Square of White Noise Unitary Evolutions on Boson Fock space

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Abstract We prove the Itô multiplication table for the stochastic differentials of the universal enveloping algebra of the square of white noise defined on Boson Fock space. Using the Itô table we derive unitarity conditions for processes satisfying quantum stochastic differential equations in terms of such noise. Computer algorithms for checking these conditions, for computing the product of stochastic differentials, and for iterating the differential of the square of white noise analogue of the Poisson- Weyl operator are also provided.

1. Introduction

The time-evolution of a classical quantum mechanical observable X (i.e a self-adjoint operator on the wave function Hilbert space $\mathcal{H}$)is described by a new observable $j_t(X) = U^*(t)\, X\, U(t)$ where $U(t) = e^{-itH}$ is a unitary operator and H is a self-adjoint operator on the wave function Hilbert space.

If the evolution is not disturbed by noise, the operator processes $U = \{U(t) \,/\, t \geq 0\}$ and $j = \{j_t(X) \,/\, t \geq 0\}$ satisfy deterministic differential equations in the "system" Hilbert space $\mathcal{H}$. In the case when the system is affected by quantum noise, described in terms of operators acting on a "noise" Fock space Γ and satisfying certain commutation relations, the equations for U and $j = \{j_t(X) \,/\, t \geq 0\}$ are replaced by stochastic differential equations driven by tha t noise (see e.g [15]), interpreted as stochastic differential equations in the tensor product $\mathcal{H} \otimes \Gamma$ and viewed as the Heisenberg picture of the Schrödinger equation in the presence of noise or as a quantum probabilistic analogue of the Langevin equation.

It is therefore important to be able to determine for specific quantum noises which processes U satisfying a quantum stochastic differential equation can be used to describe the time-evolution of an observable i.e to decide when the solution of such an equation is unitary.

The simple linear case of quantum stochastic differential equations driven by first order white noise the problem has been studied extensivelly e.g in [15] and the results are now standard.

In 1999 Accardi, Lu, and Volovich introduced nonlinearity in quantum stochastic calculus by considering the squares of the white noise functionals (see [9]). The physical motivation was provided in earlier work of Accardi and Obata related to problems arising in nonlinear quantum optics (see [10]).

Working with the square of white noise functionals led to analytical problems such as the product of distributions and forced Accardi, Lu, and Volovich to use the, well-known among physicists, method of "renormalization".

The first renormalization, corresponding to the subtraction of an infinite constant, was used by Accardi and Boukas in [1, 3] to study the problem of obtaining unitarity conditions for the solution of a quantum stochastic differential equation driven by the square of white noise (or SWN) processes.

The second renormalization, corresponding to defining the square of Dirac's delta function to be a positive constant times that function, was used in [4, 5] by Accardi, Boukas, and Kuo to study the problem of unitarity with the use of the closed Itô table of Accardi, Hida, and Kuo obtained in [8].

The natural Fock space for defining the SWN processes (see Definition 1 below) was shown by Accardi and Skeide in [11] to be related to the Finite-Difference Fock space of Feinsilver and Boukas (see [12, 13, 14]).

The SWN calculus in the framework of that Fock space was shown in [4] to be included in the representation free stochastic calculus of Accardi, Fagnola, and Quaegebeur constructed in [6]. In [7] Accardi, Franz, and Skeide realized the square of white noise in the usual Boson fock space associated with first order white noise.

The present paper addresses the problem of unitarity for processes defined as solutions of quantum stochastic differential equations in the framework of [7].

2. The Itô table for the renormalized SWN

Definition 1. *The SWN Lie algebra is the three-dimensional simple Lie algebra with basis B^+, B^-, M satisfying the commutation relations*

$$[B^-, B^+] = M,\ [M, B^+] = 2B^+,\ [M, B^-] = -2B^- \tag{2.1}$$

with involution

$$(B^-)^* = B^+,\ M^* = M \tag{2.2}$$

It was shown in [7] that the mapping ρ^+ defined by

$$\rho^+(M)\, e_n = (2n+2)\, e_n \tag{2.3}$$

$$\rho^+(B^+)\, e_n = \sqrt{(n+1)(n+2)}\, e_{n+1} \tag{2.4}$$

$$\rho^+(B^-)\, e_n = \sqrt{n(n+1)}\, e_{n-1} \tag{2.5}$$

where e_n, $n = 0, 1, 2, \cdots$ is any orthonormal basis of l_2, defines a representation of the SWN Lie algebra on l_2. Indexing B^+, B^- and M by time $t \geq 0$ and replacing M_t by $2ct + 4N_t$ we find that B_t^+, B_t^-, N_t satisfy the commutation relations of the Fock space operator realization of the SWN derived in [4], namely

$$[B_t^-, B_t^+] = 2ct + 4N_t,\ \ [N_t, B_t^+] = 2B_t^+,\ \ [N_t, B_t^-] = -2B_t^-$$

where $c > 0$ is a constant (coming from the renormalization $\delta^2(t) = c\,\delta(t)$ of [9]).

It was also shown in [7] that the quantum stochastic differentials dB_t^+, dB_t^-, dM_t are connected with the classic, first order white noise quantum stochastic differentials of [15], defined in terms of annihilation, creation, and conservation operators A_t, $A_t^\dagger$ and Λ_t respectively on the Boson Fock space $\Gamma(L^2(\mathbb{R}, l^2(\mathbb{N}))$ through

$$dM_t = d\Lambda_t(\rho^+(M)) + dt \tag{2.6}$$

$$dB_t^+ = d\Lambda_t(\rho^+(B^+)) + dA_t^\dagger(e_0) \tag{2.7}$$

$$dB_t^- = d\Lambda_t(\rho^+(B^-)) + dA_t(e_0) \tag{2.8}$$

with Itô multiplication table

$\cdot$	$dA_t^\dagger(u)$	$d\Lambda_t(F)$	$dA_t(u)$	dt
$dA_t^\dagger(v)$	0	0	0	0
$d\Lambda_t(G)$	$dA_t^\dagger(Gu)$	$d\Lambda_t(GF)$	0	0
$dA_t(v)$	$< v,u > dt$	$dA_t(F^*v)$	0	0
dt	0	0	0	0

The corresponding Itô multiplication table for the SWN quantum stochastic differentials dB_t^+, dB_t^- and dM_t is not closed and in order to discuss unitarity one should consider instead processes driven by time and the generalized square of white noise quantum stochastic differentials $d\Lambda_{n,k,l}(t)$, $dA_m(t)$ and $dA_m^\dagger(t)$, where $n,k,l,m = 0,1,...$, defined by

$$d\Lambda_{n,k,l}(t) = d\Lambda_t(\rho^+((B^+)^n M^k (B^-)^l)) \tag{2.9}$$

$$dA_m(t) = dA_t(e_m) \tag{2.10}$$

$$dA_m^\dagger(t) = dA_t^\dagger(e_m) \tag{2.11}$$

The following lemmas will be useful in obtaining the Itô table for the generalized SWN stochastic differentials.

Lemma 1. *For all* $n,k,l,m = 0,1,2,...$

$$(B^-)^n M^k = (M+2n)^k (B^-)^n \tag{2.12}$$

$$M^n (B^+)^k = (B^+)^k (M+2k)^n \tag{2.13}$$

(2.14)

$$(B^-)^n (B^+)^k = \sum_{m=0}^{n} \binom{n}{m} (k)^{(n-m)} (B^+)^{k-n+m} (M+k+m-1)^{(n-m)} (B^-)^m$$

$$\rho^+((B^+)^k)\, e_m = \frac{(m+1)_{k+1}}{\sqrt{(m+1)(m+k+1)}}\, e_{m+k} \tag{2.15}$$

$$\rho^+(M^k)\, e_m = (2m+2)^k\, e_m \tag{2.16}$$

$$\rho^+((B^-)^k)\, e_m = \begin{cases} \sqrt{\frac{m-k+1}{m+1}}(m+1)^{(k)}\, e_{m-k} & m \geq k \\ 0 & m < k \end{cases} \tag{2.17}$$

and also

$$\rho^+((B^+)^n M^k (B^-)^l)\, e_m = \theta_{n,k,l,m}\, e_{n+m-l} \tag{2.18}$$

where

$$\theta_{n,k,l,m} = H(n+m-l)\sqrt{\frac{m-l+n+1}{m+1}}\, 2^k (m-l+1)_n (m+1)^{(l)} (m-l+1)^k$$

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad \textit{is the Heaviside function}$$

$$0^0 = 1, \ \ (B^+)^n = (B^-)^n = N^n = 0, \quad \textit{for } n < 0$$

and "factorial powers" are defined by

$$x^{(n)} = x(x-1)\cdots(x-n+1)$$
$$(x)_n = x(x+1)\cdots(x+n-1)$$
$$(x)_0 = x^{(0)} = 1$$

Proof. The proof follows from (2.1) and (2.3)-(2.5) with the use of mathematical induction. We will only give the proof for (2.12) and (2.15). The proof of the rest is similar, with (2.18) following from (2.15)-(2.17) and the fact that ρ^+ is a homomorhism.

If one or both of n, k is zero then (2.12) is obviously true. So assume $n, k \geq 1$. We will first show that for all $n \geq 1$

$$(B^-)^n M = (M + 2n)(B^-)^n.$$

For $n = 1$ the above reduces to (2.1) and is therefore true. Assume it to be true for $n = n_0$. Then for $n = n_0 + 1$

$$\begin{gathered}(B^-)^{n_0+1} M = B^-(B^-)^{n_0} M = (B^-)(M + 2n_0)(B^-)^{n_0}\\ = B^- M(B^-)^{n_0} + 2n_0(B^-)^{n_0+1} = (M+2)B^-(B^-)^{n_0} + 2n_0(B^-)^{n_0+1}\\ = (M + 2(n_0+1))(B^-)^{n_0+1}.\end{gathered}$$

We will now show that for each n and all $k \geq 1$

$$(B^-)^n M^k = (M+2n)^k (B^-)^n$$

For $k = 1$ it was just proved. Assume it to be true for $k = k_0$. Then for $k = k_0+1$

$$\begin{gathered}(B^-)^n M^{k_0+1} = (B^-)^n M M^{k_0} = (M+2n)(B^-)^n M^{k_0}\\ = (M+2n)(M+2n)^{k_0}(B^-)^n = (M+2n)^{k_0+1}(B^-)^n.\end{gathered}$$

Turning to (2.15), for $k = 0$ it reduces to $\rho^+(I) = I$ which is true. For $k = 1$ it reduces to the definition of $\rho^+(B^+)$. Assume it to be true for $k = k_0$. Then for $k = k_0 + 1$

$$\begin{gathered}\rho^+((B^+)^{k_0+1})\, e_m = \rho^+((B^+)^{k_0})\rho^+(B^+)\, e_m\\ = \rho^+((B^+)^{k_0})\,\sqrt{(m+1)(m+2)}\, e_{m+1}\\ = \sqrt{(m+1)(m+2)}\frac{(m+2)_{k_0+1}}{\sqrt{(m+2)(m+2+k_0)}}\, e_{m+k_0+1}\\ = \sqrt{\tfrac{m+1}{m+2+k_0}}(m+2)_{k_0+1}\, e_{m+k_0+1}\\ = \sqrt{\tfrac{m+1}{m+2+k_0}}(m+2)(m+3)\cdots(m+2+k_0)\, e_{m+k_0+1}\\ = \sqrt{m+1}(m+2)(m+3)\cdots(m+2+k_0-1)\sqrt{m+2+k_0}\, e_{m+k_0+1}\\ = \frac{1}{\sqrt{(m+1)(m+(k_0+1)+1)}}(m+1)_{(k_0+1)+1}\, e_{m+k_0+1}.\end{gathered}$$

□

Lemma 2 (The SWN Multiplication Law). *For $\alpha, \beta, \gamma, a, b, c \in \{0, 1, 2, ...\}$*

$$(B^+)^\alpha M^\beta (B^-)^\gamma (B^+)^a M^b (B^-)^c =$$
$$\sum_{\lambda=0}^{\gamma} \sum_{\rho=0}^{\gamma-\lambda} \sum_{\sigma=0}^{\gamma-\lambda-\rho} \sum_{\omega=0}^{\beta} \sum_{\epsilon=0}^{b} c_{\beta,\gamma,a,b}^{\lambda,\rho,\sigma,\omega,\epsilon} (B^+)^{a+\alpha-\gamma+\lambda} M^{\omega+\sigma+\epsilon} (B^-)^{\lambda+c}$$

where

$$c_{\beta,\gamma,a,b}^{\lambda,\rho,\sigma,\omega,\epsilon} =$$
$$\binom{\gamma}{\lambda}\binom{\gamma-\lambda}{\rho}\binom{\beta}{\omega}\binom{b}{\epsilon} 2^{\beta+b-\omega-\epsilon} S_{\gamma-\lambda-\rho,\sigma} a^{(\gamma-\lambda)} (a+\lambda-1)^{(\rho)} (a-\gamma+\lambda)^{\beta-\omega} \lambda^{b-\epsilon}$$

Here $S_{\gamma-\lambda-\rho,\sigma}$ are the "Stirling numbers of the first kind" and $0^0 = 1$.

Proof. Recalling the binomial theorem for factorial powers of two commuting variables x, y and the connection between factorial and ordinary powers through the "Stirling numbers of the first kind" $S_{n,k}$, namely

$$(x+y)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(n-k)} y^{(k)}$$

and

$$x^{(n)} = \sum_{k=0}^{n} S_{n,k} x^k$$

the result follows using (2.12)-(2.14) to commute powers of B^+, B^- and M. In more detail, repeated use of Lemma 1 yields

$$(B^+)^\alpha M^\beta (B^-)^\gamma (B^+)^a M^b (B^-)^c =$$
$$(B^+)^\alpha M^\beta \sum_{\lambda=0}^{\gamma} \binom{\gamma}{\lambda} (a)^{(\gamma-\lambda)} (B^+)^{a-\gamma+\lambda}$$
$$(M+a+\lambda-1)^{(\gamma-\lambda)} (B^-)^\lambda M^b (B^-)^c =$$
$$\sum_{\lambda=0}^{\gamma} \binom{\gamma}{\lambda} (a)^{(\gamma-\lambda)} (B^+)^\alpha M^\beta (B^+)^{a-\gamma+\lambda}$$
$$(M+a+\lambda-1)^{(\gamma-\lambda)} (B^-)^\lambda M^b (B^-)^c =$$
$$\sum_{\lambda=0}^{\gamma} \binom{\gamma}{\lambda} (a)^{(\gamma-\lambda)} (B^+)^{\alpha+a-\gamma+\lambda} (M+2(a-\gamma+\lambda))^\beta$$
$$(M+a+\lambda-1)^{(\gamma-\lambda)} (B^-)^\lambda M^b (B^-)^c =$$
$$\sum_{\lambda=0}^{\gamma} \binom{\gamma}{\lambda} (a)^{(\gamma-\lambda)} (B^+)^{\alpha+a-\gamma+\lambda} (M+2(a-\gamma+\lambda))^\beta$$
$$(M+a+\lambda-1)^{(\gamma-\lambda)} (M+2\lambda)^b (B^-)^{\lambda+c} =$$
$$\sum_{\lambda=0}^{\gamma} \binom{\gamma}{\lambda} (a)^{(\gamma-\lambda)} (B^+)^{\alpha+a-\gamma+\lambda} \sum_{\omega=0}^{\beta} \binom{\beta}{\omega} M^\omega 2^{\beta-\omega} (a-\gamma+\lambda)^{\beta-\omega}$$
$$\sum_{\rho=0}^{\gamma-\lambda} \binom{\gamma-\lambda}{\rho} (a+\lambda-1)^{(\rho)} M^{(\gamma-\lambda-\rho)} \sum_{\epsilon=0}^{b} \binom{b}{\epsilon} M^\epsilon 2^{b-\epsilon} \lambda^{b-\epsilon} (B^-)^{\lambda+c} =$$
$$\sum_{\lambda=0}^{\gamma} \sum_{\omega=0}^{\beta} \sum_{\rho=0}^{\gamma-\lambda} \sum_{\epsilon=0}^{b} \binom{\gamma}{\lambda} (a)^{(\gamma-\lambda)} \binom{\beta}{\omega} 2^{\beta-\omega} (a-\gamma+\lambda)^{\beta-\omega}$$
$$\binom{\gamma-\lambda}{\rho} (a+\lambda-1)^{(\rho)} \binom{b}{\epsilon} 2^{b-\epsilon} \lambda^{b-\epsilon} M^\omega M^{(\gamma-\lambda-\rho)} M^\epsilon (B^-)^{\lambda+c}$$

and the result follows using the Stirling numbers to expand $M^{(\gamma-\lambda-\rho)}$ in terms of ordinary powers □

Remark 1. *By Lemma 2 the highest power of* B^+, M, B *appearing in the formula for*

$$(B^+)^\alpha M^\beta (B^-)^\gamma (B^+)^a M^b (B^-)^c$$

is $a+\alpha$, $\beta+b+\gamma$, $\gamma+c$ *respectively.*

Proposition 1 (The SWN Itô Table)**.** *For* $\alpha,\beta,\gamma,a,b,c \in \{0,1,2,...\}$

$$d\Lambda_{\alpha,\beta,\gamma}(t)\, d\Lambda_{a,b,c}(t) = \sum c^{\lambda,\rho,\sigma,\omega,\epsilon}_{\beta,\gamma,a,b}\, d\Lambda_{a+\alpha-\gamma+\lambda,\omega+\sigma+\epsilon,\lambda+c}(t) \tag{2.19}$$

$$d\Lambda_{\alpha,\beta,\gamma}(t)\, dA^\dagger_n(t) = \theta_{\alpha,\beta,\gamma,n}\, dA^\dagger_{\alpha+n-\gamma}(t) \tag{2.20}$$

$$dA_m(t)\, d\Lambda_{a,b,c}(t) = \theta_{c,b,a,m}\, dA_{c+m-a}(t) \tag{2.21}$$

$$dA_m(t)\, dA^\dagger_n(t) = \delta_{m,n}\, dt \tag{2.22}$$

where

$$\sum = \sum_{\lambda=0}^{\gamma} \sum_{\rho=0}^{\gamma-\lambda} \sum_{\sigma=0}^{\gamma-\lambda-\rho} \sum_{\omega=0}^{\beta} \sum_{\epsilon=0}^{b}$$

All other products are equal to zero.

Proof. We will only prove (2.19). The proof of (2.20), (2.21) and (2.22) is similar. By (2.9), the Itô table following (2.8), and Lemma 2

$$\begin{aligned}
d\Lambda_{\alpha,\beta,\gamma}(t)\, d\Lambda_{a,b,c}(t) &= d\Lambda_t(\rho^+((B^+)^\alpha M^\beta (B^-)^\gamma))\, d\Lambda_t(\rho^+((B^+)^a M^b (B^-)^c))\\
&= d\Lambda_t(\rho^+((B^+)^\alpha M^\beta (B^-)^\gamma)\, \rho^+((B^+)^a M^b (B^-)^c))\\
&= d\Lambda_t(\rho^+((B^+)^\alpha M^\beta (B^-)^\gamma (B^+)^a M^b (B^-)^c))\\
&= d\Lambda_t(\rho^+(\textstyle\sum c^{\lambda,\rho,\sigma,\omega,\epsilon}_{\beta,\gamma,a,b}\, (B^+)^{a+\alpha-\gamma+\lambda} M^{\omega+\sigma+\epsilon} (B^-)^{\lambda+c}))\\
&= \textstyle\sum c^{\lambda,\rho,\sigma,\omega,\epsilon}_{\beta,\gamma,a,b}\, d\Lambda_t(\rho^+((B^+)^{a+\alpha-\gamma+\lambda} M^{\omega+\sigma+\epsilon} (B^-)^{\lambda+c}))\\
&= \textstyle\sum c^{\lambda,\rho,\sigma,\omega,\epsilon}_{\beta,\gamma,a,b}\, d\Lambda_{a+\alpha-\gamma+\lambda,\omega+\sigma+\epsilon,\lambda+c}(t)
\end{aligned}$$

□

3. The Unitarity Conditions

Consider the quantum stochastic differential equation, with constant coefficients acting on a system Hilbert space H_0,

$$\begin{aligned}
dU(t) = (A\, dt &+ \textstyle\sum_{n,k,l=0}^{+\infty} B_{n,k,l}\, d\Lambda_{n,k,l}(t) + \sum_{m=0}^{+\infty} C_m\, dA_m(t)\\
&+ \textstyle\sum_{m=0}^{+\infty} D_m\, dA^\dagger_m(t))\, U(t),\;\; U(0)=I,\;\; 0\le t\le t_0<+\infty
\end{aligned} \tag{3.1}$$

interpreted as an $H_0 \otimes F$ (where F denotes Boson Fock space) quantum stochastic differential equation with infinite degrees of freedom in a way similar to that of [15], with adjoint

$$\begin{aligned}
dU^*(t) = U^*(t)(A^*\, dt &+ \textstyle\sum_{n,k,l=0}^{+\infty} B^*_{n,k,l}\, d\Lambda_{l,k,n}(t) + \sum_{m=0}^{+\infty} C^*_m\, dA^\dagger_m(t)\\
&+ \textstyle\sum_{m=0}^{+\infty} D^*_m\, dA_m(t)),\;\; U^*(0)=I,\;\; 0\le t\le t_0<+\infty
\end{aligned} \tag{3.2}$$

Under certain summability conditions on its coefficients, derived in a manner similar to that of [15], it can be shown that equation (3.1) admits a unique solution. The details will appear elsewhere.

Proposition 2 (Necessary and sufficient unitarity conditions)**.** *The solution* $U = \{U(t)\,/\,t \geq 0\}$ *of (3.1) is unitary, i.e* $U(t)\,U^*(t) = U^*(t)\,U(t) = I$ *for each* $t \geq 0$*, if and only if the coefficient operators satisfy*

$$A + A^* + \sum_{m=0}^{+\infty} D_m^* D_m = 0 \tag{3.3}$$

$$A + A^* + \sum_{m=0}^{+\infty} C_m C_m^* = 0 \tag{3.4}$$

for each $m = 0, 1, 2, ...$

$$C_m + D_m^* + \sum_{n,l=0}^{+\infty} D_{m+n-l}^* \sum_{k=0}^{+\infty} \theta_{l,k,n,m+n-l} B_{n,k,l} = 0 \tag{3.5}$$

$$C_m + D_m^* + \sum_{n,l=0}^{+\infty} C_{m+l-n} \sum_{k=0}^{+\infty} \theta_{n,k,l,m+l-n} B_{n,k,l}^* = 0 \tag{3.6}$$

and for each $n, k, l = 0, 1, 2, ...$

$$B_{n,k,l} + B_{l,k,n}^* + \sum_{\alpha,\beta,\gamma,a,b,c=0}^{n,k,min(k,l),l,k,n} B_{\alpha,\beta,\gamma} B_{a,b,c}^* \, g_{\alpha,a,\beta,\gamma,c,b}^{n,k,l} = 0 \tag{3.7}$$

$$B_{n,k,l} + B_{l,k,n}^* + \sum_{\alpha,\beta,\gamma,a,b,c=0}^{min(k,l),k,n,n,k,l} B_{\alpha,\beta,\gamma}^* B_{a,b,c} \, g_{\gamma,c,\beta,\alpha,a,b}^{n,k,l} = 0 \tag{3.8}$$

where, with δ *denoting Kronecker's delta,*

$$g_{x,y,z,X,Y,Z}^{n,k,l} = \sum_{\lambda=0}^{X} \sum_{\rho=0}^{X-\lambda} \sum_{\sigma=0}^{X-\lambda-\rho} \sum_{\omega=0}^{z} \sum_{\epsilon=0}^{Z} \delta_{x+Y-X+\lambda,n}\, \delta_{\omega+\sigma+\epsilon,k}\, \delta_{\lambda+y,l}\, c_{z,X,Y,Z}^{\lambda,\rho,\sigma,\omega,\epsilon}$$

and

$$\sum_{\alpha,\beta,\gamma,a,b,c=0}^{n,k,min(k,l),l,k,n}$$

means that α *ranges from* 0 *to* n*,* β *ranges from* 0 *to* k *e.t.c with a similar interpretation for*

$$\sum_{\alpha,\beta,\gamma,a,b,c=0}^{min(k,l),k,n,n,k,l}$$

Proof. In the theory of quantum stochastic differential equations, one obtains sufficient unitarity conditions for stochastic evolutions driven by quantum noise by starting with the definition of unitarity

$$U(t)\,U^*(t) = U^*(t)\,U(t) = I\,,\; U(0) = U^*(0) = I$$

which is equivalent to

$$d\,(U(t)U^*(t)) = dU(t)\,U^*(t) + U(t)\,dU^*(t) + dU(t)\,dU^*(t) = 0$$

and

$$d\,(U^*(t)U(t)) = dU^*(t)\,U(t) + U^*(t)\,dU(t) + dU^*(t)\,dU(t) = 0$$

replacing $dU(t)$ and $dU^*(t)$ by (in the SWN case) (3.1) and (3.2), using the Itô multiplication rule of Proposition 1 to multiply the stochastic differentials, and then equating coefficients of the time and noise differentials to zero. In the SWN case this method yields (3.3)-(3.8) as sufficient conditions for the unitarity of U. In view of the linear independence of the generalized SWN stochastic differentials (proved in Proposition 6 in the next section) conditions (3.3)-(3.8) are also necessa ry for the unitarity of U. In more detail, by (3.1) and (3.2),

$$d\,(U(t)U^*(t)) = 0$$

implies

$$\begin{gathered}(A\,dt + \textstyle\sum_{n,k,l=0}^{+\infty} B_{n,k,l}\,d\Lambda_{n,k,l}(t) + \sum_{m=0}^{+\infty} C_m\,dA_m(t) + \sum_{m=0}^{+\infty} D_m\,dA^\dagger_m(t))\\ U(t)U^*(t) + U(t)U^*(t)(A^*\,dt + \textstyle\sum_{n,k,l=0}^{+\infty} B^*_{n,k,l}\,d\Lambda_{l,k,n}(t) + \sum_{m=0}^{+\infty} C^*_m\,dA^\dagger_m(t)\\ + \textstyle\sum_{m=0}^{+\infty} D^*_m\,dA_m(t)) + (A\,dt + \sum_{n,k,l=0}^{+\infty} B_{n,k,l}\,d\Lambda_{n,k,l}(t) + \sum_{m=0}^{+\infty} C_m\,dA_m(t)\\ + \textstyle\sum_{m=0}^{+\infty} D_m\,dA^\dagger_m(t))U(t)U^*(t)(A^*\,dt + \sum_{n,k,l=0}^{+\infty} B^*_{n,k,l}\,d\Lambda_{l,k,n}(t)\\ + \textstyle\sum_{m=0}^{+\infty} C^*_m\,dA^\dagger_m(t) + \sum_{m=0}^{+\infty} D^*_m\,dA_m(t)) = 0\end{gathered}$$

which using $U(t)\,U^*(t) = I$ can be written as

$$\begin{gathered}(A + A^*)dt + \textstyle\sum_{n,k,l=0}^{+\infty} (B_{n,k,l} + B^*_{l,k,n})\,d\Lambda_{n,k,l}(t) + \sum_{m=0}^{+\infty} (C_m + D^*_m)\,dA_m(t)\\ + \textstyle\sum_{m=0}^{+\infty} (D_m + C^*_m)\,dA^\dagger_m(t) + \sum_{n,k,l=0}^{+\infty} \sum_{n_0,k_0,l_0=0}^{+\infty} B_{n,k,l}B^*_{n_0,k_0,l_0}\\ d\Lambda_{n,k,l}(t)\,d\Lambda_{l_0,k_0,n_0}(t) + \textstyle\sum_{m,n,k,l=0}^{+\infty} B_{n,k,l}C^*_m\,d\Lambda_{n,k,l}(t)\,dA^\dagger_m(t)\\ + \textstyle\sum_{m=0}^{+\infty} \sum_{m_0=0}^{+\infty} C_mC^*_{m_0}\,dA_m(t)\,dA^\dagger_{m_0}(t)\\ + \textstyle\sum_{m,n,k,l=0}^{+\infty} C_mB^*_{l,k,n}\,dA_m(t)\,d\Lambda_{n,k,l}(t) = 0\end{gathered}$$

which by Proposition 1 implies

$$\begin{gathered}(A + A^* + \textstyle\sum_{m=0}^{+\infty} C_mC^*_m)\,dt + \sum_{n,k,l=0}^{+\infty} (B_{n,k,l} + B^*_{l,k,n})\,d\Lambda_{n,k,l}(t) +\\ \textstyle\sum_{\alpha,\beta,\gamma=0}^{+\infty} \sum_{a,b,c=0}^{+\infty} B_{\alpha,\beta,\gamma}B^*_{a,b,c} \sum_{\lambda,\rho,\sigma,\omega,\epsilon} c^{\lambda,\rho,\sigma,\omega,\epsilon}_{\beta,\gamma,c,b}\,d\Lambda_{\alpha+c-\gamma+\lambda,\omega+\sigma+\epsilon,\lambda+a}(t) +\\ \textstyle\sum_{m=0}^{+\infty} (C_m + D^*_m)\,dA_m(t) + \sum_{m,n,k,l=0}^{+\infty} C_mB^*_{l,k,n}\theta_{l,k,n,m}\,dA_{l+m-n}(t) +\\ \textstyle\sum_{m=0}^{+\infty} (D_m + C^*_m)\,dA^\dagger_m(t) + \sum_{m,n,k,l=0}^{+\infty} B_{n,k,l}C^*_m\theta_{n,k,l,m}\,dA^\dagger_{n+m-l}(t) = 0\end{gathered}$$

and by reindexing we obtain

$$(A + A^* + \sum_{m=0}^{+\infty} C_m C_m^*)\, dt + \sum_{n,k,l=0}^{+\infty} (B_{n,k,l} + B_{l,k,n}^* +$$
$$\sum_{\alpha,\beta,\gamma,a,b,c=0}^{+\infty} \sum_{\lambda,\rho,\sigma,\omega,\epsilon} c_{\beta,\gamma,c,b}^{\lambda,\rho,\sigma,\omega,\epsilon} B_{\alpha,\beta,\gamma} B_{a,b,c}^*)\, d\Lambda_{n,k,l}(t) +$$
$$\sum_{m=0}^{+\infty} (C_m + D_m^* + \sum_{n,k,l=0}^{+\infty} C_{m+n-l} B_{l,k,n}^* \theta_{l,k,n,m+n-l})\, dA_m(t) +$$
$$\sum_{m=0}^{+\infty} (D_m + C_m^* + \sum_{n,k,l=0}^{+\infty} B_{n,k,l} C_{m+l-n}^* \theta_{n,k,l,m+l-n})\, dA_m^\dagger(t) = 0$$

where $\sum_{\lambda,\rho,\sigma,\omega,\epsilon}$ is over all $\lambda, \rho, \sigma, \omega, \epsilon$ such that

$$\alpha + c - \gamma + \lambda = n$$
$$\omega + \sigma + \epsilon = k$$
$$\lambda + a = l$$

In view of Remark 1 we may replace $\sum_{\alpha,\beta,\gamma,a,b,c=0}^{+\infty}$ by the finite sum appearing in (3.7). By equating coefficients to zero we obtain (3.4), (3.6) and its adjoint, and (3.7).

Similarly, starting with

$$d(U^*(t)U(t)) = 0$$

we obtain (3.3), (3.5) and its adjoint, and (3.8).

□

Proposition 3 (Matrix form of the unitarity conditions)**.** *Unitarity conditions (3.3)-(3.8) can be put in the matrix form*

$$A + A^* + D^\dagger D = 0 \tag{3.9}$$
$$A + A^* + C C^\dagger = 0 \tag{3.10}$$
$$C + D^\dagger + \theta \Delta^\dagger \hat{B} = 0 \tag{3.11}$$
$$C + D^\dagger + \theta \Gamma \hat{E} = 0 \tag{3.12}$$
$$\tilde{B} + \tilde{E} + \bar{B} G \hat{E} = 0 \tag{3.13}$$
$$\tilde{B} + \tilde{E} + \bar{E} G \hat{B} = 0 \tag{3.14}$$

where (in standard vector and matrix notation, using the notation $\delta(x_0, x_1, ...)$ for a diagonal matrix with main diagonal $x_0, x_1, ...$, denoting operator dual by using the superscript $*$, transpose by T and conjugate transpose by $\dagger$)

$$C = (C_0, C_1, ...),\ D = (D_0, D_1, ...)^T,\ \hat{B} = \delta(B, B, ...),\ \bar{B} = \delta(B^T, B^T, ...)$$

$$\Delta = \delta(\Delta_0, \Delta_1, ...),\ \hat{E} = \delta(E, E, ...),\ \Gamma = \delta(\Gamma_0, \Gamma_1, ...),\ \theta = (\theta_0, \theta_1, ...)$$

$$B = (B_0, B_1, ...)^T,\ E = (E_0, E_1, ...)^T,\ \tilde{B} = \delta(B_0, B_1, ...),\ \tilde{E} = \delta(E_0, E_1, ...)$$

$$G = \delta(g^{0,0,0}, g^{0,1,0}, ..., g^{0,0,1}, g^{0,1,1}, ..., g^{1,0,0}, g^{1,1,0}, ..., g^{1,0,1}, ..., g^{1,1,1}, ...)$$

and for $n, k, l, m, a, b, c, \alpha, \beta, \gamma \in \{0, 1, 2, ...\}$

$$\Delta_m = \delta(D_0(m), D_1(m), ...),\ \Gamma_m = \delta(C_0(m), C_1(m), ...)$$

$$C_n(m) = \delta(C_{m+n}, C_{m+n-1}, C_{m+n-2}, ...),\ D_n(m) = \delta(D_{m+n}, D_{m+n-1}, D_{m+n-2}...)$$

$$\theta_m = (\theta_0(m), \theta_1(m), ...),\ E_n = (E_{n,0}, E_{n,1}, ...)^T,\ B_n = (B_{n,0}, B_{n,1}, ...)^T$$

$$\theta_n(m) = (\theta_{0,n}(m), \theta_{1,n}(m), ...),\ \theta_{l,n}(m) = (\theta_{l,0,n,m+n-l}, \theta_{l,1,n,m+n-l}, ...)$$

$$B_{n,l} = (B_{n,0,l}, B_{n,1,l}, ...)^T,\ E_{n,l} = (E_{n,0,l}, E_{n,1,l}, ...)^T$$

$$E_{l,k,n} = B^*_{n,k,l}$$

$$g^{n,k,l} = (g_0^{n,k,l}, g_1^{n,k,l}, ...),\ g_c^{n,k,l} = (g_{0,c}^{n,k,l}, g_{1,c}^{n,k,l}, ...),\ g_{a,c}^{n,k,l} = (g_{a,c,0}^{n,k,l}, g_{a,c,1}^{n,k,l}, ...)$$

$$g_{a,c,b}^{n,k,l} = (g_{0,a,c,b}^{n,k,l}, g_{1,a,c,b}^{n,k,l}, ...)^T,\ g_{\alpha,a,c,b}^{n,k,l} = (g_{\alpha,a,0,c,b}^{n,k,l}, g_{\alpha,a,1,c,b}^{n,k,l}, ...)^T$$

$$g_{\alpha,a,\gamma,c,b}^{n,k,l} = (g_{\alpha,a,0,\gamma,c,b}^{n,k,l}, g_{\alpha,a,1,\gamma,c,b}^{n,k,l}, ...)^T$$

Proof. We will only show that (3.5) can be written as (3.11), the proof of the rest of (3.9)-(3.14) is similar. To that end, we notice that (3.5) can be written as

$$\begin{gathered} C_m + D^*_m + \sum_{n,l} D^*_{m+n-l}\theta_{l,n}(m)B_{n,l} = \\ C_m + D^*_m + \sum_n \theta_n(m)D_n^\dagger(m)B_n = \\ C_m + D^*_m + \theta_m\Delta_m^\dagger B = 0 \end{gathered}$$

for all m, which implies (3.11).

□

Corollary 1 (Compatibility of the unitarity conditions)**.** *In order for the pairs (3.9)&(3.10), (3.11)&(3.12), and (3.13)&(3.14) to be compatible it is necessary that*

$$D^\dagger D = CC^\dagger \tag{3.15}$$

$$\Delta^\dagger \hat{B} = \Gamma \hat{E} \tag{3.16}$$

$$\bar{B}G\hat{E} = \bar{E}G\hat{B}. \tag{3.17}$$

Proof. The proof follows by a direct comparison of (3.9)&(3.10), (3.11)&(3.12), and (3.13)&(3.14).

□

4. The SWN analogue of the Poisson-Weyl operator

Proposition 4. *Let $\lambda, k \in \mathbb{R}$ and $z \in \mathbb{C}$, with $|z|, |k|$ less than a sufficiently small positive number, let*

$$E(t) = \lambda\, t + z\, B_t^- + \bar{z}\, B_t^+ + k\, M_t \tag{4.1}$$
$$= (\lambda + k)\, t + z\, A_0(t) + \bar{z}\, A_0^\dagger(t) + z\, \Lambda_{0,0,1}(t) + \bar{z}\, \Lambda_{1,0,0}(t) + k\, \Lambda_{0,1,0}(t)$$

and consider $U = \{U(t) = e^{i\, E(t)},\, t \geq 0\}$. Then U is a unitary process satisfying

$$dU(t) = U(t)[\tau(\lambda, z, k)\, dt + \sum_{m=0}^{+\infty}[a_m(z, k)\, dA_m + \bar{a}_m(z, k)\, dA_m^\dagger(t)] + \sum_{0<i+j+r<+\infty} l_{i,j,r}(z, k)\, d\Lambda_{i,j,r}(t)] \tag{4.2}$$

where the coefficients $\tau(\lambda, z, k)$, $a_m(z, k)$, $\bar{a}_m(z, k)$, and $l_{i,j,r}(z, k)$ are given by

$$\tau(\lambda, z, k) = i\lambda - |z|^2/2 + \sum_{n=3}^{+\infty} \frac{i^n}{n!} \sum_{\alpha=0}^{n-2} [\sum_{\substack{j_1,\ldots,j_{n-2}\in\{-1,0,1\} \\ j_1+\ldots j_{n-2}=0}} \prod_{\epsilon=1}^{n-2} \hat{\theta}_{\epsilon, j_\epsilon}(0)](\alpha) |z|^{2(\alpha+1)} k^{n-2(\alpha+1)} \tag{4.3}$$

$$a_m(z, k) = iz + \sum_{n=2}^{+\infty} \frac{i^n}{n!} \sum_{\alpha=0}^{n-1} [\sum_{\substack{j_1,\ldots,j_{n-1}\in\{-1,0,1\} \\ m+j_1+\ldots j_{n-1}=0}} \prod_{\epsilon=1}^{n-1} \hat{\theta}_{\epsilon, j_\epsilon}(m)](\alpha) |z|^{2\alpha} z^{m+1} k^{n-2\alpha-m-1} \tag{4.4}$$

$$\bar{a}_m(z, k) = \overline{a_m(z, k)} \;\; (\text{ the complex conjugate of } a_m(z, k) \text{ }) \tag{4.5}$$

$$l_{v,j,r}(z, k) = \phi_{v,j,r}(z, k) \sum_{n=1}^{+\infty} i^n/n! \sum_1 \ldots \sum_{n-1} \prod_{s=1}^{n-1} \cdot(1 - \delta_{\epsilon_s,1}(\delta_{q_s,-1} + \delta_{q_s,0})) \cdot \phi_{v+\gamma_{n-s}-r+\sum_{\lambda=1}^{n-s} q_\lambda, \beta_{n-s}, \gamma_{n-s}}(z, k)\, \hat{c}^{\gamma_{s-1}-\delta_{q_s,1}, \rho_s, \beta_{s-1}-\omega_s-\delta_{q_s,0}\epsilon_s, \omega_s, \delta_{q_s,0}\epsilon_s}_{\beta_s, \gamma_s, \delta_{q_s,-1}, \delta_{q_s,0}} \tag{4.6}$$

where for $\xi \in \{1, 2, ..., n-1\}$

$$\sum_\xi = \sum_{\substack{q_\xi \in \{-1,0,1\},\ 0 \leq \beta_\xi \leq n-\xi \\ 0 \leq \gamma_\xi \leq n-\xi,\ 0 \leq \omega_\xi \leq \beta_\xi,\ 0 \leq \epsilon_\xi \leq 1 \\ 0 \leq \rho_\xi \leq \gamma_\xi - \gamma_{\xi-1} + \delta_{q_\xi,1}}}$$

with $\gamma_0 = r$ and $\beta_0 = j$, and

(4.7) $\hat{\theta}_{\epsilon,1}(m) = \theta_{0,0,1,m+1+\sum_{\lambda=1}^{\epsilon-1} j_\lambda} = \sqrt{(m+1+\sum_{\lambda=1}^{\epsilon-1} j_\lambda)(m+2+\sum_{\lambda=1}^{\epsilon-1} j_\lambda)}$

(4.8) $\hat{\theta}_{\epsilon,-1}(m) = \theta_{1,0,0,m-1+\sum_{\lambda=1}^{\epsilon-1} j_\lambda} = \sqrt{(m-1+\sum_{\lambda=1}^{\epsilon-1} j_\lambda)(m+\sum_{\lambda=1}^{\epsilon-1} j_\lambda)}$

(4.9) $$\hat{\theta}_{\epsilon,0}(m) = \theta_{0,1,0,m+\sum_{\lambda=1}^{\epsilon-1} j_\lambda} = 2(m+1+\sum_{\lambda=1}^{\epsilon-1} j_\lambda)$$

(4.10) $$\hat{c}^{\lambda,\rho,\sigma,\omega,\epsilon}_{\beta,\gamma,a,b} = \begin{cases} c^{\lambda,\rho,\sigma,\omega,\epsilon}_{\beta,\gamma,a,b} & \text{if } 0 \le \lambda \le \gamma, 0 \le \rho \le \gamma - \lambda, \\ & 0 \le \sigma \le \gamma - \lambda - \rho, 0 \le \omega \le \beta, 0 \le \epsilon \le b \\ 0 & \text{otherwise} \end{cases}$$

(4.11) $$\phi_{I,J,K}(z,k) = \begin{cases} \bar{z} & \text{if } I=1, J=K=0 \\ k & \text{if } J=1, I=K=0 \\ z & \text{if } K=1, I=J=0 \\ 0 & \text{otherwise} \end{cases}$$

δ denotes Kronecker's delta, $c^{\lambda,\rho,\sigma,\omega,\epsilon}_{\beta,\gamma,a,b}$ is as in Lemma 2, θ is as in Lemma 1, and the dependence on α in (4.3) (resp. (4.4)) is in the sense that α j_ϵ's are equal to 1, α (resp. $\alpha+m$) j_ϵ's are equal to -1, and $n-2\alpha-2$ (resp. $n-2\alpha-m-1$) j_ϵ's are equal to 0.

Proof. Computing the differential of $U(t)$ we find

$$\begin{aligned} dU(t) &= d(e^{i\,E(t)}) \\ &= e^{i\,E(t+dt)} - e^{i\,E(t)} \\ &= e^{i\,(E(dt)+E(t))} - e^{i\,E(t)} \\ &= e^{i\,E(dt)}\,e^{i\,E(t)} - e^{i\,E(t)} \text{ (by the commutativity of } E(dt) \text{ and } E(t)) \\ &= e^{i\,E(t)}\,[e^{i\,dE(t)} - I] \\ &= U(t)\sum_{n=1}^{\infty} \frac{(i\,dE(t))^n}{n!} \end{aligned}$$

By Proposition 1

$$dE(t)^n = \tau_n(\lambda,z,k)\,dt + \sum_{m=0}^{n-1} a_{m,n}(z,k)\,dA_m(t)$$
$$+\sum_{m=0}^{n-1} \bar{a}_{m,n}(z,k)\,dA^\dagger_m(t) + \sum_{0<i+j+k\le n} l_{i,j,k,n}(z,k)\,d\Lambda_{i,j,k}(t)$$

for some coefficients $\tau_n(\lambda,z,k)$, $a_{m,n}(z,k)$, $\bar{a}_{m,n}(z,k)$ and $l_{i,j,k,n}(z,k)$.
We will obtain recursive relations satisfied by these coefficients and by iterating these recursions we will derive explicit formulas for each one of them.
Again by Proposition 1

$$dA_{m+1}(t)\,d\Lambda_{1,0,0}(t) = \theta_{0,0,1,m+1}\,dA_m(t)$$
$$dA_m(t)\,d\Lambda_{0,1,0}(t) = \theta_{0,1,0,m}\,dA_m(t)$$
$$dA_{m-1}(t)\,d\Lambda_{0,0,1}(t) = \theta_{1,0,0,m-1}\,dA_m(t)$$

Thus, based on the right multiplication by $dE(t)$ recursive scheme,

$$dE(t)^n = dE(t)^{n-1}(t)\,dE(t)$$
$$= dE(t)^{n-1}(t)((\lambda + k)\,dt + z\,dA_0(t) + \bar{z}\,dA_0^{\dagger}(t) + z\,d\Lambda_{0,0,1}(t) + \bar{z}\,d\Lambda_{1,0,0}(t) + k\,d\Lambda_{0,1,0}(t))$$

we obtain

$$a_{m,n}(z,k) = \bar{z}\,\theta_{0,0,1,m+1}a_{m+1,n-1}(z,k) + k\,\theta_{0,1,0,m}a_{m,n-1}(z,k) + z\,\theta_{1,0,0,m-1}a_{m-1,n-1}(z,k) \tag{4.12}$$

with

$$a_{0,1}(z,k) = z \tag{4.13}$$

We can write (4.12) as

$$\begin{aligned} a_{m,n}(z,k) &= \textstyle\sum_{j_1\in\{-1,0,1\}} c_{1,j_1}a_{m+j_1,n-1}(z,k) \\ &= \textstyle\sum_{j_1,j_2\in\{-1,0,1\}} c_{1,j_1}c_{2,j_2}a_{m+j_1+j_2,n-2}(z,k) \\ &\;\;\vdots \\ &= \textstyle\sum_{j_1,\dots,j_{n-1}\in\{-1,0,1\}} c_{1,j_1}c_{2,j_2}\dots c_{n-1,j_{n-1}}a_{m+j_1+\dots+j_{n-1},1}(z,k) \end{aligned} \tag{4.14}$$

where

$$c_{1,1} = \bar{z}\,\theta_{0,0,1,m+1},\; c_{1,0} = k\,\theta_{0,1,0,m},\; c_{1,-1} = z\,\theta_{1,0,0,m-1}$$
$$c_{2,1} = \bar{z}\,\theta_{0,0,1,m+j_1+1},\; c_{2,0} = k\,\theta_{0,1,0,m+j_1},\; c_{2,-1} = z\,\theta_{1,0,0,m+j_1-1}$$
$$\vdots$$
$$c_{n-1,1} = \bar{z}\,\theta_{0,0,1,m+1+\sum_{q=1}^{n-2}j_q},\; c_{n-1,0} = k\,\theta_{0,1,0,m+\sum_{q=1}^{n-2}j_q},\; c_{n-1,-1} = z\,\theta_{1,0,0,m-1+\sum_{q=1}^{n-2}j_q}$$

In view of (4.13) we only keep $j_1,\dots,j_{n-1}$ such that $j_1+\dots+j_{n-1} = -m$ and (4.14) becomes

$$a_{m,n}(z,k) = \sum_{\substack{j_1,\dots,j_{n-1}\in\{-1,0,1\} \\ j_1+\dots+j_{n-1}=-m}} \prod_{\epsilon=1}^{n-1} c_{\epsilon,j_\epsilon}\, z \tag{4.15}$$

where

$$c_{\epsilon,1} = \bar{z}\,\theta_{0,0,1,m+1+\sum_{q=1}^{\epsilon-1}j_q},\; c_{\epsilon,0} = k\,\theta_{0,1,0,m+\sum_{q=1}^{\epsilon-1}j_q},\; c_{\epsilon-1,-1} = z\,\theta_{1,0,0,m-1+\sum_{q=1}^{\epsilon-1}j_q}$$

which can be written as

$$c_{\epsilon,1} = \bar{z}\,\hat{\theta}_{\epsilon,1}(m),\; c_{\epsilon,0} = k\,\hat{\theta}_{\epsilon,0}(m),\; c_{\epsilon,-1} = z\,\hat{\theta}_{\epsilon,-1}(m)$$

Suppose that among the $j_1,\dots,j_{n-1}$ we have α 1's, β 0's, and γ (-1)'s (corresponding to the "basic monomial" $\bar{z}^{\alpha}k^{\beta}z^{\gamma}z = \bar{z}^{\alpha}k^{\beta}z^{\gamma+1}$) where $\alpha\cdot 1+\beta\cdot 0+\gamma\cdot(-1) = -m$ i.e $\gamma = \alpha + m$. Since $\alpha+\beta+\gamma+1 = n$ it follows that $\beta = n-\alpha-\gamma-1$ and the basic monomial becomes $\bar{z}^{\alpha}k^{n-2\alpha-m-1}z^{\alpha+m+1} = |z|^{2\alpha}k^{n-2\alpha-m-1}z^{m+1}$ with coefficient (for $n \geq 2$)

$$\sum_{\alpha=0}^{n-1}[\sum_{\substack{j_1,\ldots,j_{n-1}\in\{-1,0,1\}\\ m+j_1+\ldots j_{n-1}=0}}\prod_{\epsilon=1}^{n-1}\hat{\theta}_{\epsilon,j_\epsilon}(m)](\alpha)$$

Thus, for $n \geq 2$,

$$a_{m,n}(z,k) = \sum_{\alpha=0}^{n-1}[\sum_{\substack{j_1,\ldots,j_{n-1}\in\{-1,0,1\}\\ m+j_1+\ldots j_{n-1}=0}}\prod_{\epsilon=1}^{n-1}\hat{\theta}_{\epsilon,j_\epsilon}(m)](\alpha)|z|^{2\alpha}k^{n-2\alpha-m-1}z^{m+1}$$

while

$$a_{m,1}(z,k) = \delta_{m,0}\, z$$

Thus

$$a_m(z,k) = \sum_{n=1}^{+\infty} a_{m,n}(z,k)i^n/n!$$

from which (4.4) follows.

Regarding the convergence of the above infinite series we notice that since

$$\begin{aligned}|\hat{\theta}_{\epsilon,j_\epsilon}(m)| &\leq 2(m+2+\sum_{q=1}^{\epsilon-1}1)\\ &\leq 2(m+\epsilon+1)\\ &\leq 2(m+n)\end{aligned}$$

we have that

$$\begin{aligned}&|\sum_{\alpha=0}^{n-1}[\sum_{\substack{j_1,\ldots,j_{n-1}\in\{-1,0,1\}\\ m+j_1+\ldots j_{n-1}=0}}\prod_{\epsilon=1}^{n-1}\hat{\theta}_{\epsilon,j_\epsilon}(m)](\alpha)|z|^{2\alpha}k^{n-2\alpha-m-1}z^{m+1}|\\ &\leq \sum_{\alpha=0}^{n-1}[\sum_{\substack{j_1,\ldots,j_{n-1}\in\{-1,0,1\}\\ m+j_1+\ldots j_{n-1}=0}}\prod_{\epsilon=1}^{n-1}|\hat{\theta}_{\epsilon,j_\epsilon}(m)|](\alpha)|z|^{2\alpha}|k|^{n-2\alpha-m-1}|z|^{m+1}\\ &\leq \sum_{\alpha=0}^{n-1}[\sum_{\substack{j_1,\ldots,j_{n-1}\in\{-1,0,1\}\\ m+j_1+\ldots j_{n-1}=0}}\prod_{\epsilon=1}^{n-1}|\hat{\theta}_{\epsilon,j_\epsilon}(m)|](\alpha)\ max(|z|,|k|)^n\\ &\leq n\,3^{n-1}2^{n-1}(m+n)^{n-1}\ max(|z|,|k|)^n\end{aligned}$$

and so, for each m, the ratio test yields

$$\begin{aligned}&\tfrac{|a_{m,n+1}(z,k)i^{n+1}/(n+1)!|}{|a_{m,n}(z,k)i^n/n!|}\\ &= \tfrac{6}{n}\tfrac{(m+n+1)^n}{(m+n)^{n-1}}\ max(|z|,|k|)\\ &= \tfrac{6(m+n)}{n}[(1+\tfrac{1}{m+n})^{m+n}]^{\frac{n}{m+n}}\ max(|z|,|k|)\\ &\to 6\,e\ max(|z|,|k|) < 1\end{aligned}$$

as $n \to +\infty$, provided that $max(|z|,|k|) < \frac{1}{6\,e}$.

As for (4.3), letting $\tau_n(\lambda,z,k)$ denote the coefficient of dt in dE^n, for $n \geq 2$ Proposition 1 implies

$$\tau_n(\lambda, z, k) = a_{0,n-1}(z,k)\,\bar{z} \tag{4.16}$$

with

$$\tau_1(\lambda, z, k) = \lambda \tag{4.17}$$

and since

$$\tau(\lambda, z, k) = \textstyle\sum_{n=1}^{+\infty} \tau_n(\lambda, z, k) i^n/n!$$

(4.3) follows from (4.4) which has already been proved.
Turning to (4.5), we notice that by Proposition 1

$$d\Lambda_{a,b,s}(t)\, dA_0^\dagger = \theta_{a,b,s,0}\, dA_{a-s}^\dagger$$

Letting $a - s = m$ the above becomes

$$d\Lambda_{s+m,b,s}(t)\, dA_0^\dagger = \theta_{s+m,b,s,0}\, dA_m^\dagger$$

from which we obtain the recursion

$$\bar{a}_{m,n}(z,k) = \bar{z} \textstyle\sum_{\substack{s,b \in \{0,1,\ldots,n-1\} \\ 0<2s+b+m \le n-1}} l_{s+m,b,s,n-1}(z,k)\, \theta_{s+m,b,s,0} \tag{4.18}$$

Though computationally useful the above recursion does not reveal the fact that $\bar{a}_m(z,k) = \overline{a_m(z,k)}$. To establish that we proceed as in the proof of (4.4) but this time using the left multiplication by $dE(t)$ recursive scheme

$$\begin{gathered} dE(t)^n = dE(t)\, dE(t)^{n-1}(t) \\ = ((\lambda + k)\, dt + z\, dA_0(t) + \bar{z}\, dA_0^\dagger(t) + z\, d\Lambda_{0,0,1}(t) + \bar{z}\, d\Lambda_{1,0,0}(t) \\ +k\, d\Lambda_{0,1,0}(t)) dE(t)^{n-1}(t) \end{gathered}$$

along with the following consequences of Proposition 1

$$\begin{gathered} d\Lambda_{0,0,1}(t)\, dA_{m+1}^\dagger(t) = \theta_{0,0,1,m+1}\, dA_m^\dagger(t) \\ d\Lambda_{0,1,0}(t)\, dA_m^\dagger(t) = \theta_{0,1,0,m}\, dA_m^\dagger(t) \\ d\Lambda_{1,0,0}(t)\, dA_{m-1}^\dagger(t) = \theta_{1,0,0,m-1}\, dA_m^\dagger(t) \end{gathered}$$

to obtain

$$\begin{gathered} \bar{a}_{m,n}(z,k) = z\, \theta_{0,0,1,m+1} \bar{a}_{m+1,n-1}(z,k) + k\, \theta_{0,1,0,m} \bar{a}_{m,n-1}(z,k) \\ +\bar{z}\, \theta_{1,0,0,m-1} \bar{a}_{m-1,n-1}(z,k) \end{gathered} \tag{4.19}$$

with

$$\bar{a}_{0,1}(z,k) = \bar{z} \tag{4.20}$$

which are the complex conjugates of (4.12) and (4.13) respectively.

To prove (4.6) we notice that in the notation of Proposition 1

$$d\Lambda_{\alpha,\beta,\gamma}(t)\,d\Lambda_{1,0,0}(t)=\sum c^{\lambda,\rho,\sigma,\omega,\epsilon}_{\beta,\gamma,1,0}\,d\Lambda_{1+\alpha-\gamma+\lambda,\omega+\sigma+\epsilon,\lambda}(t)$$
$$d\Lambda_{\alpha,\beta,\gamma}(t)\,d\Lambda_{0,1,0}(t)=\sum c^{\lambda,\rho,\sigma,\omega,\epsilon}_{\beta,\gamma,0,1}\,d\Lambda_{\alpha-\gamma+\lambda,\omega+\sigma+\epsilon,\lambda}(t)$$
$$d\Lambda_{\alpha,\beta,\gamma}(t)\,d\Lambda_{0,0,1}(t)=\sum c^{\lambda,\rho,\sigma,\omega,\epsilon}_{\beta,\gamma,0,0}\,d\Lambda_{\alpha-\gamma+\lambda,\omega+\sigma+\epsilon,\lambda+1}(t)$$

Thus

$$\begin{aligned} l_{i,j,r,n}&=\bar z\sum \hat c^{r,\rho,j-\omega,\omega,0}_{\beta,\gamma,1,0}\,l_{i+\gamma-r-1,\beta,\gamma,n-1}\\ &+k\sum \hat c^{r,\rho,j-\omega-\epsilon,\omega,\epsilon}_{\beta,\gamma,0,1}\,l_{i+\gamma-r,\beta,\gamma,n-1}\\ &+z\sum \hat c^{r-1,\rho,j-\omega,\omega,0}_{\beta,\gamma,0,0}\,l_{i+\gamma-r+1,\beta,\gamma,n-1}\\ &=\phi_{i,j,r}(z,k)\,\sum\nolimits_1 (1-\delta_{\epsilon_1,1}(\delta_{q_1,-1}+\delta_{q_1,0}))\\ &\cdot\hat c^{r-\delta_{q_1,1},\rho_1,j-\omega_1-\delta_{q_1,0}\epsilon_1,\omega_1,\delta_{q_1,0}\epsilon_1}_{\beta_1,\gamma_1,\delta_{q_1,-1},\delta_{q_1,0}}\,l_{i+\gamma_1-r+q_1,\beta_1,\gamma_1,n-1}\end{aligned}\tag{4.21}$$

with

$$l_{1,0,0,1}=\bar z\tag{4.22}$$
$$l_{0,1,0,1}=k\tag{4.23}$$
$$l_{0,0,1,1}=z\tag{4.24}$$

or equivalently

$$l_{i,j,r,1}=\phi_{i,j,r}\tag{4.25}$$

Iterating (4.21), using (4.25) in the last step, we obtain

$$\begin{aligned} l_{i,j,r,n}&=\phi_{i,j,r}(z,k)\,\sum\nolimits_1\sum\nolimits_2\,\phi_{i+\gamma_1-r+q_1,\beta_1,\gamma_1}(z,k)\\ &\cdot(1-\delta_{\epsilon_1,1}(\delta_{q_1,-1}+\delta_{q_1,0}))\,(1-\delta_{\epsilon_2,1}(\delta_{q_2,-1}+\delta_{q_2,0}))\\ &\cdot\hat c^{r-\delta_{q_1,1},\rho_1,j-\omega_1-\delta_{q_1,0}\epsilon_1,\omega_1,\delta_{q_1,0}\epsilon_1}_{\beta_1,\gamma_1,\delta_{q_1,-1},\delta_{q_1,0}}\cdot\hat c^{\gamma_1-\delta_{q_2,1},\rho_2,\beta_1-\omega_2-\delta_{q_2,0}\epsilon_2,\omega_2,\delta_{q_2,0}\epsilon_2}_{\beta_2,\gamma_2,\delta_{q_2,-1},\delta_{q_2,0}}\\ &\cdot l_{i+\gamma_2-r+q_1+q_2,\beta_2,\gamma_2,n-2}\\ &=\ldots=\phi_{i,j,r}(z,k)\sum\nolimits_1\ldots\sum\nolimits_{n-1}\prod_{s=1}^{n-1}(1-\delta_{\epsilon_s,1}(\delta_{q_s,-1}+\delta_{q_s,0}))\\ &\cdot\phi_{i+\gamma_{n-s}-r+\sum_{\lambda=1}^{n-s}q_\lambda,\beta_{n-s},\gamma_{n-s}}(z,k)\,\hat c^{\gamma_{s-1}-\delta_{q_s,1},\rho_s,\beta_{s-1}-\omega_s-\delta_{q_s,0}\epsilon_s,\omega_s,\delta_{q_s,0}\epsilon_s}_{\beta_s,\gamma_s,\delta_{q_s,-1},\delta_{q_s,0}}\end{aligned}$$

from which we obtain (4.6).
The convergence of the series in (4.6) is proved as before.

□

5. **Further Remarks**

Remark 2. *Let the coefficients A_i, $i=1,2,3,4$ of the quantum stochastic differential equation*

$$dU(t) = (A_1\, dt + A_2\, dB_t^- + A_3\, dB_t^+ + A_4\, dM_t)U(t) \tag{5.1}$$
$$U(0) = I$$

be time independent bounded operators on the system space. Then conditions (3.3)-(3.8) are satisfied if and only if, denoting real part by $\Re$*,*

$$\Re A_1 = A_2 = A_3 = A_4 = 0. \tag{5.2}$$

Therefore (5.1) admits a unitary solution if and only if its coefficients satisfy (5.2).

Proof. In view of (2.6)-(2.8), (5.1) can be written as

$$dU(t) = ((A_1 + A_4)\, dt + A_2\, dA_0(t) + A_3\, dA_0^+(t) + A_2\, d\Lambda_{0,0,1}(t)$$
$$+A_3\, d\Lambda_{1,0,0}(t) + A_4\, d\Lambda_{0,1,0}(t))U(t),\ U(0) = I$$

which is of the form of (3.1) with

$$A = A_1 + A_4,\ C_0 = A_2,\ D_0 = A_3,\ B_{0,0,1} = A_2,\ B_{1,0,0} = A_3,\ B_{0,1,0} = A_4$$

Attempting to satisfy (3.3)-(3.8) we find that for $(n, k, l) = (0, 2, 0)$, (3.7) implies

$$B_{0,1,0}B^*_{0,1,0} = A_4A_4^* = 0$$

i.e $A_4 = 0$. Similarly, for $(n, k, l) = (1, 0, 1)$, (3.7) and (3.8) imply

$$A_3A_3^* = 0$$
$$A_2A_2^* = 0$$

i.e $A_2 = A_3 = 0$. Finally by (3.3) $\Re A_1 = 0$. Thus (5.1) reduces to

$$dU(t) = iH\, dt\, U(t),\ U(0) = I$$

where H is self-adjoint, with solution $U(t) = e^{iHt}$. □

Remark 3. *The quantum stochastic differential equation*

$$dU(t) = (a_0\, dt + a_1\, dA_0(t) + a_2\, dA_0^\dagger(t)$$
$$+a_3\, d\Lambda_{0,0,0,}(t) + a_4\, dB_t^- + a_5\, dB_t^+ + a_6\, dM_t)U(t) \tag{5.3}$$
$$U(0) = I$$

containing first and second order white noise terms with (as in Remark 3) constant operator coefficients a_i*,* $i = 0, 1, 2, ...$*, does not admit a unitary solution unless* $a_4 = a_5 = a_6 = 0$ *in which case it is reduced to a standard, Hudson-Parthasarathy type, first order white noise quantum stochastic differential equation (see* [15]*).*

Proof. For $(n,k,l)=(1,0,1)$ (3.7) and (3.8) imply

$$a_5a_5^*=0$$
$$a_4a_4^*=0$$

i.e $a_5=a_4=0$.
Similarly, for $(n,k,l)=(2,0,0)$ (3.7) implies

$$a_6a_6^*=0$$

i.e $a_6=0$.

□

Remark 4. *Unitarity conditions (3.3)-(3.8) contain those of* [15] *for quantum stochastic differential equations with infinite degrees of freedom.*

Proof. Solving (3.5) we find

$$C_m=-\sum_{n,k,l}D^*_{m+n-l}S_{n,k,l,m} \tag{5.4}$$

where for all n,k,l,m

$$S_{n,k,l,m}=\delta_{n,0}\delta_{k,0}\delta_{l,0}+\theta_{l,k,n,m+n-l}B_{n,k,l} \tag{5.5}$$

Thus, for all n,k,l,m such that $\theta_{l,k,n,m+n-l}\neq 0$,

$$B_{n,k,l}=\frac{S_{n,k,l,m}-\delta_{n,0}\delta_{k,0}\delta_{l,0}}{\theta_{l,k,n,m+n-l}} \tag{5.6}$$

Since $\theta_{l,k,n,n-l}=0$, (5.5) implies that for all n,k,l

$$S_{n,k,l,0}=\delta_{n,0}\delta_{k,0}\delta_{l,0} \tag{5.7}$$

Letting $\hat{\theta}_{n,k,l,m}=\frac{\theta_{n,k,l,m+l-n}}{\theta_{l,k,n,m+n-l}}$ and substituting (5.4) in the left hand side of (3.6) we obtain

$$\begin{aligned}
&-\textstyle\sum_{n,k,l}D^*_{m+n-l}S_{n,k,l,m}+\sum_{n,k,l}D^*_{m+n-l}\delta_{n,0}\delta_{k,0}\delta_{l,0}+\\
&\textstyle\sum_{n,k,l}(-\sum_{N,K,L}D^*_{m+l-n+N-L}S_{N,K,L,m})\hat{\theta}_{n,k,l,m}(S^*_{n,k,l,m}-\delta_{n,0}\delta_{k,0}\delta_{l,0})\\
&=D^*_m-D^*_m\textstyle\sum_{\substack{n,k,l,N,K,L\\ l-n+N-L=0}}\hat{\theta}_{n,k,l,m}S_{N,K,L,m}S^*_{n,k,l,m}\\
&-\textstyle\sum_{\substack{n,k,l,N,K,L\\ l-n+N-L\neq 0}}\hat{\theta}_{n,k,l,m}D^*_{m+l-n+N-L}S_{N,K,L,m}S^*_{n,k,l,m}
\end{aligned}$$

which is equal to zero if

$$\sum_{\substack{n,k,l,N,K,L\\ l-n+N-L=0}}\hat{\theta}_{n,k,l,m}S_{N,K,L,m}S^*_{n,k,l,m}=I \tag{5.8}$$

and

(5.9)
$$\sum_{\substack{n,k,l,N,K,L\\ l-n+N-L\neq 0}} \hat{\theta}_{n,k,l,m} D^*_{m+l-n+N-L} S_{N,K,L,m} S^*_{n,k,l,m} = 0$$

Assuming that

(5.10)
$$S_{n,k,l,m} = 0$$

for all n, k, l, m with $n \neq l$, we find that (5.8) and (5.9) are both satisfied. In fact, since $\hat{\theta}_{n,k,n,m} = 1$, (5.8) reduces to the unitarity of $\sum_{n,k} S_{n,k,n,m}$ for all m, which is basically the Parthasarathy condition for quantum stochastic differential equations with infinite degrees of freedom [15]. However (5.10) excludes the existence of square of white noise terms in (3.1) since it implies, using (5.5), that $B_{n,k,l} = 0$ for all n, k, l with $n \neq l$.

□

Remark 5 (The first order Poisson-Weyl operator). *Let*

$$U(t) = e^{iE(t)}$$

where

$$E(t) = \lambda t + zA_0(t) + \overline{z}A_0^+(t) + k\Lambda_{0,0,0}(t)$$

with $\lambda, k \in \mathbb{R}$, $z \in \mathbb{C}$.
(a) If $k \neq 0$ *then*

$$\begin{aligned} dU(t) = U(t)[(i\lambda + \tfrac{|z|^2}{k^2} M)dt + (iz + \tfrac{z}{k} M)dA_0(t) \\ +(i\overline{z} + \tfrac{\overline{z}}{k} M)dA_0^+(t) + (ik + M)d\Lambda_0(t)] \end{aligned}$$

where

$$M = e^{ik} - 1 - ik$$

(b) If $k = 0$ *then*

$$dU(t) = U(t)[(i\lambda - \frac{|z|^2}{2})dt + izdA_0(t) + i\overline{z}dA_0^+(t)]$$

Proof. The proof is similar to that of Proposition 4 using the fact that for $k \neq 0$ and $n \geq 2$

$$dE(t)^n = |z|^2 k^{n-2} dt + zk^{n-1} dA_0(t) + \overline{z}k^{n-1} dA_0^+(t) + k^n d\Lambda_{0,0,0}(t)$$

while for $k = 0$

$$dE(t)^2 = |z|^2 \, dt$$

and for $n > 2$

$$dE(t)^n = 0$$

□

The above two equations are of the Hudson-Parthasarathy form (see [15]), namely,

$$dU(t) = U(t)[(iH - \frac{1}{2}\, L^*L)dt - L^*W dA_0(t) + L dA_0^+(t) + (W - I)d\Lambda_{0,0,0}(t)]$$

with

$$W = e^{ik} I$$

$$L = \frac{\overline{z}}{k}\,(e^{ik} - 1)I$$

$$H = (\lambda - \frac{|z|^2}{k} - \frac{i}{2}\,\frac{|z|^2}{k^2}\,[e^{ik} - e^{-ik} - 1])I$$

and

$$H = \lambda I,\; L = i\overline{z} I,\; W = -I$$

respectively.

6. **Computer algebra software algorithms**

Some of the results contained in this paper would have been very hard to obtain without the use of computer algorithms for symbolic calculations and noncommutative iterations. The computer algebra software that we used was *Mathematica 4* (see [16] for operational instuctions). A typical example of what the computer was helpful in deriving is the unitary quantum stochastic differential equation of Proposition 4.

Following are the algorithms that we used in order to derive, verify, or develop intuition for some of the results contained the previous sections.

Algorithm 1 (The Ito-table for the SWN differentials)**.** *This algorithm computes the, in general, noncommutative products of the generalized SWN stochastic differentials* $d\Lambda_{n,k,l}(t)$*,* $dA_m(t)$ *and* $dA_m^\dagger(t)$*, where* $n, k, l, m = 0, 1, ...,$ *and "time"* dt*. Each sentence corresponds to a new input. Inputs are separated by space. The first three inputs establish the notation for the SWN stochastic differentials:*

$d\Lambda[a\_, b\_, c\_]$

$dA[m\_]$

$dA^\dagger[m\_]$

dt

$p[x\_, y\_] = If\,[x == y == 0, 1, x\hat{\ }y]$

$u[x\_, n\_] = Product[x - i + 1, \{i, 1, n\}]$

$v[x\_, n\_] = Product[x + i - 1, \{i, 1, n\}]$

$\theta[n\_, k\_, l\_, m\_] = If\,[n + m - 1 <$
$0, 0, Sqrt[(m - l + n + 1)/(m + 1)2\hat{\ }k\, v[m - l + 1, n]\, u[m + 1, l]\, p[m - l + 1, k]]$

$c[\beta\_, \gamma\_, a\_, b\_, \lambda\_, \rho\_, \sigma\_, \omega\_, \epsilon\_] =$
$Binomial[\gamma, \lambda]\, Binomial[\gamma - \lambda, \rho]\, Binomial[\beta, \omega]\, Binomial[b, \epsilon]\, 2\hat{\ }(\beta + b - \omega -$
$\epsilon)\, StirlingS1[\gamma - \lambda - \rho, \sigma]\, u[a, \gamma - \lambda]\, u[a + \lambda - 1, \rho]\, p[a - \gamma + \lambda, \beta - \omega]\, p[\lambda, b - \epsilon]$

$Unprotect[NonCommutativeMultiply]$

$NonCommutativeMultiply[d\Lambda[\alpha\_, \beta\_, \gamma\_], d\Lambda[a\_, b\_, s\_]] =$
$Sum[c[\beta, \gamma, a, b, \lambda, \rho, \sigma, \omega, \epsilon]\, d\Lambda[a + \alpha - \gamma + \lambda, \omega + \sigma + \epsilon, \lambda + s], \{\lambda, 0, \gamma\}, \{\rho, 0, \gamma - \lambda\}, \{\omega, 0, \beta\}, \{\epsilon, 0, b\}]$

$NonCommutativeMultiply[d\Lambda[a\_, b\_, c\_], dA^{\dagger}[m\_]] = \theta[a, b, c, m] dA^{\dagger}[a + m - c]$

$NonCommutativeMultiply[dA[m\_], d\Lambda[a\_, b\_, c\_]] = \theta[c, b, a, m] dA[c + m - a]$

$NonCommutativeMultiply[dA[m\_], dA^{\dagger}[n\_]] = KroneckerDelta[m, n] dt$

$NonCommutativeMultiply[dA[m\_], dA[n\_]] = 0$

$NonCommutativeMultiply[dA^{\dagger}[m\_], dA^{\dagger}[n\_]] = 0$

$NonCommutativeMultiply[dA^{\dagger}[m\_], dA[n\_]] = 0$

$NonCommutativeMultiply[dA^{\dagger}[m\_], d\Lambda[\alpha\_, \beta\_, \gamma\_]] = 0$

$NonCommutativeMultiply[d\Lambda[\alpha\_, \beta\_, \gamma\_], dA[m\_]] = 0$

$NonCommutativeMultiply[d\Lambda[\alpha\_, \beta\_, \gamma\_], dt] = 0$

$NonCommutativeMultiply[dt, d\Lambda[\alpha\_, \beta\_, \gamma\_]] = 0$

$NonCommutativeMultiply[dA[m\_], dt] = 0$

$NonCommutativeMultiply[dt, dA[m\_]] = 0$

$NonCommutativeMultiply[dA^{\dagger}[m\_], dt] = 0$

$NonCommutativeMultiply[dt, dA^{\dagger}[m\_]] = 0$

$NonCommutativeMultiply[dt, dt] = 0$

For example, using the above algorithm to compute $d\Lambda_{4,1,2}(t)\, d\Lambda_{1,2,2}(t)$ we obtain

NonCommutativeMultiply$[d\Lambda[4, 1, 2], d\Lambda[1, 2, 1]] = 8d\Lambda[4, 1, 2] + 16d\Lambda[4, 2, 2] + 10d\Lambda[4, 3, 2] + 2d\Lambda[4, 4, 2] + 32d\Lambda[5, 0, 3] + 32d\Lambda[5, 1, 3] + 10d\Lambda[5, 2, 3] + d\Lambda[5, 3, 3]$

while for $d\Lambda_{4,2,1}(t)\, dA^{\dagger}_2(t)$ we obtain

NonCommutativeMultiply$[d\Lambda[4, 2, 1], dA^{\dagger}[2]] = 5760\sqrt{2}\, dA^{\dagger}[5]$

Algorithm 2 (Powers of the SWN Poisson-Weyl operator differential)**.** *To compute $dE(t)^n$, where $E(t)$ is the SWN Poisson-Weyl operator of Proposition 4 and $n = 2, 3, ...$ (the value of n must be supplied by the user), we use Algorithm 1 with the following commands attached to it:*

$NonCommutativeMultiply[0, x\_] = NonCommutativeMultiply[x\_, 0] = 0$

$NonCommutativeMultiply[(x\_\ y\_), z\_] = x\, NonCommutativeMultiply[y, z]$

$NonCommutativeMultiply[w\_\ d\Lambda[a\_, b\_, s\_], q\_\ d\Lambda[d\_, h\_, f\_]] =$
$w\, q\, NonCommutativeMultiply[d\Lambda[a, b, s], d\Lambda[d, h, f]]$

$NonCommutativeMultiply[w\_\ d\Lambda[a\_, b\_, s\_], q\_\ dA^{\dagger}[m\_]] =$
$w\, q\, NonCommutativeMultiply[d\Lambda[a, b, s], dA^{\dagger}[m]]$

$NonCommutativeMultiply[w\_\ d\Lambda[a\_, b\_, s\_], q\_\ dA[m\_]] = 0$

$NonCommutativeMultiply[q\_\; dA^\dagger[m\_], w\_\; d\Lambda[a\_, b\_, s\_]] = 0$

$NonCommutativeMultiply[q\_\; dA[m\_], w\_\; d\Lambda[a\_, b\_, s\_]] =$
$q\, w\; NonCommutativeMultiply[dA[m], d\Lambda[a, b, s]]$

$NonCommutativeMultiply[w\_\; dA^\dagger[m\_], q\_\; dA^\dagger[r\_]] = 0$

$NonCommutativeMultiply[q\_\; dA[r\_], w\_\; dA^\dagger[m\_]] =$
$q\, w\; NonCommutativeMultiply[dA[r], dA^\dagger[m]]$

$NonCommutativeMultiply[q\_\; dA[r\_], w\_\; dA[m\_]] =$
$q\, w\; NonCommutativeMultiply[dA[r], dA[m]]$

$NonCommutativeMultiply[q\_\; dA[r\_], w\_\; dt] = 0$

$NonCommutativeMultiply[q\_\; dA^\dagger[r\_], w\_\; dt] = 0$

$NonCommutativeMultiply[q\_\; d\Lambda[a\_, b\_, s\_], w\_\; dt] = 0$

$NonCommutativeMultiply[dt\, w\_, q\_\; dA[r\_]] = 0$

$NonCommutativeMultiply[dt\, w\_, q\_\; dA^\dagger[r\_]] = 0$

$NonCommutativeMultiply[dt\, w\_, q\_\; d\Lambda[a\_, b\_, s\_]] = 0$

$NonCommutativeMultiply[dt\, w\_, dt\, q\_] = 0$

$dE[1] = (\lambda + k)\, dt + z\, dA[0] + \bar{z}\, dA^\dagger[0] + \bar{z}\, d\Lambda[1, 0, 0] + k\, d\Lambda[0, 1, 0] + z\, d\Lambda[0, 0, 1]$

$n =$

$Do[Print[StringForm["dE"]\hat{}i, StringForm[" = "], dE[i] =$
$Collect\ [Expand\ [MapAll\ [Distribute,\ NonCommutativeMultiply\ [dE[i -$
$1], dE[1]]]], \{dt, dA[\_], dA^\dagger[\_], d\Lambda[\_, \_, \_]\}]], \{i, 2, n\}]$

For example, running the algorithm for $n = 2$ we obtain

$$\begin{gathered} dE^2(t) = z\bar{z}\, dt + 2kz\, dA_0(t) + \sqrt{2}z^2\, dA_1(t) + 2k\bar{z}\, dA_0^\dagger(t) + \sqrt{2}\bar{z}^2\, dA_1^\dagger(t) \\ +2kz\, d\Lambda_{0,0,1}(t) + z^2\, d\Lambda_{0,0,2}(t) + 2kz\, d\Lambda_{0,1,1}(t) + k^2\, d\Lambda_{0,2,0}(t) + z\bar{z}\, d\Lambda_{0,1,0}(t) \\ 2k\bar{z}\, d\Lambda_{1,0,0}(t) + 2z\bar{z}\, d\Lambda_{1,0,1}(t) + 2k\bar{z}\, d\Lambda_{1,1,0}(t) + \bar{z}^2\, d\Lambda_{2,0,0}(t) \end{gathered}$$

Algorithm 3 (SWN Poisson-Weyl recursions)**.** *This algorithm uses recursions (4.12), (4.13), (4.16),(4.17),(4.19)-(4.24) of Proposition 4 to compute the coefficients of* dt, $dA_m(t)$, $dA_m^\dagger(t)$, *and* $d\Lambda_{i,j,h}(t)$ *in* $dE(t)^n$, *denoted respectively by* $\tau[n\_]$, $\alpha[m\_, n\_]$, $\alpha^\dagger[m\_, n\_]$, *and* $f[i\_, j\_, h\_, n\_]$. *It also computes the* m*-th partial sum* $\sum_{n=1}^m dE^n/n!$ *where the value of* m *must be provided by the user.*

$p[x\_, y\_] = If[x == y == 0, 1, x\hat{}y]$

$upperfact[x\_, n\_] = Product[x - i + 1, \{i, 1, n\}]$

$lowerfact[x\_, n\_] = Product[x + i - 1, \{i, 1, n\}]$

$\theta[n\_, h\_, l\_, m\_] = If[n + m - l < 0, 0, Sqrt[(m - l + n + 1)/(m + 1)]$
$2\hat{}h\ lowerfact[m - l + 1, n]\ upperfact[m + 1, l]\ p[m - l + 1, h]]$

$c[\beta\_, \gamma\_, a\_, b\_, \lambda\_, \rho\_, \sigma\_, \omega\_, \epsilon\_] =\ If[0 \le \lambda \le \gamma \&\& 0 \le \rho \le \gamma - \lambda$
$\&\& 0 \le \sigma \le \gamma - \lambda - \rho \&\& 0 \le \omega \le \beta \&\& 0 \le \epsilon \le b, Binomial[\gamma, \lambda]\ Binomial[\gamma - \lambda, \rho]$
$Binomial[\beta, \omega]\ Binomial[b, \epsilon]\ 2\hat{}(\beta + b - \omega - \epsilon)\ StirlingS1[\gamma - \lambda - \rho, \sigma]\ upperfact[a, \gamma - \lambda]$
$upperfact[a + \lambda - 1, \rho]\ p[a - \gamma + \lambda, \beta - \omega]\ p[\lambda, b - \epsilon], 0]$

$\alpha[m\_, 0] = 0$

$\alpha[m\_, 1] = \mathit{If}[m == 0, z, 0]$

$\alpha[m\_, n\_] = \mathit{If}[m > n, 0, \mathit{Collect}[\bar{z}\ \mathit{Sqrt}[(m+1)\,(m+2)]\,\alpha[m+1, n-1]+$
$k\,2\,(m+1)\,\alpha[m, n-1] + z\ \mathit{Sqrt}[m\,(m+1)]\,\alpha[m-1, n-1], \{z, \bar{z}, k\}]]$

$\tau[1] = \lambda + k$

$\tau[n\_] = \mathit{Collect}[\bar{z}\ \alpha[0, n-1], \{\lambda + k, z, \bar{z}, k\}]$

$f[1, 0, 0, 1] = \bar{z}$

$f[0, 1, 0, 1] = k$

$f[0, 0, 1, 1] = z$

$f[i\_, j\_, h\_, n\_] = \mathit{If}[i + j + h > n || i + j + h == 0 || i < 0 || j < 0 || h < 0,$
$0, \bar{z}\,\mathit{Sum}[f[i + \gamma - h - 1, \beta, \gamma, n-1]\ c[\beta, \gamma, 1, 0, h, \rho, j - \omega, \omega, 0],$
$\{\gamma, 0, n-1\}, \{\beta, 0, n-1\}, \{\omega, 0, \beta\}, \{\rho, 0, \gamma - h\}]+$
$k\,\mathit{Sum}[f[i + \gamma - h, \beta, \gamma, n-1]\ c[\beta, \gamma, 0, 1, h, \rho, j - \omega - \epsilon, \omega, \epsilon],$
$\{\gamma, 0, n-1\}, \{\beta, 0, n-1\}, \{\omega, 0, \beta\}, \{\rho, 0, \gamma - h\}, \{\epsilon, 0, 1\}]+$
$z\,\mathit{Sum}[f[i + \gamma - h + 1, \beta, \gamma, n-1]\ c[\beta, \gamma, 0, 0, h-1, \rho, j - \omega, \omega, 0],$
$\{\gamma, 0, n-1\}, \{\beta, 0, n-1\}, \{\omega, 0, \beta\}, \{\rho, 0, \gamma - h + 1\}]]$

$\mathit{Unprotect}[\mathit{Power}]$

$\alpha^\dagger[m\_, 0] = 0$

$\alpha^\dagger[m\_, 1] = \mathit{If}[m == 0, \bar{z}, 0]$

$\alpha^\dagger[m\_, n\_] = \mathit{If}[m > n, 0, \mathit{Collect}[z\ \mathit{Sqrt}[(m+1)\,(m+2)]\,\alpha^\dagger[m+1, n-1]+$
$k\,2\,(m+1)\,\alpha^\dagger[m, n-1] + \bar{z}\ \mathit{Sqrt}[m\,(m+1)]\,\alpha^\dagger[m-1, n-1], \{z, \bar{z}, k\}]]$

$\mathit{Unprotect}[N]$

$M =$

$\mathit{partialsum}[M\_] = \mathit{Collect}[\mathit{Sum}[I\hat{\ }N\,/\,N!\,\tau[N]\,dt+$
$\mathit{Expand}[I\hat{\ }N\,/\,N!\ \mathit{Sum}[\alpha[m, N]\,dA[m] + \alpha^\dagger[m, N]\,dA^\dagger[m],\ \{m, 0, N-1\}] + I\hat{\ }N\,/\,N!$
$\mathit{Sum}[f[i, j, h, N]\,d\Lambda[i, j, h], \{i, 0, N\}, \{j, 0, N\}, \{h, 0, N\}]], \{N, 1, M\}], \{dt, dA[\_],$
$dA^\dagger[\_], d\Lambda[\_, \_, \_]\}]$

For example using the above algorithm we obtain

$$\alpha[2,3] = 2\sqrt{3}\, z^3$$

$$\alpha^\dagger[0,4] = 8\, k^3 \bar{z} + 16\, k z \bar{z}^2$$

$$\tau[8] = 64\, k^6 z\bar{z} + 1824\, k^4 z^2 \bar{z}^2 + 2880\, k^2 z^3 \bar{z}^3 + 272\, z^4 \bar{z}^4$$
$$f[1,0,1,3] = 12\, k z \bar{z}$$

$$\begin{gathered}\textstyle\sum_{n=1}^{2} dE^n/n! = i(k+\lambda)\, dt + (i-k)z\, dA[0] - \frac{1}{\sqrt{2}} z^2\, dA[1] \\ +(i-k)\bar{z}\, dA^\dagger[0] - \frac{1}{\sqrt{2}} \bar{z}^2\, dA^\dagger[1] \\ +(i-k)z\, d\Lambda[0,0,1] - \frac{1}{2}\, z^2\, d\Lambda[0,0,2] - kz\, d\Lambda[0,1,1] - \frac{1}{2}\, k^2\, d\Lambda[0,2,0] \\ -z\bar{z}\, d\Lambda[1,0,1] - k\bar{z}\, d\Lambda[1,1,0] - \frac{1}{2}\, \bar{z}^2\, d\Lambda[2,0,0] \\ +(i-k)\bar{z}\, d\Lambda[1,0,0] + (ik - \frac{1}{2}\, z\bar{z})\, d\Lambda[0,1,0]\end{gathered}$$

Algorithm 4 (Unitarity Conditions)**.** *This algorithm checks unitarity conditions (3.3)-(3.8) of Proposition 2 for specific coefficient processes. We present here the classical example of* [15] *where* $A = iH - \frac{1}{2}\, L^* L$*,* $C_0 = -L^* W$*,* $D_0 = L$*,* $B_{0,0,0} = W - 1$*, with* L *self-adjoint and* W *unitary, and all other coefficients are zero. Part of the algorithm deals with coding self-adjointness and unitarity for* L *and* W *respectively. For different examples different or additional commands may be needed:*

$Unprotect[NonCommutativeMultiply]$

$NonCommutativeMultiply[NonCommutativeMultiply[a\_, b\_],$
$NonCommutativeMultiply[c\_, d\_]] = NonCommutativeMultiply[a, b, c, d]$

$NonCommutativeMultiply[0, 0] = 0$

$NonCommutativeMultiply[a\_, 1] = NonCommutativeMultiply[1, a\_] = a$

$NonCommutativeMultiply[a\_, -1] = NonCommutativeMultiply[-1, a\_] = -a$

$NonCommutativeMultiply[(-1)a\_, (-1)b\_] = NonCommutativeMultiply[a, b]$

$NonCommutativeMultiply[0, a\_] = NonCommutativeMultiply[a\_, 0] = 0$

$NonCommutativeMultiply[(-1)a\_, b\_] = NonCommutativeMultiply[a\_, (-1)b\_] =$
$-NonCommutativeMultiply[a, b]$

$A = IH - 1/2\ NonCommutativeMultiply[L^*, L]$

$A^* = -IH - 1/2\ NonCommutativeMultiply[L^*, L]$

$Unprotect[C]$

$C[m\_] = If[m == 0, -NonCommutativeMultiply[L^*, W], 0]$

$C^*[m\_] = If[m == 0, -NonCommutativeMultiply[W^*, L], 0]$

$Unprotect[D]$

$D[m\_] = If[m == 0, L, 0]$

$D^*[m\_] = If[m == 0, L^*, 0]$

$B[n\_, k\_, l\_] = If[n == k == l == 0, W - 1, 0]$

$B^*[n\_, k\_, l\_] = If[n == k == l == 0, W^* - 1, 0]$

$NonCommutativeMultiply[W^*, W] = NonCommutativeMultiply[W, W^*] = 1$

$p[x\_, y\_] = If[x == y == 0, 1, x\hat{}y]$

$u[x\_, n\_] = Product[x - i + 1, \{i, 1, n\}$

$v[x\_, n\_] = Product[x + i - 1, \{i, 1, n\}]$

$\theta[n\_, k\_, l\_, m\_] = UnitStep[n + m - 1]\, Sqrt[(m - l + n + 1)/(m + 1)]$
$2\hat{}k\, v[m - l + 1, n]\, u[m + 1, l]\, p[m - l + 1, k]$

$K1 = Expand[A + A^* +$
$Sum[MapAll[Distribute,\ NonCommutativeMultiply[D^*[m], D[m]]], \{m, 0, 0\}]]$

$If[K1 == 0, Print[StringForm["\ Condition\ (3.3)\ is\ satisfied"]],$
$Print[StringForm\ "Condition\ (3.3)\ is\ not\ satisfied"]]$

$K2 = Expand[A + A^* +$
$Sum[MapAll[Distribute,\ NonCommutativeMultiply[C[m], C^*[m]]], \{m, 0, 0\}]]$

$If[K2 == 0, Print[StringForm["\ Condition\ (3.4)\ is\ satisfied"]],$
$Print[StringForm\ "Condition\ (3.4)\ is\ not\ satisfied"]]$

$m = 0;$

$K3 = Expand[C[m] + D^*[m] + Sum[MapAll[Distribute,$
$NonCommutativeMultiply[D^*[m + n - l], B[n, k, l]]\ \theta[l, k, m + n -$
$l]], \{n, 0, 0\}, \{l, 0, 0\}, \{k, 0, 0\}]$

$If[K3 == 0, Print[StringForm["Condition\ (3.5)\ is\ satisfied\ form = "], m],$
$Print[StringForm\ "Condition\ (3.5)\ is\ not\ satisfied\ for\ m = "], m]$

$c[\lambda\_, \rho\_, \sigma\_, \omega\_, \epsilon\_, \beta\_, \gamma\_, a\_, b\_] =$
$Binomial[\gamma, \lambda]\, Binomial[\gamma - \lambda, \rho]\, Binomial[\beta, \omega]\, Binomial[b, \epsilon]$
$2\hat{}(\beta + b - \omega - \epsilon)\ StirlingS1[\gamma - \lambda - \rho, \sigma]\, u[a, \gamma - \lambda]\, (a - \gamma + \lambda)\hat{}(\beta - \omega)\, \lambda\hat{}(b - \epsilon);$

$g[n\_, k\_, l\_, x\_, y\_, z\_, X\_, Y\_, Z\_] = Sum[KroneckerDelta[x + Y - X +$
$\lambda, n]KroneckerDelta[\omega + \sigma + \epsilon, k]\ KroneckerDelta[\lambda + y, l]$
$c[\lambda, \rho, \sigma, \omega, \epsilon, z, X, Y, Z], \{\lambda, 0, X\}, \{\rho, 0, X, Y - \lambda\}, \{\sigma, 0, X - \lambda -$
$\rho\}, \{\omega, 0, z\}, \{\epsilon, 0, Z\}]$

$n = 1$

$k = 1$

$l = 1$

$K4 = Expand[B[n, k, l] + B^*[l, k, n] +$
$Sum[MapAll[Distribute,\ NonCommutativeMultiply[B[\alpha, \beta, \gamma], B^*[a, b, c]]$
$g[n, k, l, \alpha, \beta, \gamma, c, b]], \{\alpha, 0, n\}, \{\beta, 0, k\},$
$\{\gamma, 0, Min[k, l]\}, \{a, 0, l\}, \{b, 0, k\}, \{c, 0, n\}]]$

$If[K4 == 0, Print[StringForm["Condition\ (3.7)\ is\ satisfied\ for\ (n, k, l) =$
$("], n, StringForm[","], k, StringForm[","], l, StringForm[")"]],$
$Print[StringForm["Condition\ (3.7)\ is\ not\ satisfied\ for(n, k, l) =$
$("], n, StringForm[","], k, StringForm[","], l, StringForm[")"]]]$

$K5 = Expand[B[n,k,l] + B^*[l,k,n] + Sum[MapAll[Distribute,$
$NonCommutativeMultiply[B^*[\alpha,\beta,\gamma], B[a,b,c]]$
$g[n,k,l,\gamma,c,\beta,\alpha,a,b]], \{\alpha,0,Min[k,l]\}, \{\beta,0,k\}, \{\gamma,0,n\}, \{a,0,n\},$
$\{b,0,k\}, \{c,0,l\}]]$

$If[K5 == 0, Print[StringForm["Condition\ (3.8)\ is\ satisfied\ for\ (n,k,l) =$
$("], n, StringForm[","], k, StringForm[","], l, StringForm[")"]],$
$Print[StringForm["Condition\ (3.8)\ is\ not\ satisfied\ for (n,k,l) =$
$("], n, StringForm[","], k, StringForm[","], l, StringForm[")"]]]$

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