} b_t^{k-l} b_s^K \delta^l(t-s) + b_t^{\dagger n}\, b_s^{\dagger N}\, b_t^k\, b_s^K
\end{aligned}
$$

□

Lemma 3. *For all $t, s \in \mathbb{R}_+$ and $n, k, N, K \geq 0$*

$$
\begin{aligned}
[b_t^{\dagger n} b_t^k, b_s^{\dagger N} b_s^K] = {}& \epsilon_{k,0}\epsilon_{N,0} \textstyle\sum_{l\geq 1} \binom{k}{l} N^{(l)}\, b_t^{\dagger n}\, b_s^{\dagger N-l}\, b_t^{k-l}\, b_s^K\, \delta^l(t-s) \\
&- \epsilon_{K,0}\epsilon_{n,0} \textstyle\sum_{L\geq 1} \binom{K}{L} n^{(L)}\, b_s^{\dagger N}\, b_t^{\dagger n-L}\, b_s^{K-L}\, b_t^k\, \delta^L(t-s)
\end{aligned} \tag{1.3}
$$

Proof. The first term on the right hand side of (1.2) is

$$\begin{aligned}
b_t^{\dagger n} b_s^{\dagger N} b_t^k b_s^K &= b_s^{\dagger N} b_t^{\dagger n} b_s^K b_t^k \\
&= b_s^{\dagger N} \left([b_t^{\dagger n}, b_s^K] + b_s^K b_t^{\dagger n} \right) b_t^k \\
&= b_s^{\dagger N} \left(-[b_s^K, b_t^{\dagger n}] + b_s^K b_t^{\dagger n} \right) b_t^k \\
&= b_s^{\dagger N} \left(-\epsilon_{n,0}\epsilon_{K,0} \sum_{l\geq 1} \binom{K}{l} n^{(l)}\, b_t^{\dagger n-l}\, b_s^{K-l}\, \delta^l(t-s) + b_s^K b_t^{\dagger n} \right) b_t^k \\
&= -\epsilon_{n,0}\epsilon_{K,0} \sum_{l\geq 1} \binom{K}{l} n^{(l)}\, b_s^{\dagger N} b_t^{\dagger n-l}\, b_s^{K-l} b_t^k\, \delta^l(t-s) + b_s^{\dagger N} b_s^K b_t^{\dagger n} b_t^k
\end{aligned}$$

from which (1.3) follows by substituting into (1.2). $\square$

For reasons explained in the following section, Accardi, Volovich and Lu introduced in [18] the renormalization

$$\delta^2(t-s) = c \cdot \delta(t-s) \tag{1.4}$$

where $c > 0$ is an arbitrary real number. This particular renormalization of the square of the delta function turned out to be very fruitful in relation to the study of the squares of the Hida white noise functionals (cf. [2], [7], [19],[20]). It has found applications to quantum optics and to the control of quantum systems described by quantum stochastic differential equations ([8]-[13],[21]-[25]). For a test function $f : \mathbb{R} \to \mathbb{C}$ we define the symbols

$$B_k^n(f) = \int_{\mathbb{R}} f(s)\, b_s^{\dagger n}\, b_s^k\, ds$$

with involution

$$(B_k^n(f))^* = B_n^k(\bar{f})$$

and with

$$B_0^0(f) = \int_{\mathbb{R}} f(s)\, ds\, 1$$

which implies

$$B_0^0(\bar{g}f) = \int_{\mathbb{R}} \bar{g}(s)\, f(s)\, ds\, 1 = < g, f > 1$$

where $< g, f >$ is the usual L_2 inner product and 1 is the identity operator. The renormalization formula (1.4) has the obvious generalization

$$\delta^n(t-s) = c^{n-1} \cdot \delta(t-s) \tag{1.5}$$

where $n \geq 2$. Multiplying both sides of (1.3) by $f(t)\bar{g}(s)$ and formally integrating the resulting identity (i.e. taking $\int\int \ldots dsdt$), we obtain the commutation relations

$$\begin{aligned} & [B_K^N(\bar{g}), B_k^n(f)] \\ & = \sum_{L=1}^{K\wedge n} b_L(K,n)\, B_{K+k-L}^{N+n-L}(\bar{g}f) - \sum_{l=1}^{k\wedge N} b_l(k,N)\, B_{K+k-l}^{N+n-l}(\bar{g}f) \\ & = \sum_{L=1}^{(K\wedge n)\vee(k\wedge N)} \theta_L(N,K;n,k)\, c^{L-1}\, B_{K+k-L}^{N+n-L}(\bar{g}f) \end{aligned} \tag{1.6}$$

where

$$b_x(y,z) := \epsilon_{y,0}\, \epsilon_{z,0} \binom{y}{x} z^{(x)}\, c^{x-1},$$

$n, k, N, K \in \{0, 1, 2, ...\}$ and

$$\theta_L(N,K;n,k) := \epsilon_{L,0} \left(\epsilon_{K,0}\, \epsilon_{n,0} \binom{K}{L} n^{(L)} - \epsilon_{k,0}\, \epsilon_{N,0} \binom{k}{L} N^{(L)} \right).$$

In particular (1.6) contains the Heisenberg or Canonical Commutation Relations (CCR) of [28], namely

$$[B_1^0(\bar{g}), B_0^1(f)] =< g, f >$$

and

$$[B_1^0(\bar{g}), B_1^1(f)] = B_1^0(\bar{g}f),\ [B_1^1(\bar{g}), B_0^1(f)] = B_0^1(\bar{g}f),$$

as well as the commutation relations of the Renormalized Square of White Noise (RSWN) of [18], i.e

$$[B_2^0(\bar{g}), B_0^2(f)] = 4\, B_1^1(\bar{g}f) + 2c < g, f >$$

and

$$[B_1^1(\bar{g}), B_2^0(f)] = -2\, B_2^0(\bar{g}f), \quad [B_1^1(\bar{g}), B_0^2(f)] = 2\, B_0^2(\bar{g}f).$$

From the point of view of Probability Theory, the Heisenberg commutation relations lead to Brownian Motion and the Poisson process ([28]). The Renormalized Square of White Noise (RSWN) Lie algebra, on the other hand, leads to the Gamma process and the Meixner polynomials ([1], [15],[6]). The generalized renormalization formula (1.5) and the commutation relations (1.6) have been the focus of recent efforts aiming at examining the possibility of existence of a Fock space representation for the Lie algebra associated with the higher powers of the white noise functionals ([3],[4], [16]). This amounts to establishing the positive semi-definiteness of the kernel

$$\langle B_0^{K_N}(f_N)\dots B_0^{K_1}(f_1)\Phi, B_0^{n_M}(g_M)\dots B_0^{n_1}(g_1)\Phi\rangle$$

where f_i, $g_j \in \mathcal{H}$ are suitably chosen test functions (containing the characteristic functions of intervals), K_i, $n_j \in \mathbb{N}\cup\{0\}$ for all $i = 1,2,...,N$, $j = 1,2,...,M$, and Φ is the Fock vacuum vector defined by

$$\begin{aligned} B_k^0\Phi &= 0 \quad ; \qquad \forall\, k\in\mathbb{N} \\ B_k^h\Phi &= 0 \quad ; \qquad \forall\, k>0, h\geq 0. \end{aligned}$$

So far, most results have been in the direction of non-existence of a Fock representation. The main counter-example is that if a common Fock representation of the B_k^n existed, one should be able to define inner products of the form

$$< (a\, B_0^{2n}(\chi_I) + b\,(B_0^n(\chi_I))^2)\Phi, (a\, B_0^{2n}(\chi_I) + b\,(B_0^n(\chi_I))^2)\Phi >$$

where $a, b \in \mathbb{R}$ and I is an arbitrary interval of finite measure $\mu(I)$. Using the notation $< x >=< \Phi, x\,\Phi >$ this amounts to the positive semi-definiteness of the quadratic form

$$a^2 \,< B_{2n}^0(\chi_I)\, B_0^{2n}(\chi_I) > +2\,a\,b\, < B_{2n}^0(\chi_I)(B_0^n(\chi_I))^2 > +a^2\, < (B_n^0(\chi_I))^2\,(B_0^n(\chi_I))^2 >$$

or equivalently of the matrix

$$A = \begin{bmatrix} < B_{2n}^0(\chi_I)\, B_0^{2n}(\chi_I) > & < B_{2n}^0(\chi_I)\,(B_0^n(\chi_I))^2 > \\ < B_{2n}^0(\chi_I)\,(B_0^n(\chi_I))^2 > & < (B_n^0(\chi_I))^2\,(B_0^n(\chi_I))^2 > \end{bmatrix}.$$

Using the commutation relations (1.6) we find that

$$A = \begin{bmatrix} (2n)!c^{2n-1}\mu(I) & (2n)!c^{2n-2}\mu(I) \\ (2n)!c^{2n-2}\mu(I) & 2(n!)^2c^{2n-2}\mu(I)^2 + ((2n)! - 2(n!)^2)c^{2n-3}\mu(I) \end{bmatrix}.$$

A is a symmetric matrix, so it is positive semi-definite if and only if its minors are non-negative. The minor determinants of A are

$$d_1 = (2n)! c^{2n-1} \mu(I) \geq 0$$

and

$$d_2 = 2c^{4(n-1)} \mu(I)^2 (n!)^2 (2n)! (c\,\mu(I) - 1) \geq 0 \Leftrightarrow \mu(I) \geq \frac{1}{c}.$$

Thus the interval I cannot be arbitrarily small. The counter-example extends to the q-deformed case

$$b_t\, b_s^\dagger - q\, b_s^\dagger\, b_t \quad = \quad \delta(t-s)$$

where $q \in (-1, 1), q \neq 0$. As it turns out,

$$\mu(I) \geq \frac{1}{c}$$

is a universal sort of bound for these "no-go" theorems ([3], [4],[5], [6], [15], [16]). The problem of choosing a good renormalization of the powers of the delta function, that is, one that will lead to a Fock representation of the operator commutation relations obtained by multiplying both sides of (1.3) by the product of test functions $f(t)\bar{g}(s)$ and integrating the resulting identity, is therefore still open and very challenging. It is also very interesting to see what kind of probability distributions one could obtain in this way. In this paper we will show how by suitably truncating the commutation relations (1.6) we can be reduced to either the Heisenberg (CCR) or the Renormalized Square of White Noise (RSWN) commutation relations, which are known to admit a Fock representation.

2. The Renormalized Square Of The Dirac Delta Function

It is well known ([26]) that the square of the Dirac delta function cannot be rigorously defined as a generalized function. Accardi, Volovich and Lu renormalized the square of the Dirac delta function in [18] motivated as follows: Let $\mathcal{S} = \mathcal{S}(\mathbb{R})$ be the Schwartz space on the real line, let

$$\mathcal{S}_0 = \{\phi \in \mathcal{S} \,:\, \phi(0) = 0\} = \{x\,\psi(x) \,:\, \psi \in \mathcal{S}\}$$

and, for $n \in \{1, 2, ...\}$ define

$$f_n(x) = \begin{cases} \frac{n}{2} & \text{if } -\frac{1}{n} \leq x \leq \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$$

For all $\phi \in \mathcal{S}$,

$$\lim_{n\to+\infty} \int_{\mathbb{R}} f_n(x)\,\phi(x)\,dx = \lim_{n\to+\infty} \tfrac{n}{2} \int_{-\frac{1}{n}}^{\frac{1}{n}} \phi(x)\,dx$$
$$= \lim_{n\to+\infty} \tfrac{1}{2} \int_{-1}^{1} \phi(\tfrac{1}{n}\,y)\,dy = \phi(0)$$

where we have used the substitution $x = \frac{1}{n}\,y$, and so

$$\lim_{n\to+\infty} f_n(x) = \delta(x)$$

in the sense of generalized functions. To give a meaning to $\delta^2(x)$ we notice that for $\phi \in \mathcal{S}_0$

$$\lim_{n\to+\infty} \int_{\mathbb{R}} f_n^2(x)\,\phi(x)\,dx = \lim_{n\to+\infty} \tfrac{n^2}{4} \int_{-\frac{1}{n}}^{\frac{1}{n}} x\,\psi(x)\,dx$$
$$= \lim_{n\to+\infty} \tfrac{1}{4} \int_{-1}^{1} y\,\psi(\tfrac{1}{n}\,y)\,dy = \tfrac{1}{4}\,\psi(0) \int_{-1}^{1} y\,dy = 0$$

thus, as a distribution on $\mathcal{S}_0$,

$$\lim_{n\to+\infty} f_n^2(x) = 0.$$

Let F be an extension of $\lim_{n\to+\infty} f_n^2(x)$ to all of $\mathcal{S}$. For any $\phi \in \mathcal{S}$ we have

$$\phi(x) = \phi(x) - \phi(0)\,\psi(x) + \phi(0)\,\psi(x) \tag{2.1}$$

where $\psi \in \mathcal{S}$ is arbitrary with $\psi(0) = 1$. Since $\phi(x) - \phi(0)\,\psi(x) \in \mathcal{S}_0$ and F is zero as a distribution on $\mathcal{S}_0$, applying F to both sides of (2.1) we obtain

$$F(\phi) = \phi(0)\,F(\psi) \tag{2.2}$$

which is satisfied by

$$F = c \cdot \delta$$

i.e

$$\delta^2 = c \cdot \delta$$

where $c \in \mathcal{C}$ is arbitrary.

In the remaining part of this section we provide some formal calculations which indicate that the renormalization constant c should be somehow allowed to go to infinity or, at least, be thought of as a "very large" positive number: From (2.2) we see that the renormalization constant c could be taken to be equal to

$$c =< F, \psi >:= F(\psi) = \int_{\mathbb{R}} F(s)\, \psi(s)\, ds$$

where $F = \delta^2$ and $\psi \in \mathcal{S}$ is arbitrary, but such that

$$\psi(0) = 1.$$

This choice of c has only one possible value, namely, $c = +\infty$. To see this, let $\{f_n\}_{n=1}^{+\infty}$ be any sequence that can be used to define the delta function. For such a sequence we have

$$\lim_{n \to +\infty} f_n(0) = +\infty.$$

Therefore,

$$c =< F, \psi >=< \delta^2, \psi >= \lim_{n\to+\infty} < f_n\, \delta, \psi >$$
$$= \lim_{n\to+\infty} < \delta, f_n\, \psi >= \lim_{n\to+\infty} f_n(0)\, \psi(0) = +\infty \cdot 1 = +\infty.$$

A different approach, based on distributions with compact support, is the following: We know ([27]) that every distribution with support $\{0\}$ can be written as a linear combination of the Dirac delta function and its derivatives. Therefore

$$\delta^2 = \sum_{\alpha \leq N} c_\alpha \cdot \delta^{(\alpha)}$$

for some $N \geq 0$ where ([27])

$$c_\alpha = \frac{(-1)^\alpha}{\alpha!} < \delta^2, x^\alpha\, \psi >,$$

and $\psi \in C_0^\infty(\mathbb{R})$ with $\psi(x) = 1$ for $|x| < 1/2$ and $\psi(x) = 0$ for $|x| > 1$. Since, for any delta-sequence $\{f_k\}_{k=1}^{+\infty}$

$$< \delta^2, x^\alpha\, \psi >= \lim_{k\to+\infty} < f_k\, \delta, x^\alpha\, \psi >= \lim_{k\to+\infty} < \delta, f_k\, x^\alpha\, \psi >$$
$$= \lim_{k\to+\infty} f_k(0) \cdot 0 \cdot \psi(0) = +\infty \cdot 0 \cdot 1 = 0$$

for $\alpha > 0$, where we have used the convention $0 \cdot +\infty = 0$, while for $\alpha = 0$

$$< \delta^2, x^\alpha\, \psi >= \lim_{k\to+\infty} < f_k\, \delta, \psi >= \lim_{k\to+\infty} < \delta, f_k\, \psi >$$
$$= \lim_{k\to+\infty} f_k(0) \cdot \psi(0) = +\infty \cdot 1 = +\infty$$

it follows that,

$$\delta^2 = c \cdot \delta$$

where $c = +\infty$. In addition, since for the delta function, apart from its pointwise properties, the fact that

$$\int_{\mathbb{R}} \delta(x)\, dx = 1$$

is very important, the following formal calculation also indicates that c should be allowed to go to infinity or be considered to be a very large positive number: Let H denote the Heaviside function. We know that $H' = \delta$. Then, we formally have

$$\begin{gathered} \int_{\mathbb{R}} \delta^2(x)\, dx = \int_{\mathbb{R}} \delta(x)\, \delta(x)\, dx = \int_{\mathbb{R}} H'(x)\, \delta(x)\, dx \\ = H(x)\, \delta(x)|_{x=-\infty}^{x=+\infty} - \int_{\mathbb{R}} H(x)\, \delta'(x)\, dx \\ = H(+\infty)\, \delta(+\infty) - H(-\infty)\, \delta(-\infty) - \int_0^{+\infty} \delta'(x)\, dx \\ = 0 - (\delta(+\infty) - \delta(0)) = \delta(0). \end{gathered}$$

Since

$$\delta^2 = c\, \delta,$$

we have

$$\int_{\mathbb{R}} \delta^2(x)\, dx = \int_{\mathbb{R}} c\, \delta(x)\, dx$$

and so

$$\delta(0) = c \int_{\mathbb{R}} \delta(x)\, dx = c \cdot 1$$

which implies that

$$c = \delta(0).$$

3. Heisenberg Type Commutators Associated With The RPQWN

Definition 1. *For $n, k, N, K \in \{0, 1, 2, ...\}$ with $(K \wedge n) \vee (k \wedge N) \geq 1$ we define the commutator*

$$[B_K^N(\bar{g}), B_k^n(f)]_\infty := \lim_{c \to +\infty} \frac{1}{c^{(K \wedge n) \vee (k \wedge N) - 1}} [B_K^N(\bar{g}), B_k^n(f)]$$

i.e $[B^N_K(\bar g), B^n_k(f)]_\infty$ is the coefficient of the leading term $c^{(K\wedge n)\vee(k\wedge N)-1}$, corresponding to the most singular term, i.e the highest power of δ, in the expansion of $[B^N_K(\bar g), B^n_k(f)]$ as a polynomial in c. We should interpret it as

$$B^N_K(\bar g)\,B^n_k(f) = B^n_k(f)\,B^N_K(\bar g) + [B^N_K(\bar g), B^n_k(f)]_\infty$$

Proposition 1. *For $n \geq 1$,*

$$[B^0_n(\bar g), B^n_0(f)]_\infty = n!\ < g, f >,$$

and for all $1 \leq k \leq n$,

$$[B^k_k(\bar f), B^n_0(g)]_\infty = n^{(k)}\,B^n_0(\bar f g)$$

and

$$[B^0_n(\bar g)), B^k_k(f)]_\infty = n^{(k)}\,B^0_n(\bar g f).$$

i.e B^0_n, B^n_0 and B^k_k satisfy CCR type commutation relations with respect to $[\cdot,\cdot]_\infty$.

Proof.

$$\begin{aligned} &[B^0_n(\bar g), B^n_0(f)]_\infty \\ &= \theta_{(n\wedge n)\vee(0\wedge 0)}(0,n;n,0)\,B^{0+n-(n\wedge n)\vee(0\wedge 0)}_{n+0-(n\wedge n)\vee(0\wedge 0)}(\bar g f) \\ &= \theta_n(0,n;n,0)\,B^{0+n-n}_{n+0-n}(\bar g f) = n!\,B^0_0(\bar g f) = n!\ < g, f > . \end{aligned}$$

Moreover,

$$\begin{aligned} &[B^k_k(\bar g), B^n_0(f)]_\infty \\ &= \theta_{(0\wedge k)\vee(k\wedge n)}(k,k;n,0)\,B^{k+n-(0\wedge k)\vee(k\wedge n)}_{k+0-(0\wedge k)\vee(k\wedge n)}(\bar g f) \\ &= \theta_k(k,k;n,0)\,B^{k+n-k}_{k+0-k}(\bar g f) = n^{(k)}\,B^n_0(\bar g f) \end{aligned}$$

from which by taking adjoints we find

$$[B^0_n(\bar f), B^k_k(g)]_\infty = n^{(k)}\,B^0_n(\bar f g).$$

□

Corollary 1. *For $n \geq 1$, if*

$$A_n(f) := B^0_n(f),\ \ A^\dagger_n(f) := B^n_0(f),\ \ \Lambda_n(f) := \sum_{k=1}^{n} B^k_k(f)$$

then

$$[A_n(\bar{g}), A_n^\dagger(f)]_\infty = n! < g, f >$$

$$[\Lambda_n(\bar{g}), A_n^\dagger(f)]_\infty = \left(\sum_{k=1}^{n} n^{(k)} \right) A_n^\dagger(\bar{g} f)$$

and

$$[A_n(\bar{f}), \Lambda_n(g)]_\infty = \left(\sum_{k=1}^{n} n^{(k)} \right) A_n(\bar{f} g)$$

i.e $A_n, A_n^\dagger$ and Λ_n satisfy CCR type commutation relations with respect to $[\cdot, \cdot]_\infty$.

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Proof. The proof follows from Proposition 1 and the linearity of $[\cdot, \cdot]_\infty$. □

The RPQWN commutator $[B_K^N(\bar{g}), B_k^n(f)]$ of (1.6) is a polynomial in c of degree $(K \wedge n) \vee (k \wedge N) - 1$. If c is very large, then $[B_K^N(\bar{g}), B_k^n(f)]$ is dominated by the $c^{(K \wedge n) \vee (k \wedge N) - 1}$ term which corresponds to B_0^0. We can then obtain the Heisenberg commutation relations as follows.

Definition 2. *For $n, k, N, K \in \{0, 1, 2, ...\}$ with $(K \wedge n) \vee (k \wedge N) \geq 1$, we define the commutator*

$$[B_K^N(\bar{g}), B_k^n(f)]_1 :=$$
$$\theta_{(K \wedge n) \vee (k \wedge N)}(N, K; n, k)\, c^{(K \wedge n) \vee (k \wedge N) - 1}\, B_{K+k-(K \wedge n) \vee (k \wedge N)}^{N+n-(K \wedge n) \vee (k \wedge N)}(\bar{g} f)$$

i.e $[B_K^N(\bar{g}), B_k^n(f)]_1$ is the leading term in the expansion of $[B_K^N(\bar{g}), B_k^n(f)]$ as a polynomial in c. We should interpret it as

$$B_K^N(\bar{g})\, B_k^n(f) = B_k^n(f)\, B_K^N(\bar{g}) + [B_K^N(\bar{g}), B_k^n(f)]_1$$

Proposition 2. *For $n \geq 1$,*

$$[B_n^0(\bar{g}), B_0^n(f)]_1 = n!\, c^{n-1} < g, f >,$$

$$[B_1^1(\bar{f}), B_0^n(g)]_1 = n\, B_0^n(\bar{f} g)$$

and

$$[B_n^0(\bar{g}), B_1^1(f)]_1 = n\, B_n^0(\bar{g} f)$$

i.e B^0_n, B^n_0 *and* B^1_1 *satisfy CCR type commutation relations with respect to* $[\cdot,\cdot]_1$.

Proof.

$$[B^0_n(\bar g), B^n_0(f)]_1$$
$$= \theta_{(n\wedge n)\vee(0\wedge 0)}(0,n;n,0)\ c^{(n\wedge n)\vee(0\wedge 0)-1}\ B^{0+n-(n\wedge n)\vee(0\wedge 0)}_{n+0-(n\wedge n)\vee(0\wedge 0)}(\bar g f)$$
$$= \theta_n(0,n;n,0)\ c^{n-1}\ B^0_0(\bar g f) = n!\, c^{n-1}\ B^0_0(\bar g f) = n!\, c^{n-1}\ < g, f >$$

and

$$[B^1_1(\bar g), B^n_0(f)]_1$$
$$= \theta_{(1\wedge n)\vee(1\wedge 0)}(1,1;n,0)\ c^{(1\wedge n)\vee(1\wedge 0)-1}\ B^{1+n-(1\wedge n)\vee(1\wedge 0)}_{1+0-(1\wedge n)\vee(1\wedge 0)}(\bar g f)$$
$$= \theta_1(1,1;n,0)\ c^0\ B^n_0(\bar g f) = n\, B^n_0(\bar g f) = n\, B^n_0(\bar g f)$$

from which by taking adjoints we obtain

$$[B^0_n(\bar f), B^1_1(g)]_1 = n\, B^0_n(\bar f g).$$

□

4. Square Of White Noise Commutators Associated With The RPQWN

Definition 3. *For* $n, k, N, K \in \{0,1,2,...\}$ *with* $(K\wedge n)\vee(k\wedge N) \geq 1$, *we define the commutator*

$$[B^N_K(\bar g), B^n_k(f)]_2 :=$$
$$\theta_{(K\wedge n)\vee(k\wedge N)}(N,K;n,k)\, c^{(K\wedge n)\vee(k\wedge N)-1}\ B^{N+n-(K\wedge n)\vee(k\wedge N)}_{K+k-(K\wedge n)\vee(k\wedge N)}(\bar g f)$$
$$+\theta_{(K\wedge n)\vee(k\wedge N)-1}(N,K;n,k)\, c^{(K\wedge n)\vee(k\wedge N)-2}\ B^{N+n-(K\wedge n)\vee(k\wedge N)+1}_{K+k-(K\wedge n)\vee(k\wedge N)+1}(\bar g f)$$

i.e $[B^N_K(\bar g), B^n_k(f)]_2$ is the sum of the two leading terms in the expansion of $[B^N_K(\bar g), B^n_k(f)]$ as a polynomial in c. We should interpret it as

$$B^N_K(\bar g)\, B^n_k(f) = B^n_k(f)\, B^N_K(\bar g) + [B^N_K(\bar g), B^n_k(f)]_2$$

Proposition 3. *For* $n \geq 2$,

$$[B^0_n(\bar g), B^n_0(f)]_2 = n!\left(c^{n-1}\ < g, f > +n\ c^{n-2}\, B^1_1(\bar g f)\right),$$

$$[B^1_1(\bar g), B^n_0(f)]_2 = n\, B^n_0(\bar g f)$$

and

$$[B_n^0(\bar{f}), B_1^1(g)]_2 = n\, B_n^0(\bar{f}g).$$

i.e B_n^0, B_0^n *and* B_1^1 *satisfy Renormalized Square of White Noise (RSWN) type commutation relations with respect to* $[\cdot,\cdot]_2$.

Proof.

$$\begin{aligned}
&[B_n^0(\bar{g}), B_0^n(f)]_2 \\
&= \theta_{(n\wedge n)\vee(0\wedge 0)}(0,n;n,0)\; c^{(n\wedge n)\vee(0\wedge 0)-1}\, B^{0+n-(n\wedge n)\vee(0\wedge 0)}_{n+0-(n\wedge n)\vee(0\wedge 0)}(\bar{g}f) \\
&+\theta_{(n\wedge n)\vee(0\wedge 0)-1}(0,n;n,0)\; c^{(n\wedge n)\vee(0\wedge 0)-2}\, B^{0+n-(n\wedge n)\vee(0\wedge 0)+1}_{n+0-(n\wedge n)\vee(0\wedge 0)+1}(\bar{g}f) \\
&= \theta_n(0,n;n,0)\; c^{n-1}\, B_0^0(\bar{g}f) + \theta_{n-1}(0,n;n,0)\; c^{n-2}\, B_1^1(\bar{g}f) \\
&= n!\, c^{n-1}\, B_0^0(\bar{g}f) + n\,\cdot\, n!\, c^{n-2}\, B_1^1(\bar{g}f) \\
&= n!\,(c^{n-1}\, B_0^0(\bar{g}f) + n\, c^{n-2}\, B_1^1(\bar{g}f)) \\
&= n!\,(c^{n-1}\, < g,f > + n\, c^{n-2}\, B_1^1(\bar{g}f))
\end{aligned}$$

and

$$\begin{aligned}
[B_1^1(\bar{g}), B_0^n(f)]_2 &= \theta_{(1\wedge n)\vee(1\wedge 0)}(1,1;n,0)\; c^{(1\wedge n)\vee(1\wedge 0)-1}\, B^{1+n-(1\wedge n)\vee(1\wedge 0)}_{1+0-(1\wedge n)\vee(1\wedge 0)}(\bar{g}f) \\
&+\theta_{(1\wedge n)\vee(1\wedge 0)-1}(1,1;n,0)\; c^{(1\wedge n)\vee(1\wedge 0)-2}\, B^{1+n-(1\wedge n)\vee(1\wedge 0)+1}_{1+0-(1\wedge n)\vee(1\wedge 0)+1}(\bar{g}f) \\
&= \theta_1(1,1;n,0)\; c^0\, B_0^n(\bar{g}f) + \theta_0(1,1;n,0)\; c^{-1}\, B_1^{1+n}(\bar{g}f) \\
&= n\, B_0^n(\bar{g}f) + 0 = n\, B_0^n(\bar{g}f)
\end{aligned}$$

from which by taking adjoints we obtain

$$[B_n^0(\bar{f})), B_1^1(g)]_2 = n\, B_n^0(\bar{f}g).$$

□

References

[1] Accardi L., *Meixner classes and the square of white noise*, Talk given at the: AMS special session “Analysis on Infinite Dimensional Spaces (in honor of L. Gross)” during the AMS-AMA Joint Mathematics Meetings in New Orleans, LA, January 10-13, 2001. AMS Contemporary Mathematics 317, Kuo H.–H., A. Sengupta (eds.) (2003) 1–13.

[2] Accardi L., Amosov G., Franz U., *Second quantized automorphisms of the renormalized square of white noise (RSWN) algebra*, Infinite Dimensional Analysis, Quantum Probability and Related Topics (IDA–QP) 7 (2) (2004) 183–194 Preprint Volterra (2002)

[3] Accardi L., Boukas A., Franz U., *Renormalized powers of quantum white noise*,to appear in Infinite Dimensional Analysis, Quantum Probability, and Related Topics, Vol.9, No. 1 (2006)

[4] Accardi L., Boukas A., *Higher Powers of q-deformed White Noise* , to appear in Methods of Functional and Topology (2005).

[5] Accardi L., Boukas A., *On the Fock Representation of the Renormalized Powers of Quantum White Noise*, to appear in the Levico Proceedings of the "26th Conference: Quantum Probability and Infinite Dimensional Analysis" (2005).

[6] Accardi L., Boukas A., *Ito calculus and quantum white noise calculus*, to appear in the Proceedings of "The 2005 Abel Symbosium, Stochastic Analysis and Applications -A Symposium in Honor of Kiyosi Ito", July 29 - August 4, Oslo, Norway.

[7] Accardi L., Boukas A., *The unitarity conditions for the square of white noise*, Infinite Dimensional Anal. Quantum Probab. Related Topics , Vol. 6, No. 2 (2003) 1-26.

[8] Accardi L., Boukas A., *Control of elementary quantum flows*, Proceedings of the 5th IFAC symposium on nonlinear control systems, July 4-6, 2001, St. Petersburg, Russia".

[9] Accardi L., Boukas A., *Quadratic control of quantum processes*, Russian Journal of Mathematical Physics, vol.9, no. 4, pp. 381-400, 2002, MR 1966015.

[10] Accardi L., Boukas A., *Control of elementary quantum flows*, Proceedings of the 5th IFAC symposium on nonlinear control systems, July 4-6, 2001, St. Petersburg, Russia".

[11] Accardi L., Boukas A., *Control of quantum Langevin equations*, Open Systems and Information Dynamics, 10 (2003), no. 1, 89-104, MR 1965 608.

[12] Accardi L., Boukas A., *From classical to quantum quadratic control*, Proceedings of the International Winter School on Quantum Information and Complexity, pp.106-117, January 6-10, 2003, Meijo university, Japan, World Scientific 2004, ISBN 981-256-047-5

[13] Accardi L., Boukas A., *Control of quantum stochastic differential equations*, Proceedings of the Fourth International Conference on System Identification and Control Problems (SICPRO'05), January 25-28, 2005. The Conference is sponsored by the Institute of Control Sciences of the Russian Academy of Sciences and co-sponsored by the Russian National Committee of Automatic Control and the Russian Academy of Sciences.

[14] Accardi L., Boukas A., *Powers of the delta function*, to appear in the Levico Proceedings of the "26th Conference: Quantum Probability and Infinite Dimensional Analysis" (2005).

[15] Accardi L., Boukas A., *White noise calculus and stochastic calculus*, Stochastic Analysis: Classical and Quantum- Perspectives of White Noise Theory. Meijo University, Nagoya, Japan 1 - 5 November 2004, edited by Takeyuki Hida (Meijo University, Nagoya, Japan), pp.260-300, World Scientific (2005)

[16] Accardi L., Boukas A., *Non-Existence of a Fock Representation of the Renormalized Powers of Quantum White Noise*, submitted (2005).

[17] Accardi L., Franz U., Skeide M., *Renormalized squares of white noise and non-Gaussian noises as Levy processes on real Lie algebras*, Comm. Math. Phys. 228 (2002), no. 1, 123-150.

[18] Accardi L., Lu Y.G., Volovich I.V., *White noise approach to classical and quantum stochastic calculi*, Lecture Notes of the Volterra International School of the same title, Trento, Italy, 1999, Volterra Center preprint 375.

[19] Accardi L., Obata N., *Towards a non-linear extension of stochastic calculus*, Publications of the Research Institute for Mathematical Sciences, Kyoto, RIMS Kokyuroku 957, Obata N. (ed.), (1996), 1-15.

[20] Accardi L., Roschin R., *Renormalized squares of Boson fields* Infinite Dimensional Analysis, Quantum Probability and Related Topics (IDA–QP) 8 (2) (2005)

[21] Boukas A.,*Linear Quantum Stochastic Control*, Quantum Probability and related topics, 105-111, QP -PQ IX, World Scientific Publishing, Riuer Edge NJ,1994.

[22] Boukas A., *Application of Operator Stochastic Calculus to an Optimal Control problem*, Mat. Zametki 53, (1993), no5, 48-56 Russian).Translation in Math.Notes 53 (1993), No 5-6, 489 -494, MR 96a 81070.

[23] Boukas A., *Operator valued stochastic control in Fock space with applications to noise filtering and orbit tracking*, Journal of Probability and Mathematical Statistics, Vol .16, 1, 1994.

[24] Boukas A., *Stochastic Control of operator-valued processes in Boson Fock space*, Russian Journal of Mathematical Physics ,4 (1996) , no. 2, 139-150, MR 97j 81178

[25] Boukas A., *Application of Quantum Stochastic Calculus to Feedback Control*, Global Journal of Pure and Applied Mathematics, Vol. 1, no.1, (2005).

[26] Folland G. B., *Fourier analysis and its applications*, Brooks/Cole Publishing Company, 1992.

[27] Friedlander F. G.,*Introduction to the theory of distributions*, Cambridge University Press, 1983.

[28] Parthasarathy K. R., *An introduction to quantum stochastic calculus*, Birkhauser Boston Inc., 1992.