The centrally extended Heisenberg algebra and its connection with the Schrödinger, Galilei and Renormalized Higher Powers of Quantum White Noise Lie algebras

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Thanks We thank Stoimen Stoimenov for pointing out to us the connection with the Galilei algebra.

Abstract We study the non-trivial central extensions $CE(Heis)$ of the Heisenberg algebra $Heis$, prove that a real form of $CE(Heis)$ is the Galilei Lie algebra and obtain a matrix representation of $CE(Heis)$. We also show that $CE(Heis)$ can be realized (i) as a sub–Lie–algebra of the Schrödinger algebra and (ii) in terms of two independent copies of the canonical commutation relations (CCR). This gives a natural family of unitary representations of $CE(Heis)$ and allows an explicit determination of the associated group by exponentiation. In contrast with $Heis$, the group law for $CE(Heis)$ is given by nonlinear (quadratic) functions of the coordinates. The vacuum characteristic and moment generating functions of the classical random variables canonically associated to $CE(Heis)$ are computed. The second quantization of $CE(Heis)$ is also considered.

1. Central extensions of the Heisenberg algebra

The one mode Heisenberg algebra $Heis$ is the 3–dimensional $*$–Lie algebra with generators a, $a^\dagger$, h (central element) satisfying the commutation relations

$$[a,a^\dagger]_{Heis} = h \ ; \ [a,h]_{Heis} = [h,a^\dagger]_{Heis} = [a,a]_{Heis} = 0 \tag{1.1}$$

and the duality relations

$$(a)^* = a^\dagger \ ; \ h^* = h \tag{1.2}$$

In [4] we proved that this algebra admits non trivial central extensions. More precisely, all 2-cocycles ϕ on $Heis \times Heis$ are defined through bilinear skew-symmetric extension of the functionals

$$\phi(a,a^\dagger) = \lambda \quad ; \quad \phi(h,a^\dagger) = z \quad ; \quad \phi(a,h) = \bar{z} \tag{1.3}$$

where $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}$. Each 2-cocycle (1.3) defines a central extension $CE(Heis)$ of $Heis$ and the corresponding central extension is trivial if and only if $z = 0$. In what follows we will always assume that $z \neq 0$.

The centrally extended Heisenberg relations are (1.2) and

$$[a,a^\dagger]_{CE(Heis)} = h + \lambda E \ ; \ [h,a^\dagger]_{CE(Heis)} = z\,E \ ; \ [a,h]_{CE(Heis)} = \bar{z}\,E \tag{1.4}$$

where $E \not\equiv 0$ is the self-adjoint central element and where, here and in the following, all omitted commutators are assumed to be equal to zero.

Renaming $h + \lambda E$ by just h in (1.4) we obtain the 4–dimensional $*$–Lie algebra $CE(Heis)$ with generators a, $a^\dagger$, h, E (central element) satisfying the relations (1.2) and

$$[a,a^\dagger]_{CE(Heis)} = h \quad ; \quad [h,a^\dagger]_{CE(Heis)} = z\,E \quad ; \quad [a,h]_{CE(Heis)} = \bar{z}\,E \tag{1.5}$$

Moreover, the rescaling

$$a \to \frac{|z|^{2/3}}{\bar{z}}\,a \ ; \ a^\dagger \to \frac{|z|^{2/3}}{z}\,a^\dagger \ ; \ h \to \frac{1}{|z|^{2/3}}\,h \tag{1.6}$$

shows that we may take $z = 1$. We therefore obtain the (canonical) $CE(Heis)$ commutation relations

$$[a,a^\dagger]_{CE(Heis)} = h \quad ; \quad [h,a^\dagger]_{CE(Heis)} = E \quad ; \quad [a,h]_{CE(Heis)} = E \tag{1.7}$$

Proposition 1. *Commutation relations (1.7) define a nilpotent (therefore solvable) four–dimensional $*$–Lie algebra $CE(Heis)$ with generators $a, a^\dagger, h$ and E.*

Proof. Let $l_1 = a, l_2 = a^\dagger, l_3 = h, l_4 = E$. Using (1.7) we have that

$$[l_2, l_3]_{CE(Heis)} = -E \ ; \ [l_3, l_1]_{CE(Heis)} = -E \ ; \ [l_1, l_2]_{CE(Heis)} = h$$

Hence

$$[l_1, [l_2, l_3]_{CE(Heis)}]_{CE(Heis)} = [l_2, [l_3, l_1]_{CE(Heis)}]_{CE(Heis)} = [l_3, [l_1, l_2]_{CE(Heis)}]_{CE(Heis)} = 0$$

which implies that

$$[l_1, [l_2, l_3]_{CE(Heis)}]_{CE(Heis)} + [l_2, [l_3, l_1]_{CE(Heis)}]_{CE(Heis)} + [l_3, [l_1, l_2]_{CE(Heis)}]_{CE(Heis)} = 0$$

i.e. the Jacobi identity is satisfied. To show that $a, a^\dagger, h$ and E are linearly independent, suppose that

$$\alpha\, a + \beta\, a^\dagger + \gamma\, h + \delta\, E = 0 \tag{1.8}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Taking the commutator of (1.8) with $a^\dagger$ we find that

$$\alpha\, h + \gamma E = 0$$

which, after taking its commutator with $a^\dagger$, implies that $\alpha\, E = 0$. Since $E \not\equiv 0$, it follows that $\alpha = 0$ and (1.8) is reduced to

$$\beta\, a^\dagger + \gamma\, h + \delta\, E = 0 \tag{1.9}$$

Taking the commutator of (1.9) with h we find that $\beta\, E = 0$. Hence $\beta = 0$ and (1.9) is reduced to

$$\gamma\, h + \delta\, E = 0 \tag{1.10}$$

Taking the commutator of (1.10) with $a^\dagger$ we find that $\gamma\ E = 0$. Hence $\gamma = 0$ and (1.10) is reduced to

$$\delta\, E = 0$$

which implies that $\delta = 0$ as well. Finally

$$CE(Heis)^{\ 2} := [CE(Heis), CE(Heis)] = \{\gamma\, h + \delta\, E \ \ : \ \ \gamma, \delta \in \mathbb{C}\}$$

and

$$CE(Heis)^{\ 3} := [CE(Heis)^{\ 2}, CE(Heis)] = \{0\}$$

Therefore $CE(Heis)$ is nilpotent and thus solvable. □

Proposition 2. *Define p, q and H by*

$$a^\dagger = p + i\, q \qquad ; \qquad a = p - i\, q \qquad ; \qquad H = -ih/2 \tag{1.11}$$

Then p, q, E are self-adjoint and H is skew-adjoint. Moreover p, q, E and H are the generators of a real four-dimensional solvable $*$–Lie algebra with central element E and commutation relations

$$[p, q] = H \ ; \ [q, H] = \frac{1}{2}\, E \ ; \ [H, p] = 0 \tag{1.12}$$

Conversely, let p, q, H, E be the generators (with p, q, E self-adjoint and H skew-adjoint) of a real four-dimensional solvable $*$–Lie algebra with central element E and commutation relations (1.12). Then, the operators defined by (1.11) are the generators of the nontrivial central extension $CE(Heis)$ of the Heisenberg algebra defined by (1.7), (1.2).

Proof. The proof consists of a simple algebraic verification. □

There is a large literature on the classification of low dimensional Lie algebras (see [17]). In particular real four–dimensional solvable Lie algebras were fully classified by Kruchkovich in 1954 (see [15]). There are exactly fifteen isomorphism classes and they are listed, for example, in Proposition 2.1 of [16] (see references therein for additional information). One of the fifteen Lie algebras that appear in the above mentioned classification list is the Galilei Lie algebra denoted by η_4 with generators $\xi_1, \xi_2, \xi_3, \xi_4$ and (non-zero) commutation relations among generators

$$[\xi_4, \xi_1] = \xi_2 \qquad ; \qquad [\xi_4, \xi_2] = \xi_3 \tag{1.13}$$

Corollary 1. *The real form of $CE(Heis)$, described in Proposition 2, can be identified to the Galilei algebra η_4 defined above.*

Proof. We may take

$$\xi_4 = q \quad ; \quad \xi_1 = p \quad ; \quad \xi_2 = -H \quad ; \quad \xi_3 = -\frac{1}{2}\,E$$

□

The Galilei Lie algebra η_4 has also been studied by Feinsilver and Schott in [10]. In section 6 we study the connection between our work and that of Feinsilver and Schott in detail.

2. Boson realization of $CE(Heis)$

In this section we show how the generators a, $a^\dagger, h$ and E of $CE(Heis)$ can be expressed in terms of a subset of generators of the Schrödinger algebra.

Definition 1. *Let $b^\dagger$, b and 1 be the generators of the Schrödinger representation of the Heisenberg algebra, so that*

$$[b, b^\dagger] = 1 \qquad ; \qquad (b^\dagger)^\dagger = b \tag{2.1}$$

The Schrödinger algebra, denoted by Schroed, is the six–dimensional complex $*$–Lie algebra generated by b, $b^\dagger$, b^2, ${b^\dagger}^2$, $b^\dagger\, b$ and 1.

Remark 1.

In the notation of Definition 1 it is well known that the following commutation relations take place:

$$[b - b^\dagger, b + b^\dagger] = 2 \;;\; [(b - b^\dagger)^2, b + b^\dagger] = 4\,(b - b^\dagger) \;;\; [b\,f(b^\dagger), f(b^\dagger)\,b] = f'(b^\dagger) \tag{2.2}$$

for any analytic function f defined, weakly on the number vectors, by its series expansion.

Theorem 1. *(Boson representation of $CE(Heis)$) Let $\{a^+, a, h, E = 1\}$ be the generators of $CE(Heis)$.*

For arbitrary $\rho, r \in \mathbb{R}$ with $r \neq 0$, define the map:

$$a \in CE(Heis) \;\; \mapsto \;\; -\left(\frac{r^2}{4} - i\,\rho\right)(b - b^\dagger)^2 - \frac{i}{2\,r}\,(b + b^\dagger) \;\in\; Schroed \tag{2.3}$$

$$a^\dagger \in CE(Heis) \;\; \mapsto \;\; -\left(\frac{r^2}{4} + i\,\rho\right)(b - b^\dagger)^2 + \frac{i}{2\,r}\,(b + b^\dagger) \;\in\; Schroed \tag{2.4}$$

$$h \in CE(Heis) \;\mapsto\; i\,r\,(b^\dagger - b) \;\in\; Schroed \quad ; \quad 1 \in CE(Heis) \;\mapsto\; 1 \in Schroed \tag{2.5}$$

Then the maps defined above extend by linearity to injective $*$–Lie algebra homomorphisms.

Proof. Both maps are injective because of the linear independence of the generators. Moreover in both cases $(a^\dagger)^* = a$ and $h^* = h$. The statement follows from the identities

$$\begin{aligned}
[a, a^\dagger] &= \left(\frac{-r^2}{4} + i\,\rho\right)\frac{i}{2\,r}\,[(b-b^\dagger)^2, b+b^\dagger] - \frac{i}{2\,r}\left(\frac{-r^2}{4} - i\,\rho\right)[b+b^\dagger, (b-b^\dagger)^2] \\
&= \left(\frac{-r^2}{4} + i\,\rho\right)\frac{i}{2\,r}\,4\,(b-b^\dagger) - \frac{i}{2\,r}\left(\frac{-r^2}{4} - i\,\rho\right)4\,(b^\dagger - b) = -i\,r\,(b-b^\dagger) = h
\end{aligned}$$

$$[a, h] = -\frac{i}{2\,r}\,(-i\,r)\,[b+b^\dagger, b-b^\dagger] = -\frac{r}{2\,r}\,(-2) = 1$$

and

$$[h, a^\dagger] = (-i\,r)\,\frac{i}{2\,r}\,[b-b^\dagger, b+b^\dagger] = \frac{r}{2\,r}\,2 = 1$$

□

Remark 2.

Using the results of Theorem 1 we can obtain a simpler Boson representation, of $CE(Heis)$. In fact, since

(2.6) $$a = A\,(b-b^\dagger)^2 + B\,(b+b^\dagger)\ ;\ a^\dagger = \bar{A}\,(b-b^\dagger)^2 + \bar{B}\,(b+b^\dagger)\ ;\ h = C\,(b^\dagger - b)$$

where

(2.7) $$A = -\frac{r^2}{4}\ ;\ B = -\frac{i}{2\,r}\ ;\ C = i\,r$$

we see that, since we have assumed for non–triviality that $C \neq 0$, $r \neq 0$ and also

(2.8) $$A\,\bar{B} - \bar{A}\,B = -\frac{i\,r}{4} \neq 0$$

by defining

(2.9) $$p := b - b^\dagger \qquad ; \qquad q := b + b^\dagger$$

we find that

(2.10) $$p^2 = \frac{\bar{B}\,a - B\,a^\dagger}{A\,\bar{B} - \bar{A}\,B}\ ;\ q = \frac{A\,a^\dagger - \bar{A}\,a}{A\,\bar{B} - \bar{A}\,B}$$

which implies that the Lie algebra generated by $\{p^2, p, q, 1\}$ coincides with $CE(Heis)$.

3. Random variables in $CE(Heis)$

Denote $\mathcal{F}$ the Hilbert space of the Schrödinger representation of $b, b^\dagger$, Φ the vacuum vector, so that $b\,\Phi = 0$ and $||\Phi|| = 1$, and $y(\lambda) = e^{\lambda\,b^\dagger}\,\Phi$ the exponential vector with parameter $\lambda \in \mathbb{C}$. Self-adjoint operators X on $\mathcal{F}$ correspond to classical random variables with moment generating function $\langle \Phi, e^{s\,X}\,\Phi\rangle$ and characteristic function $\langle \Phi, e^{i\,s\,X}\,\Phi\rangle$, where $s \in \mathbb{R}$. In this section we compute the moment generating and characteristic functions of the self-adjoint operator $X = a + a^\dagger + h$.

Lemma 1. *(i) Let $L \in \mathbb{R}$ and $M, N \in \mathbb{C}$. Then for all $s \in \mathbb{R}$ such that $2\,L\,s + 1 > 0$*

$$e^{s\,(L\,b^2 + L\,{b^\dagger}^2 - 2\,L\,b^\dagger\,b - L + M\,b + N\,b^\dagger)}\,\Phi = e^{w_1(s)\,{b^\dagger}^2}\,e^{w_2(s)\,b^\dagger}\,e^{w_3(s)}\,\Phi$$

where

$$w_1(s) = \frac{L\,s}{2\,L\,s + 1} \tag{3.1}$$

$$w_2(s) = \frac{L\,(M+N)\,s^2 + N\,s}{2\,L\,s + 1} \tag{3.2}$$

$$w_3(s) = \frac{(M+N)^2\,(L^2\,s^4 + 2\,L\,s^3) + 3\,M\,N\,s^2}{6\,(2\,L\,s + 1)} - \frac{\ln(2\,L\,s + 1)}{2} \tag{3.3}$$

(ii) Let $L \in \mathbb{R}$ and $M, N \in \mathbb{C}$. Then for all $s \in \mathbb{R}$

$$e^{i\,s\,(L\,b^2 + L\,{b^\dagger}^2 - 2\,L\,b^\dagger\,b - L + M\,b + N\,b^\dagger)}\,\Phi = e^{\hat{w}_1(s)\,{b^\dagger}^2}\,e^{\hat{w}_2(s)\,b^\dagger}\,e^{\hat{w}_3(s)}\,\Phi$$

where

$$\hat{w}_1(s) = \frac{L\,s}{2\,L\,s - i} \tag{3.4}$$

$$\hat{w}_2(s) = \frac{i\,L\,(M+N)\,s^2 + N\,s}{2\,L\,s - i} \tag{3.5}$$

$$\hat{w}_3(s) = \frac{(M+N)^2\,(L^2\,s^4 - 2\,i\,L\,s^3) - 3\,M\,N\,s^2}{6\,(2\,i\,L\,s + 1)} - \frac{\ln(2\,i\,L\,s + 1)}{2} \tag{3.6}$$

Proof. We will use the differential method of Proposition 4.1.1, Chapter 1 of [9]. To prove part (i) of the Lemma, let

$$\begin{aligned} F(s) &= e^{s\,(L\,b^2 + L\,{b^\dagger}^2 - 2\,L\,b^\dagger\,b - L + M\,b + N\,b^\dagger)}\,\Phi \\ &= e^{w_1(s)\,{b^\dagger}^2}\,e^{w_2(s)\,b^\dagger}\,e^{w_3(s)}\,\Phi \\ \text{(since } b^\dagger, {b^\dagger}^2 \text{ and 1 commute)} &= e^{w_1(s)\,{b^\dagger}^2 + w_2(s)\,b^\dagger + w_3(s)}\,\Phi \end{aligned} \tag{3.7}$$

where w_1, w_2, w_3 are scalar-valued functions with $w_1(0) = w_2(0) = w_3(0) = 0$. Then

$$\frac{\partial}{\partial s}\,F(s) = (w_1(s)\,{b^\dagger}^2 + w_2(s)\,b^\dagger + w_3(s))\,F(s) \tag{3.8}$$

and also

$$\begin{aligned} \frac{\partial}{\partial s}\,F(s) &= (L\,b^2 + L\,{b^\dagger}^2 - 2\,L\,b^\dagger\,b - L + M\,b + N\,b^\dagger)\,F(s) \\ &= (L\,b^2 + L\,{b^\dagger}^2 - 2\,L\,b^\dagger\,b - L + M\,b + N\,b^\dagger)\,e^{w_1(s)\,{b^\dagger}^2 + w_2(s)\,b^\dagger + w_3(s)}\,\Phi \end{aligned} \tag{3.9}$$

Using (2.2) with $f(b^\dagger) = e^{w_1(s)\,{b^\dagger}^2 + w_2(s)\,b^\dagger + w_3(s)}$ and the fact that $b\,\Phi = 0$ we find that

$$b\,F(s) = b\,f(b^\dagger)\,\Phi = f'(b^\dagger)\,\Phi = (2\,w_1(s)\,b^\dagger + w_2(s))\,f(b^\dagger)\,\Phi = (2\,w_1(s)\,b^\dagger + w_2(s))\,F(s)$$

and

$$\begin{aligned} b^2\,F(s) &= b\,(2\,w_1(s)\,b^\dagger + w_2(s))\,F(s) = (2\,w_1(s)\,(1 + b^\dagger b) + w_2(s)\,b)\,F(s) \\ &= (2\,w_1(s) + w_2(s)^2 + 4\,w_1(s)\,w_2(s)\,b^\dagger + 4\,w_1(s)^2\,{b^\dagger}^2)\,F(s) \end{aligned}$$

and so (3.9) becomes

$$\frac{\partial}{\partial s} F(s) = \{2 L w_1(s) + L w_2(s)^2 - L + M w_2(s) + (4 L w_1(s) w_2(s) - 2 L w_2(s) + 2 M w_1(s) + N) b^\dagger + (4 L w_1(s)^2 + L - 4 L w_1(s)) {b^\dagger}^2\} F(s) \tag{3.10}$$

From (3.8) and (3.10), after equating coefficients of 1, $b^\dagger$ and ${b^\dagger}^2$, we obtain

$$\begin{aligned} w_1'(s) &= 4 L w_1(s)^2 - 4 L w_1(s) + L \text{ (Riccati differential equation)} \\ w_2'(s) &= (4 L w_1(s) - 2 L) w_2(s) + 2 M w_1(s) + N \text{ (Linear differential equation)} \\ w_3'(s) &= 2 L w_1(s) + L w_2(s)^2 - L + M w_2(s) \end{aligned}$$

with $w_1(0) = w_2(0) = w_3(0) = 0$. Therefore w_1, w_2 and w_3 are given by (3.1)-(3.3).

The proof of part (ii) is similar. It can also be obtained by replacing in (i), L, M, N by $i L, i M, i N$ respectively. □

Remark 3.

For $L \neq 0$ the Riccati equation

$$w_1'(s) = 4 L w_1(s)^2 - 4 L w_1(s) + L$$

appearing in the proof of Lemma 1 can be put in the canonical form

$$V'(s) = 1 + 2 \alpha V(s) + \beta V(s)^2$$

of the theory of Bernoulli systems of chapters 5 and 6 of [9], where $V(s) = \frac{w_1(s)}{L}$, $\alpha = -2 L$ and $\beta = 4 L^2$. Then $\delta^2 := \alpha^2 - \beta = 0$ which is characteristic of exponential and Gaussian systems ([9], Proposition 5.3.2).

Proposition 3. *(Moment Generating Function) For all $s \in \mathbb{R}$ such that $2 L s + 1 > 0$*

$$\langle \Phi, e^{s (a + a^\dagger + h)} \Phi \rangle = (2 L s + 1)^{-1/2} e^{\frac{(M+N)^2 (L^2 s^4 + 2 L s^3) + 3 M N s^2}{6 (2 L s + 1)}} \tag{3.11}$$

where in the notation of Theorem 1

$$L = -\frac{r^2}{2} \; ; \; M = -i r \; ; \; N = i r \tag{3.12}$$

Proof. We have that

$$a + a^\dagger + h = L b^2 + L {b^\dagger}^2 - 2 L b^\dagger b - L + M b + N b^\dagger$$

Therefore, in the notation of Lemma 1, using $\left(e^{f(b^\dagger)}\right)^\dagger = e^{\bar{f}(b)}$ and the fact that for all scalars λ we have that $e^{\lambda b} \Phi = \Phi$, we obtain

$$\begin{aligned} \langle \Phi, e^{s (a + a^\dagger + h)} \Phi \rangle &= \langle \Phi, e^{s (L b^2 + L {b^\dagger}^2 - 2 L b^\dagger b - L + M b + N b^\dagger)} \Phi \rangle \\ &= \langle \Phi, e^{w_3(s)} \Phi \rangle \\ &= (2 L s + 1)^{-1/2} e^{\frac{(M+N)^2 (L^2 s^4 + 2 L s^3) + 3 M N s^2}{6 (2 L s + 1)}} \langle \Phi, \Phi \rangle \\ &= (2 L s + 1)^{-1/2} e^{\frac{(M+N)^2 (L^2 s^4 + 2 L s^3) + 3 M N s^2}{6 (2 L s + 1)}} \end{aligned}$$

□

Remark 4.

The term $(2\,L\,s+1)^{-1/2}$ appearing in (3.11) is the moment generating function of a gamma random variable.

Proposition 4. *(Characteristic Function) For all $s \in \mathbb{R}$*

$$\langle \Phi, e^{i\,s\,(a+a^\dagger+h)}\,\Phi\rangle = (2\,i\,L\,s+1)^{-1/2}\,e^{\frac{(M+N)^2\,(L^2\,s^4-2\,i\,L\,s^3)-3\,M\,N\,s^2}{6\,(2\,i\,L\,s+1)}} \tag{3.13}$$

where L, M, N are as in Proposition 3.

Proof. The proof is similar to that of Proposition 3 with the use of Lemma 1 (ii). As expected, the result can be obtained from (3.11) by replacing s by $i\,s$. □

Remark 5.

The possibility of a direct Fock representation of $CE(Heis)$, i.e. such that $a\,\Phi = 0$, was considered in [6].

4. Representation of $CE(Heis)$ in terms of two independent copies of the CCR

Theorem 2. *For $j, k \in \{1, 2\}$ let $[q_j, p_k] = \frac{i}{2}\,\delta_{j,k}$ and $[q_j, q_k] = [p_j, p_k] = 0$ with $p_j^\dagger = p_j$, $q_j^\dagger = q_j$ and $i^2 = -1$. Then, for arbitrary $r \in \mathbb{R}$ and $c \in \mathbb{C}$*

$$a := i\,q_1 + (1+i\,r)\;p_1^2 + c\,q_2^2 \tag{4.1}$$

$$a^\dagger := -i\,q_1 + (1-i\,r)\;p_1^2 + \bar{c}\,q_2^2 \tag{4.2}$$

$$h := -2\,p_1 \tag{4.3}$$

and $E := 1$ satisfy the commutation relations (1.7) and the duality relations (1.1) of $CE(Heis)$.

Proof. (i) It is easy to see that $[q_j, p_j^2] = i\,p_j$, $[q_j^2, p_j] = i\,q_j$ and $[q_j^2, p_j^2] = 2\,i\,p_j\,q_j$. Then

$$[a, a^\dagger] = i\,[q_1, p_1^2] - i\,[p_1^2, q_1] = i\,(i\,p_1) - i\,(-i\,p_1) = -2\,p_1 = h$$

$$[a, h] \quad = \quad [i\,q_1 + p_1^2 - 2\,(p_1+q_2)] = i\,(-2)[q_1, p_1] = i\,(-2)\left(\frac{i}{2}\right) = 1$$

and

$$[h, a^\dagger] = [-2\,(p_1+q_2), -i\,q_1 + p_1^2 = 2\,i\,[p_1, q_1] = 2\,i\left(-\frac{i}{2}\right) = 1$$

Clearly $(a^\dagger)^* = a$ and $h^* = h$. The proofs of (ii) and (iii) are similar.

□

Remark 6.

In the notation of Theorem 2 we may take

$$q_1 = \frac{b_1 + b_1^\dagger}{2}\;\;;\;\; p_1 = \frac{i\,(b_1^\dagger - b_1)}{2}\;\;;\;\; q_2 = \frac{b_2 + b_2^\dagger}{2}\;\;;\;\; p_2 = \frac{i\,(b_2^\dagger - b_2)}{2} \tag{4.4}$$

where

$$[b_1, b_1^\dagger] = [b_2, b_2^\dagger] = 1 \ ; \ [b_1^\dagger, b_2^\dagger] = [b_1, b_2] = [b_1, b_2^\dagger] = [b_1^\dagger, b_2] = 0 \tag{4.5}$$

In that case Theorem 3 would extend to the product of the moment generating functions of two independent random variables defined in terms of the generators of two mutually commuting Schrödinger algebras.

5. The centrally extended Heisenberg group

To derive the group law of the group associated with $CE(Heis)$ we will use the following:

(i) For all $X, Y \in span\{a, a^\dagger, h, E\}$

$$e^{X+Y} = e^X \, e^Y \, e^{-\frac{1}{2}[X,Y]} \, e^{\frac{1}{6}(2[Y,[X,Y]]+[X,[X,Y]])} \tag{5.1}$$

This is a special case of the general Zassenhaus formula (converse of the BCH formula, see for example [18] and [14]) and follows from the fact that all triple commutators of elements of $span\{a, a^\dagger, h, E\}$ are in the center.

(ii) If x, D and h are three operators satisfying the Heisenberg commutation relations,then (see [9])

$$[D, x]_{Heis} = h, \ \ [D, h]_{Heis} = [x, h]_{Heis} = 0 \tag{5.2}$$

then

$$e^{s\,D} \, e^{b\,x} = e^{b\,x} \, e^{s\,D} \, e^{b\,s\,h} \qquad ; \qquad \forall s, b, c \in \mathbb{C} \tag{5.3}$$

(iii) For all $\lambda, \mu \in \mathbb{C}$ (see [6] for the proofs)

$$e^{\lambda\,a} \, e^{\mu\,a^\dagger} = e^{\mu\,a^\dagger} \, e^{\lambda\,a} \, e^{\lambda\,\mu\,h} \, e^{\frac{\lambda\mu}{2}(\mu-\lambda)} \tag{5.4}$$

$$a \, e^{\mu\,a^\dagger} = e^{\mu\,a^\dagger} \, (a + \mu\,h + \frac{\mu^2}{2}) \tag{5.5}$$

$$e^{\lambda\,a} \, e^{\mu\,h} = e^{\mu\,h} \, e^{\lambda\,a} \, e^{\lambda\,\mu} \tag{5.6}$$

$$e^{\mu\,h} \, e^{\lambda\,a^\dagger} = e^{\lambda\,a^\dagger} \, e^{\mu\,h} \, e^{\lambda\,\mu} \tag{5.7}$$

$$a \, e^{\mu\,h} = e^{\mu\,h} \, (a + \mu) \tag{5.8}$$

$$h \, e^{\lambda\,a^\dagger} = e^{\lambda\,a^\dagger} \, (h + \lambda) \tag{5.9}$$

Corollary 2. *(Group Law) For $u, v, w, y \in \mathbb{C}$ define*

$$g(u, v, w, y) := e^{u\,a^\dagger} \, e^{v\,h} \, e^{w\,a} e^{yE} \tag{5.10}$$

Then the family of operators of the form (5.10) is a group with group law given by

$$g(a, b, c, d) \, g(A, B, C, D) = \tag{5.11}$$
$$= g(a + A, b + B + c\,A, c + C, \left(\frac{c\,A^2}{2} + b\,A\right) + \left(\frac{c^2\,A}{2} + c\,B\right) + d + D)$$

The family of operators of the form (5.10) with $u, v, w \in \mathbb{R}$ and $y \in \mathbb{C}$ is a sub–group. The group $\mathbb{R}^3 \times \mathbb{C}$ endowed with the composition law (5.11) is called the ***centrally extended Heisenberg group****.*

Proof.

$$\begin{aligned}
g(\alpha,\beta,\gamma,\delta)\,g(A,B,C,D) &= e^{\alpha\,a^\dagger}\,e^{\beta\,h}\,e^{\gamma\,a}\,e^{A\,a^\dagger}\,e^{B\,h}\,e^{C\,a}e^{(\delta+D)E}\\
&= e^{\alpha\,a^\dagger}\,e^{\beta\,h}\,e^{A\,a^\dagger}\,e^{\gamma\,a}\,e^{\gamma\,A\,h}\,e^{\frac{\gamma\,A}{2}\,(A-\gamma)}\,e^{B\,h}\,e^{C\,a}e^{(\delta+D)E}\\
&= e^{\alpha\,a^\dagger}\,e^{\beta\,h}\,e^{A\,a^\dagger}\,e^{\gamma\,a}\,e^{(\gamma\,A+B)\,h}\,e^{C\,a}e^{(\delta+D+\frac{\gamma\,A}{2}\,(A-\gamma))E}\\
&= e^{\alpha\,a^\dagger}\,e^{A\,a^\dagger}\,e^{\beta\,h}\,e^{\beta\,A}\,e^{(\gamma\,A+B)\,h}\,e^{\gamma\,a}\,e^{\gamma\,(\gamma\,A+B)}\,e^{C\,a}e^{(\delta+D+\frac{\gamma\,A}{2}\,(A-\gamma))E}\\
&= e^{(\alpha+A)\,a^\dagger}\,e^{(\beta+B+\gamma\,A)\,h}\,e^{(\gamma+C)\,a}e^{\{\left(\frac{\gamma\,A^2}{2}+\beta\,A\right)+\left(\frac{\gamma^2\,A}{2}+\gamma\,B\right)+\delta+D\}E}
\end{aligned}$$

$$= g\left(\alpha+A,\beta+B+\gamma\,A,\gamma+C,\left(\frac{\gamma\,A^2}{2}+\beta\,A\right)+\left(\frac{\gamma^2\,A}{2}+\gamma\,B\right)+\delta+D\right)$$

□

6. Matrix representation of $CE(Heis)$

In example (ix) of [10], Feinsilver and Schott considered the Galilei Lie algebra η_4 mentioned in corollary 1 and gave the explicit form of its adjoint representation:

$$\sum_{i=1}^{4}\alpha_i\,\xi_i = \begin{bmatrix} 0 & \alpha_4 & 0 & \alpha_3\\ 0 & 0 & \alpha_4 & \alpha_2\\ 0 & 0 & 0 & \alpha_1\\ 0 & 0 & 0 & 0\end{bmatrix} \tag{6.1}$$

where $\alpha_i \in \mathbb{C}$ for $i = 1,2,3,4$. From this they deduced that

$$e^{\sum_{i=1}^{4}\alpha_i\,\xi_i} = \begin{bmatrix} 1 & \alpha_4 & \frac{\alpha_4^2}{2} & \frac{\alpha_4^2\,\alpha_1}{6}+\frac{\alpha_4\,\alpha_2}{2}+\alpha_3\\ 0 & 1 & \alpha_4 & \frac{\alpha_4\,\alpha_1}{2}+\alpha_2\\ 0 & 0 & 1 & \alpha_1\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{6.2}$$

Defining the group elements

$$g(a,b,c,d) = e^{a\,\xi_1}\,e^{b\,\xi_2}\,e^{c\,\xi_3}\,e^{d\,\xi_4} \tag{6.3}$$

the group law

$$g(a,b,c,d)\,g(A,B,C,D) = g(a+A,b+B+d\,A,c+C+d\,B+\frac{1}{2}\,d^2\,A,d+D) \tag{6.4}$$

can easily be verified through matrix multiplication since both sides are equal to

$$\begin{bmatrix} 1 & d+D & \frac{d^2}{2}+d\,D+\frac{D^2}{2} & c+\frac{d^2\,A}{2}+d\,B+C\\ 0 & 1 & d+D & b+d\,A+B\\ 0 & 0 & 1 & a+A\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{6.5}$$

Using the identification of η_4 with $CE(Heis)$ one can then deduce the corresponding matrix representation for $CE(Heis)$. Following the proof of Corollary 1, we take $q = \xi_4$, $p = \xi_1$, $H = -\xi_2$ and $E = -2\,\xi_3$ and define $a = p - i\,q$, $a^\dagger = p + i\,q$ and $h = 2\,i\,H$. Then p, q, H and E satisfy commutation relations (1.12) and, as in Corollary 1, a^+, a, h and E satisfy the commutation relations (1.7) of $CE(Heis)$. If we introduce the duality relations

$$\xi_1^\dagger = \xi_1\,;\,\xi_2^\dagger = -\xi_2\,;\,\xi_3^\dagger = \xi_3\,;\,\xi_4^\dagger = \xi_4 \tag{6.6}$$

we conclude that a^+, a, h and E also satisfy the duality relations (1.1) of $CE(Heis)$.

Using (6.1) we will obtain a matrix representation of $CE(Heis)$ (satisfying commutation relations (1.7) but not duality relations (1.1)). We have

$$\alpha_1\, a + \alpha_2\, a^\dagger + \alpha_3\, h + \alpha_4\, E = (\alpha_1 + \alpha_2)\, \xi_1 - 2\, i\, \alpha_3\, \xi_2 - 2\, \alpha_4\, \xi_3 + i\, (\alpha_2 - \alpha_1)\, \xi_4 \tag{6.7}$$

$$= \begin{bmatrix} 0 & i\,(\alpha_2-\alpha_1) & 0 & -2\,\alpha_4 \\ 0 & 0 & i\,(\alpha_2-\alpha_1) & -2\,i\,\alpha_3 \\ 0 & 0 & 0 & \alpha_1+\alpha_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$e^{\alpha_1\, a+\alpha_2\, a^\dagger+\alpha_3\, h+\alpha_4\, E} = e^{(\alpha_1+\alpha_2)\, \xi_1-2\, i\, \alpha_3\, \xi_2-2\, \alpha_4\, \xi_3+i\, (\alpha_2-\alpha_1)\, \xi_4} \tag{6.8}$$

$$= \begin{bmatrix} 1 & i\,(\alpha_2-\alpha_1) & -\frac{1}{2}\,(\alpha_2-\alpha_1)^2 & \frac{1}{6}(\alpha_2-\alpha_1)\,(\alpha_1^2-\alpha_2^2+6\,a_3)-2\,a_4 \\ 0 & 1 & i\,(\alpha_2-\alpha_1) & \frac{i}{2}\,(\alpha_2^2-\alpha_1^2-4\,a_3) \\ 0 & 0 & 1 & \alpha_1+\alpha_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Remark 7.

As mentioned in [10], the generators $\xi_1, \xi_2, \xi_3, \xi_4$ can be represented on the space of smooth functions $f(x)$ as $x^2/2$, x, 1 and $D = d/dx$ respectively, with $[D, x] = 1$. Using the duality between x and D, given by the Fourier transform, one sees that this representation is unitarily equivalent to the one discussed in Remark 2.

7. Second quantization of $CEHeis$

The quantum white noise functionals $b_t^\dagger$ (creation density) and b_t (annihilation density) satisfy the Boson commutation relations

$$[b_t, b_s^\dagger] = \delta(t-s)\ ;\ [b_t^\dagger, b_s^\dagger] = [b_t, b_s] = 0 \tag{7.1}$$

where $t, s \in \mathbb{R}$ and δ is the Dirac delta function, as well as the duality relation

$$(b_s)^* = b_s^\dagger \tag{7.2}$$

In order to consider the smeared fields defined by the higher powers of b_t and $b_t^\dagger$, for a test function f and $n, k \in \{0, 1, 2, ...\}$, the sesquilinear forms

$$B_k^n(f) = \int_{\mathbb{R}} f(t)\, {b_t^\dagger}^n\, b_t^k\, dt \tag{7.3}$$

with involution

$$(B_k^n(f))^* = B_n^k(\bar{f}) \tag{7.4}$$

were defined in [8]. In [1] and [2] we introduced the convolution type renormalization

$$\delta^l(t-s) = \delta(s)\,\delta(t-s) \;\; ; \quad l = 2, 3, ... \tag{7.5}$$

of the higher powers of the Dirac delta function, and by restricting to test functions $f(t)$ such that $f(0) = 0$ we obtained the RHPWN $*$–Lie algebra commutation relations

$$[B^n_k(f), B^N_K(g)]_{RHPWN} = (k\,N - K\,n)\; B^{n+N-1}_{k+K-1}(f\,g) \tag{7.6}$$

The easily checked commutation relations

$$[B^0_1(f) - B^1_0(f), B^0_1(g) + B^1_0(g)] \;=\; 2\,B^0_0(f\,g) \tag{7.7}$$

$$[B^0_2(f) + B^2_0(f) - 2\,B^1_1(f) - B^0_0(f), B^0_1(g) + B^1_0(g)] \;=\; 4\,\left(B^0_1(f\,g) - B^1_0(f\,g)\right) \tag{7.8}$$

allow us to immediately extend the proof of Theorem 1 and realize the *second quantized* $CEHeis$ *commutation relations*

$$[a(f), a^\dagger(g)]_{CE(Heis)} \;=\; h(f\,g) \tag{7.9}$$

$$[h(f), a^\dagger(g)]_{CE(Heis)} \;=\; E(f\,g) \tag{7.10}$$

$$[a(f), h(g)]_{CE(Heis)} \;=\; E(f\,g) \tag{7.11}$$

as well as the duality relations

$$a(f)^* = a^\dagger(\bar{f}) \;;\; h(f)^* = h(\bar{f}) \;;\; E(f)^* = E(\bar{f}) \tag{7.12}$$

in terms of the first and second order $RHPWN$ generators B^0_2, B^2_0, B^1_1, B^0_1, B^1_0, and B^0_0.

Theorem 3. *($RHPWN$ representation of $CE(Heis)$) Let f, g be arbitrary test functions as in (7.6). For arbitrary $\rho, r \in \mathbb{R}$ with $r \neq 0$:*

$$a(f) \;=\; -\left(\frac{r^2}{4} - i\,\rho\right)\left(B^0_2(f) + B^2_0(f) - 2\,B^1_1(f) - B^0_0(f)\right) - \frac{i}{2\,r}\left(B^0_1(f) + B^1_0(f)\right) \tag{7.13}$$

$$a^\dagger(f) \;=\; -\left(\frac{r^2}{4} + i\,\rho\right)\left(B^0_2(f) + B^2_0(f) - 2\,B^1_1(f) - B^0_0(f)\right) + \frac{i}{2\,r}\left(B^0_1(f) + B^1_0(f)\right) \tag{7.14}$$

$$h(f) \;=\; i\,r\,\left(B^1_0(f) - B^0_1(f)\right) \tag{7.15}$$

$$E(f) \;=\; B^0_0(f) \tag{7.16}$$

satisfy the second quantized $CEHeis$ commutation relations (7.10)-(7.11) and the duality relations (7.12) above.

Proof. The proof is similar to that of theorem 1. □

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