# The noncommutative markovian property

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# 1 Introduction

The notion of $(d)$–Markov property was introduced for discrete random fields by R.L. Dobrushin [1]. E. Nelson [2] formulated the Markov property in the continuous case and showed that this notion plays a significant role in the theory of Euclidean Bose fields. The attempt of extending Nelson's method to the case of Fermi fields naturally leads to the problem of defining a noncommutative Markov property.

On the other hand, in connection with the results obtained by H. Araki [3] on quantum lattice systems, Ya.G. Sinai pointed out (appendix to the Russian edition of D. Ruelle's book [4]) that an investigation of such systems naturally leads to the problem of defining the concept of "noncommutative Markov chains" (i.e. a class of states on the algebra of quasilocal obsevables on a one–dimensional quantum lattice system, analogue to the classical Markov chains).

The present paper solves this problem by introducing a general definition of noncommutative Markov property and showing that, in the uniformly hyperfinite case, the structure and properties of the corresponding states have noteworthy analogies with the usual Markov chains.

The noncommutative analogue of the Chapman–Kolmogorov equation is deduced and it is proved that it generalizes the Schrödinger equation for the density matrix.

A relationship is established between our noncommutative Markov states and the Gibbs states constructed by H. Araki [3].

The author thanks Ya.G. Sinai for his fruitful discussion of the present paper.

# 2 General definitions

**Definition 1** *Let $d(B) \subseteq B \subseteq A$ be $C^*$–algebras. The quasiconditional expectation with respect to the triplet $d(B) \subseteq B \subseteq A$ is called a linear mapping $E : A \to B$ with the following properties:*

*1) $E(A) \geq 0$, if $a \in A$, $a \geq 0$; 2) $E(c \cdot a) = c \cdot E(a)$ $\forall\, c \in d(B)$, $\forall\, a \in A$; 3) $\|E(c')\| \leq \|c'\|$ $\forall\, c' \in d(B')$, where $(\cdot)'$ is the commutant in $A$.*

For example, if $P : A \to B$ is a conditional expectation (see [5]) and $H \in d(B)'$, $\|H\| \leq 1$, then $E(a) = P(H^*aH)$ defines a quasiconditional expectation with respect to the triple $d(B) \subseteq B \subseteq A$.

**Definition 2** *Let $d(B) \subseteq B \subseteq A$ be the same as they are above, and let $E$ be the quasiconditional expectation with respect to this triplet. It is said that $E$ has the $(d)$–Markovian property if $E(d(B)' \cap A) \subseteq d(B)' \cap B$.*

The quasiconditional expectation given in the example of Definition 1 has the $(d)$–Markovian property[1].

Let $\{A_\alpha\}_{\alpha \in \mathcal{F}}$ be a filtering family of $C^*$–algebras, and let $d : \mathcal{F} \to \mathcal{F}$ be a mapping such that $d(a) \prec \alpha$, $\alpha \prec \beta \Rightarrow d(\alpha) \prec d(\beta)$. It is said that the family $\{E_{\beta,\alpha}\}_{\alpha \prec \beta}$ of quasiconditional expectations relative to the triplets $A_{d(\alpha)} \subseteq A_\alpha \subset A_\beta$ has the $(d)$–Markovian property if each $E_{\beta,\alpha}$ has this property (i.e., if $E_{\beta,\alpha}\ (A_{a(d)} \cap A_\beta) \subseteq A'_{d(\alpha)} \cap A_\alpha$, $\alpha \prec \beta$ (the commutant is understood in relation to $A = C^* - \lim\limits_{\to} A_\alpha$ which is the $C^*$–inductive limit of the family $\{A_\alpha\}_{\alpha \in \mathcal{F}}$)).

**Definition 3** *Let $\{A_\alpha\}_{\alpha \in \mathcal{F}}$ and $d$ be the same as they are above. The state $\varphi$ on $A = C^* - \lim\limits_{\to} A_\alpha$ is called $(d)$–Markovian if there exists a family $\{E_{\beta\alpha}\}$ of quasiconditional expectations relative to the triples $A_{d(\alpha)} \subseteq A_\alpha \subseteq A_\beta$ which is such that $\varphi(\alpha_\beta) = \varphi(E_{\beta,\alpha}(\alpha_\beta))\ \forall\, \alpha_\beta \in A_\beta$, $\alpha \prec \beta$.*

*Remark 1.* If $\varphi$ is a $(d)$–Markovian state and $\{E_{\beta,\alpha}\}$ is the corresponding family of quasiconditional expectations, then

$$E_{\beta,\alpha}(a_\alpha) = a_\alpha (\text{mod } \varphi)\ \forall\, a_\alpha \in A_\alpha,\ \alpha \prec \beta$$

The latter equation should be understood in the sense that

$$\varphi(E_{\beta,\alpha}(a_\alpha)) = \varphi(a_\alpha) \forall\, a_\alpha \in A_\alpha,\ \alpha \prec \beta$$

We shall continue to hold to this agreement further on. Specifically, if $\{E'_{\beta,\alpha}\}$ is another family of quasi–conditional expectations satisfying the conditions of Definition 3, then $E_{\beta,\alpha} = E_{\beta,\alpha}(\text{mod } \varphi)$. Therefore, the $(d)$–Markovian state defines the corresponding family of quasiconditional expectations uniquely. Moreover, $E_{\gamma,\alpha} = E_{\beta,\alpha} \circ E_{\gamma,\beta}(\text{mod } \varphi)$, $\alpha \prec \beta \prec \gamma$.

*Remark 2.* The property of being a $(d)$–Markovian state depends essentially on the family of local algebras $\{A_\alpha\}_{\alpha \in \mathcal{F}}$. Further on the dependence will be assumed in the general case.

[1] It may be proved that each quasiconditional expectation has the $(d)$–Markovian property.

Let $S$ be a topological space and $\mathcal{F}$ a family of a closed subsets of $S$ such that

I) the union of all sets in $\mathcal{F}$ is equal to $S$;

II) if $F \in \mathcal{F}$, then $S - F$ and $\partial F$ (the boundary of $F$) belong to $\mathcal{F}$.

The family of local algebras on $S$ is the family of $C^*$–algebras $\{A_F\}_{F\in\mathcal{F}}$ which is such that $\mathcal{F}$ satisfies I), II), and

III) $F \subseteq G \Rightarrow A_F \subseteq A_G$ (isotonicity),

IV) $A'_F = A_{S-F}$ (duality).

(For the open subset $U \subset S$, $A_U$ is defined as a $C^*$–algebra generated by all $A_F$, $F \in \mathcal{F}$, contained in $U$, while the commutant is understood in relation to $A = C^* - \lim\limits_{\rightarrow} A_F$). The context of the local algebras is natural for formulation of the noncommutative Markovian property.

**Definition 4** *The Markovian state $\varphi$ on $A = C^* - \lim\limits_{\rightarrow} A_F$ is a $(d)$–Markovian state in which the mapping of $d$ is defined as $d(F) = F$ (the interior of $F$), $F \in \mathcal{F}$.*

If $\varphi$ is a Markovian state and $\{E_{G,F}\}_{F\subseteq G}$ is the corresponding family of quasiconditional expectations relative to the triple $A_F \subseteq A_F \subseteq A_G$, then the Markovian state can be expressed by the relationship

$$E_{G,F}(A_{S-F} \cap A_G) \subseteq A_{S-F} \cap A_F, \;\; F \subseteq G, F, G \in \mathcal{F}$$

In the general case the relationship $A_{S-F} \cap A_F = A_{\partial F}$ does not hold. Therefore, the relationship $E_{G,F}(A_{S-F} \cap A_G) \subseteq A_{\partial F}$, $F \subseteq G$, $F$, $G \in \mathcal{F}$, will be called "the strong Markovian proeprty". Assume now thata $\{A_F\}_{F\in\mathcal{F}}$ is such that if $F \subseteq G$, then $A_G$ is generated by $A_F$ and $A_{G-F}$. In this case we shall write $A_G = A_F \vee A_{G-F}$ and say that the family $\{A_F\}$ is factorizable. Moreover, let $E_{G,F}$ be a conditional expectation; then it can easily be seen that for $G = S$, $E_{S,F} = E_F$ is a strong Markovian property equivalent to the relationship $E_F(A_{S-F}) \subseteq A_{\partial F}$.In the commutative case the family of local algebras is factorizable (see [6]) and the relationship given, as can easily be shown, coincides with the Markovian property in the Nelson formulation, is being true that the $(d)$–Markovian property generalizes the analogous concept formulated by Dobuschin [1].

# 3 The Uniformly Hyperfinite Case

Let $A = C^* - \lim\limits_{\rightarrow} M_{[0,n]}$, where $M_{[0,n]}$ is a factor of the type $I_{p_n}$, $p_n \in N$, while all $M_{[0,n]}$ are assumed to have one and the same unity. Fro $m \le n$ we place $M_{[m,n]} = M'_{[0,m-1]} \cap M_{[0,n]}$. If $\varphi$ is a state on $A$, then we use $\varphi_{[0,n]}$ to denote the constriction of $\varphi$ on $M_{[0,n]}$ and $\varphi_n$ the constriction of $\varphi$ on $M_{[n,n]} = M_n$ (which is $I_{q_n}$–factor). The Markovian state will be a $(d)$–Markovian state, where the function $d$ is defined as $d : [0, n] \to [0, n-1]$. The Markovian property for the quasiconditional expectation $E_{n+1,n}$: $M_{[0,n+1]} \to M_{[0,n]}$ is expressed thus: $E_{n+1}(M_{[n,n+1]} \subseteq M_n$. The family $\{M_{[0,n]}\}$ is factorizable, and the Markovian property coincides with the strong Markovian property.

**Theorem 1** *Let $\varphi$ be a Markovian state on $A$. Then $\varphi$ defines the pair $\{(\sigma_n); \varphi_0\}$ such that the following hold: (i) $\varphi_0$ is a state on $M_0$; (iii) $\sigma_n : M_n \to \mathcal{L}(M_{n+1}, M_n)$ is a linear operator such that the mapping $a_n \cdot a_{n+1} \in M_{[n,n+1]} \mapsto \sigma_n(a_n)[a_{n+1}] \in M_n$ is $p_{n-1}$–positive in the sense of [7] with a norm not exceeding 1. ($\mathcal{L}(M_{n+1}, M_n)$ is the space of linear operators from $M_{n+1}$ into $M_n$). (iii) Let $b_i \in M_i$, $\sigma_i(b_i)*$ be conjugate with respect to $\sigma_i(b_i)$, $0 \le i \le n$, for each $n \in N$. Then the equation*

$$\varphi_{[0,n]}(b_0 \cdot \ldots \cdot b_n) = [\sigma_n(b_n) \cdot \ldots \cdot \sigma_0(b_0) \cdot \varphi_0] \tag{1}$$

*completely defines the projective family $(\varphi_{[0,n]})$. Conversely, each such pair defines a unique Markovian state on $A$.*

*Proof*. Let $\varphi$ be a Markovian state. Then there exists a family $\{E_{n,n-1}\}$ of quasiconditional expectations relative to the triples $M_{[0,n-2]} \subseteq M_{[0,n]}$, which has the Markovian property, and $\varphi$ is completely defined by the inductive relationships

$$\varphi_{[0,n]}(a_{[0,n]}) = \varphi_{[0,n-1]}(E_{n,n-1}(a_{[0,n]}))\ \forall\, a_{[0,n]} \in M_{[0,n]} \tag{2}$$

Since $\{M_{[0,n]}\}$ is factorizable, the quasiconditional expectation $E_{n,n-1}$ is defined by its values on $M_{[n-1,n]}$. Let $\sigma_n$ be defined by the equation

$$\sigma_n(b_n)[b_{n+1}] = E_{n+1,n}(b_n \cdot b_{n+1})\ , \quad b_n \in M_n, b_{n+1} \in M_{n+1} \tag{3}$$

Then the first statement in (ii) and (iii) derive, respectively, from the Markovian property and from Eq. (2). From factorizability it follows that $M_{[0,n+1]} W_{p_{n-1}}(M_{[n,n+1]})^2$

[2] $W_n(A)$ is a matrix algebra of order $n \times n$ having coefficients in $A$.

and $M_{[0,n]}W_{p_{n-1}}(M_n)$; therefore, positiveness of $E_{n+1,n}$ is equivalent to $p_{n-1}$–positiveness of the mapping $a_n \cdot a_{\underline{n}+1} \in M_{[n,n+1]} \mapsto \sigma_n(a_n)[a_{n+1}] \in M_n$, and this proves (ii).

Assume conversely that $\{(\sigma_n); \varphi_0\}$ is a pair satisfying (i), (ii), (iii). The family $(\varphi_{[0,n]})$ is projected and defines a unique state $\varphi$ on $A$. Let $E_{n+1,n}$: $M_{[0,n+1]} \to M_{[0,n]}$ be a linear mapping that is defined by means of (3) and the equation

$$E_{n+1,n}(b_{[0,n-1]} \cdot b_{[n,n+1]}) = b_{[0,n-1]} \cdot E_{n+1,n}(b_{[n,n+1]}),\ b_{[0,n-1]} \in M_{[0,n-1]}$$

The concepts presented above prove that $E_{n+1,n}$ is a quasiconditional expectation and that the Markovian property derives from (ii). The quasiconditional expectation $E_{m,n+1}$, is defined by a composition for $m \leq n$; the state $\varphi$ satisfies the relationship (2) and is consequently Markovian. The theorem has been proved.

*Remark 1.* The fact that Eq. (1) defines a projective family of states may be expressed by the equation

$$\sigma_n(b_n)[1] = b_n \pmod{\varphi} \tag{4}$$

*Remark 2.* In the commutative case, (1) takes the form

$$\varphi_{[0,n]}(b_0 \cdot \ldots \cdot b_n) = [\,{}^tP_n b_n \cdot {}^tP_{n-1}b_{n-1} \cdot \ldots \cdot {}^tP_0 b_n \cdot w_0] \tag{5}$$

where ${}^tP_k$ is a transposed stochastic matrix; $w_0$ is a stochastic vector; $b_k$ is a diagonal matrix, and $w_0(u) = \sum_i m_i u_i$, $w_0 = (w_i)$, $u = (u_i)$. If $b_k$ are projectors, then the right side of (5) yields the expression for joint probabilities in a conventional nonuniform Markovian chian.

Assume now that $Z_n = \sigma_n(1)$ for each $n \in \mathbb{N}$.The sequence $(Z_n)$ is called a sequence of transitional matrices for the Markovian state $\varphi$. The following concept justifies this name.

*Corollary 1.* The operator $Z_n \in \mathcal{L}(M_{n+1}, M_n)$ is defined by the matrix $\left(\xi^{(n)}_{ij,\alpha\beta}\right)$, $1 \leq i,\ j \leq q_n$, $1 \leq \alpha, \beta \leq q_{n+1}$, whose coefficients satisfy the relationships

$$\xi^{(n)}_{ij,\alpha\beta} = \overline{\xi^{(n)}_{ji,\beta\alpha}} \tag{6}$$

$$\sum_{\alpha=1}^{q_{n+1}} \xi^{(n)}_{ij,\alpha\alpha} = \delta_{ij} (\mathrm{mod}\ \varphi) \tag{7}$$

*Proof*. From the property (ii) in Theorem 1 it follows that $Z_n$ is positive and therefore transforms Hermite operators into Hermite operators, which proves (6). The relationship (7) is particular case of Eq. (4).

Using $W_n$ to denote the density matrix of $\varphi_n$, we derive the relationship $W_{n+1} = W_n Z_n$, from (1); this relationship represents the analog of the well–known relationship $v_{n+1} = v_n P_n$ ($P_n$ is a stochastic matrix; $v_n$ is a stochastic vector) for a conventional Markov chain. One may write the equation

$$W_t = W_s Z(s', t),\ \ s \leq t \tag{8}$$

in a more general way, where $Z(s;s) = 1$, $Z(s;s+1) = Z_s$, and $Z(s;t)$ satisfy the noncommutative Chapman–Kolmogorov equation $Z(r;t) = Z(r;s) \cdot Z(s;t)$, $r \leq s \leq t$. It may be proved that Theorem 1 also holds for continuous parameters of these equations; then applying reasoning which is analogous to the reasoning used in the commutative case, we derive the noncommutative direct Kolmogorov equation $(d/dt)W\ W(t)S(t)$, where the operator $B \to BS(t)$ transforms Hermite operators with a zero trace for each $t$. A simple example of an operator of this form is $B \to i[B, H(t)] = i(BH(t) - H(t)B)$, where $H(t) = H(t)^*$. Substituting this operator into the noncommutative direct Kolmogorov equation, we obtain $(d/dt)W(t) = i[W(t); H(t)]$ (i.e., we obtain the Schrödinger equation for the density matrix). Conversely, starting from the Schrödinger equation, we obtain the semigroup $K(s,t)$ of matrices whose coefficients satisfy the relationships (6) and (7) which define a noncommutative stochasti matrix.

# 4 The Uniform Case

Unlike the commutative case, the Markovian state is not defined by just the initial distribution $\varphi_0$ ??? the sequence $(Z_n)$ of transition matrices; it is necessary to know the sequence $(\sigma_n)$. In this section it is proved that nevertheless, the ergodic behavior of $\varphi$ depends solely on the transition matrices. ???? the notation in the preceding section, let us consider the case when $M_n M$ does not depend on $n$. In this case $A \otimes_N M$, where $M$ is a fixed $I_q$–factor. We use $J_n$ to denote the insertion of $M$ into the $n$–th factor and products. The shift operator $T$ in $A$ is an algebra endomorphism, which is defined by the property $T \circ J_k = J_{j+1}(k \geq 0)$. It is said that $\varphi$ is stationary if $\varphi \circ T = \varphi$. Let $\varphi \equiv \{(\sigma_n); \varphi_0\}$ be a Markovian state. We shall consider linear operators $\sigma_n : M \to \mathcal{L}(M)$ which arae such that $E_{n+1,n}(J_n(a_n) \cdot J_{n+1}(a_{n+1})) = J_n[\sigma_n(a_n)[a_{n+1}]]$.

**Lemma 1** *Let $\varphi \equiv \{(\sigma_n); \varphi_0\}$ be a Markovian state on A, and let $Z_n = \sigma_n(1)$ for each $n$. Then $\varphi$ is stationary if and only if 1) $Z_n^* \varphi_0 = \varphi_0$, 2) $\sigma_0(mod\ \varphi)$, $\forall\, n \in \mathbb{N}$.*

*Proof*. The sufficiency is obvious. If $\varphi$ is stationary, then for each $b \in M$ the equation

$$\varphi(J_1(b)) = [\sigma_1(b)^* Z_0 \varphi_0](1) = [\sigma_0(b)^*(1) = \varphi_0(b)$$

holds, whence $Z_0^* \varphi_0$, $\sigma_1 = \sigma_0(\text{mod } \varphi)$. The properties 1) and 2) derive from this by induction.

Thus, the stationary Markovian state is defined by the pair $\{\sigma; \varphi_0\}$, where $\sigma(1)^* \varphi_0 = \varphi_0$. Since we shall consider Markovian states for different initial data $\varphi_0$, it is assumed in this section (in accordance with the agreement adopted in the commutative case) that Eqs. (1) and (2) in Lemma 1 hold absolutely are not only for modulo $\varphi$.

For a stipulated $\varphi = \{\sigma; \varphi_0\}$ let the linear transform $S_{[m,n]} : M_{[m,n]} \to \mathcal{L}(M)$, $m \leq n$, be defined as follows:

$$J_m(b_m) \cdot \ldots \cdot J_n(b_n) \mapsto \sigma(b_m)[\sigma(b_{m+1})[\ldots \sigma(b_n)[\cdot] \ldots]] \ , \quad b_i \in M \ , \quad m \leq i \leq n$$

Let us place $\rho_k = S_{[0,k]}(M_{[0,k]})^* \varphi_0 \subseteq M^*$ for $k \in \mathbb{N}$.

**Theorem 2** *Let $\varphi = \{\sigma; \varphi_0\}$ be a stationary Markovian state with the transition matrix $\sigma(1) = Z$. Then if 1 is the sole unitary eigenvalue of $Z$ and at the same time is prive, it follows that $\varphi$ is a factor state. Conversely, if $\varphi$ is a factor–state and $\bigcup_{k=1}^{\infty} S_k^* = M^*$, then 1 is the sole unitary eigenvalue of $Z$ and prime.*

*Proof. Necessity*. First of all note that if $k \leq m \leq n$, are stipulated, then for each $b \in M_{[0,k]}$, $c \in M_{[m,n]}$ we have $\varphi(b{\cdot}c) = [S^*_{[0,k]}(b)\varphi_0](Z^{m-k}S_{[m,n]}(c)[1])$. Moreover, from the properties of quasiconditions expectations it follows that $\|Z\| \leq 1$ and $\|S_{[m,n]}(c)[1]\| \leq \|c\|$. From the fact that $V \to VZ$ conserves ??? trace it follows that $Z(1) = 1$. Therefore, from stationary in the results obtained by S. Kakutani and K. Yoshida [8] it follows that $\lim_{\nu\to\infty} Z^\nu = 1 \otimes \varphi_0$, where $(1 \otimes \varphi_0)(a) = 1 \cdot \varphi(a)$, $a \in M$. Moreover, from stationary it follows that $\varphi_0(S_{[m,n]}(c)[1]) = \varphi_0(Z^m S_{[m,n]}(c)[1]) = \varphi(c)$. Therefore, if $k \in \mathbb{N}$ and $B \in M_{[0,k]}$ are stipulated there exists a $m_0 \in N$ such that for $n \geq m \geq m_0$ and $\forall\, c \in M_{[m,n]}$ we have

$$|\varphi(b \cdot c) - \varphi(b)\varphi(c)| \leq \|c\| \qquad (9)$$

¿From the arbitrariness of $n$, it follows that the inequality (9) is equivalent to the factorizability derived by R.T. Powers [9], and therefore $\varphi$ is a factor–state.

Assume conversely that $\varphi$ is a factor–state. Then Eq. (9) holds, and using the compactness of the unit sphere in $M_{[0,k]}$, one may write it in equivalent form

$$|\psi * ([Z^{m-k} - 1 \otimes \varphi_0](c))| \leq \|c\|\ \forall\, c \in M$$

for each $\psi = S^*_{[0,k]}(b)\varphi_0 c\|b\| \leq 1$. But from the inequality presented above and from the statement of the theorem it derives that $\lim_{\nu\to\infty} Z^\nu = 1 \otimes \varphi_0$ with respect to the norm. From this it follows (see [8]) that 1 is a prime eigenvalue of $Z$, being unique modulo 1.

# 5 Gibbsian States

In this section we prove the following theorem.

**Theorem 3** *Each one–dimensional Gibbsian state is a limit of the inverse (d)–Markovian states for $d \to \infty$ in the H. Araki sense [3]. Under these conditions convergence is exponentially fast.*

The proof of Theorem 3 will be split into three steps:

(1) the structure of the inverse (d)–Markovian space is described;
(2) the class of state which are examples of inverse (d)–Markovian states is

formulated;
(3) it is proved that by means of states constructed in (2) one may approximate the arbitrary Gibbsian state constructed by H. Araki [3].

**Definition 5** *Let $M$ be a matrix algebra of the type $I_q$, $A = \otimes_N M$; let $\varphi$ be a state on $A$. It is said that $\varphi$ is an inverse (d)–Markovian state if a family $\{E_{[0,n],[1,n]}\}_{n\in\mathbb{N}}$ exists which is such that*

*1) $E_{[0,n],[1,n]} : M_{[0,n]} \to M_{[1,n]}$ is a quasiconditional expectation having the (d)–Markovian property, where d is defined on the et sof all segments of the type $[1,n] (n \in \mathbb{N})$ by the formula $d : [1.n] \to [d+2, n]$.*

*2) For each $n \geq d+1$ and $a_{[0,n]} \in M_{[0,n]}$ the equation*

$$\varphi(a_{[0,n]}) = \varphi(T_c E_{[0,n],[1,n]}(a_{[0,n]})) \tag{10}$$

*holds, where $T_c : M_{[1,\infty]} \to A$ is an algebra homomorphism that is defined by the equation $T_c \circ J_k = J_{k-1}$ $(k \geq 1)$.*

*According to the general Definition 2 (see Section 1) the (d)–Markovian property can be expressed in this case by the relationships $E_{[0,n],[1,n]}(M_{[0,d+1]}) \subseteq M_{[1,d+1]}$ for each $n \in \mathbb{N}$.*

The following theorem determines the structure of inverse (d)–Markovian states.

**Theorem 4** *Let $\varphi$ be an inverse (d)–Markovian state on $A = \otimes_N M$. Then a pair $\{\sigma; \varphi_{[0,d]}\}$ exists which is such that: 1) $\varphi_{[0,d]}$ is the state on $M_{[0,d]}$; 2) $\sigma : M \to \mathcal{L}_{d+1}(M_{[0,d]})$ is the linear operator such that the mapping $a \otimes a_{[0,d]} \in M \otimes M_{[0,d]} \to \sigma(a)[a_{[0,d]}] \in M_{[0,d]}$ is $q^{d+1}$–positive (in the sense of [7]) with a norm not exceeding 1; 3) for each $a_i \in M_i$, $0 \leq i \leq n$, the equation*

$$\varphi_{[0,n]}(J_0(a_0) \cdot \ldots \cdot J_n(a_n)) = [\sigma(a_{d+1})^* \cdot \ldots \cdot \sigma(a_n)^* \varphi_{[0,d]}](J_0(a_0) \cdot \ldots \cdot J_d(a_d))$$

*defines a projected family $(\varphi_{[0,n]})$. Conversely, each such pair defines a unique inverse (d)–Markovian state.*

*Remark*. If one compares Eq. (3) in the theorem cited above to Eq. (1) which describes the general structure of Markovian states, it is immediately evident that for $d = 0$ the latter is derived formally from the former by inverting the sequence of the indies $\{d+1, \ldots, n\}$. It is this which justifies the name "inverse Markovian state".

*Proof of Theorem 4*. Let $\varphi$ be an inverse (d)–Markovian state on $A$, and let $\{E_{[0,n],[1,n]}\}_{n\in\mathbb{N}}$ be the corresponding family of quasiconditional expectations. Then if $a_i \in M$, $0 \le i \le d+1$, it follows that for each $n \ge d+1$

$$\varphi(T_c E_{[0,n],[1,n]}(J_0(a_0)\cdot\ldots\cdot J_{d+1}(a_{d+1}))) = \varphi(J_0(a_0)\cdot\ldots\cdot J_{d+1}(a_{d+1})) \quad (11)$$

Let us define the mapping $\sigma_1^{(n)} : M \to \mathcal{L}(M_{[0,d]})$:

$$\sigma_1^{(n)}(a_{d+})(a_{d+1})[a_{[0,d]}] = T_c E_{[0,n],[1,n]}(a_{[0,d]} \cdot J_{d+1}(a_{d+1}))$$

Then by virtue of the (d)–Markovian property

$$E_{[0,n],[1,n]}(a_{[0,d]} \cdot J_{d+1}(a_{d+1})) \in M_{[1,d+1]} \;\forall\, n \in \mathbb{N}$$

for each $a_{[0,d]} \in M_{[0,d]}$, $a_{d+1} \in M$. Therefore, the mappings $\sigma_1^{(n)}$ are correctly defined. But then from (11), it follows that

$$\varphi_{[0,d]}(\sigma_1^{(d+1)}(a_{d+1})[a_{[0,d]}]) = \varphi_{[0,d]}(\sigma_1^{(n)}(a_{d+1})[a_{[0,d]}])$$

for each $a_{d+1} \in M$ and $a_{[0,d]} \in M_{[0,d]}$. In this case we write, as usual, $\sigma_1^{(n)} = \sigma_1^{(d+1)} = \sigma(mod\varphi)\; \forall\, n \in \mathbb{N}$. Finally, the equation in the statement 3) of the theorem derives from the properties of quasiconditional expectations for repetition of the procedure described above.

Conversely, let the pair $\{\sigma; \varphi_{[0,d]}\}$, satisfying the conditions 1), 2), 3) be stipulated. Then the projective family $(\varphi_{[0,n]})$ defines a unique state on $A$. Let us define the family $\{E_{[0,n],[1,n]}\}_{n\in\mathbb{N}}$ by means of the formula

$$E_{[0,n],[1,n]}(a_{[0,d+1]} \cdot a_{[d+2,n]}) = a_{[d+2,n]} \cdot E_{[0,n],[1,n]}(a_{[0,d+1]})$$

$$T\sigma(a_{d+1})[a_{[0,d]}] = E_{[0,n],[1,n]}(a_{[0,d]} \cdot J_{d+1}(a_{d+1}))$$

where $T$ denotes the endomorphism of a rightward shift and $a_{[\alpha,\beta]} \in M_{[\alpha,\beta]}$, $a_{d+1} \in M$. Then, by virtue of the factorizability of the family $(M_{[0,n]})$, each $E_{[0,n],[1,n]}$ is a quasiconditional expectation satisfying the (d)–Markovian property, where the function $d$ is defined above. Moreover, Eq. (10) derives from the condition of the theorem. Therefore, $\varphi$ is an inverse (d)–Markovian state. The theorem has been proved.

Note that the congruence condition for the family $(\varphi_{[0,n]})$ is equivalent to the equation $\sigma(1)^*\varphi_{[0,d]} = \varphi_{[0,d]}$.

In order to formulate specific examples of inverse (d)–Markovian states the following lemma is useful.

**Lemma 2** *Let $\psi$ (a state on $A$) be defined by the equation $\psi(Q) = \varphi(K_0^* Q K_0)/\varphi(K_0^* K)$, $Q \in A$, where $\varphi$ is a state on $A$. Assume that the following conditions are satisfied: 1) $K_0 \in M_{[0,d]}$ (where $d \in \mathbb{N}$ is fixed). 2) An operator $K \in M_{[0,d]}$ and a number $\lambda > 0$ exist which are such that $\varphi_0 \mathcal{L} = \lambda \varphi$, where $\mathcal{L}$ denotes a linear operator $A \to A$, defined as $\mathcal{L}(Q) = T_c \overline{\tau}_0 (K^* Q K)$; ($\tau_0 : A \to M_{[1,\infty]}$ is defined as $\overline{\tau}_0(J_0(a)b) = b \cdot \tau(a)$; $a \in M$; $b \in M_{[1,\infty]}$). Then $\psi$ is an inverse (d)–Markovian state.*

*Proof*. Let $a_{[0,d]} \in M_{[0,d]}$, $a_{d+1} \in M$. We place

$$\sigma(a_{d+1})[a_{[0,d]}] = \lambda^{-1} T_c \{\overline{\tau}_0 (K^* a_{[0,d]} K)\} J_d (K_0^* a_{d+1} K_0)$$

Then for $n > d$

$$\psi(J_0(a_0) \cdot \ldots \cdot J_n(a_n)) = [\sigma(a_{d+1})^* \sigma(a_{d+2})^* \sigma(a_{d+2})^* \cdot \ldots \cdot \sigma(a_n)^* \psi_{[0,d]}](J_0(a_0) \ldots J_d(a_d))$$

Moreover, the mapping $J_{d+1}(a_{d+1}) \cdot a_{[0,d]} \to \sigma(a_{d+1})[a_{[0,d]}]$ is completely positive. From Theorem 4 it then follows that $\varphi$ is an inverse (d)–Markovian state.

From Lemma 2 it is not difficult to derive the following.

*Proof of Theorem 3*. Let $\varphi$ be a Gibbsian state on $A$ corresponding to the finite potential $\Phi$. H. Araki [3] proved that such always exists and ha the form $\psi(Q) = \varphi(K_0^* Q K_0)/\varphi(K_0^* K_0)$, $Q \in A$, where $K_0 \in A$, and $\varphi$ satisfies the relationship $\varphi_0 \mathcal{L} = \lambda \varphi$, where $\mathcal{L} : A \to A$ is the linear operator defined by the equation $\mathcal{L}(Q) = T_c \overline{\tau}_0 (K^* Q K)$, $Q \in A$, for a certain $K \in A$. The operators $K$, $K_0$ can be inverted, and therefore they may be approximated in the norm by the sequences $(K_d)$, $(K_{0,d})$ and inverse opertors which are such that $K_d$, $K_{0,d} \in M_{[0,d]}$. From the reasoning presented by H. Araki ([3], Section 7) it then follows that states $\varphi^{(d)}$ on $A$ and a number $\lambda_d > 0$ exist which are such that $\varphi_0^{(d)} \mathcal{L}_d = \lambda_d \varphi^{(d)}$, and $\mathcal{L}_d(Q) = T_c \overline{\tau}_0 (K_d^* Q K_d)$. Consequently, by virtue of Lemma 2 the state

$$\psi_d(Q) = \varphi^{(d)}(K_{0,d}^* Q K_{0,d}) / \varphi^{(d)}(K_{0,d}^* K_{0,d})$$

is an inverse (d)–Markovian state for each $d \in \mathbb{N}$. But $\mathcal{L}$, and consequently $\varphi$ also, depend continuously on $K$ (see [3], Section 5). Hence, it follows that $\lim_{d \to \infty} \psi_d = \psi$ (in the norm). This proves the first statement of the theorem. The second statement derives from the fact that the approximating sequences may be determined by truncating (starting with the $d$–th term) all series in the expression for $K$ and $K_0$ means of the Tomonaga–Schwinger–Dyson formula (see [3], Section 6). The theorem has been proved.

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