A global version of the quantum duality principle

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The quantum duality principle states that a quantization of a Lie bialgebra provides also a quantisation of the dual formal Poisson group and, conversely, a quantisation of a formal Poisson group yields a quantisation of the dual Lie bialgebra as well. We extend this to a much more general result: namely, for any principal ideal domain R and for each prime $p \in R$ we establish an inner Galois' correspondence on the category $\mathcal{HA}$ of torsionless Hopf algebras over R, using two functors (from $\mathcal{HA}$ to itself) such that the image of the first, resp. of the second, is the full subcategory of those Hopf algebras which are commutative, resp. cocommutative, modulo p (i.e. they are *quantum function algebras (=QFA)*, resp. *quantum universal enveloping algebras (=QUEA)*, at p). In particular we provide a machine to get two quantum groups — a QFA and a QUEA — out of *any* Hopf algebra H over a field k: apply the functors to $k[\nu] \otimes_k H$ for $p = \nu$.

A relevant example occurring in quantum electro-dynamics is studied in some detail.

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1 Introduction

The quantum duality principle is known in literature in several formulations. One of these, due to Drinfeld ([1], §7, and [2]), states that any quantisation $F_h[[G]]$ of $F[[G]]$ yields also a quantisation of $U(\mathfrak{g}^*)$, and, conversely, any quantisation $U_h(\mathfrak{g})$ of $U(\mathfrak{g})$ provides a quantisation of $F[[G^*]]$: here G^*, resp. $\mathfrak{g}^*$, is a Poisson group, resp. a Lie bialgebra, dual to G, resp. to $\mathfrak{g}$). Namely, Drinfeld defines two functors, inverse to each other, from the category of quantum enveloping algebras to the category of quantum formal series Hopf algebras and viceversa such that (roughly) $U_h(\mathfrak{g}) \mapsto F_h[[G^*]]$ and $F_h[[G]] \mapsto U_h(\mathfrak{g}^*)$. The *global version* of the above principle is an improvement of Drinfeld's result, which put it more in purely Hopf algebra theoretical terms and make it much more manageable.

The general idea is the following. Quantisation of groups and Lie algebras is a matter of dealing with suitable Hopf algebras. Roughly, the classical Hopf algebras of interest are the commutative and the connected cocommutative ones: the first are function algebras of affine algebraic groups, the second are universal enveloping algebras of Lie algebras. A quantisation of such an object H_0 will be a Hopf algebra H depending on some parameter, say p, such that setting $p = 0$, i.e. taking the quotient $H \big/ p\,H$, one gets back the original Hopf algebra H_0 . When a quantisation H is given the classical object H_0 inherits an additional structure, that of a Poisson algebra, if $H_0 = F[G]$, or that of a co-Poisson algebra, if $H_0 = U(\mathfrak{g})$; correspondingly, G is an affine *Poisson* group, $\mathfrak{g}$ is a *Lie bialgebra, and then also* its dual space

$\mathfrak{g}^*$ is a Lie bialgebra; we'll denote by G^* any affine Poisson group with tangent Lie bialgebra $\mathfrak{g}^*$, and we say G^* is *dual* to G. In conclusion, one is lead to consider such quantum groups, namely p–depending Hopf algebras which are either *commutative modulo* p or *cocommutative modulo* p. In detail, I focus on the category $\mathcal{HA}$ of all Hopf algebras which are torsion-free modules over a PID, say R; the role of the quantisation parameter will be played by any prime element $p \in R$. For any such p, I introduce well-defined Drinfeld's-like functors from $\mathcal{HA}$ to itself, I show that their image is contained in a category of quantum groups — quantised function algebras in one case, quantised enveloping algebras in the other — and that when restricted to quantum groups these functors are inverse to each other and they exchange the type of the quantum group — switching function to enveloping — and the underlying group — switching G to G^*. The global picture is much richer, giving indeed — from the mathematical point of view — sort of a (inner) Galois' correspondence on $\mathcal{HA}$. I wish to stress the fact that, compared with Drinfeld's result, mine is global in several respects. First, I deal with functors applying to general Hopf algebras (not only quantum groups, i.e. I do not require them to be commutative up to specialisation or cocommutative up to specialisation). Second, I work with more global objects, namely algebraic Poisson groups rather than *formal* algebraic Poisson groups. Third, I do *not require* the geometric objects — Poisson groups and Lie bialgebras — to be finite dimensional. Fourth, the ground ring R is any PID, not necessarily $k[[h]]$ as in Drinfeld's approach: therefore one may have several primes $p \in R$, and to each of them the machinery applies. In particular we have a method to get, out of any Hopf algebra over a PID, several quantum groups, namely two of them (of both types) for each prime $p \in R$. As an application, one can start from any Hopf algebra H over a field k and then take $H_x := k[x] \otimes_k H$, (x being an indeterminate): this is a Hopf algebra over the PID $k[x]$, to which Drinfeld's functors at any prime $p \in k[x]$ may be applied to give quantum groups. In this note, I state the result and illustrate an application to a nice example occurring in the study of renormalisation of quantum electrodynamics.

2 The global quantum duality principle

2.1 The classical setting.

Let k be a field of zero characteristic. We call (affine) algebraic group the maximal spectrum $G := Hom_{k\text{-}Alg}(H,k)$ of any commutative Hopf k–algebra H; then H is called the algebra of regular function on G, denoted with $F[G]$. We say that G is connected if $F[G]$ has no non-trivial idempotents. We denote by $\mathfrak{m}_e$ the defining ideal of the unit element $e \in G$, and by $\overline{\mathfrak{m}_e{}^2}$ the closure of $\mathfrak{m}_e{}^2$ w.r.t. the weak topology; the cotangent space of G at e is $\mathfrak{g}^\times := \mathfrak{m}_e \Big/ \overline{\mathfrak{m}_e{}^2}$, endowed with its weak topology; by $\mathfrak{g}$ we mean the tangent space of G at e, realized as the topological dual $\mathfrak{g} := \left(\mathfrak{g}^\times\right)^\star$ of $\mathfrak{g}^\times$: this is *the tangent Lie algebra of* G. By $U(\mathfrak{g})$ we mean the universal enveloping algebra of $\mathfrak{g}$: this is a connected cocommutative Hopf algebra, and there is a natural Hopf pairing between $F[G]$ and $U(\mathfrak{g})$. Now assume G is a

Poisson algebraic group: then $\mathfrak{g}$ is a Lie bialgebra, $U(\mathfrak{g})$ is a co-Poisson Hopf algebra, $F[G]$ is a Poisson Hopf algebra. Then $\mathfrak{g}^\times$ and $\mathfrak{g}^\star$ (the topological dual of $\mathfrak{g}$ w.r.t. the weak topology) are (topological) Lie bialgebras too: in both cases, the Lie bracket is induced by the Poisson bracket of $F[G]$, and the non-degenerate natural pairings $\mathfrak{g} \times \mathfrak{g}^\times \longrightarrow k$ and $\mathfrak{g} \times \mathfrak{g}^\star \longrightarrow k$ are compatible with these Lie bialgebra structures: so $\mathfrak{g}$ and $\mathfrak{g}^\times$ or $\mathfrak{g}^\star$ are Lie bialgebras *dual to each other*. We let $G^\star$ be any connected algebraic Poisson group with tangent Lie bialgebra $\mathfrak{g}^\star$, and say it is *dual* to G.

2.2 The quantum setting.

Let R be a principal ideal domain, let $Q(R)$ be its quotient field, let $p \in R$ be a fixed prime element and $k_p := R\big/(p)$ be the residue field, that we assume have zero characteristic. Let $\mathcal{A}$ the category of torsion-free R–modules, and $\mathcal{HA}$ the subcategory of all Hopf algebras in $\mathcal{A}$. Let $\mathcal{A}_F$ the category of $Q(R)$–vector spaces, and $\mathcal{HA}_\mathcal{F}$ be the subcategory of all Hopf algebras in $\mathcal{A}_F$. For $M \in \mathcal{A}$, set $M_F := Q(R) \otimes_R M$, $M_p := M\big/p\,M = k_p \otimes_R M$ (the *specialisation* of M at $p = 0$). For any $H \in \mathcal{HA}$, let $I_H := {\epsilon_H}^{-1}(p\,R)$: set $I_H^\infty := \bigcap_{n=0}^{+\infty} I_H^{\,n}$, $H_\infty := \bigcap_{n=0}^{+\infty} p^n H$.

Given $\mathbb{H}$ in $\mathcal{HA}_\mathcal{F}$, a subset H of $\mathbb{H}$ is called *an R–integer form of* $\mathbb{H}$ if:

(a) H is an R–Hopf subalgebra of $\mathbb{H}$; *(b)* H is torsion-free as an R–module (hence $H \in \mathcal{HA}$); *(c)* $H_F := Q(R) \otimes_R H = \mathbb{H}$.

Definition 2.1. *(Global quantum groups [or algebras]) Fix a prime $p \in R$.*

(a) We call quantized universal enveloping algebra *(in short, QUEA) any pair $\big(\mathbb{U}, U\big)$ such that $U \in \mathcal{HA}$, $\mathbb{U} \in \mathcal{HA}_\mathcal{F}$, U is an R–integer form of $\mathbb{U}$, and $U_p := U\big/p\,U$ is (isomorphic to) the universal enveloping algebra of a Lie algebra. We denote by $\mathcal{QUEA}$ the subcategory of $\mathcal{HA}$ whose objects are all the QUEA's.*

(b) We call quantized function algebra *(in short, QFA) any pair $\big(\mathbb{F}, F\big)$ such that $F \in \mathcal{HA}$, $\mathbb{F} \in \mathcal{HA}_\mathcal{F}$, F is an R–integer form of $\mathbb{F}$, $F_\infty = I_F^\infty$ (notation of §2.2) and $F_p := F\big/p\,F$ is (isomorphic to) the algebra of regular functions of a connected algebraic group. We call $\mathcal{QFA}$ the subcategory of $\mathcal{HA}$ of all QFA's.*

If $\big(\mathbb{U}, U\big)$ is a QUEA (at p), then U_p is a co-Poisson Hopf algebra, so $U_p \cong U(\mathfrak{g})$ for $\mathfrak{g}$ a *Lie bialgebra*; in this situation we shall write $\mathbb{U} = \mathbb{U}_p(\mathfrak{g})$, $U = U_p(\mathfrak{g})$. Similarly, if $\big(\mathbb{F}, F\big)$ is a QFA then F_p is a Poisson Hopf algebra, so $F_p \cong F[G]$ for G a *Poisson algebraic group*: thus we shall write $\mathbb{F} = \mathbb{F}_p[G]$, $F = F_p[G]$.

2.3 Drinfeld's functors ([1], §7).

Let $H \in \mathcal{HA}$. Define $\Delta^n \colon H \longrightarrow H^{\otimes n}$ by $\Delta^0 := \epsilon$, $\Delta^1 := id_H$, and $\Delta^n := \big(\Delta \otimes id_H^{\otimes (n-2)}\big) \circ \Delta^{n-1}$ for every $n > 2$. Then set $\delta_n := (id_H - \epsilon)^{\otimes n} \circ \Delta^n$, for all $n \in \mathbb{N}$. Set $H' := \big\{\, a \in H \,\big|\, \delta_n(a) \in p^n H^{\otimes n}\ \forall\, n \in \mathbb{N} \,\big\}$ $(\subseteq H)$, $H^\vee := \sum_{n \geq 0} p^{-n} I_H^{\,n}$ $(\subseteq H_F)$.

Theorem 2.1. *(The global quantum duality principle)*

(a) $H \mapsto H'$ and $H \mapsto H^\vee$ defines functors $(\ )' \colon \mathcal{HA} \longrightarrow \mathcal{HA}$ and $(\ )^\vee \colon \mathcal{HA} \longrightarrow \mathcal{HA}$, respectively, whose images lie in $\mathcal{QFA}$ and $\mathcal{QUEA}$, respectively.

(b) For all $H \in \mathcal{HA}$, we have $H \subseteq \big(H^\vee\big)'$ and $H \supseteq \big(H'\big)^\vee$. Moreover

$$H = \big(H^\vee\big)' \iff \big(H_F, H\big) \in \mathcal{QFA} \quad \text{and} \quad \mathcal{H} = \big(\mathcal{H}'\big)^\vee \iff \big(\mathcal{H}_{\mathcal{F}}, \mathcal{H}\big) \in \mathcal{QUEA}\,,$$

thus we have induced functors $(\ )' \colon \mathcal{QUEA} \longrightarrow \mathcal{QFA}\,,\ \big(\mathbb{H}, H\big) \mapsto \big(\mathbb{H}, H'\big)$ *and* $(\ )^\vee \colon \mathcal{QFA} \longrightarrow \mathcal{QUEA}\,,\ \big(\mathbb{H}, H\big) \mapsto \big(\mathbb{H}, H^\vee\big)$ *which are inverse to each other.*

(c) With notation of § 1, we have

$$U_p(\mathfrak{g})' \Big/ p\, U_p(\mathfrak{g})' \, = \, F\big[G^\star\big]\,, \qquad F_p[G]^\vee \Big/ p\, F_p[G]^\vee \, = \, U(\mathfrak{g}^\times)$$

where the choice of the group $G^\star$ *(among all the connected algebraic Poisson groups with tangent Lie bialgebra* $\mathfrak{g}^\star$*) depends on the choice of the QUEA* $\Big(\mathbb{U}_p(\mathfrak{g}), U_p(\mathfrak{g})\Big)$. *In short, if* $\Big(\mathbb{U}_p(\mathfrak{g}),\, U_p(\mathfrak{g})\Big)$ *is a QUEA for the Lie bialgebra* $\mathfrak{g}$*, then* $\Big(\mathbb{U}_p(\mathfrak{g}),\, U_p(\mathfrak{g})'\Big)$ *is a QFA for the Poisson group* $G^\star$*, and if* $\Big(\mathbb{F}_p[G],\, F_p[G]\Big)$ *is a QFA for the Poisson group* G*, then* $\Big(\mathbb{F}_p[G],\, F_p[G]^\vee\Big)$ *is a QUEA for the Lie bialgebra* $\mathfrak{g}^\times$. $\square$

In particular, part *(c)* of the claim above shows that $\mathbb{U}_p(\mathfrak{g})$ may be thought of — roughly — as a quantum function algebra, as well as a quantum universal enveloping algebra, for at the same time it has an integer form which is a quantisation of a universal enveloping algebra and *also* an integer form which is a quantisation of a function algebra. Similarly, $\mathbb{F}_p[G]$ may be seen as a quantum function algebra *and* as a quantum universal enveloping algebra.

3 Example: the global quantum duality principle in massless QED

3.1 Classical data.

Let $\mathcal{L} = \mathcal{L}(\mathbb{N}_+)$ be the free Lie algebra over $\mathbb{C}$ with countably many generators $\{x_n\}_{n \in \mathbb{N}_+}$, and $U = U(\mathcal{L})$ its universal enveloping algebra; let $V = V(\mathbb{N}_+)$ be the $\mathbb{C}$–vector space with basis $\{x_n\}_{n \in \mathbb{N}_+}$, and $T = T(V)$ the tensor algebra of V. We shall use the canonical identifications $U(\mathcal{L}) = T(V) = \mathbb{C}\langle x_1, x_2, \dots, x_n, \dots\rangle$ (the unital $\mathbb{C}$–algebra of non-commutative polynomials in the x_n's), $\mathcal{L}$ being the Lie subalgebra of $U = T$ generated by $\{x_n\}_{n \in \mathbb{N}_+}$. Note also that $U(\mathcal{L})$ has a standard Hopf algebra structure given by $\Delta(x) = x \otimes 1 + 1 \otimes x\,,\ \epsilon(x) = 0\,,\ S(x) = -x$ for all $x \in \mathcal{L}\,,$ which is also determined by the same formulæ for $x \in \{x_n\}_{n \in \mathbb{N}_+}$ only.

3.2 The Hopf algebra of Brouder-Frabetti and its deformations.

Let $\mathcal{H}^{\text{dif}}$ be the complex Hopf algebra defined in [4], §5.1: as a $\mathbb{C}$–algebra it is simply $\mathcal{H}^{\text{dif}} := \mathbb{C}\langle \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n, \dots\rangle$, and its Hopf algebra structure is given by

$$\Delta(\mathbf{a}_n) = \mathbf{a}_n \otimes 1 + 1 \otimes \mathbf{a}_n + \sum_{m=1}^{n-1} Q_m^{n-m}(\mathbf{a}_*) \otimes \mathbf{a}_{n-m}\,, \qquad \epsilon(\mathbf{a}_n) = 0 \tag{1}$$

$$S(\mathbf{a}_n) = -\mathbf{a}_n - \sum_{m=1}^{n-1} S\big(Q_m^{n-m}(\mathbf{a}_*)\big)\, \mathbf{a}_{n-m} = -\mathbf{a}_n - \sum_{m=1}^{n-1} Q_m^{n-m}(\mathbf{a}_*)\, S(\mathbf{a}_{n-m}) \tag{2}$$

(for all $n \in \mathbb{N}_+$) where the latter formula gives the antipode by recursion. Hereafter $P_m^{(k)}(\mathbf{a}_*) := \sum_{\substack{j_1,\dots,j_k>0 \\ j_1+\cdots+j_k=m}} \mathbf{a}_{j_1} \cdots \mathbf{a}_{j_k}$, $Q_m^\ell(\mathbf{a}_*) := \sum_{k=1}^m \binom{\ell+1}{k} P_m^{(k)}(\mathbf{a}_*)$ $(m,k,\ell \in \mathbb{N}_+)$.

The very definitions imply that mapping $\mathbf{a}_n \mapsto x_n$ (for all n) yields an isomorphism $\mathcal{H}^{\mathrm{dif}} \xrightarrow{\cong} U(\mathcal{L})$ of unital associative $\mathbb{C}$–algebras, but *not* of *Hopf* algebras.

Now we build up quantum groups. Pick an indeterminate ν, and consider $\mathcal{H}^{\mathrm{dif}}[\nu] := \mathbb{C}[\nu] \otimes_{\mathbb{C}} \mathcal{H}^{\mathrm{dif}} = \mathbb{C}[\nu]\big\langle \mathbf{a}_1, \dots, \mathbf{a}_n, \dots \big\rangle$: this is a Hopf $\mathbb{C}[\nu]$–algebra, with Hopf structure given by (1–2) again; by construction, $\mathcal{H}^{\mathrm{dif}}[\nu] \in \mathcal{HA}$. Set also $\mathcal{H}^{\mathrm{dif}}(\nu) := \mathbb{C}(\nu) \otimes_{\mathbb{C}[\nu]} \mathcal{H}^{\mathrm{dif}}[\nu] = \mathbb{C}(\nu) \otimes_{\mathbb{C}} \mathcal{H}^{\mathrm{dif}} = \mathbb{C}(\nu)\big\langle \mathbf{a}_1, \dots, \mathbf{a}_n, \dots \big\rangle$, a Hopf algebra over $\mathbb{C}(\nu)$.

We consider $\mathcal{H}_\nu := \Big(\mathcal{H}^{\mathrm{dif}}[\nu]\Big)^\vee$ at $p = \nu$ and describe it explicitly. For all $n \in \mathbb{N}_+$, set $\mathbf{x}_n := \nu^{-1}\mathbf{a}_n$: then $\mathcal{H}_\nu$ is the $\mathbb{C}[\nu]$–subalgebra of $\mathcal{H}^{\mathrm{dif}}(\nu)$ generated by the $\mathbf{x}_n$'s, so $\mathcal{H}_\nu = \mathbb{C}[\nu]\big\langle \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n, \dots \big\rangle$. Moreover, formulæ (1–2) give

$$\Delta(\mathbf{x}_n) = \mathbf{x}_n \otimes 1 + 1 \otimes \mathbf{x}_n + \sum_{m=1}^{n-1} \sum_{k=1}^{m} \nu^k \binom{n-m+1}{k} P_m^{(k)}(\mathbf{x}_*) \otimes \mathbf{x}_{n-m} , \qquad \epsilon(\mathbf{x}_n) = 0$$

$$S(\mathbf{x}_n) = -\mathbf{x}_n - \sum\nolimits_{m=1}^{n-1} \sum\nolimits_{k=1}^{m} \nu^k \binom{n-m+1}{k} S\big(P_m^{(k)}(\mathbf{x}_*)\big)\, \mathbf{x}_{n-m} =$$

$$= -\mathbf{x}_n - \sum\nolimits_{m=1}^{n-1} \sum\nolimits_{k=1}^{m} \nu^k \binom{n-m+1}{k} P_m^{(k)}(\mathbf{x}_*)\, S(\mathbf{x}_{n-m})$$

(for all $n \in \mathbb{N}_+$) whence we see by hands that the following holds:

Lemma 3.1. *Formulæ above make $\mathcal{H}_\nu = \mathbb{C}[\nu]\big\langle \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n, \dots \big\rangle$ into a Hopf $\mathbb{C}[\nu]$–subalgebra, embedded into $\mathcal{H}^{dif}(\nu) := \mathbb{C}(\nu) \otimes_{\mathbb{C}} \mathcal{H}^{dif}$ as a Hopf subalgebra. Moreover, $\mathcal{H}_\nu$ is a quantisation of $\mathcal{H}^{dif}$ at $(\nu{-}1)$, for its specialisation at $\nu = 1$ is isomorphic to $\mathcal{H}^{dif}$, i.e. $\mathcal{H}_1 := \mathcal{H}_\nu \Big/ (\nu{-}1)\, \mathcal{H}_\nu \cong \mathcal{H}^{dif}$ via $\mathbf{x}_n \bmod (\nu{-}1)\, \mathcal{H}_\nu \mapsto \mathbf{a}_n$ ($\forall\ n \in \mathbb{N}_+$) as Hopf algebras over $\mathbb{C}$.* □

The previous result shows that $\mathcal{H}_\nu$ *is a deformation of* $\mathcal{H}^{\mathrm{dif}}$, *which is recovered as specialisation limit (of* $\mathcal{H}_\nu$*) at* $\nu = 1$. The next result shows that $\mathcal{H}_\nu$ *is also a deformation of* $U(\mathcal{L})$, *which is recovered as specialisation limit at* $\nu = 0$.

Lemma 3.2. *$\mathcal{H}_\nu$ is a quantized universal enveloping algebra at $\nu = 0$. More precisely, the specialisation limit of $\mathcal{H}_\nu$ at $\nu = 0$ is $\mathcal{H}_0 := \mathcal{H}_\nu \Big/ \nu\, \mathcal{H}_\nu \cong U(\mathcal{L})$ via $\mathbf{x}_n \bmod \nu\, \mathcal{H}_\nu \mapsto x_n\ \forall\ n \in \mathbb{N}_+$, thus inducing on $U(\mathcal{L})$ the structure of co-Poisson Hopf algebra uniquely provided by the Lie bialgebra structure on $\mathcal{L}$ given by $\delta(x_n) = \sum_{\ell=1}^{n-1} (\ell+1)\, x_{n-\ell} \wedge x_\ell$ (for all $n \in \mathbb{N}_+$), where $a \wedge b := a \otimes b - b \otimes a$.* □

Next, we can consider ${\mathcal{H}_\nu}'$, again at $p = \nu$. By direct computation one finds

Theorem 3.3. *(a) Let $\tilde{\mathbf{x}} := \nu\, \mathbf{x}$ for all $\mathbf{x} \in \mathcal{H}_\nu$, and let $[a,c] := a\,c - c\,a$. Then*

$${\mathcal{H}_\nu}' = \Big\langle \widetilde{\mathcal{L}} \Big\rangle = \mathbb{C}[\nu] \left\langle \big\{ \widetilde{\mathbf{b}}_b \big\}_{b \in B} \right\rangle \bigg/ \bigg(\Big\{ \big[\widetilde{\mathbf{b}}_{b_1}, \widetilde{\mathbf{b}}_{b_2}\big] - \nu\, \widetilde{\big[\mathbf{b}_{b_1}, \mathbf{b}_{b_2}\big]} \;\Big|\; \forall\, b_1, b_2 \in B \Big\} \bigg) .$$

(b) ${\mathcal{H}_\nu}'$ *is a Hopf* $\mathbb{C}[\nu]$*–subalgebra of* $\mathcal{H}_\nu$ *, and* $\mathcal{H}^{dif}$ *is naturally embedded into* ${\mathcal{H}_\nu}'$ *as a Hopf subalgebra via* $\mathbf{a}_n \mapsto \tilde{\mathbf{x}}_n$ *(for all* $n \in \mathbb{N}_+$*).*

(c) ${\mathcal{H}_0}' := {\mathcal{H}_\nu}' \Big/ \nu \, {\mathcal{H}_\nu}' = F\big[G_{\mathcal{L}}^\star\big]$ *, where* $G_{\mathcal{L}}^\star$ *is an infinite dimensional connected Poisson algebraic group with cotangent Lie bialgebra (isomorphic to)* $\mathcal{L}$*.* □

3.3 Specialisation limits.

So far, I have already pointed out (by Lemmas 3.1–2 and Theorem 3.3*(c)*) the following specialisation limits: $\mathcal{H}_\nu \xrightarrow{\nu\to1} \mathcal{H}^{\mathrm{dif}}$, $\mathcal{H}_\nu \xrightarrow{\nu\to0} U(\mathcal{L})$, ${\mathcal{H}_\nu}' \xrightarrow{\nu\to0} F\big[G_{\mathcal{L}}^\star\big]$. In addition, Theorem 3.3*(a)* implies that ${\mathcal{H}_\nu}' \xrightarrow{\nu\to1} \mathcal{H}^{\mathrm{dif}}$ as well.

To summarize, $\mathcal{H}^{\mathrm{dif}}$ can be thought of as a deformation of $U(\mathcal{L})$, because we can see $\mathcal{H}_\nu$ as a one-parameter family of Hopf algebras linking $U(\mathcal{L}) = \mathcal{H}_0$ to $\mathcal{H}_1 = \mathcal{H}^{\mathrm{dif}}$. Similarly, $\mathcal{H}^{\mathrm{dif}}$ can also be seen as a deformation of $F\big[G_{\mathcal{L}}^\star\big]$, in that ${\mathcal{H}_\nu}'$ acts as a one-parameter family of Hopf algebras linking $F\big[G_{\mathcal{L}}^\star\big] = {\mathcal{H}_0}'$ to ${\mathcal{H}_1}' = \mathcal{H}^{\mathrm{dif}}$. The two families match at the value $\nu = 1$, corresponding (in both families) to the common element $\mathcal{H}^{\mathrm{dif}}$. At a glance, we have

$$U(\mathcal{L}) = \mathcal{H}_0 \xleftrightarrow{0\leftarrow\nu\to1} \mathcal{H}_1 = \mathcal{H}^{\mathrm{dif}} = {\mathcal{H}_1}' \xleftrightarrow{1\leftarrow\nu\to0} {\mathcal{H}_0}' = F\big[G_{\mathcal{L}}^\star\big] \; .$$

Note that the leftmost and the rightmost term above are Hopf algebras of classical type, having a very precise geometrical meaning: in particular, they both have a Poisson structure, arising as semiclassical limit of their deformation. Since $\mathcal{H}^{\mathrm{dif}}$ *is intermediate between them, their geometrical meaning should shed some light on it. In turn, the physical meaning of* $\mathcal{H}^{\mathrm{dif}}$ *should have some reflect on the meaning of the semiclassical (geometrical) objects* $U(\mathcal{L})$ *and* $F\big[G_{\mathcal{L}}^\star\big]$*.*

Finally, I wish to stress that the discussion about this example applies to a much more general framework, namely to every Hopf $\mathbb{C}$–algebra $\mathcal{H}$ (instead of $\mathcal{H}^{\mathrm{dif}}$) which is *graded* and is generated by a set of *generators of positive degree.*

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