Random variables and positive definite kernels associated with the Schrödinger algebra

Luigi Accardi
Centro Vito Volterra, Università di Roma Tor Vergata
via Columbia 2, 00133 Roma, Italy
accardi@volterra.mat.uniroma2.it
http://volterra.mat.uniroma2.it
Andreas Boukas
Department of Mathematics, The American College of Greece
Aghia Paraskevi, Athens 15342, Greece
andreasboukas@acg.edu

Contents

Abstract We show that the Feinsilver-Kocik-Schott (FKS) kernel for the Schrödinger algebra is not positive definite. We show how the FKS Schrödinger kernel can be reduced to a positive definite one through a restriction of the defining parameters of the exponential vectors. We define the Fock space associated with the reduced FKS Schrödinger kernel. We compute the characteristic functions of quantum random variables naturally associated with the FKS Schrödinger kernel and expressed in terms of the renormalized higher powers of white noise (or RHPWN) Lie algebra generators.

1. The Schrödinger and RHPWN Algebras

The quantum white noise functionals $b_t^\dagger$ (creation density) and b_t (annihilation density) satisfy the Boson commutation relations

$$[b_t, b_s^\dagger] = \delta(t-s) \; ; \; [b_t^\dagger, b_s^\dagger] = [b_t, b_s] = 0 \tag{1.1}$$

where $t, s \in \mathbb{R}$ and δ is the Dirac delta function, as well as the duality relation

$$(b_s)^* = b_s^\dagger \tag{1.2}$$

In order to consider the smeared fields defined by the higher powers of b_t and $b_t^\dagger$, for a test function f and $n, k \in \{0, 1, 2, ...\}$ we define the sesquilinear form

$$B_k^n(f) = \int_{\mathbb{R}} f(t)\, {b_t^\dagger}^n\, b_t^k\, dt \tag{1.3}$$

with involution

$$(B_k^n(f))^* = B_n^k(\bar{f}) \tag{1.4}$$

In [?] and [?] we introduced the convolution type renormalization

$$\delta^l(t-s) = \delta(s)\,\delta(t-s) \quad ; \quad l = 2, 3, ... \tag{1.5}$$

of the higher powers of the Dirac delta function, and by restricting to test functions $f(t)$ such that $f(0) = 0$ we obtained the renormalized higher powers of white noise (or RHPWN) $*$–Lie algebra commutation relations

$$[B_k^n(f), B_K^N(g)]_{RHPWN} = (k\,N - K\,n)\; B_{k+K-1}^{n+N-1}(f\,g) \tag{1.6}$$

Let $I \subset \mathbb{R}$ be a set of finite positive measure $\mu(I)$. In the notation of [?], the operators $M := B_0^0(\chi_I) = \mu(I)\, I$, $K := 2\, B_0^2(\chi_I)$, $G := 2\, B_0^1(\chi_I)$, $D := B_1^1(\chi_I)$, $P_x := \frac{1}{2}\, B_1^0(\chi_I)$, and $P_t := \frac{1}{8}\, B_2^0(\chi_I)$, satisfy the commutation relations of the Schrödinger algebra $\mathcal{S}$ given in the table

(1.7)

	M	K	G	D	P_x	P_t
M	0	0	0	0	0	0
K	0	0	0	$-2\,K$	$-G$	$-D$
G	0	0	0	$-G$	$-M$	$-P_x$
D	0	$2\,K$	G	0	$-P_x$	$-2\,P_t$
P_x	0	G	M	P_x	0	0
P_t	0	D	P_x	$2\,P_t$	0	0

The Feinsilver-Kocik-Schott (FKS) kernel (or Leibniz function), i.e. the inner product of the exponential vectors

$$\psi_{a,b}(\chi_I) := e^{a\,B_0^2(\chi_I)}\,e^{b\,B_0^1(\chi_I)}\,\Phi = e^{\frac{a}{2}\,K}\,e^{\frac{b}{2}\,G}\,\Phi \tag{1.8}$$

where $a, b \in \mathbb{C}$ with $|a| < 2$ and Φ is the Fock vacuum such that $B_2^0(\chi_I)\,\Phi = B_1^0(\chi_I)\,\Phi = 0$, is given by (see Proposition 6.1 of [?] for $c = \mu(I)\,/\,2$, $m = \mu(I)$ and $a, b \in \mathbb{R}$; a detailed proof can be found in [?])

$$\begin{aligned} \langle \psi_{a,b}(\chi_I), \psi_{A,B}(\chi_I)\rangle &= \left(1 - \frac{\bar{a}\,A}{4}\right)^{-\frac{\mu(I)}{2}} e^{\frac{\mu(I)}{4}\left(\frac{\bar{a}\,B^2 + 4\,\bar{b}\,B + \bar{b}^2\,A}{4 - \bar{a}\,A}\right)} \\ &= e^{-\frac{\mu(I)}{2}\,\ln\left(1 - \frac{\bar{a}\,A}{4}\right)}\, e^{\frac{\mu(I)}{4}\left(\frac{\bar{a}\,B^2 + 4\,\bar{b}\,B + \bar{b}^2\,A}{4 - \bar{a}\,A}\right)} \end{aligned} \tag{1.9}$$

For $a = A = 0$ (??) reduces to the usual inner product of the Heisenberg algebra exponential vectors, and for $b = B = 0$ it reduces to the inner product of the $sl(2)$ (Square of White Noise Lie algebra) exponential vectors.

Using the commutativity of the Schrödinger algebra generators on disjoint sets, we may extend the FKS kernel (??) to exponential vectors of the form

$$\Psi(f, g) := \prod_i e^{a_i\,B_0^2(\chi_{I_i})}\,e^{b_i\,B_0^1(\chi_{I_i})}\,\Phi = e^{B_0^2(f)}\,e^{B_0^1(g)}\,\Phi \tag{1.10}$$

where $f = \sum_i a_i\,\chi_{I_i}$ and $g = \sum_i b_i\,\chi_{I_i}$, with $I_i \cap I_j = \oslash$ for $i \neq j$, are simple functions with $|f| < 2$ and we obtain

$$\langle \Psi(f_1, g_1), \Psi(f_2, g_2)\rangle = e^{-\frac{1}{2}\int \ln\left(1 - \frac{\bar{f}_1\,f_2}{4}\right)\,d\mu}\, e^{\frac{1}{4}\int\left(\frac{\bar{f}_1\,g_2^2 + 4\,\bar{g}_1\,g_2 + \bar{g}_1^2\,f_2}{4 - \bar{f}_1\,f_2}\right)\,d\mu} \tag{1.11}$$

Proposition 1. *The kernel (??) (and so also (??)) is not positive definite for arbitrary $\mu(I)$. This, in particular, implies the non-positive definiteness of the kernel of Proposition 6.1 of* [?] *for arbitrary c.*

Proof. Denoting $\| \ldots \|^2 := \langle \ldots, \ldots \rangle$, we examine the non–negativity of

$$N_k := \| \sum_{i=1}^{k} c_i \psi_{a_i,b_i} \|^2$$

where $k \in \{1, 2, ...\}$ and (for simplicity, as in [?]), $a_i, b_i, c_i \in \mathbb{R}$ with $|a_i| < 2$ for all $i \in \{1, ..., k\}$.

For $k = 1$ we clearly have

$$N_1 := |c_1|^2 \, \|\psi_{a_1,b_1}\|^2 = |c_1|^2 \, e^{-\frac{\mu(I)}{2} \ln\left(1-\frac{a_1^2}{4}\right)} \, e^{\frac{\mu(I)}{4}\left(\frac{a_1\, b_1^2+4\, b_1^2+b_1^2\, a_1}{4-a_1^2}\right)} \geq 0$$

for all intervals $I \subset \mathbb{R}$.

For $k = 2$ we have

$$\begin{aligned} N_2 &= \sum_{i,j=1}^{2} c_i\, c_j\, e^{-\frac{\mu(I)}{2} \ln\left(1-\frac{a_i\, a_j}{4}\right)} \, e^{\frac{\mu(I)}{4}\left(\frac{a_i\, b_j^2+4\, b_i\, b_j+b_i^2\, a_j}{4-a_i\, a_j}\right)} \\ &= |c_1|^2\, e^{-\frac{\mu(I)}{2} \ln\left(1-\frac{a_1^2}{4}\right)} \, e^{\frac{\mu(I)}{4}\left(\frac{a_1\, b_1^2+4\, b_1^2+b_1^2\, a_1}{4-a_1^2}\right)} + |c_2|^2\, e^{-\frac{\mu(I)}{2} \ln\left(1-\frac{a_2^2}{4}\right)} \, e^{\frac{\mu(I)}{4}\left(\frac{a_2\, b_2^2+4\, b_2^2+b_2^2\, a_2}{4-a_2^2}\right)} \\ &+ 2\, c_1\, c_2\, e^{-\frac{\mu(I)}{2} \ln\left(1-\frac{a_1\, a_2}{4}\right)} \, e^{\frac{\mu(I)}{4}\left(\frac{a_1\, b_2^2+4\, b_1\, b_2+b_1^2\, a_2}{4-a_1\, a_2}\right)} \end{aligned}$$

If $c_1\, c_2 \geq 0$ then, clearly, $N_2 \geq 0$. Suppose that $c_1\, c_2 < 0$. The positive definiteness of N_2 can be studied by checking the non-negativity of the determinants d_1 and d_2 of the principal minors of the real symmetric matrix

$$A_2 = \begin{bmatrix} e^{-\frac{\mu(I)}{2} \ln\left(1-\frac{a_1^2}{4}\right)} \, e^{\frac{\mu(I)}{4}\left(\frac{a_1\, b_1^2+4\, b_1^2+b_1^2\, a_1}{4-a_1^2}\right)} & e^{-\frac{\mu(I)}{2} \ln\left(1-\frac{a_1\, a_2}{4}\right)} \, e^{\frac{\mu(I)}{4}\left(\frac{a_1\, b_2^2+4\, b_1\, b_2+b_1^2\, a_2}{4-a_1\, a_2}\right)} \\ e^{-\frac{\mu(I)}{2} \ln\left(1-\frac{a_1\, a_2}{4}\right)} \, e^{\frac{\mu(I)}{4}\left(\frac{a_1\, b_2^2+4\, b_1\, b_2+b_1^2\, a_2}{4-a_1\, a_2}\right)} & e^{-\frac{\mu(I)}{2} \ln\left(1-\frac{a_2^2}{4}\right)} \, e^{\frac{\mu(I)}{4}\left(\frac{a_2\, b_2^2+4\, b_2^2+b_2^2\, a_2}{4-a_2^2}\right)} \end{bmatrix}$$

corresponding to N_2. As in the case $k = 1$, we have

$$d_1 = |c_1|^2\, e^{-\frac{\mu(I)}{2} \ln\left(1-\frac{a_1^2}{4}\right)} \, e^{\frac{\mu(I)}{4}\left(\frac{a_1\, b_1^2+4\, b_1^2+b_1^2\, a_1}{4-a_1^2}\right)} \geq 0$$

Moreover, since the case $\mu(I) = 0$ is trivial, we may divide out $\mu(I) > 0$ and proving $d_2 \geq 0$ is equivalent to showing that

$$-\frac{1}{2} \ln\left(1 - \frac{a_1^2}{4}\right) + \frac{1}{4}\left(\frac{a_1\, b_1^2 + 4\, b_1^2 + b_1^2\, a_1}{4 - a_1^2}\right) - \frac{1}{2} \ln\left(1 - \frac{a_2^2}{4}\right) + \frac{1}{4}\left(\frac{a_2\, b_2^2 + 4\, b_2^2 + b_2^2\, a_2}{4 - a_2^2}\right)$$

$$\geq -\ln\left(1-\frac{a_1\,a_2}{4}\right)+\frac{1}{2}\left(\frac{a_1\,b_2^2+4\,b_1\,b_2+b_1^2\,a_2}{4-a_1\,a_2}\right)$$

which is equivalent to showing that

$$\ln\left(\frac{4-a_1\,a_2}{\sqrt{(a_1^2-4)\,(a_2^2-4)}}\right)\geq\frac{((a_2-2)\,b_1-(a_1-2)\,b_2)^2}{(a_1-2)\,(a_2-2)\,(a_1\,a_2-2)}$$

which is true since

$$\frac{((a_2-2)\,b_1-(a_1-2)\,b_2)^2}{(a_1-2)\,(a_2-2)\,(a_1\,a_2-2)}\leq 0$$

and

$$\frac{4-a_1\,a_2}{\sqrt{(a_1^2-4)\,(a_2^2-4)}}\geq 1\Leftrightarrow 4\,(a_1-a_2)^2\geq 0$$

Thus $N_1 \geq 0$ and $N_2 \geq 0$. However, for $k = 3$ the situation is different. We have

$$N_3=\sum_{i,j=1}^{3} c_i\,c_j\,e^{-\frac{\mu(I)}{2}\,\ln\left(1-\frac{a_i\,a_j}{4}\right)}\,e^{\frac{\mu(I)}{4}\left(\frac{a_i\,b_j^2+4\,b_i\,b_j+b_i^2\,a_j}{4-a_i\,a_j}\right)} \tag{1.12}$$

with associated real symmetric matrix

$$A_3=\begin{bmatrix}\langle\psi_{a_1,b_1}(\chi_I),\psi_{a_1,b_1}(\chi_I)\rangle & \langle\psi_{a_1,b_1}(\chi_I),\psi_{a_2,b_2}(\chi_I)\rangle & \langle\psi_{a_1,b_1}(\chi_I),\psi_{a_3,b_3}(\chi_I)\rangle\\ \langle\psi_{a_2,b_2}(\chi_I),\psi_{a_1,b_1}(\chi_I)\rangle & \langle\psi_{a_2,b_2}(\chi_I),\psi_{a_2,b_2}(\chi_I)\rangle & \langle\psi_{a_2,b_2}(\chi_I),\psi_{a_3,b_3}(\chi_I)\rangle\\ \langle\psi_{a_3,b_3}(\chi_I),\psi_{a_1,b_1}(\chi_I)\rangle & \langle\psi_{a_3,b_3}(\chi_I),\psi_{a_2,b_2}(\chi_I)\rangle & \langle\psi_{a_3,b_3}(\chi_I),\psi_{a_3,b_3}(\chi_I)\rangle\end{bmatrix}$$

As before, $d_1 \geq 0$ and $d_2 \geq 0$. However, for $\mu(I) \approx 0.21$, $(a_1, a_2, a_3) = (1, -1, -1)$ and $(b_1, b_2, b_3) = (0, 10, 0)$ we find that $d_3 \approx -3.4 < 0$ and, for $(c_1, c_2, c_3) = (3, -0.1, -2.82)$, we have $N_3 \approx -0.29 < 0$. In fact, for this choice of (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_1, c_2, c_3) we find that $d_3, N_3 < 0$ for $0 < \mu(I) < m_0$, and $d_3, N_3 \geq 0$ for $\mu(I) \geq m_0$, where $m_0 \approx 0.275$. This is in analogy with the no–go theorems of [**?**]–[**?**] and [**?**] for the impossibility of a Fock representation of $RHPWN$, where the existence of negative–norm (or "ghost") vectors was proved for $\mu(I)$ below a certain threshold.

□

2. Random variables associated with the Schrödinger algebra

For a fixed interval I, we will compute the vacuum characteristic function $\langle \Phi, e^{i\,s\,X}\,\Phi\rangle$ of the self-adjoint operator

$$X := x\,B_0^2(\chi_I) + \bar{x}\;B_2^0(\chi_I) + y\,B_0^1(\chi_I) + \bar{y}\,B_1^0(\chi_I) \tag{2.1}$$

where $x, y \in \mathbb{C}$ with $x \neq 0$.

For reasons explained in [?] we define the action of the RHPWN generators $B_k^n(f)$ on the Fock vacuum vector Φ by

$$B_k^n(f)\,\Phi := \begin{cases} 0 & \text{if } n < k \text{ or } n\cdot k < 0 \\ B_0^{n-k}(f)\,\Phi & \text{if } n > k \geq 0 \\ \frac{1}{n+1}\int_{\mathbb{R}} f(t)\,dt\,\Phi & \text{if } n = k \end{cases} \tag{2.2}$$

Thus, in particular, $B_2^0(\chi_I)\,\Phi = B_1^0(\chi_I)\,\Phi = 0$ and $B_1^1(\chi_I)\,\Phi = \mu(I)\,\Phi$.

Lemma 1. *(i) If x, d, N and h satisfy the oscillator algebra commutation relations*

$$[d, x] = h\ ;\ [d, h] = [x, h] = 0\ ;\ [N, x] = x\ ;\ [d, N] = d \tag{2.3}$$

then for all analytic functions f and for all $a \in \mathbb{C}$

$$[N, f(x)] = x\,f'(x)\ ;\ a^N\,f(x) = f(a\,x)\,a^N \tag{2.4}$$

Moreover, for all $n, m \in \mathbb{N}$

$$d^n\,x^m = \sum_{j=0}^{n\wedge m} \binom{n, m}{j} x^{m-j}\,d^{n-j}\,h^j \tag{2.5}$$

where

$$\binom{n, m}{j} = \binom{n}{j}\binom{m}{j} j! \tag{2.6}$$

(ii) If Δ, R and ρ satisfy the $sl(2)$ algebra commutation relations

$$[\Delta, R] = \rho\ ;\ [\rho, R] = 2\,R\ ;\ [\Delta, \rho] = 2\,\Delta \tag{2.7}$$

then

$$e^{t\,\Delta}\,e^{a\,R} = e^{\frac{a\,R}{1-a\,t}}\,(1 - a\,t)^{-\rho}\,e^{\frac{t\,\Delta}{1-a\,t}} \tag{2.8}$$

Proof. The proofs of (i) and (ii) can be found in [**?**]. □

In the context of the $RHPWN$ generators, in a manner consistent with the RHPWN representation of the Schrödinger algebra generators M, K, G, D, P_x, P_t given in the previous section,

$$R = K = 2\,B_0^2(\chi_{_I}) \;;\; \Delta = P_t = \frac{1}{8}\,B_2^0(\chi_{_I}) \;;\; \rho = D = B_1^1(\chi_{_I}) \tag{2.9}$$

and

$$x = G = 2\,B_0^1(\chi_{_I}) \;;\; d = P_x = \frac{1}{2}\,B_1^0(\chi_{_I}) \;;\; N = D = B_1^1(\chi_{_I}) \;;\; h = M = B_0^0(\chi_{_I}) \tag{2.10}$$

Lemma 2. *For all $a, b \in \mathbb{C}$*

(i)

$$\begin{aligned} B_2^0(\chi_{_I})\, e^{a\,B_0^2(\chi_I)} &= 4\,a^2\,B_0^2(\chi_{_I})\,e^{a\,B_0^2(\chi_I)} + \\ &\quad 4\,a\,e^{a\,B_0^2(\chi_I)}\,B_1^1(\chi_{_I}) + e^{a\,B_0^2(\chi_I)}\,B_2^0(\chi_{_I}) \end{aligned} \tag{2.11}$$

(ii)

$$B_1^1(\chi_{_I})\,e^{b\,B_0^1(\chi_I)}\,\Phi = \left(b\,B_0^1(\chi_{_I}) + \frac{\mu(I)}{2}\right)\,e^{b\,B_0^1(\chi_I)}\,\Phi \tag{2.12}$$

(iii) For all $n \geq 0$

$$B_2^0(\chi_{_I})\,(B_0^1(\chi_{_I}))^n\,\Phi = n\,(n-1)\,\mu(I)\,(B_0^1(\chi_{_I}))^{n-2}\,\Phi \tag{2.13}$$

(iv)

$$B_2^0(\chi_{_I})\,e^{b\,B_0^1(\chi_I)}\,\Phi = b^2\,\mu(I)\,e^{b\,B_0^1(\chi_I)}\,\Phi \tag{2.14}$$

(v) For all $n \geq 0$

$$B_1^0(\chi_{_I})\,(B_0^2(\chi_{_I}))^n = 2\,n\,B_0^1(\chi_{_I})\,(B_0^2(\chi_{_I}))^{n-1} + (B_0^2(\chi_{_I}))^n\,B_1^0(\chi_{_I}) \tag{2.15}$$

(vi)

$$B_1^0(\chi_{_I})\,e^{a\,B_0^2(\chi_I)} = 2\,a\,B_0^1(\chi_{_I})\,e^{a\,B_0^2(\chi_I)} + e^{a\,B_0^2(\chi_I)}\,B_1^0(\chi_{_I}) \tag{2.16}$$

Proof. (i) By (**??**) and lemma **??**(ii)

$$\begin{aligned} B_2^0(\chi_{_I})\, e^{a\, B_0^2(\chi_I)} &= \frac{\partial}{\partial\, t}|_{t=0} \left(e^{t\, B_2^0(\chi_I)}\, e^{a\, B_0^2(\chi_I)} \right) \\ &= \frac{\partial}{\partial\, t}|_{t=0} \left(e^{\frac{a}{1-4\,t\,a}\, B_0^2(\chi_I)}\, (1-4\,t\,a)^{-B_1^1(\chi_I)}\; e^{\frac{t}{1-4\,t\,a}\, B_2^0(\chi_I)} \right) \\ &= 4\,a\, e^{a\, B_0^2(\chi_I)}\, B_1^1(\chi_{_I}) + e^{a\, B_0^2(\chi_I)}\, B_2^0(\chi_{_I}) \end{aligned}$$

(ii) By (**??**), (**??**) and lemma **??**(i)

$$\begin{aligned} B_1^1(\chi_{_I})\, e^{b\, B_0^1(\chi_I)}\, \Phi &= \left([B_1^1(\chi_{_I}), e^{b\, B_0^1(\chi_I)}] + e^{b\, B_0^1(\chi_I)}\, B_1^1(\chi_{_I}) \right)\, \Phi \\ &= \left(b\, B_0^1(\chi_{_I}) + \frac{\mu(I)}{2} \right)\, e^{b\, B_0^1(\chi_I)}\, \Phi \end{aligned}$$

(iii) For $n = 0$, both sides of (**??**) are zero. For $n = 1$, since

$$B_2^0(\chi_{_I})\, B_0^1(\chi_{_I}) = 2\, B_1^0(\chi_{_I}) + B_0^1(\chi_{_I})\, B_2^0(\chi_{_I})$$

again both sides of (**??**) are zero. For $n \geq 2$ we have

$$\begin{aligned} a_n &:= B_2^0(\chi_{_I})\, (B_0^1(\chi_{_I}))^n\, \Phi \\ &= B_2^0(\chi_{_I})\, B_0^1(\chi_{_I})\, (B_0^1(\chi_{_I}))^{n-1}\, \Phi \\ &= (2\, B_1^0(\chi_{_I}) + B_0^1(\chi_{_I})\, B_2^0(\chi_{_I}))\, (B_0^1(\chi_{_I}))^{n-1}\, \Phi \\ &= 2\, B_1^0(\chi_{_I})\, (B_0^1(\chi_{_I}))^{n-1}\, \Phi + B_0^1(\chi_{_I})\, a_{n-1} \end{aligned}$$

By (**??**)

$$B_1^0(\chi_{_I})\, (B_0^1(\chi_{_I}))^{n-1} = (n-1)\, (B_0^1(\chi_{_I}))^{n-2}\, B_0^0(\chi_{_I}) + (B_0^1(\chi_{_I}))^{n-1}\, B_1^0(\chi_{_I})$$

Thus

$$\begin{aligned}
a_n &= 2\,\mu(I)\,(n-1)\,(B_0^1(\chi_I))^{n-2}\,\Phi + B_0^1(\chi_I)\,a_{n-1} \\
&= 2\,\mu(I)\,((n-1)+(n-2))\,(B_0^1(\chi_I))^{n-2}\,\Phi + (B_0^1(\chi_I))^2\,a_{n-2} \\
&= 2\,\mu(I)\,((n-1)+(n-2)+(n-3))\,(B_0^1(\chi_I))^{n-2}\,\Phi + (B_0^1(\chi_I))^3\,a_{n-3} \\
&= \ldots \\
&= 2\,\mu(I)\,((n-1)+(n-2)+(n-3)+\ldots+(n-(n-1)))\,(B_0^1(\chi_I))^{n-2}\,\Phi + (B_0^1(\chi_I))^{n-1}\,a_1 \\
&= 2\,\mu(I)\,((n-1)+(n-2)+(n-3)+\ldots+(n-(n-1)))\,(B_0^1(\chi_I))^{n-2}\,\Phi \\
&= 2\,\mu(I)\,(n\,(n-1)-(1+2+\ldots+(n-1))\,(B_0^1(\chi_I))^{n-2}\,\Phi \\
&= 2\,\mu(I)\left(n\,(n-1)-\frac{n\,(n-1)}{2}\right)(B_0^1(\chi_I))^{n-2}\,\Phi \\
&= \mu(I)\,n\,(n-1)\,(B_0^1(\chi_I))^{n-2}\,\Phi
\end{aligned}$$

(iv) Using (iii)

$$\begin{aligned}
B_2^0(\chi_I)\,e^{b\,B_0^1(\chi_I)}\,\Phi &= B_2^0(\chi_I)\sum_{n=0}^{\infty}\frac{b^n}{n!}\,(B_0^1(\chi_I))^n\,\Phi \\
&= \sum_{n=0}^{\infty}\frac{b^n}{n!}\,B_2^0(\chi_I)\,(B_0^1(\chi_I))^n\,\Phi \\
&= \sum_{n=0}^{\infty}\frac{b^n}{n!}\,n\,(n-1)\,\mu(I)\,(B_0^1(\chi_I))^{n-2}\,\Phi \\
&= \sum_{n=2}^{\infty}\frac{b^n}{n!}\,n\,(n-1)\,\mu(I)\,(B_0^1(\chi_I))^{n-2}\,\Phi \\
&= b^2\,\mu(I)\sum_{n=2}^{\infty}\frac{b^{n-2}}{(n-2)!}\,(B_0^1(\chi_I))^{n-2}\,\Phi \\
&= b^2\,\mu(I)\,e^{b\,B_0^1(\chi_I)}\,\Phi
\end{aligned}$$

(v) As in the proof of (iii)

$$\begin{aligned} b_n &:= B_1^0(\chi_{_I})\,(B_0^2(\chi_{_I}))^n \\ &= B_1^0(\chi_{_I})\,B_0^2(\chi_{_I})\,(B_0^2(\chi_{_I}))^{n-1} \\ &= (2\,B_0^1(\chi_{_I}) + B_0^2(\chi_{_I})\,B_1^0(\chi_{_I}))\,(B_0^2(\chi_{_I}))^{n-1} \\ &= 2\,B_0^1(\chi_{_I})\,(B_0^2(\chi_{_I}))^{n-1} + B_0^2(\chi_{_I})\,b_{n-1} \\ &= 4\,B_0^1(\chi_{_I})\,(B_0^2(\chi_{_I}))^{n-1} + (B_0^2(\chi_{_I}))^2\,b_{n-2} \\ &= \ldots \\ &= 2\,n\,B_0^1(\chi_{_I})\,(B_0^2(\chi_{_I}))^{n-1} + (B_0^2(\chi_{_I}))^n\,b_0 \\ &= 2\,n\,B_0^1(\chi_{_I})\,(B_0^2(\chi_{_I}))^{n-1} + (B_0^2(\chi_{_I}))^n\,B_1^0(\chi_{_I}) \end{aligned}$$

(vi)

$$\begin{aligned} B_1^0(\chi_{_I})\,e^{a\,B_0^2(\chi_{_I})} &= B_1^0(\chi_{_I})\sum_{n=0}^{\infty}\frac{a^n}{n!}\,(B_0^2(\chi_{_I}))^n \\ &= \sum_{n=0}^{\infty}\frac{a^n}{n!}\,B_1^0(\chi_{_I})\,(B_0^2(\chi_{_I}))^n \\ &= \sum_{n=0}^{\infty}\frac{a^n}{n!}\,(2\,n\,B_0^1(\chi_{_I})\,(B_0^2(\chi_{_I}))^{n-1} + (B_0^2(\chi_{_I}))^n\,B_1^0(\chi_{_I})) \\ &= 2\,a\,B_0^1(\chi_{_I})\sum_{n=0}^{\infty}\frac{a^{n-1}}{(n-1)!}\,(B_0^2(\chi_{_I}))^{n-1} + \sum_{n=0}^{\infty}\frac{a^n}{n!}\,(B_0^2(\chi_{_I}))^n\,B_1^0(\chi_{_I}) \\ &= 2\,a\,B_0^1(\chi_{_I})\,e^{a\,B_0^2(\chi_{_I})} + e^{a\,B_0^2(\chi_{_I})}\,B_1^0(\chi_{_I}) \end{aligned}$$

$\square$

Lemma 3. *For all* $s \in \mathbb{R}$

$$e^{i\,s\,X}\,\Phi = e^{w_1(s)\,B_0^2(\chi_{_I})}\,e^{w_2(s)\,B_0^1(\chi_{_I})}\,e^{w_3(s)}\,\Phi$$

where X *is as in (**??**) and*

$$w_1(s) = \frac{i}{2}\sqrt{\frac{x}{\bar{x}}}\,tanh\,(2\,s\,|x|) \tag{2.17}$$

$$w_2(s) = \frac{i}{2}\,\frac{\bar{y}}{\bar{x}}\,(1 - sech\,(2\,s\,|x|)) + \frac{y}{2\,|x|}\,tanh(2\,s\,|x|) \tag{2.18}$$

and

(2.19)
$$w_3(s) = \left(\frac{|y|^2 + i\,s\,(y^2\,\bar{x} + x\,\bar{y}^2)}{4\,|x|^2} - \frac{i\,(y^2\,\bar{x} + x\,\bar{y}^2)}{8\,|x|^3}\, tanh\,(2\,s\,|x|) - \frac{|y|^2}{4\,|x|^2}\, sech\,(2\,s\,|x|) \right) \mu(I)$$

Proof. We will show that $w_1(s), w_2(s)$ and $w_3(s)$ satisfy the differential equations

$$\begin{aligned}
w_1'(s) &= 4\,i\,\bar{x}\,w_1(s)^2 + i\,x \;\; ; \;\; w_1(0) = 0 \text{ (Riccati ODE)} && (2.20)\\
w_2'(s) &= 4\,i\,\bar{x}\,w_1(s)\,w_2(s) + 2\,\bar{y}\,w_1(s) + y \;\; ; \;\; w_2(0) = 0 \text{ (Linear ODE)} && (2.21)\\
w_3'(s) &= (i\,\bar{x}\,w_2(s)^2 + \bar{y}\,w_2(s) + 2\,i\,\bar{x}\,w_1(s))\,\mu(I) \;\; ; \;\; w_3(0) = 0 && (2.22)
\end{aligned}$$

whose solutions are given by (**??**), (**??**) and (**??**) respectively.

In so doing, let

$$E\,\Phi := e^{s(x\,B_0^2(\chi_I) + \bar{x}\,B_2^0(\chi_I) + y\,B_0^1(\chi_I) + \bar{y}\,B_1^0(\chi_I))}\,\Phi \tag{2.23}$$

and also

$$E\,\Phi := e^{w_1(s)\,B_0^2(\chi_I)}\,e^{w_2(s)\,B_0^1(\chi_I)}\,e^{w_3(s)}\,\Phi \tag{2.24}$$

where $w_i(0) = 0$ for $i \in \{1,2,3\}$. Then, (**??**) implies that

$$\frac{\partial}{\partial s}\,E\,\Phi = \left(w_1'(s)\,B_0^2(\chi_I) + w_2'(s)\,B_0^1(\chi_I) + w_3'(s)\right)\,E\,\Phi \tag{2.25}$$

and, since by lemma **??**,

$$\begin{aligned}
B_2^0(\chi_I)\,E\,\Phi &= (4\,w_1(s)^2\,B_0^2(\chi_I) + 4\,w_1(s)\,w_2(s)\,B_0^1(\chi_I) \\
&\quad + (2\,w_1(s) + w_2(s)^2)\,\mu(I))\,E\,\Phi
\end{aligned} \tag{2.26}$$

and

$$B_1^0(\chi_I)\,E\,\Phi = \left(2\,w_1(s)\,B_0^1(\chi_I) + \mu(I)\,w_2(s)\right)\,E\,\Phi \tag{2.27}$$

from (**??**) we also obtain

$$\begin{aligned}
\frac{\partial}{\partial s}\,E\,\Phi &= ((4\,i\,\bar{x}\,w_1(s)^2 + i\,x)\,B_0^2(\chi_I) + (4\,i\,\bar{x}\,w_1(s)\,w_2(s) + 2\,\bar{y}\,w_1(s) + y)\,B_0^1(\chi_I) \\
&\quad + (i\,\bar{x}\,w_2(s)^2 + \bar{y}\,w_2(s) + 2\,i\,\bar{x}\,w_1(s))\,\mu(I))\,E\,\Phi
\end{aligned} \tag{2.28}$$

From (**??**) and (**??**), by equating coefficients of $B_0^2(\chi_I), B_0^1(\chi_I)$ and 1, we obtain (**??**), (**??**) and (**??**) thus completing the proof.

□

Proposition 2. *(Characteristic Function) For all $s \in \mathbb{R}$*

$$\langle \Phi, e^{i\,s\,X}\,\Phi\rangle = e^{\left(\frac{|y|^2+i\,s\,\left(y^2\,\bar{x}+x\,\bar{y}^2\right)}{4\,|x|^2}-\frac{i\,\left(y^2\,\bar{x}+x\,\bar{y}^2\right)}{8\,|x|^3}\,tanh\,(2\,s\,|x|)-\frac{|y|^2}{4\,|x|^2}\,sech\,(2\,s\,|x|)\right)\,\mu(I)} \tag{2.29}$$

Proof. Using lemma **??** and the fact that for any $z \in \mathbb{C}$

$$e^{z\,B_2^0(\chi_I)}\,\Phi = e^{z\,B_1^0(\chi_I)}\,\Phi = \Phi \tag{2.30}$$

we have

$$\begin{aligned}
\langle \Phi, e^{i\,s\,X}\,\Phi\rangle &= \langle \Phi, e^{w_1(s)\,B_0^2(\chi_I)}\,e^{w_2(s)\,B_0^1(\chi_I)}\,e^{w_3(s)}\,\Phi\rangle \\
&= \langle e^{\bar{w}_2(s)\,B_1^0(\chi_I)}\,e^{\bar{w}_1(s)\,B_2^0(\chi_I)}\,\Phi, e^{w_3(s)}\,\Phi\rangle \\
&= \langle \Phi, e^{w_3(s)}\,\Phi\rangle \\
&= e^{w_3(s)}\,\langle \Phi, \Phi\rangle \\
&= e^{w_3(s)} \\
&= e^{\left(\frac{|y|^2+i\,s\,\left(y^2\,\bar{x}+x\,\bar{y}^2\right)}{4\,|x|^2}-\frac{i\,\left(y^2\,\bar{x}+x\,\bar{y}^2\right)}{8\,|x|^3}\,tanh\,(2\,s\,|x|)-\frac{|y|^2}{4\,|x|^2}\,sech\,(2\,s\,|x|)\right)\,\mu(I)}
\end{aligned} \tag{2.31}$$

□

3. Positive definite restrictions of the Schrödinger kernel

To overcome the problem of the non-positive definiteness of the FKS kernel (**??**) we look for a restriction of the parameters $a, b \in \mathbb{C}$, $|a| < 2$, in the definition (**??**) of the exponential vectors $\psi_{a,b}(\chi_I)$. We notice that if $a, b, A, B \in \mathbb{C}$ are such that

$$\bar{a}\,B^2 + \bar{b}^2\,A = 0 \tag{3.1}$$

i.e. if $(a, b), (A, B) \in S_\lambda$ where, for given $\lambda \in \mathbb{R}$,

$$S_\lambda := \{(a, b) \in \mathbb{C}^2 \;/\; a = i\,\lambda\,b^2,\; |\lambda|\,|b|^2 < 2\} \tag{3.2}$$

then the FKS kernel (**??**) reduces to

$$\begin{aligned}
\langle \psi_{a,b}(\chi_I), \psi_{A,B}(\chi_I)\rangle &= \left(1-\frac{\bar{a}\,A}{4}\right)^{-\frac{\mu(I)}{2}}\,e^{\frac{\mu(I)}{4}\,\frac{\bar{b}\,B}{1-\frac{\bar{a}\,A}{4}}} \\
&= \left(1-\frac{\bar{a}\,A}{4}\right)^{-\frac{\mu(I)}{2}}\,e^{\frac{\mu(I)}{4}\,\left(\bar{b}\,B\right)\,\sum_{n=0}^{\infty}\left(\frac{\bar{a}\,A}{4}\right)^n}
\end{aligned} \tag{3.3}$$

The kernels $K_i : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ defined for $i \in \{1, 2, 3\}$ by

$$K_1((a,b),(A,B)) := \left(1 - \frac{\bar{a}\,A}{4}\right)^{-\frac{\mu(I)}{2}} \tag{3.4}$$

$$K_2((a,b),(A,B)) := \frac{\mu(I)}{4}\,\bar{b}\;B \tag{3.5}$$

$$K_3((a,b),(A,B)) := \lim_{\rho\to\infty} K_{3,\rho}((a,b),(A,B)) \tag{3.6}$$

where, for $\rho \geq 1$,

$$K_{3,\rho}((a,b),(A,B)) := \sum_{n=0}^{\rho} \left(\frac{\bar{a}\,A}{4}\right)^n \tag{3.7}$$

are positive definite, since K_1 is the well-known $sl(2)$ kernel, K_2 and $K_{3,\rho}$ are Heisenberg-type kernels and K_3 is a limit of positive definite kernels . Therefore $K_1\, e^{K_2\, K_3}$, i.e. (**??**), is also a positive definite kernel.

For given $\lambda \in \mathbb{R}$ and $I \subset \mathbb{R}$ we introduce the notation

$$\psi(b) := \psi_{i\,\lambda\,b^2,b}(\chi_I) = e^{i\,\lambda\,b^2\,B_0^2(\chi_I)}\,e^{b\,B_0^1(\chi_I)}\,\Phi \tag{3.8}$$

$$\langle\psi(b),\psi(B)\rangle_\lambda := \langle\psi_{i\,\lambda\,b^2,b}(\chi_I),\psi_{i\,\lambda\,B^2,B}(\chi_I)\rangle = \left(1 - \frac{\lambda^2}{4}\,(\bar{b}\,B)^2\right)^{-\frac{\mu(I)}{2}} e^{\frac{\bar{b}\;B}{4-\lambda^2\,(\bar{b}\,B)^2}\,\mu(I)} \tag{3.9}$$

and its extension to simple functions $f = \sum_j b_j\,\chi_{I_j}$

$$\Psi(f) := \Psi_{i\,\lambda\,f^2,f} = \prod_j e^{i\,\lambda\,b_j^2\,B_0^2(\chi_{I_j})}\,e^{b_j\,B_0^1(\chi_{I_j})}\,\Phi = e^{i\,\lambda\,B_0^2(f^2)}\,e^{B_0^1(f)}\,\Phi \tag{3.10}$$

$$\langle\Psi(f),\Psi(g)\rangle_\lambda := \langle\Psi(i\,\lambda\,f^2,f),\Psi(i\,\lambda\,g^2,g)\rangle = e^{-\frac{1}{2}\int \ln\left(1-\frac{\lambda^2\,(\bar{f}\,g)^2}{4}\right)d\mu}\,e^{\int \frac{\bar{f}\,g}{4-\lambda^2\,(\bar{f}\,g)^2}\,d\mu} \tag{3.11}$$

If $f = b\,\chi_I$ then $\Psi(f) = \psi(b)$.

For $\lambda \in \mathbb{R}$ we define the (restricted) Schrödinger Fock space $\mathcal{F}_{\mathcal{S}}(\lambda)$ as the closure of the linear span of the exponential vectors $\Psi(f)$ defined in (**??**) with respect to the inner product $\langle\Psi(f),\Psi(g)\rangle_\lambda$ defined in (**??**).

The study of $\mathcal{F}_{\mathcal{S}}(\lambda)$ and the random variables that can be defined on it, is in progress.

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