Optimal Hamiltonian of Fermion Flows

Luigi Accardi
Centro Vito Volterra, Università di Roma Tor Vergata
via Columbia, 2– 00133 Roma, Italy
accardi@Volterra.mat.uniroma2.it
Andreas Boukas
Department of Mathematics and Natural Sciences, American College of Greece
Aghia Paraskevi, Athens 15342, Greece
andreasboukas@acgmail.gr

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Abstract. We consider the problem of determining the noise coefficients of the Hamiltonian associated with a Fermion flow so as to minimize a naturally associated quadratic performance functional. This extends the results of [4] obtained for Boson flows to Fermion flows. We also provide a general formulation of Fermion flows.

1. Quantum Stochastic Calculus

The Boson Fock space $\Gamma := \Gamma(L^2(\mathbb{R}_+, \mathbb{C}))$ over $L^2(\mathbb{R}_+, \mathbb{C})$ is the Hilbert space completion of the linear span of the exponential vectors $\psi(f)$ under the inner product

$$< \psi(f), \psi(g) >:= e^{<f,g>}$$

where $f, g \in L^2(\mathbb{R}_+, \mathbb{C})$ and $< f, g >= \int_0^{+\infty} \bar{f}(s)\, g(s)\, ds$ where, here and in what follows, $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. .

The annihilation, creation and conservation operator processes A_t, $A_t^\dagger$ and Λ_t respectively, are defined on the exponential vectors $\psi(g)$ of Γ by

$$\begin{aligned} A_t \psi(g) &:= \int_0^t g(s)\, ds\, \psi(g) \\ A_t^\dagger \psi(g) &:= \frac{\partial}{\partial \epsilon}|_{\epsilon=0}\, \psi(g + \epsilon \chi_{[0,t]}) \\ \Lambda_t \psi(g) &:= \frac{\partial}{\partial \epsilon}|_{\epsilon=0}\, \psi(e^{\epsilon \chi_{[0,t]}} g) \end{aligned}$$

The basic quantum stochastic differentials dA_t, $dA_t^\dagger$, and $d\Lambda_t$ are defined by

$$\begin{aligned} dA_t &:= A_{t+dt} - A_t \\ dA_t^\dagger &:= A_{t+dt}^\dagger - A_t^\dagger \\ d\Lambda_t &:= \Lambda_{t+dt} - \Lambda_t \end{aligned}$$

Hudson and Parthasarathy defined in [13] stochastic integration with respect to the noise differentials dA_t, $dA_t^\dagger$ and $d\Lambda_t$ and obtained the Itô multiplication table

$\cdot$	$dA_t^\dagger$	$d\Lambda_t$	dA_t	dt
$dA_t^\dagger$	0	0	0	0
$d\Lambda_t$	$dA_t^\dagger$	$d\Lambda_t$	0	0
dA_t	dt	dA_t	0	0
dt	0	0	0	0

We couple Γ with a "system" Hilbert space $\mathcal{H}$ and consider processes defined on $\mathcal{H} \otimes \Gamma$. The fundamental theorems of the Hudson-Parthasarathy quantum stochastic calculus give formulas for expressing the matrix elements of quantum stochastic integrals in terms of ordinary Riemann-Lebesgue integrals.

Theorem 1. *Let*

$$M(t) := \int_0^t E(s)\, d\Lambda_s + F(s)\, dA_s + G(s)\, dA_s^\dagger + H(s)\, ds$$

where E, F, G, H are (in general) time dependent adapted processes. Let also $u \otimes \psi(f)$ and $u \otimes \psi(g)$ be in the "exponential domain" of $\mathcal{H} \otimes \Gamma$. Then

$$\begin{cases} < u \otimes \psi(f), M(t)\, u \otimes \psi(g) >= \\ \int_0^t < u \otimes \psi(f), \big(\bar{f}(s)\, g(s)\, E(s) + g(s)\, F(s) + \bar{f}(s)\, G(s) + H(s)\big)\, u \otimes \psi(g) > \, d \end{cases}$$

Proof. See Theorem 4.1 of [13].

□

Theorem 2. *Let*

$$M(t) := \int_0^t \, E(s)\, d\Lambda_s + F(s)\, dA_s + G(s)\, dA_s^{\dagger} + H(s)\, ds$$

and

$$M'(t) := \int_0^t \, E'(s)\, d\Lambda_s + F'(s)\, dA_s + G'(s)\, dA_s^{\dagger} + H'(s)\, ds$$

where E, F, G, H, E', F', G', H' are (in general) time dependent adapted processes. Let also $u \otimes \psi(f)$ and $u \otimes \psi(g)$ be in the exponential domain of $\mathcal{H} \otimes \Gamma$. Then

$$\begin{gathered} < M(t)\, u \otimes \psi(f), M'(t)\, u \otimes \psi(g) >= \\ \int_0^t \{ < M(s)\, u \otimes \psi(f), \big(\bar{f}(s)\, g(s) E'(s) + g(s)\, F'(s) + \bar{f}(s)\, G'(s) + H'(s)\big)\, u \otimes \psi(g) > \\ + < (\bar{g}(s)\, f(s)\, E(s) + f(s)\, F(s) + \bar{g}(s)\, G(s) + H(s))\, u \otimes \psi(f), M'(s)\, u \otimes \psi(g) > \\ + < (f(s) E(s) + G(s))\, u \otimes \psi(f), (g(s) E'(s) + G'(s))\, u \otimes \psi(g) > \} \, ds \end{gathered}$$

Proof. See Theorem 4.3 of [13].

□

The connection between classical and quantum stochastic analysis is given in the following:

Theorem 3. *The processes $B = \{B_t \,/\, t \geq 0\}$ and $P = \{P_t \,/\, t \geq 0\}$ defined by*

$$B_t := A_t + A_t^{\dagger}$$

and

$$P_t := \Lambda_t + \sqrt{\lambda}\, (A_t + A_t^{\dagger}) + \lambda\, t$$

are identified with Brownian motion and Poisson process of intensity λ respectively, in the sense that their vacuum characteristic functionals are given by

$$< \psi(0), e^{i\, s\, B_t}\, \psi(0) >= e^{-\frac{s^2}{2}\, t}$$

and

$$< \psi(0), e^{i\, s\, P_t}\, \psi(0) >= e^{\lambda\, (e^{i\, s} - 1)\, t}.$$

Proof. See Theorem 5 of [11]. □

The processes A_t, $A_t^\dagger$ satisfy the Boson Commutation Relations

$$\left[A_t, A_t^\dagger\right] := A_t\, A_t^\dagger - A_t^\dagger\, A_t = t\, I$$

In [14] Hudson and Parthasarathy showed that the processes F_t and $F_t^\dagger$ defined on the Boson Fock space by

$$\begin{aligned} F_t &:= \int_0^t J_s\, dA_s \\ F_t^\dagger &:= \int_0^t J_s\, dA_s^\dagger \end{aligned}$$

satisfy the Fermion anti-commutation relations

$$\{F_t, F_t^\dagger\} := F_t\, F_t^\dagger + F_t^\dagger\, F_t = t\, I$$

It follows that

$$dF_t = J_t\, dA_t \tag{1.1}$$

$$dF_t^\dagger = J_t\, dA_t^\dagger \tag{1.2}$$

Here J_t is the self-adjoint, unitary-valued, adapted, so called "reflection" process, acting on the noise part of the Fock space and extended as the identity on the system part, defined by

$$J_t := \gamma(-P_{[0,t]} + P_{(t,+\infty)})$$

where P_S denotes the multiplication operator by χ_S and γ is the second quantization operator defined by

$$\gamma(U)\, \psi(f) := \psi(U\, f)$$

The reflection process J_t commutes with system space operators and satisfies the differential equation (cf. Lemma 3.1 of [14])

$$\begin{aligned} dJ_t &= -2\, J_t\, d\Lambda_t \\ J_0 &= 1. \end{aligned} \tag{1.3}$$

2. Fermion Evolutions and Flows

As shown in [14], Fermion unitary evolution equations have the form

$$\begin{aligned} dU_t &= -\left(\left(iH + \frac{1}{2}\, L^* L\right)\, dt + L^*\, W\, dF_t - L\, dF_t^\dagger + (1 - W)\ d\Lambda_t\right) U_t \\ U_0 &= 1 \end{aligned} \tag{2.1}$$

with adjoint

$$\begin{aligned} dU_t^* &= -U_t^*\,((-iH + \frac{1}{2}\, L^* L)\, dt - L^* dF_t + W^*\, L\, dF_t^\dagger + (1 - W^*)\, d\Lambda_t) \\ U_0^* &= 1 \end{aligned} \tag{2.2}$$

where, for each $t \geq 0$, U_t is a unitary operator defined on the tensor product $\mathcal{H} \otimes \Gamma(L^2(\mathbb{R}_+, \mathbb{C}))$ of the system Hilbert space $\mathcal{H}$ and the noise Fock space Γ. Here H, L, W are in $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators on $\mathcal{H}$, with W unitary and H self-adjoint. We identify time-independent, bounded, system space operators X with their ampliation $X \otimes 1$ to $\mathcal{H} \otimes \Gamma(L^2(\mathbb{R}_+, \mathbb{C}))$.

Proposition 1. *Let*

$$\phi_t(T,S) := V_t^* \, (T + S\, J_t)\, V_t$$

where T, S are bounded system space operators and V_t, V_t^* satisfy the quantum stochastic differential equations

$$dV_t = \Big(\alpha\, dt + \beta\, dF_t + \gamma\, dF_t^\dagger + \delta\, d\Lambda_t\Big)\, V_t = \Big(\alpha\, dt + \beta\, J_t\, dA_t + \gamma\, J_t\, \; dA_t^\dagger + \delta\, d\Lambda_t\Big)\, V_t$$

and

$$dV_t^* = V_t^* \, \Big(\alpha^*\, dt + \beta^*\, dF_t^\dagger + \gamma^*\, dF_t + \delta^*\, d\Lambda_t\Big) = V_t^* \, \Big(\alpha^*\, dt + \beta^*\, J_t\, dA_t^\dagger + \gamma^*\, J_t\, dA_t + \delta^*\, d\Lambda_t\Big)$$

where $\alpha, \beta, \gamma, \delta$ are bounded system space operators. Then

$$\begin{aligned} d\phi_t(T,S) &= \phi_t(\alpha^*\, T + T\, \alpha + \gamma^*\, T\, \gamma, \alpha^*\, S + S\, \alpha + \gamma^*\, S\, \gamma)\, dt \\ &+ \phi_t(\gamma^*\, S + S\, \beta - \gamma^*\, S\, (2+\delta), \gamma^*\, T + T\, \beta + \gamma^*\, T\, \delta)\, dA_t \\ &+ \phi_t(\beta^*\, S + S\, \gamma - \delta^*\, S\, \gamma, \beta^*\, T + T\, \gamma + \delta^*\, T\, \gamma)\, dA_t^\dagger \\ &+ \phi_t(\delta^*\, T + T\, \delta + \delta^*\, T\, \delta, -(2\, S + S\, \delta + \delta^*\, S + \delta^*\, S\, \delta))\, d\Lambda_t \end{aligned} \tag{2.3}$$

with

$$\phi_0(T,S) = T + S \tag{2.4}$$

Proof. Making use of the algebraic rule

$$d(x\, y) = dx\, y + x\, dy + dx\, dy$$

we find

$$\begin{aligned} d\phi_t(T,S) &= dV_t^*\, (T + S\, J_t)\, V_t + V_t^*\, d((T + S\, J_t)\, V_t) + dV_t^*\, d((T + S\, J_t)\, V_t) \\ &= dV_t^*\, (T + S\, J_t)\, V_t + V_t^*\, T\, dV_t + V_t^*\, S\, d(J_t\, V_t) + dV_t^*\, T\, dV_t + dV_t^*\, S\, d(J_t\, V_t). \end{aligned}$$

But, by (1.3), the Itô table for the Boson stochastic differentials and the fact that $J_t^2 = 1$

$$\begin{aligned} d(J_t\, V_t) &= dJ_t\, V_t + J_t\, dV_t + dJ_t\, dV_t \\ &= -2\, J_t\, d\Lambda_t\, V_t + J_t\, \Big(\alpha\, dt + \beta\, J_t\, dA_t + \gamma\, J_t\, dA_t^\dagger + \delta\, d\Lambda_t\Big)\, V_t \\ &- 2\, J_t\, d\Lambda_t\, \Big(\alpha\, dt + \beta\, J_t\, dA_t + \gamma\, J_t\, dA_t^\dagger + \delta\, d\Lambda_t\Big)\, V_t \\ &= \Big(\alpha\, J_t\, dt + \beta\, dA_t - \gamma\, dA_t^\dagger - (2+\delta)\, J_t\, d\Lambda_t\Big)\, V_t \end{aligned}$$

Thus

$$
\begin{aligned}
& d\phi_t(T,S) = V_t^* \left(\alpha^*\,dt + \beta^*\,J_t\,dA_t^\dagger + \gamma^*\,J_t\,dA_t + \delta^*\,d\Lambda_t\right)(T + S\,J_t)\,V_t \\
+ \quad & V_t^*\,T\left(\alpha\,dt + \beta\,J_t\,dA_t + \gamma\,J_t\,dA_t^\dagger + \delta\,d\Lambda_t\right) V_t \\
+ \quad & V_t^*\,S\left(\alpha\,J_t\,dt + \beta\,dA_t - \gamma\,dA_t^\dagger - (2+\delta)\,J_t\,d\Lambda_t\right) V_t \\
+ \quad & V_t^*\left(\alpha^*\,dt + \beta^*\,J_t\,dA_t^\dagger + \gamma^*\,J_t\,dA_t + \delta^*\,d\Lambda_t\right) T\left(\alpha\,dt + \beta\,J_t\,dA_t + \gamma\,J_t\,dA_t^\dagger + \delta\,d\Lambda_t\right) V_t \\
+ \quad & V_t^*\left(\alpha^*\,dt + \beta^*\,J_t\,dA_t^\dagger + \gamma^*\,J_t\,dA_t + \delta^*\,d\Lambda_t\right) S\left(\alpha\,J_t\,dt + \beta\,dA_t - \gamma\,dA_t^\dagger - (2+\delta)\,J_t\,d\Lambda_t\right) V_t \\
= \quad & \phi_t(\alpha^*\,T + T\,\alpha + \gamma^*\,T\,\gamma, \alpha^*\,S + S\,\alpha + \gamma^*\,S\,\gamma)\,dt + \phi_t(\gamma^*\,S + S\,\beta - \gamma^*\,S\,(2+\delta), \gamma^*\,T + T\,\beta + \gamma^*\,T\,\delta)\,dA_t \\
+ \quad & \phi_t(\beta^*\,S + S\,\gamma - \delta^*\,S\,\gamma, \beta^*\,T + T\,\gamma + \delta^*\,T\,\gamma)\,dA_t^\dagger + \phi_t(\delta^*\,T + T\,\delta + \delta^*\,T\,\delta, -(2\,S + S\,\delta + \delta^*\,S + \delta^*\,S\,\delta))\,d\Lambda_t
\end{aligned}
$$

□

With the processes U_t and U_t^* of (2.1) and (2.2) we associate the Fermion flow

$$j_t(X) := U_t^*\,X\,U_t = \phi(X,0) \tag{2.5}$$

and the reflected flow

$$r_t(X) := j_t(X\,J_t) = \phi(0,X) \tag{2.6}$$

where X is a bounded system space operator.

Corollary 1. *The Fermion flow $j_t(X)$ and the reflected flow $r_t(X)$ defined in (2.5) and (2.6) satisfy the system of quantum stochastic differential equations*

$$
\begin{aligned}
dj_t(X) \quad = \quad & j_t\left(i\,[H,X] - \frac{1}{2}\,(L^*LX + XL^*L - 2L^*XL)\right) dt \\
+ \quad & r_t\,([L^*,X]\;W)\;dA_t + r_t\,(W^*\;[X,L])\;dA_t^\dagger + j_t\,(W^*\,X\,W - X)\;d\Lambda_t
\end{aligned} \tag{2.7}
$$

and

$$
\begin{aligned}
dr_t(X) \quad = \quad & r_t\left(i\,[H,X] - \frac{1}{2}\,(L^*LX + XL^*L - 2L^*XL)\right) dt \\
- \quad & j_t\,(\{L^*,X\}\,W)\;dA_t - j_t\,(W^*\{X,L\} - 2\,X\,L)\;dA_t^\dagger - r_t\,(W^*\,X\,W + X)\;d\Lambda_t
\end{aligned} \tag{2.8}
$$

with

$$j_0(X) = r_0(X) = X.$$

Here, as usual, $[x,y] := x\,y - y\,x$ *and* $\{x,y\} := x\,y + y\,x$.

Proof. We replace V_t and V_t^* in Proposition 1 by U_t and U_t^*. Then, equation (2.7) is a special case of (2.3) for $T = X$, $S = 0$ and

$$\begin{aligned}\alpha &= -(iH + \frac{1}{2}L^*L)\\ \beta &= -L^*W\\ \gamma &= L\\ \delta &= W-1\end{aligned}$$

while equation (2.8) is a special case of (2.3) for $T=0$, $S=X$ and $\alpha,\beta,\gamma,\delta$ as above.

□

3. Generalized Fermion Flows

Fermion flows can be formulated and studied in a manner similar to Boson (also called Evans-Hudson) flows (cf. [12] and [15]). Let $\mathcal{B}(\mathcal{H})$ denote the space of bounded system operators. We define $\mathcal{B}(\mathcal{H})$-valued operations $\bigtriangledown$ and $\bigtriangleup$ on $\mathcal{B}(\mathcal{H})\times\mathcal{B}(\mathcal{H})$ by

$$(T_1,S_1)\bigtriangledown(T_2,S_2) := T_1\,T_2+S_1\,S_2 \tag{3.1}$$

$$(T_1,S_1)\bigtriangleup(T_2,S_2) := T_1\,S_2+S_1\,T_2 \tag{3.2}$$

Notice that if

$$\rho(x,y) := (y,x)$$

is the reflection map, then

$$(T_1,S_1)\bigtriangleup(T_2,S_2) = (T_1,S_1)\bigtriangledown\rho(T_2,S_2).$$

We also define the $\mathcal{B}(\mathcal{H})\times\mathcal{B}(\mathcal{H})$-valued product map $\circ$ on $(\mathcal{B}(\mathcal{H})\times\mathcal{B}(\mathcal{H}))^2$ by

$$x\circ y := (x\bigtriangledown y, x\bigtriangleup y) = (T_1\,T_2+S_1\,S_2, T_1\,S_2+S_1\,T_2) \tag{3.3}$$

where $x=(T_1,S_1)$ and $y=(T_2,S_2)$.

Lemma 1. *The $\circ$-product is associative with unit $id := (1,0)$ where 1 and 0 are the identity and zero operators in $\mathcal{B}(\mathcal{H})$.*

Proof. Let $x=(T_1,S_1)$, $y=(T_2,S_2)$ and $z=(T_3,S_3)$. Then

$$\begin{aligned}(x\circ y)\circ z &= (T_1\,T_2+S_1\,S_2, T_1\,S_2+S_1\,T_2)\circ(T_3,S_3)\\ &= ((T_1\,T_2+S_1\,S_2, T_1\,S_2+S_1\,T_2)\bigtriangledown(T_3,S_3), (T_1\,T_2+S_1\,S_2, T_1\,S_2+S_1\,T_2)\bigtriangleup(T_3,S_3))\\ &= ((T_1\,T_2+S_1\,S_2)\,T_3+(T_1\,S_2+S_1\,T_2)\,S_3, (T_1\,T_2+S_1\,S_2)\,S_3+(T_1\,S_2+S_1\,T_2)\,T_3)\\ &= (T_1\,T_2\,T_3+S_1\,S_2\,T_3+T_1\,S_2\,S_3+S_1\,T_2\,S_3, T_1\,T_2\,S_3+S_1\,S_2\,S_3+T_1\,S_2\,T_3+S_1\,T_2\,T_3)\end{aligned}$$

and

$$\begin{aligned}x\circ(y\circ z) &= (T_1,S_1)\circ(T_2\,T_3+S_2\,S_3, T_2\,S_3+S_2\,T_3)\\ &= ((T_1,S_1)\bigtriangledown(T_2\,T_3+S_2\,S_3, T_2\,S_3+S_2\,T_3), (T_1,S_1)\bigtriangleup(T_2\,T_3+S_2\,S_3, T_2\,S_3+S_2\,T_3))\\ &= (T_1\,(T_2\,T_3+S_2\,S_3)+S_1\,(T_2\,S_3+S_2\,T_3), T_1\,(T_2\,S_3+S_2\,T_3)+S_1\,(T_2\,T_3+S_2\,S_3))\\ &= (T_1\,T_2\,T_3+T_1\,S_2\,S_2+S_1\,T_2\,S_3+S_1\,S_2\,T_3, T_1\,T_2\,S_3+T_1\,S_2\,T_3+S_1\,T_2\,T_3+S_1\,S_2\,S_3)\end{aligned}$$

Thus

$$(x \circ y) \circ z = x \circ (y \circ z).$$

Finally

$$x \circ id = (T_1, S_1) \circ (1, 0) = (T_1\, 1 + S_1\, 0, T_1\, 0 + S_1\, 1) = (T_1, S_1) = x$$

and

$$id \circ x = (1, 0) \circ (T_1, S_1) = (1\, T_1 + 0\, S_1, 1\, S_1 + 0\, T_1) = (T_1, S_1) = x$$

□

Let the flow $\phi_t(T, S) := V_t^* \,(T + S\, J_t)\, V_t$ be as in Proposition 1 and let $x = (T_1, S_1)$ and $y = (T_2, S_2)$. Then

$$\begin{aligned}
\phi_t(x)\, \phi_t(y) &= \phi_t(T_1, S_1)\phi_t(T_2, S_2) \\
&= V_t^* \,(T_1 + S_1\, J_t)\, V_t\, V_t^* \,(T_2 + S_2\, J_t)\, V_t \\
&= V_t^* \,(T_1 + S_1\, J_t)\,(T_2 + S_2\, J_t)\, V_t \\
&= V_t^* \,(T_1\, T_2 + S_1\, S_2 + (T_1\, S_2 + S_1\, T_2)\, J_t)\, V_t \\
&= \phi_t(T_1\, T_2 + S_1\, S_2, T_1\, S_2 + S_1\, T_2) \\
&= \phi_t(x \circ y)
\end{aligned}$$

i.e ϕ_t is a homomorphism with respect to the $\circ$-product. Since ϕ_t is the solution of (2.3), (2.4) this suggests considering flows satisfying quantum stochastic differential equations of the form

$$d\phi_t(x) = \phi_t(\theta_1(x))\, dt + \phi_t(\theta_2(x))\, dA_t + \phi_t(\theta_3(x))\, dA_t^\dagger + \phi_t(\theta_4(x))\, d\Lambda_t \tag{3.4}$$

with

$$\phi_0(x) = x \bigtriangledown id + x \bigtriangleup id$$

where $x = (T, S) \in \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$ and for $i = 1, 2, 3, 4$, $\theta_i : \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$ are the "structure maps". In the case of (2.3), (2.4)

$$\begin{aligned}
\theta_1(T, S) &= (\alpha^*\, T + T\, \alpha + \gamma^*\, T\, \gamma, \alpha^*\, S + S\, \alpha + \gamma^*\, S\, \gamma) \\
\theta_2(T, S) &= (\gamma^*\, S + S\, \beta - \gamma^*\, S\,(2 + \delta), \gamma^*\, T + T\, \beta + \gamma^*\, T\, \delta) \\
\theta_3(T, S) &= (\beta^*\, S + S\, \gamma - \delta^*\, S\, \gamma, \beta^*\, T + T\, \gamma + \delta^*\, T\, \gamma) \\
\theta_4(T, S) &= (\delta^*\, T + T\, \delta + \delta^*\, T\, \delta, -(2\, S + S\, \delta + \delta^*\, S + \delta^*\, S\, \delta))
\end{aligned}$$

where

$$\begin{aligned}
\alpha &= -(iH + \frac{1}{2}\, L^* L) \\
\beta &= -L^*\, W \\
\gamma &= L \\
\delta &= W - 1
\end{aligned}$$

The general conditions on the θ_i in order for ϕ_t to be an identity preserving $\circ$-product homomorphism are given in the following.

Proposition 2. *Let ϕ_t be the solution of (3.4). Then*

$$\begin{aligned} \phi_t(x)\,\phi_t(y) &= \phi_t(x\circ y) \\ \phi_t(x)^* &= \phi_t(x^*) \\ \phi_t(id) &= 1 \end{aligned}$$

if and only if the θ_i satisfy the structure equations

$$\theta_1(x)\circ y + x\circ\theta_1(y) + \theta_2(x)\circ\theta_3(y) = \theta_1(x\circ y) \tag{3.5}$$
$$\theta_2(x)\circ y + x\circ\theta_2(y) + \theta_2(x)\circ\theta_4(y) = \theta_2(x\circ y) \tag{3.6}$$
$$\theta_3(x)\circ y + x\circ\theta_3(y) + \theta_3(x)\circ\theta_4(y) = \theta_3(x\circ y) \tag{3.7}$$
$$\theta_4(x)\circ y + x\circ\theta_4(y) + \theta_4(x)\circ\theta_4(y) = \theta_4(x\circ y) \tag{3.8}$$

and, with $*$ denoting "adjoint" and $x=(T_1,S_1)\Leftrightarrow x^*=(T_1^*,S_1^*)$,

$$\theta_1(x^*) = (\theta_1(x))^* \tag{3.9}$$
$$\theta_2(x^*) = (\theta_3(x))^* \tag{3.10}$$
$$\theta_3(x^*) = (\theta_2(x))^* \tag{3.11}$$
$$\theta_4(x^*) = (\theta_4(x))^* \tag{3.12}$$

with

$$\theta_1(id)=\theta_2(id)=\theta_3(id)=\theta_4(id)=(0,0). \tag{3.13}$$

Proof.

$$\phi_t(x\circ y)=\phi_t(x)\,\phi_t(y)\Leftrightarrow d\phi_t(x\circ y)=d\phi_t(x)\,\phi_t(y)+\phi_t(x)\,d\phi_t(y)+d\phi_t(x)\,d\phi_t(y)\Leftrightarrow$$

$$\begin{gathered}\phi_t(\theta_1(x\circ y))\,dt+\phi_t(\theta_2(x\circ y))\,dA_t+\phi_t(\theta_3(x\circ y))\,dA_t^\dagger+\phi_t(\theta_4(x\circ y))\,d\Lambda_t=\\ \phi_t(\theta_1(x))\,\phi_t(y)\,dt+\phi_t(\theta_2(x))\,\phi_t(y)\,dA_t+\phi_t(\theta_3(x))\,\phi_t(y)\,dA_t^\dagger+\phi_t(\theta_4(x))\,\phi_t(y)\,d\Lambda_t+\\ \phi_t(x)\phi_t(\theta_1(y))\,dt+\phi_t(x)\,\phi_t(\theta_2(y))\,dA_t+\phi_t(x)\,\phi_t(\theta_3(y))\,dA_t^\dagger+\phi_t(x)\,\phi_t(\theta_4(y))\,d\Lambda_t+\\ \phi_t(\theta_2(x))\,\phi_t(\theta_3(y))\,dt+\phi_t(\theta_2(x))\,\phi_t(\theta_4(y))\,dA_t+\phi_t(\theta_3(x))\,\phi_t(\theta_4(y))\,dA_t^\dagger+\phi_t(\theta_4(x))\,\phi_t(\theta_4(y))\,d\Lambda_t\end{gathered}$$

from which collecting the $dt, dA_t, dA_t^\dagger$ and $d\Lambda_t$ terms on each side, using the homomorphism property and then equating the coefficients of $dt, dA_t, dA_t^\dagger$ and $d\Lambda_t$ on both sides we obtain (3.5)-(3.8). Similarly

$$\phi_t(x)^*=\phi_t(x^*)\Leftrightarrow d\phi_t(x)^*=d\phi_t(x^*)$$

i.e

$$\begin{gathered}\phi_t(\theta_1(x)^*)\,dt+\phi_t(\theta_2(x)^*)\,dA_t^\dagger+\phi_t(\theta_3(x)^*)\,dA_t+\phi_t(\theta_4(x)^*)\,d\Lambda_t=\\ \phi_t(\theta_1(x^*))\,dt+\phi_t(\theta_2(x^*))\,dA_t+\phi_t(\theta_3(x^*))\,dA_t^\dagger+\phi_t(\theta_4(x^*))\,d\Lambda_t\end{gathered}$$

and (3.9)-(3.12) follow by equating the coefficients of $dt, dA_t, dA_t^\dagger$ and $d\Lambda_t$ on both sides. Finally

$$\phi_t(id) = 1 \Leftrightarrow d\phi_t(id) = 0$$

which by the linear independence of dt and the stochastic differentials implies (3.13).

□

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4. Optimal Noise Coefficients

As in [2] and [4], motivated by classical linear system control theory, we think of the self-adjoint operator H appearing in (2.1) as fixed and we consider the problem of determining the coefficients L and W of the noise part of the Hamiltonian of the evolution equation (2.1) that minimize the "evolution perfomance functional"

$$Q_{\xi,T}(u) = \int_0^T [\|X\,U_t\,\xi\|^2 + \frac{1}{4}\,\|L^*\,L\,U_t\,\xi\|^2]\,dt + \frac{1}{2}\,\|L\,U_T\,\xi\|^2 \tag{4.1}$$

where $T \in [0, +\infty)$, ξ is an arbitrary vector in the exponential domain of $\mathcal{H} \otimes \Gamma$ and X is a bounded self-adjoint system operator. By the unitarity of U_t, U_T,J_t, and J_T, (4.1) is the same as the "Fermion flow performance functional"

$$J_{\xi,T}(L, W) = \int_0^T [\,\|j_t(X)\,\xi\|^2 + \frac{1}{4}\|j_t(L^*L)\,\xi\|^2\,]\,dt + \frac{1}{2}\|j_T(L)\,\xi\|^2 \tag{4.2}$$

and the "reflected flow performance functional"

$$R_{\xi,T}(L, W) = \int_0^T [\,\|r_t(X)\,\xi\|^2 + \frac{1}{4}\|r_t(L^*L)\,\xi\|^2\,]\,dt + \frac{1}{2}\|r_T(L)\,\xi\|^2 \tag{4.3}$$

We consider the problem of minimizing the functionals $J_{\xi,T}(L, W)$ and $R_{\xi,T}(L, W)$ over all system operators L, W where L is bounded and W is unitary. The motivation behind the definition of the performance functionals (4.1), (4.2) and (4.3) can be found in the following theorem which is the quantum stochastic analogue of the classical linear regulator theorem.

Theorem 4. *Let $U = \{U_t\,/\,t \geq 0\}$ be a process satisfying the quantum stochastic differential equation*

$$dU_t = (F\,U_t + u_t)\,dt + \Psi\,U_t\,dF_t + \Phi\,U_t\,dF_t^\dagger + Z\,U_t\,d\Lambda_t,\; U_0 = 1,\; t \in [0, T] \tag{4.4}$$

with adjoint

$$dU_t^* = (U_t^*\,F^* + u_t^*)\,dt + U_t^*\,\Psi^*\,dF_t^\dagger + U_t^*\,\Phi^* dF_t + U_t^*\,Z^*\,d\Lambda_t,\; U_0^* = 1 \tag{4.5}$$

where $0 < T < +\infty$, the coefficients F, Ψ, Φ, Z are bounded operators on the system space $\mathcal{H}$ and $u_t := -\Pi\,U_t$ for some positive bounded system operator Π. Then the functional

$$Q_{\xi,T}(u) = \int_0^T [< U_t\,\xi, X^2\,U_t\,\xi > + < u_t\,\xi, u_t\,\xi >]\,dt - < u_T\,\xi, U_T\,\xi > \tag{4.6}$$

where X is a system space observable, identified with its ampliation $X \otimes I$ to $\mathcal{H} \otimes \Gamma$, is minimized over the set of feedback control processes of the form $u_t = -\Pi\,U_t$, by choosing Π to be a bounded, positive, self-adjoint system operator satisfying

$$\Pi F+F^*\Pi+\Phi^*\Pi\Phi-\Pi^2+X^2 = 0 \quad (4.7)$$
$$\Pi\Psi+\Phi^*\Pi+\Phi^*\Pi Z = 0 \quad (4.8)$$
$$\Pi Z+Z^*\Pi+Z^*\Pi Z = 0 \quad (4.9)$$

The minimum value is $<\xi,\Pi\xi>$. We recognize (4.7) as the algebraic Riccati equation.

Proof. Let

$$\theta_t=<\xi,U_t^*\,\Pi\,U_t\,\xi> \quad (4.10)$$

Then

$$d\theta_t=<\xi,d(U_t^*\,\Pi\,U_t)\,\xi>=<\xi,(dU_t^*\,\Pi\,U_t+U_t^*\,\Pi\,dU_t+dU_t^*\,\Pi\,dU_t)\,\xi> \quad (4.11)$$

which, after replacing dU_t and dU_t^* by (4.4) and (4.5) respectively, and using (1.1), (1.2) and the Itô table, becomes

$$\begin{aligned} d\theta_t=&<\xi,U_t^*\,((F^*\,\Pi+\Pi\,F+\Phi^*\,\Pi\,\Phi)\,dt+(\Phi^*\,\Pi+\Pi\,\Psi+\Phi^*\,\Pi\,Z)\,J_t\,dA_t\\ &+(\Psi\,\Pi^*+\Pi\,\Phi+Z^*\,\Pi\,\Phi)\,J_t\,dA_t^\dagger+(Z^*\,\Pi+\Pi\,Z+Z^*\,\Pi\,Z)\,d\Lambda_t)\,U_t\,\xi>\\ &+<\xi,(u_t^*\,\Pi\,U_t+U_t^*\,\Pi\,u_t)\,dt\,\xi> \end{aligned} \quad (4.12)$$

and by (4.7)-(4.9)

$$d\theta_t=<\xi,U_t^*\,(\Pi^2-X^2)\,U_t\,dt\,\xi>+<\xi,(u_t^*\,\Pi\,U_t+U_t^*\,\Pi\,u_t)\,dt\,\xi> \quad (4.13)$$

By (4.10)

$$\theta_T-\theta_0=<\xi,U_T^*\,\Pi\,U_T\,\xi>-<\xi,\Pi\,\xi> \quad (4.14)$$

while by (4.13)

$$\theta_T-\theta_0=\int_0^T(<\xi,U_t^*\,(\Pi^2-X^2)\,U_t\,\xi>+<\xi,(u_t^*\,\Pi\,U_t+U_t^*\,\Pi\,u_t)\,\xi>)\,dt \quad (4.15)$$

By (4.14) and (4.15)

$$\begin{aligned} &<\xi,U_T^*\,\Pi\,U_T\,\xi>-<\xi,\Pi\,\xi>=\\ &\int_0^T(<\xi,U_t^*\,(\Pi^2-X^2)\,U_t\,\xi>+<\xi,(u_t^*\,\Pi\,U_t+U_t^*\,\Pi\,u_t)\,\xi>)\,dt \end{aligned} \quad (4.16)$$

Thus

$$\begin{aligned} Q_{\xi,T}(u)=&(<\xi,U_T^*\,\Pi\,U_T\,\xi>-<\xi,\Pi\,\xi>)+Q_{\xi,T}(u)\\ &-(<\xi,U_T^*\,\Pi\,U_T\,\xi>-<\xi,\Pi\,\xi>) \end{aligned} \quad (4.17)$$

Replacing the first parenthesis on the right hand side of (4.17) by (4.16), and $Q_{\xi,T}(u)$ by (4.6) we obtain after cancellations

$$\begin{aligned} Q_{\xi,T}(u)&=\int_0^T(<\xi,(U_t^*\,\Pi^2\,U_t+u_t^*\,\Pi\,U_t+U_t^*\,\Pi\,u_t+u_t^*\,u_t)\,\xi>\,dt+<\xi,\Pi\,\xi>\\ &=\int_0^T\,||(u_t+\Pi\,U_t)\,\xi||^2\,dt+<\xi,\Pi\,\xi> \end{aligned} \quad (4.18)$$

which is clearly minimized by $u_t = -\Pi\, U_t$ and the minimum is $< \xi, \Pi\, \xi >$.

□

Theorem 5. *Let X be a bounded self-adjoint system operator such that the pair $(i\, H,\, X)$ is stabilizable i.e there exists a bounded system operator K such that $i\, H + K X$ is the generator of an asymptotically stable semigroup. Then, the quadratic performance functionals (4.2) and (4.3) associated with the Fermion flow $\{j_t(X) := U_t^*\, X\, U_t \,/\, t \geq 0\}$ and the reflected flow $\{r_t(X) := j_t(X\, J_t) \,/\, t \geq 0\}$, where $U = \{U_t \,/\, t \geq 0\}$ is the solution of (2.1), are minimized by*

$$L = \sqrt{2}\, \Pi^{1/2}\, W_1 \tag{4.19}$$

and

$$W = W_2 \tag{4.20}$$

where Π is a positive self-adjoint solution of the "algebraic Riccati equation"

$$i\, [H, \Pi] + \Pi^2 + X^2 = 0 \tag{4.21}$$

and W_1, W_2 are bounded unitary system operators commuting with Π. It is known (see [16]*) that if the pair $(i\, H,\, X)$ is stabilizable, then (4.21) has a positive self-adjoint solution Π. Moreover*

$$\min_{L,W}\; J_{\xi,T}(L, W) = \min_{L,W}\; R_{\xi,T}(L, W) = < \xi, \Pi\, \xi > \tag{4.22}$$

independently of T.

Proof. Looking at (2.1) as (4.4) with $u_t = -\frac{1}{2}\, L^*\, L\, U_t$, $F = -i\, H$, $\Psi = -L^*\, W$, $\Phi = L$, and $Z = W - 1$, (4.6) is identical to (4.1). Moreover, equations (4.7)-(4.9) become

$$i\, [H, \Pi] + L^*\, \Pi\, L - \Pi^2 + X^2 = 0 \tag{4.23}$$

$$L^*\, \Pi - \Pi\, L^*\, W + L^*\, \Pi\, (W - 1) = 0 \tag{4.24}$$

$$(W^* - 1)\, \Pi + \Pi\, (W - 1) + (W^* - 1)\, \Pi\, (W - 1) = 0 \tag{4.25}$$

By the self-adjointness of Π, (4.24) implies that

$$[L, \Pi] = [L^*, \Pi] = 0 \tag{4.26}$$

while (4.25) implies that

$$[W, \Pi] = [W^*, \Pi] = 0 \tag{4.27}$$

i.e (4.20). By (4.24) and the fact that in this case

$$\Pi = \frac{1}{2}\, L^*\, L \text{ i.e } L^*\, L = 2\, \Pi \tag{4.28}$$

equation (4.23) implies (4.21). Equations (4.26) and (4.28) also imply that

$$[L, L^*] = 0 \tag{4.29}$$

which implies (4.19).

□

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