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Entangled Quantum Markov Chain satisyfying Entanglement Condition

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Abstract

The entropic criterion of entanglement is applied to prove that entangled Markov chain with unitarily implementable transition operator is indeed an entangled state on infinite multiple algebras.

1 Introduction and Preliminaries

Accardi and Fidaleo [2] proposed a construction to relate, based on classical Markov chain with discrete state space, to a quantum Markov chain (in the sense of [1]) on infinite tensor products of type I factors. They called *entangled Markov chain* (EMC) the special class of quantum Markov chains obtained in this way.

Using the PPT entanglement criterion [13, 8] (positivity of the partial transpose of the density matrix) Miyadera showed [9] that the finite volume restriction of a class of EMC on infinite tensor products of 2×2 matrix algebras is entangled.

In our previous paper [3], using the entropic type of entanglement criterion for pure states [11, 3], which is based on the notion of *degree of entanglement*, we proved that the vector states defining the EMC's on infinite tensor products of $d \times d$ matrix algebras ($d \in \mathbb{N}$) "generically" are entangled (see Definition (3) below).

Our result did not include Miyadear's one because, by restricting an EMC to some local algebra, one obtains a mixed state to which the above criterion for a pure state is not applicable. However our entanglement criterion gives the sufficient condition for entanglement in the case of mixtures (for pure states this condition is necessary and sufficient) [4]. Moreover our entanglement criterion, being based on a numerical inequality, is in many cases easier to verify than the positivity condition required by the PPT criterion.

In this note we will show some results obtained in [4] with proof for the reader's convenience. Our entanglement condition is applied to the restriction of EMC's, generated by a unitarily implementable $d \times d$ stochastic matrix, to algebras localized which is obtained as a mixed state. This allows to prove that the above EMC induce an entangled state on infinite tensor products of $d \times d$ matrix algebras for any $d \in \mathbb{N}$.

We consider a classical Markov chain (S_n) with state space $S = \{1, 2, \cdots, d\}$, initial distribution $p = (p_j)$ and transition probability matrix $P = (p_{ij})$

$$p_{ij} \geq 0 \quad ; \quad \sum_j p_{ij} = 1$$

Let $\{e_i\}_{i \leq d}$ be an orthonormal basis (ONB) of $\mathbb{C}^{|S|}$. For a fixed vector e_0 in this basis, denote

$$\mathcal{H}_{\mathbb{N}} := \bigotimes_{\mathbb{N}}^{(e_0)} \mathbb{C}^{|S|} \tag{1}$$

the infinite tensor product of $\mathbb{N}$-copies of the Hilbert space $\mathbb{C}^{|S|}$ with respect to the constant sequence (e_0). An orthonormal basis of $\mathcal{H}_{\mathbb{N}}$ is given by the vectors

$$|e_{j_0}, \cdots, e_{j_n}\rangle := \left(\otimes_{\alpha \in [0,\, n]} e_{j_\alpha}\right) \otimes \left(\otimes_{\alpha \in [0,\, n]^c} e_0\right).$$

For any Hilbert space $\mathcal{H}$ we denote $\mathcal{H}^*$ its dual and $\xi \in \mathcal{H} \mapsto \xi^* \in \mathcal{H}^*$ the canonical embedding. Thus, if $\xi \in \mathcal{H}$ is a unit vector, $\xi\xi^*$ denotes the projection onto the subspace generated by ξ.

Let M_d denote the algebra of complex $d \times d$ matrices and let $\mathcal{A} := \otimes_{\mathbb{N}} M_d = M_d \otimes M_d \otimes \cdots$ be the C*–infinite tensor product of $\mathbb{N}$–copies of M_d.

An element $A_\Lambda \in \mathcal{A}$ (observable) will be said to be *localized* in a finite region $\Lambda \subset \mathbb{N}$ if there exists an operator $\overline{A}_\Lambda \in \otimes_\Lambda M_d$ such that

$$A_\Lambda = \overline{A}_\Lambda \otimes 1_{\Lambda^c}$$

In the following we will identify $A_\Lambda = \overline{A}_\Lambda$ and we denote $\mathcal{A}_\Lambda$ the local algebra at Λ. Let $\sqrt{p_i}$ (resp. $\sqrt{p_{ij}}$) be any complex square root of p_i (resp. p_{ij}) (i.e. $|\sqrt{p_i}|^2 = p_i$(resp. $|\sqrt{p_{ij}}|^2 = p_{ij}$)) and define the vector

$$\Psi_n = \sum_{j_0, \cdots, j_n} \sqrt{p_{j_0}} \prod_{\alpha=0}^{n-1} \sqrt{p_{j_\alpha j_{\alpha+1}}}\, |e_{j_0}, \cdots, e_{j_n}\rangle \tag{2}$$

Although the limit $\lim_{n\to\infty} \Psi_n$ will not exist, the basic property of Ψ_n is the following [2].

Proposition 1 *There exists a unique quantum Markov chain ψ on $\mathcal{A}$ such that, for every $k \in \mathbb{N}$ and for every $A \in \mathcal{A}_{[0,\, k]}$, one has*

$$\langle \Psi_{k+1},\ A\Psi_{k+1}\rangle = \lim_{n\to\infty} \langle \Psi_n,\ A\Psi_n\rangle =: \psi(A) \tag{3}$$

Moreover ψ is stationary if and only if the associated classical Markov chain $\{p := (p_i)\ ,\ P = (p_{ij})\}$ is stationary, i.e.

$$\sum_i p_i p_{ij} = p_j \quad ; \quad \forall j \tag{4}$$

2 Notions of entanglement and degree of entanglement

Definition 2 *Let $\mathcal{A}_j$ $(j \in \{1, 2, \cdots, n\})$ with $n < \infty$ be C^*-algebras and let $\mathcal{A} = \overset{n}{\underset{j=1}{\otimes}} \mathcal{A}_j$ be a tensor product of C^*-algebras. A state $\omega \in \mathcal{S}\left(\overset{n}{\underset{j=1}{\otimes}} \mathcal{A}_j\right)$ is called separable if*

$$\omega \in \overline{\mathbf{Conv}}\left\{\overset{n}{\underset{j=1}{\otimes}} \omega_j \ ; \ \omega_j \in \mathcal{S}(\mathcal{A}_j), j \in \{1, 2, \cdots, n\}\right\}$$

where $\overline{\mathbf{Conv}}$ denotes norm closure of the convex hull.
A nonseparable state is called entangled.

Notice that the notion of separability may depend on the choice of the tensor product of $\mathbf{C}^*$-algebras. Unless otherwise specified, one realizes the $\mathbf{C}^*$-algebras on Hilbert spaces and one considers the induced tensor product. In any case a separable pure state must be a product of pure states.

Definition 3 *[3] In the notations of Definition (2) a state $\omega \in \mathcal{S}(\mathcal{A})$ is called 2-separable if*

$$\omega \in \overline{\mathbf{Conv}}\left\{\omega_{k]} \otimes \omega_{(k} \ : \ \omega_{k]} \in \mathcal{S}\left(\mathcal{A}_{k]}\right) \ , \ \omega_{(k} \in \mathcal{S}\left(\mathcal{A}_{(k}\right), \ \forall k \in \{1, 2, \cdots, n\}\right\}$$

where $\mathcal{A} = \mathcal{A}_{k]} \otimes \mathcal{A}_{(k} := \mathcal{A}_{[1,k]} \otimes \mathcal{A}_{(k,n]}$.
A non-2-separable state is called 2-entangled.

Remark Notice that, for $n = 2$, 2–entanglement is equivalent to usual entanglement. For $n > 2$, 2–entanglement is a strictly stronger property than usual entanglement.

Definition 4 *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be separable Hilbert spaces and let θ be density matrices in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with its marginal densities denoted by ρ and σ in $\mathcal{B}(\mathcal{H}_1)$, $\mathcal{B}(\mathcal{H}_2)$ respectively.*
The quasi mutual entropy of ρ and σ w.r.t θ is defined by [10]

$$I_\theta(\rho, \sigma) \equiv tr\theta(\log\theta - \log\rho \otimes \sigma) \tag{5}$$

The degree of entanglement of θ, denoted by $D_{EN}(\theta)$, is defined by [11]

$$D_{EN}(\theta) \equiv \frac{1}{2}\{S(\rho) + S(\sigma)\} - I_\theta(\rho, \sigma) \tag{6}$$

where $S(\cdot)$ is the von-Neumann entropy.

In the following we identify normal states on $\mathcal{B}(\mathcal{H})$ ($\mathcal{H}$ some separable Hilbert space) with their density matrices and, if θ is such a state, we will use indifferently the notations

$$\theta(x) = tr(\theta x) \qquad ; \qquad x \in \mathcal{B}(\mathcal{H}) \tag{7}$$

Recalling that, for density operators θ, δ in $\mathcal{B}(\mathcal{H})$, the relative entropy of δ with respect to θ is defined by:

$$R(\theta|\delta) := tr\theta(\log\theta - \log\delta) \tag{8}$$

(see [5, 12] for a more general discussion) we see that $I_\theta(\rho,\sigma)$ is the relative entropy of the tensor product of its marginal densities with respect to θ itself. Since it is known that the relative entropy is a kind of distance between states, it is clear why the degree of entanglement of θ by (6) is a measure of how far θ is from being a product state. Moreover we see also that D_{EN} is a kind of symmetrized quantum conditional entropy. In the classical case the conditional entropy always takes non-negative value, however our new criterion can be negative according to the strength of quantum correlation between ρ and σ [4].

Theorem 5 *A necessary condition for a (normal) state $\boldsymbol{\theta}$ on $\mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2)$ to be separable is that*

$$D_{EN}(\theta) \geq 0 \tag{9}$$

Equivalently: a sufficient condition for θ to be entangled is that

$$D_{EN}(\theta) < 0. \tag{10}$$

Proof. *Let θ be a state on $\mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2)$. If θ is separable, there exist density matrices ρ_n, σ_n respectively in $\mathcal{B}(\mathcal{H}_1)$, $\mathcal{B}(\mathcal{H}_2)$ such that*

$$\theta = \sum_n p_n\rho_n\otimes\sigma_n$$

with

$$p_n \geq 0\ ,\ \forall n \qquad ; \qquad \sum_n p_n = 1$$

Let $\{x_n\}$ be an ONB in $\mathcal{H}_1$ and define the completely positive unital $(CP1)$ map $\Lambda_0 : \mathcal{B}(\mathcal{H}_1)\to\mathcal{B}(\mathcal{H}_1)$ by

$$\Lambda_0(A) = \sum_n tr(A\rho_n)x_nx_n^* \quad ; \quad A\in\mathcal{B}(\mathcal{H}_1) \tag{11}$$

Then its dual is

$$\Lambda_0^*(\delta) = \sum_n \langle x_n,\delta x_n\rangle\rho_n \quad ; \quad \delta\in\mathcal{B}(\mathcal{H}_1)_* \tag{12}$$

so that defining the $CP1$ map

$$\Lambda := \Lambda_0\otimes id\ :\ \mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2)\to\mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2)$$

and the density matrix

$$\theta_d := \sum_n p_nx_nx_n^*\otimes\sigma_n$$

one easily verifies that

$$\Lambda^*(\theta_d) = \theta$$

Moreover, denoting

$$\rho = \sum_n p_n \rho_n \quad \textit{and} \quad \sigma = \sum_n p_n \sigma_n$$

the marginal densities of θ *and* $\rho_d = \sum_n p_n |x_n\rangle\langle x_n|$ *the first marginal density of* θ_d*, one has:*

$$\Lambda^*(\rho_d \otimes \sigma) = \rho \otimes \sigma$$

Recall now that the monotonicity property of the relative entropy (see [12] for proof and history) that for any pair of von Neumann algebras $\mathcal{M}$, $\mathcal{M}^0$, *for any normal* $CP1$ *map* $\Lambda : \mathcal{M} \to \mathcal{M}^0$ *and for any pair of normal states* ω_0, φ_0 *on* $\mathcal{M}^0$ *one has*

$$R\left(\Lambda^*(\omega_0)|\Lambda^*(\varphi_0)\right) \leq R\left(\omega_0|\varphi_0\right) \tag{13}$$

Using this property one finds:

$$I_\theta(\rho,\sigma) = R\left(\theta|\rho\otimes\sigma\right) = R\left(\Lambda^*(\theta_d)|\Lambda^*(\rho_d\otimes\sigma)\right) \leq R\left(\theta_d|\rho_d\otimes\sigma\right) = I_{\theta_d}(\rho_d,\sigma)$$

so that

$$S(\sigma) - I_\theta(\rho,\sigma) \geq S(\sigma) - I_{\theta_d}(\rho_d,\sigma) = -\sum_n p_n tr(\sigma_n \log\sigma_n) \geq 0 \tag{14}$$

Introducing the density operator

$$\hat{\theta}_d = \sum_n p_n \rho_n \otimes y_n y_n^*$$

where $\{y_n\}$ *is an ONB in* $\mathcal{H}_2$, *and using a variant of the above argument (in which the density* θ_d *is replaced by* $\hat{\theta}_d$*) one proves the analogue inequality*

$$S(\rho) - I_\theta(\rho,\sigma) \geq S(\rho) - I_{\hat{\theta}_d}(\rho,\sigma_d) = -\sum_n p_n tr(\rho_n \log\rho_n) \geq 0 \tag{15}$$

Combining (14) and (15) one obtains:

$$D_{EN}(\theta;\rho,\sigma) = \frac{1}{2}\left(\left(S(\sigma) - I_\theta(\rho,\sigma)\right) + \left(S(\rho) - I_\theta(\rho,\sigma)\right)\right) \geq \tag{16}$$

$$\geq \frac{1}{2}\left(-\sum_n p_n tr(\rho_n \log\rho_n) - \sum_n p_n tr(\sigma_n \log\sigma_n)\right) \geq 0$$

which is (9). ■

Remark For pure (normal) states θ on $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ condition (10) is also necessary for entanglement (see [11, 3]).

3 The localized EMC and its marginal states

We discuss the entanglement of the finite volume restrictions of a class of EMC on infinite tensor products of $d \times d$ matrix algebras. By restricting an EMC to some local algebra one obtains a mixed state to which our entanglement criterion D_{EN} is applicable because of theorem 5. In the following arguments we will denote $u_{ij} = \sqrt{p_{ij}}$ any (fixed) complex square root of p_{ij} so that

$$|u_{ij}|^2 = p_{ij} \qquad ; \qquad \forall i, j$$

and we assume that $U = (u_{ij})$ is a unitary matrix.

Let denote the unitarily implementable EMC state restricted to a finite region $[0, \nu]$ by $\rho_{[0,\nu]}$, then for every local observable $A \in \mathcal{A}_{[0,\nu]}$ one has $\rho_{[0,\nu]}(A) = \langle \Psi_{\nu+1}, (A \otimes I)\, \Psi_{\nu+1} \rangle$. Hence the density operator $\rho_{[0,\nu]}$ is given by:

$$\begin{aligned}
\rho_{[0,\nu]} &= tr_{\mathcal{H}_{\nu+1}} |\Psi_{\nu+1}\rangle \langle \Psi_{\nu+1}| \\
&= \sum_{i_0,\cdots,i_{\nu+1},j_0,\cdots,j_{\nu+1},l_{\nu+1}} \sqrt{p_{i_0}}^* \sqrt{p_{j_0}} \prod_{\alpha=0}^{\nu} u^*_{i_\alpha i_{\alpha+1}} u_{j_\alpha j_{\alpha+1}} \\
&\quad \langle e_{l_{\nu+1}}, e_{j_{\nu+1}} \rangle \langle e_{i_{\nu+1}}, e_{l_{\nu+1}} \rangle |e_{j_0}, \cdots, e_{j_\nu}\rangle \langle e_{i_0}, \cdots, e_{i_\nu}| \\
&= \sum_{j_0,\cdots,j_\nu,l,i_0,\cdots,i_\nu} \sqrt{p_{i_0}}^* \sqrt{p_{j_0}} \prod_{\alpha=0}^{\nu-1} u^*_{i_\alpha i_{\alpha+1}} u_{j_\alpha j_{\alpha+1}} \\
&\quad u^*_{i_\nu l} u_{j_\nu l} |e_{j_0}, \cdots, e_{j_\nu}\rangle \langle e_{i_0}, \cdots, e_{i_\nu}|
\end{aligned}$$

From the unitarity of $U = (u_{ij})$ one has $\sum_l u^*_{i_\nu l} u_{j_\nu l} = (UU^*)_{j_\nu i_\nu} = \delta_{i_\nu, j_\nu}$. Using this unitarity one has

$$\begin{aligned}
\rho_{[0,\nu]} &= \sum_{j_0,j_1,\cdots,j_\nu,i_0,i_1,\cdots,i_\nu,k} \sqrt{p_{i_0}}^* \sqrt{p_{j_0}} \prod_{\alpha=0}^{\nu-2} u^*_{i_\alpha i_{\alpha+1}} u_{j_\alpha j_{\alpha+1}} \\
&\quad u^*_{i_{\nu-1}k} u_{j_{\nu-1}k} \left|e_{j_0}, e_{j_1}, \cdots, e_{j_{\nu-1}}, e_\nu(k)\right\rangle \left\langle e_{i_0}, e_{i_1}, \cdots, e_{j_{\nu-1}}, e_\nu(k)\right| \\
&= \sum_k p_k e_{[0,\nu]}(k)\, e_{[0,\nu]}(k)^*, \qquad (17)
\end{aligned}$$

where

$$e_{[0,\nu]}(k) := \frac{1}{\sqrt{p_k}} \sum_{j_0,\cdots,j_{\mu-1}} \sqrt{p_{j_0}} \left(\prod_{\alpha=0}^{\nu-2} u_{j_\alpha j_{\alpha+1}}\right) u_{j_{\mu-1}k} \left|e_{j_0}, \cdots, e_{j_{\nu-1}}, e_\nu(k)\right\rangle.$$

The vectors $\{e_{[0,\nu]}(k)\}_k$ are normalized and orthogonal each other. In fact

$$\begin{aligned}\left\|e_{[0,\nu]}(k)\right\|^2 &= \frac{1}{p_k}\sum_{j_{\mu+1},j_1,\cdots,j_{\nu-1}} p_{j_{\mu+1}}\prod_{\alpha=\mu+1}^{\nu-2} p_{j_\alpha j_{\alpha+1}} p_{j_{\nu-1}k} \\ &= \frac{1}{p_k}\sum_{j_{\nu-1}} p_{j_{\nu-1}} p_{j_{\nu-1}k} = \frac{p_k}{p_k} = 1,\end{aligned}$$

and the orthogonality of $\{e_{[0,\nu]}(k)\}_k$ is clear because of the orthogonality of $\{e_\nu(k)\}_k$. We see that the decomposition (17) gives a Schatten decomposition.

Let us consider the marginal states of density $\rho_{[0,\nu]}$ for each $\mu \in [0,\nu-1]$ given by

$$\rho_{\mu]} \equiv tr_{H_{(\mu,\nu]}}\rho_{[0,\nu]}, \quad \rho_{(\mu} \equiv tr_{H_{[0,\mu]}}\rho_{[0,\nu]} \tag{18}$$

Since, by Proposition (1), the family $(\rho_{[0,\nu]})_\nu$ is projective, for each $\mu \in [0,\nu-1]$ the restriction of $\rho_{[0,\nu]}$ to the algebra localized on $[0,\mu]$ is $\rho_{[0,\mu]}$. This implies

$$\rho_{\mu]} \equiv tr_{H_{(\mu,\nu]}}\rho_{[0,\nu]} = \rho_{[0,\mu]}. \tag{19}$$

On the other hand the marginal state $\rho_{(\mu}$ is given by

$$\begin{aligned}\rho_{(\mu} &= tr_{\mathcal{H}_{[0,\mu]}}\rho_{[0,\nu]} \\ &= \sum_{j_0,\cdots,j_\mu,j_{\mu+1},\cdots,j_{\nu-1},i_{\mu+1},\cdots,i_{\nu-1},k} p_{j_0}\left(\prod_{\alpha=0}^{\mu-1} p_{j_\alpha j_{\alpha+1}}\right) u^*_{j_\mu i_{\mu+1}} u_{j_\mu j_{\mu+1}} \\ &\quad \left(\prod_{\alpha=\mu+1}^{\nu-2} u^*_{i_\alpha i_{\alpha+1}} u_{j_\alpha j_{\alpha+1}}\right) u^*_{i_{\nu-1}k} u_{j_{\nu-1}k} \\ &\quad \left|e_{j_{\mu+1}},\cdots,e_{j_{\nu-1}},e_\nu(k)\right\rangle\left\langle e_{i_{\mu+1}},\cdots,e_{i_{\nu-1}},e_\nu(k)\right| \\ &= \sum_{n,j_{\mu+1},\cdots\cdots,j_{\nu-1},i_{\mu+1},\cdots,i_{\nu-1},k} p_n u^*_{n i_{\mu+1}} u_{n j_{\mu+1}} \prod_{\alpha=\mu+1}^{\nu-2} u^*_{i_\alpha i_{\alpha+1}} u_{j_\alpha j_{\alpha+1}} \\ &\quad u^*_{i_{\nu-1}k} u_{j_{\nu-1}k}\left|e_{j_{\mu+1}},\cdots,e_{j_{\nu-1}},e_\nu(k)\right\rangle\left\langle e_{i_{\mu+1}},\cdots,e_{i_{\nu-1}},e_\nu(k)\right| \\ &= \sum_{n,k} p_n e^n_{(\mu,\nu]}(k)\, e^n_{(\mu,\nu]}(k)^*\end{aligned} \tag{20}$$

where

$$e^n_{(\mu,\nu]}(k) = \sum_{j_{\mu+1},\cdots,j_{\nu-1}} u_{n j_{\mu+1}}\left(\prod_{\alpha=\mu+1}^{\nu-2} u_{j_\alpha j_{\alpha+1}}\right) u_{j_{\nu-1}k}\left|e_{j_{\mu+1}},\cdots,e_{j_{\nu-1}},e_\nu(k)\right\rangle.$$

Remark If we put

$$\rho_{(\mu,\nu]}(n) := \sum_k e^n_{(\mu,\nu]}(k)\, e^n_{(\mu,\nu]}(k)^*, \tag{21}$$

then it is shown that (21) can be recognized as an orthogonal decompositions of a density operator. In fact we can show the following properties of $\rho_{(\mu,\nu]}(n)$.

(i) Orthogonality:

$$\left\langle e^n_{(\mu,\nu]}(k), e^n_{(\mu,\nu]}(l)\right\rangle = \delta_{k,l},$$

(ii) Density:

$$\begin{aligned}\left\|e^n_{(\mu,\nu]}(k)\right\|^2 &= \sum_{j_{\mu+1},\cdots,j_{\nu-1}} p_{nj_{\mu+1}}\left(\prod_{\alpha=\mu+2}^{\nu-2} p_{j_\alpha j_{\alpha+1}}\right) p_{j_{\nu-1}k} \\ &\equiv \left(P^{\nu-(\mu+1)}\right)_{nk}.\end{aligned}$$

This matrix $\left(P^{\nu-(\mu+1)}\right)$ can be recognized as a transition probability matrix generated by $P=(p_{ij})$ (i.e. a classical ergodic Markov chain). This implies

$$tr\rho_{(\mu,\nu]}(n) = \sum_k \left(P^{\nu-(\mu+1)}\right)_{nk} = 1.$$

Let denote $\widehat{e}^n_{(\mu,\nu]}(k)$ the normalized vector i.e.

$$\widehat{e}^n_{(\mu,\nu]}(k) = \frac{1}{\sqrt{\left(P^{\nu-(\mu+1)}\right)_{nk}}} e^n_{(\mu,\nu]}(k).$$

Then $\rho_{(\mu,\nu]}(n)$ is represented by

$$\rho_{(\mu,\nu]}(n) = \sum_k \left(P^{\nu-(\mu+1)}\right)_{nk} \widehat{e}^n_{(\mu,\nu]}(k)\,\widehat{e}^n_{(\mu,\nu]}(k)^* \tag{22}$$

which is a Schatten decomposition.

4 The DEN of EMC generated by unitarily implementable channel

We can define the entanglement criterion of EMC φ via the DEN of a localized EMC $\rho_{[0,\nu]}$. According to the definition of DEN one can compute the DEN of $\rho_{[0,\nu]}$ as follows:

$$\begin{aligned} D_{EN}\left(\rho_{[0,\nu]};\rho_{\mu]},\rho_{(\mu}\right) &= \frac{1}{2}\left\{S\left(\rho_{\mu]}\right)+S\left(\rho_{(\mu}\right)\right\} - I_{\rho_{[0,\nu]}}\left(\rho_{\mu]},\rho_{(\mu}\right) \\ &= \frac{1}{2}\left\{S\left(\rho_{\mu]}\right)+S\left(\rho_{(\mu}\right)\right\} - \left\{S\left(\rho_{\mu]}\right)+S\left(\rho_{(\mu}\right)-S\left(\rho_{[0,\nu]}\right)\right\} \\ &= S\left(\rho_{[0,\nu]}\right) - \frac{1}{2}\left\{S\left(\rho_{\mu]}\right)+S\left(\rho_{(\mu}\right)\right\}. \end{aligned}$$

Definition 6 *For a fixed $\mu \in \mathbb{N}$ we define the 2-entangled DEN of EMC φ by*

$$D_{EN}\left(\varphi;\rho_{\mu]},\rho_{(\mu}\right) \equiv \lim_{\nu\to\infty} D_{EN}\left(\rho_{[0,\nu]};\rho_{\mu]},\rho_{(\mu}\right), \tag{23}$$

where $\nu > \mu$. The D_{EN} of EMC φ is defined by the infimum of the 2-entangled DEN.

$$D_{EN}(\varphi) \equiv \inf_{\mu\in\mathbb{N}} D_{EN}\left(\varphi;\rho_{\mu]},\rho_{(\mu}\right). \tag{24}$$

Then we have the following result [4].

Theorem 7

$$D_{EN}(\varphi) = -\frac{1}{2}H(P) < 0 \tag{25}$$

where $H(P)$ is a Shannon entropy of a initial distribution of P.
Proof. *The localized state $\rho_{[0,\nu]}$ is decomposed to (17) and its marginal state $\rho_{\mu]}$ has a similar decomposition because of (19) which implies*

$$S\left(\rho_{[0,\nu]}\right) = S\left(\rho_{\mu]}\right) = -\sum_{n=1}^{d} p_n \log p_n = H(P). \tag{26}$$

On the other hand the another marginal state $\rho_{(\mu}$ is decomposed to (20) which can not be recognized as a orthogonal decomposition in general. However we can estimate the orthogonality of the vectors $e^n_{(\mu,\nu]}(k)$ and $e^m_{(\mu,\nu]}(k)$ $(n \neq m)$ asymptotically as follows:

$$\begin{aligned}
\left\langle e^n_{(\mu,\nu]}(k), e^m_{(\mu,\nu]}(k)\right\rangle &= \sum_{j_{\mu+1},\cdots,j_{\nu-1}} u^*_{nj_{\mu+1}} u_{mj_{\mu+1}} \left(\prod_{\alpha=\mu+1}^{\nu-2} p_{j_\alpha j_{\alpha+1}}\right) p_{j_{\nu-1}k} \\
&= \sum_{j_{\mu+1}} u^*_{nj_{\mu+1}} u_{mj_{\mu+1}} \sum_{j_{\mu+2},\cdots,j_{\nu-1}} p_{j_{\mu+1}j_{\mu+2}} \left(\prod_{\alpha=\mu+2}^{\nu-2} p_{j_\alpha j_{\alpha+1}}\right) p_{j_{\nu-1}k} \\
&= \sum_{j_{\mu+1}} u^*_{nj_{\mu+1}} u_{mj_{\mu+1}} \left(P^{\nu-\mu-2}\right)_{j_{\mu+1}k}
\end{aligned}$$

From the ergodic property of $\left(P^{\nu-\mu-2}\right)$ we have

$$\lim_{\nu\to\infty} \left(P^{\nu-\mu-2}\right)_{j_{\mu+1}k} = p_k$$

Therefore

$$\begin{aligned}
\lim_{\nu\to\infty} \left\langle e^n_{(\mu,\nu]}(k), e^m_{(\mu,\nu]}(k)\right\rangle &= p_k \sum_{j_{\mu+1}} u^*_{nj_{\mu+1}} u_{mj_{\mu+1}} \\
&= p_k \delta_{n,m}.
\end{aligned} \tag{27}$$

In large $\nu \gg 0$ *we can estimate the orthogonality of* $\left\{\rho_{(\mu,\nu]}(n)\right\}_n$ *approximately*

$$\rho_{(\mu,\nu]}(n)\,\rho_{(\mu,\nu]}(m) \simeq 0 \quad (n \neq m). \tag{28}$$

It is known (see [12]) that, if a density operator ρ *is a convex combination of densities* ρ_n*,*

$$\rho = \sum_n \lambda_n \rho_n \quad , \quad \lambda_n \geq 0 \quad , \quad \sum_n \lambda_n = 1$$

then the following inequality holds:

$$S(\rho) \leq \sum_n \lambda_n S(\rho_n) - \sum_n \lambda_n \log \lambda_n \tag{29}$$

and the equality holds if $\rho_n \perp \rho_m$ *for* $n \neq m$*. Thanks to (28) we can apply the equality of (29) to* $\rho_{(\mu} = \sum_n p_n \rho_{(\mu,\nu]}(n)$*.*

$$\begin{aligned}
\lim_{\nu\to\infty} S\left(\rho_{(\mu}\right) &= \lim_{\nu\to\infty} S\left(\sum_n p_n \rho_{(\mu,\nu]}(n)\right) \\
&= -\sum_{n=1}^{d} p_n \sum_{k=1}^{d} p_k \log p_k - \sum_{n=1}^{d} p_n \log p_n \\
&= 2H(P).
\end{aligned} \tag{30}$$

From the above arguments we have

$$\begin{aligned}
\lim_{\nu\to\infty} D_{EN}\left(\rho_{[0,\nu]};\rho_{\mu]},\rho_{(\mu}\right) &= H(P) - \frac{1}{2}\{H(P) + 2H(P)\} \\
&= -\frac{1}{2} H(P).
\end{aligned} \tag{31}$$

It is clear that the equation (31) holds for any $\mu \in \mathbb{N}$*. This fact shows that the equation (25) holds.* ■

This theorem says that the unitary implementable EMC is entangled state in the sense of definition 6. On the base of theorem 7 we can compute another entropic criteria, introduced in [6, 7], for EMC. As a result of such computations we can conclude that EMC gives an example of maximal entangled state on infinite multiple algebras. The detailed discussion will appear in a forthcoming paper [4].

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