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*Edited by:*  
Alberto Collino  
Alberto Conte  
Marina Marchisio

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## On the Geometric Genus of Reducible Surfaces and Degenerations of Surfaces to Unions of Planes.

ALBERTO CALABRI - CIRO CILIBERTO - FLAMINIO FLAMINI - RICK MIRANDA (\*)

### 1. - Introduction.

In this paper we study some properties of degenerations of surfaces whose general fibre is a smooth projective algebraic surface and whose central fibre is a reduced, connected surface  $X \subset \mathbb{P}^r$ ,  $r \geq 3$ , which is assumed to be a union of planes. Here we present a first set of results on the subject; other aspects are still work in progress and will appear later (see [4]).

Our original motivation has been a series of papers by Guido Zappa which appeared in the 1940-50's regarding degenerations of scrolls to unions of planes and the computation of bounds for the topological invariants of an arbitrary smooth projective surface which is assumed to degenerate to a union of planes (see [18, 19, 20, 21, 22, 23, 24] and [4]).

Zappa was in turn motivated by earlier papers by Francesco Severi concerning *Zeuthen's problem*, i.e. the existence of degenerations of smooth projective (space) curves to unions of lines with only nodes as singularities (now called *stick curves*).

Zeuthen's problem has been studied by several authors, also recently (see e.g. [14]); on the contrary, unions of planes have been studied only in terms of degenerations of a few types of smooth surfaces, e.g. K3 surfaces (see [5, 6, 7, 9]).

In this paper, we first study the geometry and the combinatorics of a union of planes  $X$  considered as a reduced, connected surface on its own (cf. § 3 and 4); then, we focus on the case in which  $X$  is the central fibre of an embedded degeneration  $\mathcal{X} \rightarrow \Delta$ , where  $\Delta$  is the complex unit disk and where  $\mathcal{X} \subseteq \Delta \times \mathbb{P}^r$ ,  $r \geq 3$ , is a closed subscheme of relative dimension two. In this case, we deduce some properties of the general fibre  $\mathcal{X}_t$ ,  $t \neq 0$ , of the degeneration from the ones of its central fibre  $\mathcal{X}_0 = X$  (see § 5).

It is well-known that, in dimension one, for any integer  $g \geq 2$  any smooth projective curve of genus  $g$  with general moduli and sufficiently

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general degree can be degenerated to a suitable stick curve (see, e.g. [1] and [17]).

On the contrary, in dimension two, worse singularities than normal crossings are needed in order to degenerate as many surfaces as possible to unions of planes (cf. [4]).

Here we shall focus on the case of  $X$  a union of planes — or more generally a union of smooth projective surfaces — whose singularities are:

- in codimension one, double curves which are smooth and irreducible;
- multiple points, which are locally analytically isomorphic to the vertex of a cone over a stick curve with arithmetic genus either zero or one and which is projectively normal in the projective space it spans.

These multiple points will be called *Zappatic singularities*, whereas a surface like  $X$  will be called a *Zappatic surface*. If moreover  $X \subset \mathbb{P}^r$ , for some positive  $r$ , and if all its irreducible components are planes, then  $X$  is said to be a *planar Zappatic surface*.

Actually we will concentrate on the so called *good Zappatic surfaces*, i.e. Zappatic surfaces having only Zappatic singularities whose associated stick curve has one of the following dual graphs (cf. Examples 2.8 and 2.9, Definition 3.6, Figures 2 and 4):

- $R_n$ : a chain of length  $n$ , with  $n \geq 3$ ;
- $S_n$ : a fork with  $n - 1$  teeth, with  $n \geq 4$ ;
- $E_n$ : a cycle of order  $n$ , with  $n \geq 3$ .

Let us call  $R_n$ -,  $S_n$ -,  $E_n$ -*point* the corresponding multiple point of the Zappatic surface  $X$ . These singularities play a major role in the whole subject (cf. [4]).

We remark that a Zappatic surface  $X$  is locally Gorenstein (i.e. its dualizing sheaf  $\omega_X$  is invertible) if and only if its Zappatic singularities are only  $E_n$ -points, for any  $n \geq 3$ .

We associate to a good Zappatic surface  $X$  a graph  $G_X$  (see Definition 3) which encodes the configuration of the irreducible components of  $X$  as well as of its Zappatic singularities.

We shall see (cf. Sections 3 and 4) how to combinatorially compute from the associated graph  $G_X$  some intrinsic and extrinsic invariants of  $X$ , e.g. the Euler-Poincaré characteristic  $\chi(\mathcal{O}_X)$ , the geometric genus  $p_g(X)$  (cf. Remark 3.4), as well as — when  $X \subset \mathbb{P}^r$ ,  $r \geq 3$  — the degree  $d = \deg(X)$ , the sectional genus  $g$ , and so on.

When  $X$  is further assumed to be the central fibre of a degeneration  $\mathcal{X} \rightarrow \Delta$  (resp., of an embedded degeneration  $\mathcal{X} \rightarrow \Delta$ , where  $\mathcal{X} \subset \Delta \times \mathbb{P}^r$ ,  $r \geq 3$ ) we will then compute intrinsic (resp., intrinsic and extrinsic) invariants of the general fibre  $\mathcal{X}_t$ , for  $t \neq 0$ .

We shall see how to directly compute some of the invariants of  $X$  by means of the associated graph  $G_X$ . Determining formulas for a few invariants (e.g.  $d$  and  $g$ ) is quite easy, whereas for other invariants, like  $\chi(\mathcal{O}_X)$ , it requires some more work.

Actually the computation of the geometric genus is still an open question, in general. Indeed, we first prove the following (cf. Theorem 4.15):

**THEOREM.** – *Let  $X = \bigcup_i X_i$  be a Zappatic surface with global normal crossings, i.e. with only  $E_3$ -points as Zappatic singularities. Denote by  $\omega_X$  the dualizing sheaf of  $X$  and by  $G_X$  the associated graph of  $X$ . Consider the natural map:*

$$\Phi : \bigoplus_i H^1(X_i, \mathcal{O}_{X_i}) \rightarrow \bigoplus_{i,j} H^1(C_{ij}, \mathcal{O}_{C_{ij}}),$$

where  $C_{ij} = X_i \cap X_j$  (cf. formula (4.8)). Then, the following inequality holds:

$$(1.1) \quad p_g(X) := h^0(X, \omega_X) \leq h_2(G_X, \mathbb{C}) + \sum_{i=1}^v p_g(X_i) + \dim(\text{coker}(\Phi)).$$

Furthermore, a sufficient condition for the equality in (1.1) to hold is either that

(i) each irreducible component  $X_i$  is a regular surface (i.e.  $h^1(\mathcal{O}_{X_i}) = 0$ ), or that

(ii) for any irregular component  $X_j$  of  $X$ , the divisor  $C_j := X_j \cap \overline{(X \setminus X_j)}$  is ample on  $X_j$ .

The proof of the above theorem also shows the following:

**COROLLARY.** – *Let  $X$  be a planar Zappatic surface with global normal crossings (i.e. only  $E_3$ -points) and  $G_X$  be its associated graph. Then, there exists an explicit isomorphism*

$$(1.2) \quad H^0(X, \omega_X) \cong H_2(G_X, \mathbb{C}),$$

where  $\omega_X$  is the dualizing sheaf of  $X$ . Therefore

$$p_g(X) := h^0(X, \omega_X) = h_2(G_X, \mathbb{C}).$$

From the proof of Theorem 4.15, it will be clear that the isomorphism (1.2) essentially follows from evaluation of residues at the  $E_3$ -points, with a suitable use of signs.

We show that equality holds in (1.1) when  $X$  is smoothable, i.e. when  $X$  is the central fibre of a *semistable degeneration*  $\mathcal{X} \rightarrow \Delta$ : this follows by the computation of the geometric genus  $p_g(\mathcal{X}_t)$  of the general fibre  $\mathcal{X}_t$ , for  $t \neq 0$ , via the Clemens-Schmid exact sequence (cf. Theorem 5.12).

We remark that our computation of the geometric genus is independent of the fact that  $X$  is the central fibre of a semistable degeneration. We deal with this particular case in § 5, where we show that in a semistable degeneration, whose central fibre is a Zappatic surface with only  $E_3$ -points, the geometric genus of the fibres is constant (see Corollary 5.15).

It is still an open problem to find an example for which the strict inequality holds in (1.1).

Finally, we will see that the above results can be generalized to a smoothable good Zappatic surface, i.e. with  $R_n$ -,  $S_n$ - and  $E_n$ -points, for any  $n \geq 3$  (see Theorem 5.20).

A natural question to ask is which Zappatic singularities are needed in order to degenerate as many surfaces as possible. Results and some examples contained in [4] suggest that, even if a given projective surface  $X$  needs  $E_n$ -,  $R_n$ -, or  $S_n$ -points with large  $n$ , there might be a birational model of  $X$  which needs just  $R_3$ - and  $E_n$ -points, with  $n \leq 6$ . For example, in [6] there are interesting examples of K3 surfaces degenerating to a Zappatic surface with at most  $R_3$ - and  $E_6$ -points, called *pillow* degenerations. However, we do not have enough evidence to state a reasonable conjecture in this direction.

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## 2. – Reducible curves and associated graphs.

Let  $C$  be a projective curve and let  $C_i$ ,  $i = 1, \dots, n$ , be its irreducible components. We will assume that:

- $C$  is connected and reduced;
- $C$  has at most nodes as singularities;
- the curves  $C_i$ ,  $i = 1, \dots, n$ , are smooth.

If two components  $C_i$ ,  $C_j$ ,  $i < j$ , intersect at  $m_{ij}$  points, we will denote by  $P_{ij}^h$ ,  $h = 1, \dots, m_{ij}$ , the corresponding nodes of  $C$ .

We can associate to this situation a simple (i.e. with no loops), connected graph  $G_C$ :

- whose vertices  $v_1, \dots, v_n$ , correspond to the components  $C_1, \dots, C_n$ ;
- whose edges  $\eta_{ij}^h$ ,  $i < j$ ,  $h = 1, \dots, m_{ij}$ , joining the vertices  $v_i$  and  $v_j$ , correspond to the nodes  $P_{ij}^h$  of  $C$ .

We will assume the graph to be *lexicographically oriented*, i.e. each edge is assumed to be oriented from the vertex with lower index to the one with higher index.

We will use the following notation:

- $v$ : the number of vertices of  $G_C$ , i.e.  $v = n$ ;
- $e$ : the number of edges of  $G_C$ , i.e. the number of nodes of  $C$ ;
- $g_i$ : the genus of the curve  $C_i$ , which we consider as the *weight* of the vertex  $v_i$ ;
- $\chi(G_C) = v - e$  is the Euler-Poincaré characteristic of  $G_C$ ;
- $h_1(G_C) = 1 - \chi(G_C)$  is the first Betti number of  $G_C$ .

Remark that conversely, given any simple, connected, weighted (oriented) graph  $G$ , there is some curve  $C$  such that  $G = G_C$ .

One has the following basic result:

**THEOREM 2.1.** – *In the above situation*

$$(2.2) \quad \chi(\mathcal{O}_C) = \chi(G_C) - \sum_{i=1}^v g_i = v - e - \sum_{i=1}^v g_i.$$

**PROOF.** – Let  $\nu: \tilde{C} \rightarrow C$  be the normalization morphism; this defines the exact sequence of sheaves on  $C$ :

$$(2.3) \quad 0 \rightarrow \mathcal{O}_C \rightarrow \nu_*(\mathcal{O}_{\tilde{C}}) \rightarrow \underline{\tau} \rightarrow 0,$$

where  $\underline{\tau}$  is a sky-scraper sheaf supported at  $\text{Sing}(C)$ . Since the singularities of  $C$  are only nodes, one easily determines  $H^0(C, \underline{\tau}) \cong C^e$ . Therefore, by the exact sequence (2.3), one gets

$$\chi(\mathcal{O}_C) = \chi(\nu_*(\mathcal{O}_{\tilde{C}})) - e.$$

By the Leray isomorphism and by the fact that  $\nu$  is finite, one has  $\chi(\nu_*(\mathcal{O}_{\tilde{C}})) = \chi(\mathcal{O}_{\tilde{C}})$ . Since  $\tilde{C}$  is a disjoint union of the  $v = n$  irreducible components of  $C$ , one has  $\chi(\mathcal{O}_{\tilde{C}}) = v - \sum_{i=1}^v g_i$ , which proves (2.2). (Cf. also [2] for another proof.) ■

We remark that formula (2.2) is equivalent to

$$(2.4) \quad p_a(C) = h_1(G_C) + \sum_{i=1}^v g_i,$$

(cf. Proposition 3.12).

Notice that  $C$  is locally Gorenstein, i.e. the dualizing sheaf  $\omega_C$  is invertible. One defines the *geometric genus* of  $C$  to be

$$(2.5) \quad p_g(C) := h^0(C, \omega_C).$$

By the Riemann-Roch Theorem, one has

$$(2.6) \quad p_g(C) = p_a(C) = h_1(G_C) + \sum_{i=1}^v g_i = e - v + 1 + \sum_{i=1}^v g_i.$$

However, one can prove the previous formula combinatorially by showing that there is a natural short exact sequence:

$$(2.7) \quad 0 \rightarrow \bigoplus_{i=1}^v H^0(C_i, \omega_{C_i}) \rightarrow H^0(C, \omega_C) \rightarrow H_1(G_C, \mathbb{C}) \rightarrow 0.$$

We will not dwell on this now, since we shall show the existence of an analogous sequence in the surface case in § 4.

If we have a flat family  $\mathcal{C} \rightarrow \Delta$  over a disc  $\Delta$  with general fibre  $\mathcal{C}_t$  a smooth and irreducible curve of genus  $g$  and special fibre  $\mathcal{C}_0 = C$ , then we can combinatorially compute  $g$  via the formula:

$$g = p_a(C) = h_1(G_C) + \sum_{i=1}^v g_i = e - v + 1 + \sum_{i=1}^v g_i.$$

Usually we will consider a curve  $C$  embedded in a projective space  $\mathbb{P}^r$ . In this situation each curve  $C_i$  will have a certain degree  $d_i$ , and we will consider the graph  $G_C$  as *double weighted*, by attaching to each vertex the pair of weights  $(g_i, d_i)$ . Moreover we will attribute to the graph a further marking number, i.e. the embedding dimension  $r$  of  $C$ .

The total degree of  $C$  is

$$d = \sum_{i=1}^v d_i$$

which is also invariant by flat degeneration.

If each curve  $C_i$  is a line, the curve  $C$  is called a *stick curve*. In this case the double weighting is  $(0, 1)$  for each vertex, and it will be omitted if no confusion arises.

It should be stressed that it is not true that for any simple, connected, double weighted graph  $G$  there is a curve  $C$  in a projective space such that  $G_C = G$ . For example there is no stick curve corresponding to the graph of Figure 1.

We now give two examples of stick curves which will be frequently used in this paper.

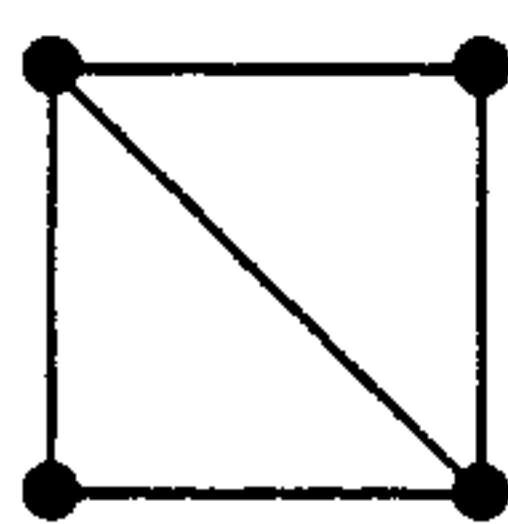


Figure 1. - Dual graph of an «impossible» stick curve.

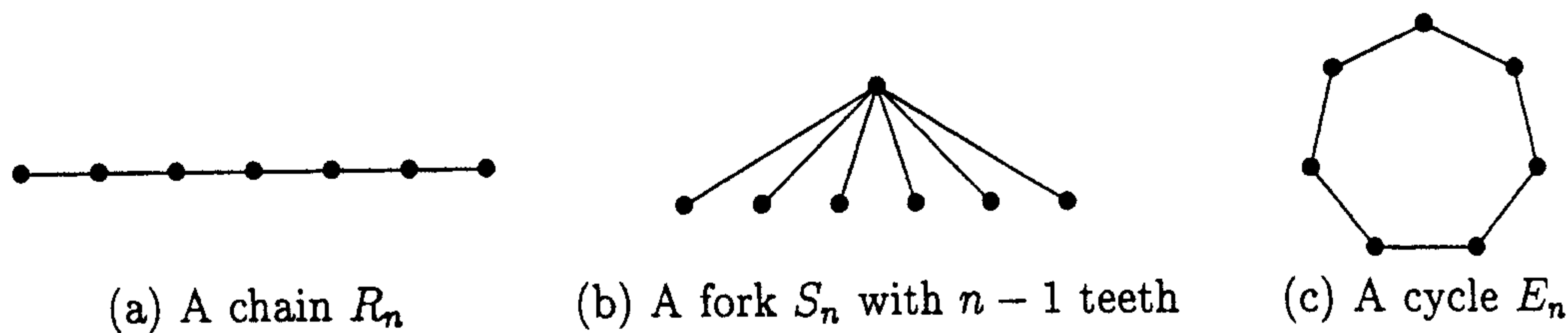


Figure 2. - Examples of dual graphs.

EXAMPLE 2.8. - Let  $T_n$  be any connected tree with  $n \geq 3$  vertices. This corresponds to a non-degenerate stick curve of degree  $n$  in  $\mathbb{P}^n$ , which we denote by  $C_{T_n}$ . Indeed one can check that, taking a general point  $p_i$  on each component of  $C_{T_n}$ , the line bundle  $\mathcal{O}_{C_{T_n}}(p_1 + \dots + p_n)$  is very ample. Of course  $C_{T_n}$  has arithmetic genus 0 and is a flat limit of rational normal curves in  $\mathbb{P}^n$ .

We will often consider two particular trees  $T_n$ : a chain  $R_n$  of length  $n$  and the fork  $S_n$  with  $n - 1$  teeth, i.e. a tree consisting of  $n - 1$  vertices joining a further vertex (see Figures 2.(a) and (b)). The curve  $C_{R_n}$  is the union of  $n$  lines  $l_1, l_2, \dots, l_n$  spanning  $\mathbb{P}^n$ , such that  $l_i \cap l_j = \emptyset$  if and only if  $1 < |i - j|$ . The curve  $C_{S_n}$  is the union of  $n$  lines  $l_1, l_2, \dots, l_n$  spanning  $\mathbb{P}^n$ , such that  $l_1, \dots, l_{n-1}$  all intersect  $l_n$  at distinct points (see Figure 3).

EXAMPLE 2.9. - Let  $Z_n$  be any simple, connected graph with  $n \geq 3$  vertices and  $h^1(Z_n, \mathbb{C}) = 1$ . This corresponds to a projectively normal stick curve of degree  $n$  in  $\mathbb{P}^{n-1}$ , which we denote by  $C_{Z_n}$  (as in Example 2.8). The curve  $C_{Z_n}$  has arithmetic genus 1 and it is a flat limit of elliptic normal curves in  $\mathbb{P}^{n-1}$ .

We will often consider the particular case of a cycle  $E_n$  of order  $n$  (see Figure 2.c). The curve  $C_{E_n}$  is the union of  $n$  lines  $l_1, l_2, \dots, l_n$  spanning  $\mathbb{P}^{n-1}$ , such that  $l_i \cap l_j = \emptyset$  if and only if  $1 < |i - j| < n - 1$  (see Figure 3).

We remark that  $C_{E_n}$  is projectively Gorenstein, because  $\omega_{C_{E_n}}$  is trivial, since there is an everywhere non-zero global section of  $\omega_{C_{E_n}}$ , given by the meromorphic 1-form on each component with residues 1 and  $-1$  at the nodes (in a suitable order).

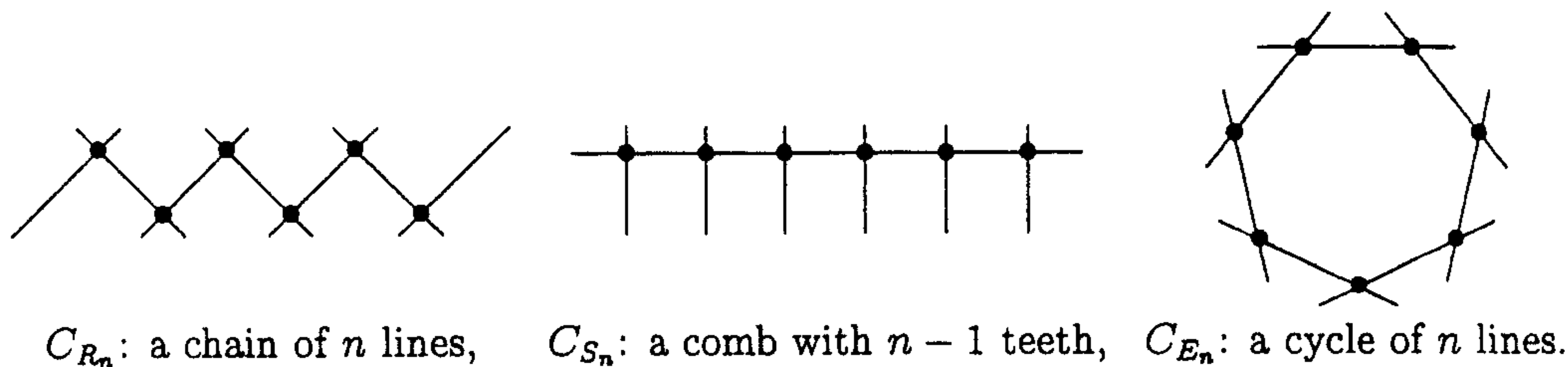


Figure 3. - Examples of stick curves.



Furthermore, if  $X \subset \mathbb{P}^r$ , for some  $r$ , we denote by

- $d$ : the degree of  $X$ ;
- $d_i$ : the degree of  $X_i$ ,  $1 \leq i \leq v$ ;
- $c_{ij}$ : the degree of  $C_{ij}$ ,  $1 \leq i \neq j \leq v$ ;
- $D$ : a general hyperplane section of  $X$ ;
- $g$ : the arithmetic genus of  $D$ ;
- $D_i$ : the (smooth) irreducible component of  $D$  lying in  $X_i$ , which is a general hyperplane section of  $X_i$ ,  $1 \leq i \leq v$ ;
- $g_i$ : the genus of  $D_i$ ,  $1 \leq i \leq v$ .

If moreover  $X$  is a planar Zappatic surface, then  $d = v$ , each non-empty set  $\Sigma_{ijk}$  is a singleton and  $m_{ijk} = 1$ , for each  $i \neq j \neq k$ .

REMARK 3.4. – A Zappatic surface  $X$  is locally Cohen-Macaulay. Thus the dualizing sheaf  $\omega_X$  is well-defined. If  $X$  has only  $E_n$ -points as Zappatic singularities, then  $X$  is locally Gorenstein, hence  $\omega_X$  is an invertible sheaf. If  $X$  has global normal crossings, i.e. if  $X$  has only  $E_3$ -points as Zappatic singularities, we define the *geometric genus* of  $X$  as:

$$(3.5) \quad p_g(X) := h^0(X, \omega_X).$$

If  $X$  is smoothable, namely if  $X$  is the central fibre of a degeneration, we will define its geometric genus later in Definition 5.18.

DEFINITION 3.6 (Good Zappatic surface). – The *good Zappatic singularities* are the

- $R_n$ -points, for  $n \geq 3$ ,
- $S_n$ -points, for  $n \geq 4$ ,
- $E_n$ -points, for  $n \geq 3$ ,

which are the Zappatic singularities whose associated stick curves are respectively  $C_{R_n}$ ,  $C_{S_n}$ ,  $C_{E_n}$  (see Examples 2.8 and 2.9, Figures 2, 3 and 4).

A *good Zappatic surface* is a Zappatic surface with only good Zappatic singularities.

To a good Zappatic surface  $X$  we can associate a complex  $G_X$ , which we briefly call the *associated graph* to  $X$ .

DEFINITION 3.7 (The associated graph to  $X$ ). – Let  $X$  be a good Zappatic surface with Notation 3.3. The graph  $G_X$  associated to  $X$  is defined as follows (cf. Figure 5):

- each surface  $X_i$  corresponds to a vertex  $v_i$ ;
- each double curve  $C_{ij}$  corresponds to an edge  $e_{ij}$  joining  $v_i$  and  $v_j$ . The edge  $e_{ij}$ ,  $i < j$ , is oriented from the vertex  $v_i$  to the one  $v_j$ ;

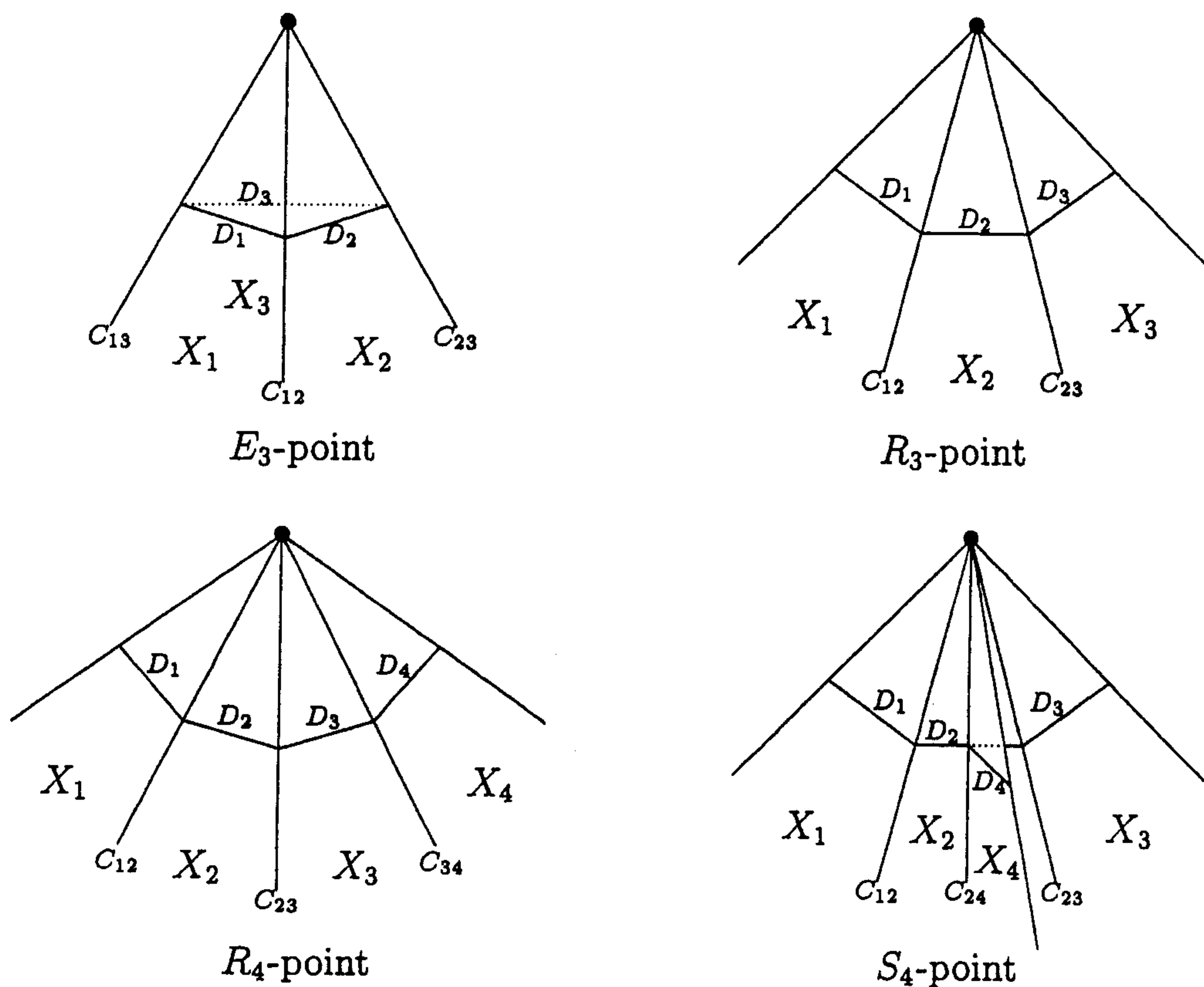


Figure 4. – Examples of good Zappatic singularities.

- each  $E_n$ -point  $P$  of  $X$  is a face of the graph whose  $n$  edges correspond to the double curves concurring at  $P$ . This is called a  $n$ -face of the graph;
- for each  $R_n$ -point  $P$ , with  $n \geq 3$ , if  $P \in X_{i_1} \cap X_{i_2} \cap \dots \cap X_{i_n}$ , where  $X_{i_j}$  meets  $X_{i_k}$  along a curve  $C_{i_j i_k}$  only if  $1 = |j - k|$ , we add in the graph a *dashed edge* joining the vertices corresponding to  $X_{i_1}$  and  $X_{i_n}$ . The dashed edge  $e_{i_1, i_n}$ , together with the other  $n - 1$  edges  $e_{i_j, i_{j+1}}$ ,  $j = 1, \dots, n - 1$ , bound an *open  $n$ -face* of the graph;
- for each  $S_n$ -point  $P$ , with  $n \geq 4$ , if  $P \in X_{i_1} \cap X_{i_2} \cap \dots \cap X_{i_n}$ , where  $X_{i_1}, \dots, X_{i_{n-1}}$  all meet  $X_{i_n}$  along curves  $C_{i_j i_n}$ ,  $j = 1, \dots, n - 1$ , concurring at  $P$ , we mark this in the graph by an *angle spanned* by the edges corresponding to the curves  $C_{i_j i_n}$ ,  $j = 1, \dots, n - 1$ .

In the sequel, when we speak of *faces* of  $G_X$  we always mean closed faces. Of course each vertex  $v_i$  is weighted with the relevant invariants of the corresponding surface  $X_i$ . We will usually omit these weights if  $X$  is planar, i.e. if all the  $X_i$ 's are planes.

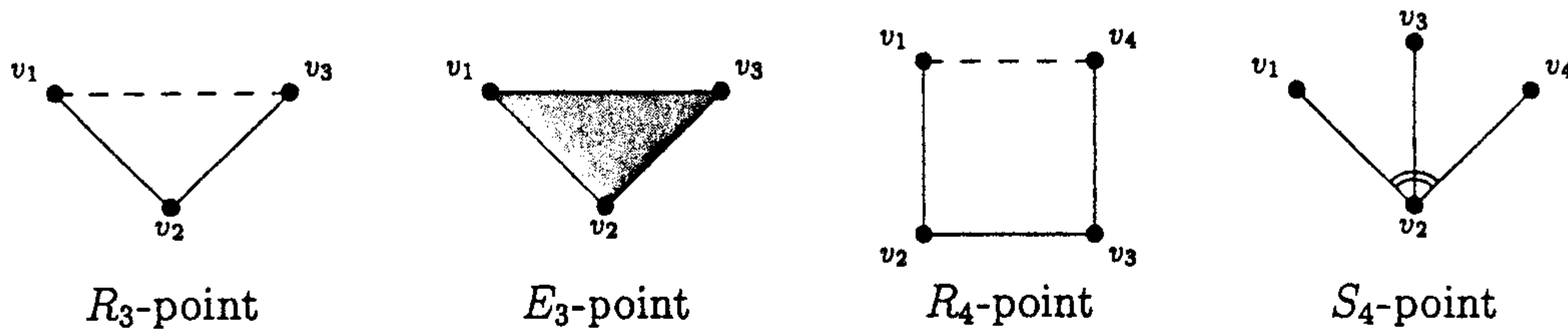


Figure 5. - Associated graphs of  $R_3$ -,  $E_3$ -,  $R_4$ - and  $S_4$ -points (cf. Figure 4).

Since each  $R_n$ -,  $S_n$ -,  $E_n$ -point is an element of some set of points  $\Sigma_{ijk}$  (cf. Notation 3.3), we remark that there can be different faces (as well as open faces and angles) of  $G_X$  which are incident on the same set of vertices and edges. However this cannot occur if  $X$  is planar.

Notice that angles, open and closed faces of  $G_X$  have been defined in order to encode the good Zappatic singularities of  $X$ . In other words, the associated graph  $G_X$  uniquely determines the configuration of the good Zappatic singularities of  $X$ .

Consider three vertices  $v_i, v_j, v_k$  of  $G_X$  in such a way that  $v_i$  is joint with  $v_j$  and  $v_k$ . Any point in  $C_{ij} \cap C_{ik}$  is either a  $R_n$ -, or a  $S_n$ -, or an  $E_n$ -point, and the curves  $C_{ij}$  and  $C_{ik}$  intersect transversally, by definition of Zappatic singularities. Hence we can compute the intersection number  $C_{ij} \cdot C_{ik}$  by adding the number of closed and open faces and of angles involving the edges  $e_{ij}, e_{ik}$ . In particular, if  $X$  is planar, for every pair of adjacent edges only one of the following possibilities occur: either they belong to an open face, or to a closed one, or to an angle. Therefore for good, planar Zappatic surfaces we can avoid to mark open 3-faces without losing any information (see Figure 6, cf. Figure 5).

REMARK 3.8. - We also notice that, given our choices, if  $X$  is good Zappatic and has only  $E_3$ -points, the graph  $G_X$  comes with a lexicographic orientation of the faces; indeed, let  $\Sigma_{ijk} = X_i \cap X_j \cap X_k = \{P_{ijk}^1, P_{ijk}^2, \dots, P_{ijk}^{m_{ijk}}\}$ ; thus, each face of  $G_X$  corresponds to a sequence of three vertices  $i, j, k$  with  $i < j < k$ , together with an integer  $t$  such that  $1 \leq t \leq m_{ijk}$ , hence it will be denoted by  $f_{\sigma(i), \sigma(j), \sigma(k)}^t$ , with  $\sigma$  any permutation of  $i, j, k$ , and will be oriented according to the orientation of its boundary determined by the sequence of vertices  $v_i, v_j, v_k$ .

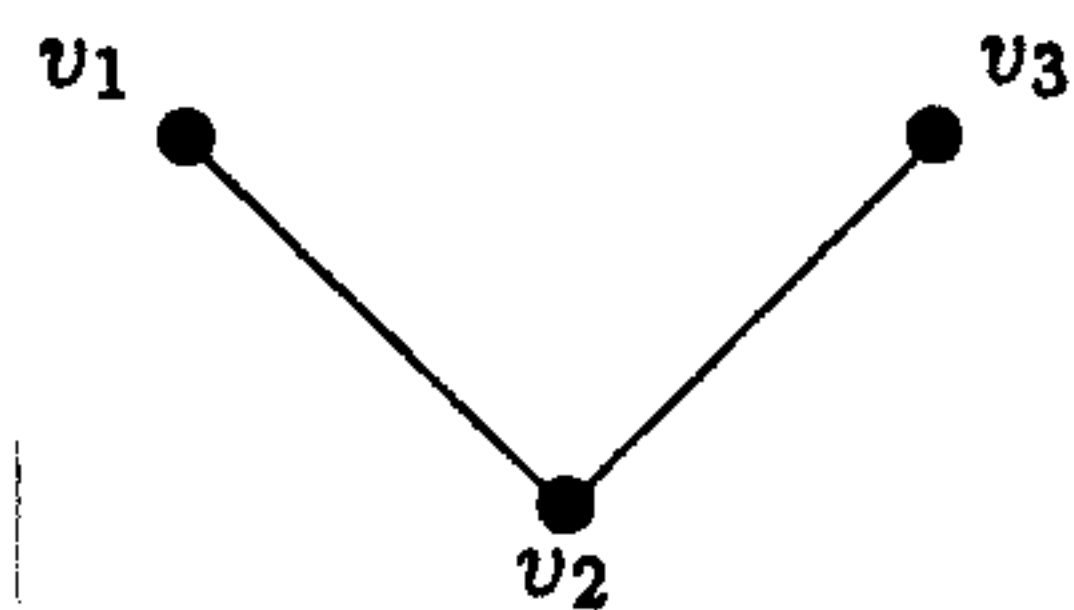


Figure 6. - Associated graph of a  $R_3$ -point in a good, planar Zappatic surface.

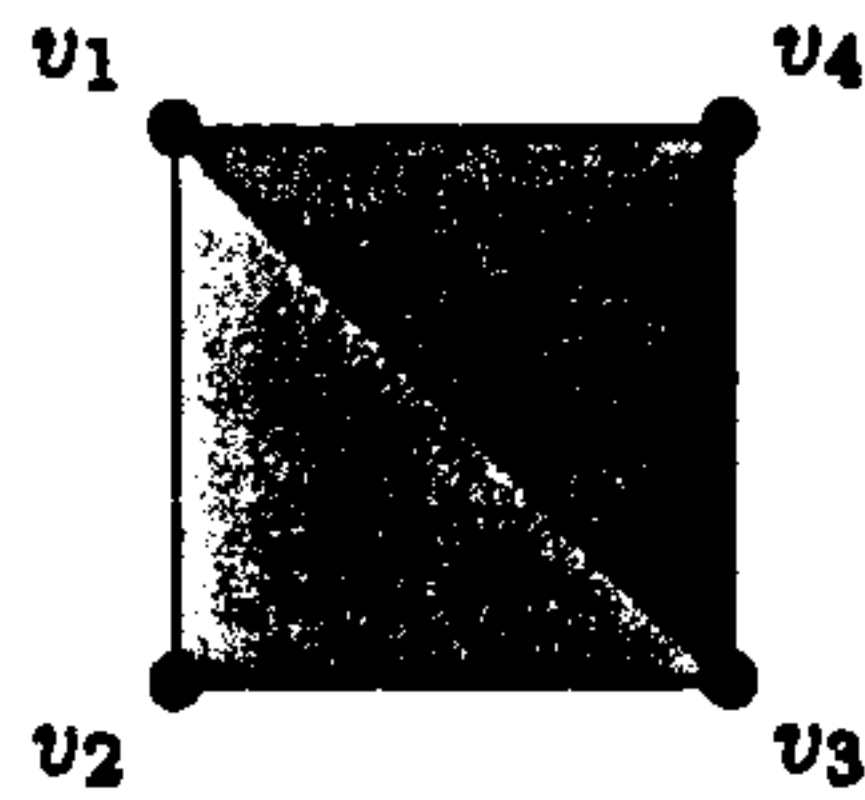


Figure 7. – Graph associated to an impossible planar Zappatic surface.

As for stick curves, if  $G$  is a given graph as above, there does not necessarily exist a good planar Zappatic surface  $X$  such that its associated graph is  $G = G_X$ .

EXAMPLE 3.9. – Consider the graph  $G$  of Figure 7. If  $G$  were the associated graph of a good planar Zappatic surface  $X$ , then  $X$  should be a global normal crossing union of 4 planes with 5 double lines and two  $E_3$  points,  $P_{123}$  and  $P_{134}$ , both lying on the double line  $C_{13}$ . Since the lines  $C_{23}$  and  $C_{34}$  (resp.  $C_{14}$  and  $C_{12}$ ) both lie on the plane  $X_3$  (resp.  $X_1$ ), they should intersect. This means that the planes  $X_2, X_4$  also should intersect along a line, therefore the edge  $e_{24}$  should appear in the graph.

Before going on, we need some notation.

NOTATION 3.10. – Let  $X$  be a good Zappatic surface (with invariants as in Notation 3.3) and let  $G = G_X$  be its associated graph. We denote by

- $v$ : the number of vertices of  $G$  (i.e. the number of irreducible components of  $X$ );
- $V$ : the (indexed) set of vertices of  $G$ ;
- $e$ : the number of edges of  $G$  (i.e. the number of double curves in  $X$ );
- $E$ : the set of edges of  $G$ ; this is indexed by the ordered pairs  $(i, j) \in V \times V$ ,  $i < j$ , such that the corresponding surfaces  $X_i, X_j$  meet along the curve  $C_{ij} = C_{ji}$ ;
- $f_n$ : the number of  $n$ -faces of  $G$ , i.e. the number of  $E_n$ -points of  $X$ , for  $n \geq 3$ ;
- $f := \sum_{n \geq 3} f_n$ , the number of faces of  $G$ , i.e. the total number of  $E_n$ -points of  $X$ , for all  $n \geq 3$ ;
- $r_n$ : the number of open  $n$ -faces of  $G$ , i.e. the number of  $R_n$ -points of  $X$ , for  $n \geq 3$ ;
- $s_n$ : the number of  $n$ -angles of  $G$ , i.e. the number of  $S_n$ -points of  $X$ , for  $n \geq 4$ ;
- $\chi(G) := v - e + f$ , i.e. the Euler-Poincaré characteristic of  $G$ ;

- $G^{(1)}$ : the 1-skeleton of  $G$ , i.e. the graph obtained from  $G$  by forgetting all the faces, dashed edges and angles;
- $\chi(G^{(1)}) = v - e$ , i.e. the Euler-Poincaré characteristic of  $G^{(1)}$ .

REMARK 3.11. – Observe that, when  $X$  is a good, planar Zappatic surface, the 1-skeleton  $G_X^{(1)}$  of  $G_X$  coincides with the dual graph  $G_D$  of the general hyperplane section  $D$  of  $X$ .

Now we can compute some of the invariants of good Zappatic surfaces.

PROPOSITION 3.12. – Let  $X = \bigcup_{i=1}^v X_i \subset \mathbb{P}^r$  be a good Zappatic surface and let  $G = G_X$  be its associated graph. Let  $C$  be the double locus of  $X$ , i.e. the union of the double curves of  $X$ ,  $C_{ij} = C_{ji} = X_i \cap X_j$  and let  $c_{ij} = \deg(C_{ij})$ . Let  $D_i$  be a general hyperplane section of  $X_i$ , and denote by  $g_i$  its genus. Then the arithmetic genus of a general hyperplane section  $D$  of  $X$  is:

$$(3.13) \quad g = \sum_{i=1}^v g_i + \sum_{e_{ij} \in E} c_{ij} - v + 1.$$

In particular, when  $X$  is a good, planar Zappatic surface, then

$$(3.14) \quad g = e - v + 1 = 1 - \chi(G^{(1)}).$$

PROOF. – Denote by  $d_i$  the degree of  $X_i$ ,  $1 \leq i \leq v$ . Then,  $D$  is the union of the  $v$  irreducible components  $D_i$ ,  $1 \leq i \leq v$ , such that  $\deg(D_i) = d_i$  and  $d := \deg(D) = \sum_{i=1}^v d_i$ . Consider its associated graph  $G_D$ , defined as in § 2.

Take  $G$ , whose indexed set of edges is denoted by  $E$ , and consider  $e_{ij} \in E$  joining its vertices  $v_i$  and  $v_j$ ,  $i < j$ , which correspond to the irreducible components  $X_i$  and  $X_j$ , respectively. Since  $e_{ij}$  in  $G$  correspond to the double curve  $C_{ij}$ , we have exactly  $c_{ij}$  oriented edges in the graph  $G_D$  joining its vertices  $v_i$  and  $v_j$ , which now correspond to the irreducible components of  $D$ ,  $D_i$  and  $D_j$ , respectively. These  $c_{ij}$  oriented edges correspond to the  $c_{ij}$  nodes of the reducible curve  $D_i \cup D_j$ , which is part of the hyperplane section  $D$ .

Now, recall that the Hilbert polynomial of  $D$  is, with our notation,  $P_D(t) = dt + 1 - g$ . On the other hand,  $P_D(t)$  equals the number of independent conditions imposed on hypersurfaces  $\mathcal{H}$  of degree  $t \gg 0$  to contain  $D$ .

From what observed above on  $G_D$ , it follows that the number of singular points of  $D$  is  $\sum_{e_{ij} \in E} c_{ij}$ . These points impose independent conditions on hypersurfaces  $\mathcal{H}$  of degree  $t \gg 0$ .

Since  $t \gg 0$  by assumption, we get that the map

$$H^0(\mathcal{O}_{\mathbb{P}^r}(t)) \rightarrow H^0(\mathcal{O}_{D_i}(t))$$

is surjective and that the line bundle  $\mathcal{O}_{D_i}(t)$  is non-special on  $D_i$ , for each  $1 \leq i \leq v$ . Thus, in order for  $\mathcal{H}$  to contain  $D_i$  we have to impose  $d_i t - g_i + 1 - \sum_{j \text{ s.t. } e_{ij} \in E} c_{ij}$  conditions. Therefore the total number of conditions for  $\mathcal{H}$  to contain  $D$  is:

$$\begin{aligned} \sum_{e_{ij} \in E} c_{ij} + \sum_{i=1}^v \left( d_i t - g_i + 1 - \sum_{j, e_{ij} \in E} c_{ij} \right) &= \sum_{e_{ij} \in E} c_{ij} + dt - \sum_{i=1}^v g_i + v - \sum_{i=1}^v \sum_{j, e_{ij} \in E} c_{ij} = \\ &= dt + v - \sum_{i=1}^v g_i - \sum_{e_{ij} \in E} c_{ij}, \end{aligned}$$

since  $\sum_{i=1}^v \sum_{j, e_{ij} \in E} c_{ij} = 2 \sum_{e_{ij} \in E} c_{ij}$ . This proves (3.13) (cf. formula (2.4)).

The second part of the statement directly follows from the above computations and from the fact that, in the good planar Zappatic case  $g_i = 0$  and  $c_{ij} = 1$ , for each  $i < j$ , i.e.  $G_D$  coincides with  $G^{(1)}$  (cf. Remark 3.11). ■

By recalling Notation 3.10, one also has:

PROPOSITION 3.15. – Let  $X = \bigcup_{i=1}^v X_i$  be a good Zappatic surface and  $G_X$  be its associated graph. Let  $C$  be the double locus of  $X$ , which is the union of the curves  $C_{ij} = C_{ji} = X_i \cap X_j$ . Then:

$$(3.16) \quad \chi(\mathcal{O}_X) = \sum_{i=1}^v \chi(\mathcal{O}_{X_i}) - \sum_{e_{ij} \in E} \chi(\mathcal{O}_{C_{ij}}) + f.$$

In particular, when  $X$  is a good, planar Zappatic surface, then

$$(3.17) \quad \chi(\mathcal{O}_X) = \chi(G_X) = v - e + f.$$

PROOF. – We can consider the sheaf morphism:

$$(3.18) \quad \bigoplus_{i=1}^v \mathcal{O}_{X_i} \xrightarrow{\lambda} \bigoplus_{1 \leq i < j \leq v} \mathcal{O}_{C_{ij}},$$

defined in the following way: if

$$\pi_{ij}: \bigoplus_{1 \leq i < j \leq v} \mathcal{O}_{C_{ij}} \rightarrow \mathcal{O}_{C_{ij}}$$

denotes the projection on the  $(ij)^{\text{th}}$ -summand, then

$$(\pi_{ij} \circ \lambda)(h_1, \dots, h_v) := h_i - h_j.$$

Notice that the definition of  $\lambda$  is consistent with the lexicographic order of the indices and with the lexicographic orientation of the edges of the graph  $G_X$ .

Observe that, if  $\tilde{X}$  denotes the desingularization of  $X$ , then  $\tilde{X}$  is isomorphic to the disjoint union of the smooth, irreducible components  $X_i$ ,  $1 \leq i \leq v$ , of  $X$ .

Therefore, by the very definition of  $\mathcal{O}_X$ , we see that

$$\ker(\lambda) \cong \mathcal{O}_X.$$

We claim that the morphism  $\lambda$  is not surjective and that its cokernel is a sky-scraper sheaf supported at the  $E_n$ -points of  $X$ . To show this, we focus on any irreducible component of  $C = \bigcup_{1 \leq i < j \leq v} C_{ij}$ , the double locus of  $X$ .

Fix any index pair  $(i, j)$ , with  $i < j$ , and consider the generator

$$(3.19) \quad (0, \dots, 0, 1, 0, \dots, 0) \in \bigoplus_{1 \leq l < m \leq v} \mathcal{O}_{C_{lm}},$$

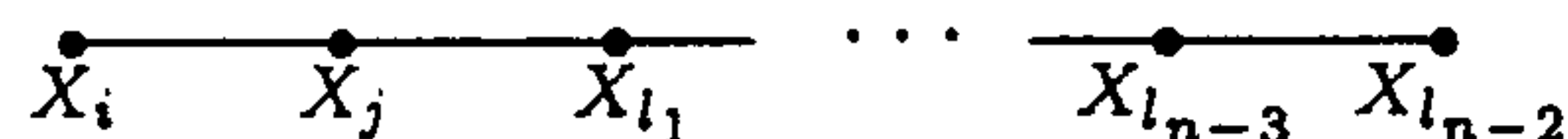
where  $1 \in \mathcal{O}_{C_{ij}}$ , the  $(ij)^{\text{th}}$ -summand. The obstructions to lift up this element to an element of  $\bigoplus_{1 \leq t \leq v} \mathcal{O}_{X_t}$  are given by the presence of good Zappatic singularities of  $X$  along  $C_{ij}$ .

For what concerns the irreducible components of  $X$  which are not involved in the intersection determining a good Zappatic singularity on  $C_{ij}$ , the element in (3.19) trivially lifts-up to 0 on each of them. Thus, in the sequel, we shall focus only on the irreducible components involved in the Zappatic singularity, which will be denoted by  $X_i, X_j, X_{l_t}$ , for  $1 \leq t \leq n - 2$ .

We have to consider different cases, according to the good Zappatic singularity type lying on the curve  $C_{ij} = X_i \cap X_j$ .

• Suppose that  $C_{ij}$  passes through a  $R_n$ -point  $P$  of  $X$ , for some  $n$ ; we have two different possibilities. Indeed:

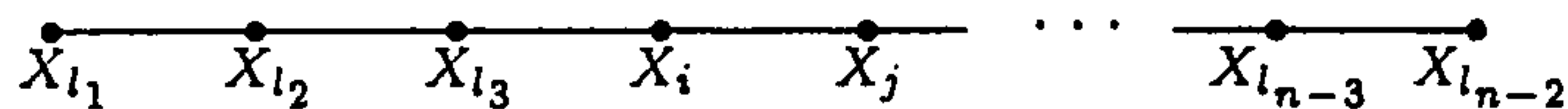
(a) let  $X_i$  be an «external» surface for  $P$  — i.e.  $X_i$  corresponds to a vertex of the associated graph of  $P$  which has valence 1. Therefore, we have:



In this situation, the element in (3.19) lifts up to

$$(1, 0, \dots, 0) \in \mathcal{O}_{X_i} \oplus \mathcal{O}_{X_j} \oplus \bigoplus_{1 \leq t \leq n-2} \mathcal{O}_{X_{l_t}}.$$

(b) let  $X_i$  be an «internal» surface for  $P$  — i.e.  $X_i$  corresponds to a vertex of the associated graph to  $P$  which has valence 2. Thus, we have a picture like:



In this case, the element in (3.19) lifts up to the  $n$ -tuple having components:

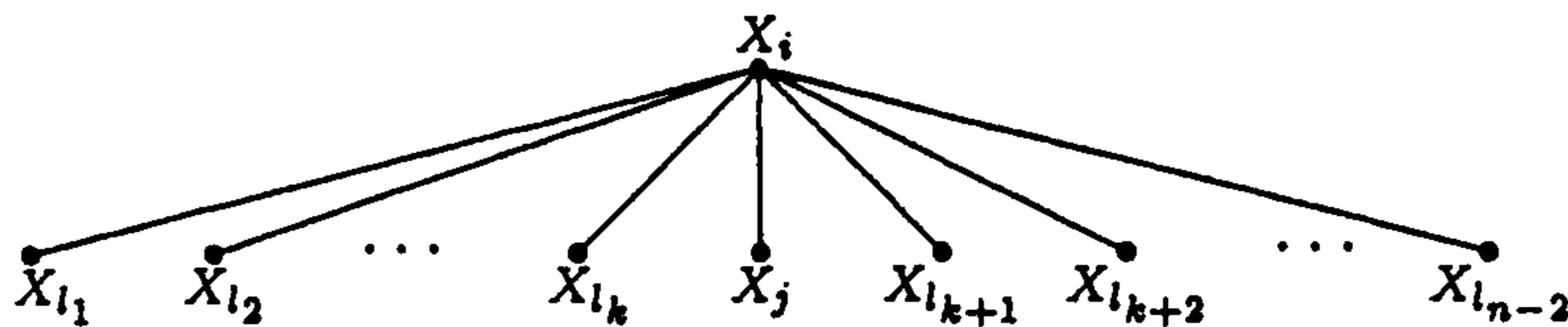
$$\begin{aligned} 1 &\in \mathcal{O}_{X_i}, \\ 0 &\in \mathcal{O}_{X_j}, \end{aligned}$$

$1 \in \mathcal{O}_{X_k}$ , for those  $X_k$ 's corresponding to vertices in the graph associated to  $P$  which are on the left of  $X_i$  and,

$0 \in \mathcal{O}_{X_k}$  for those  $X_k$ 's corresponding to vertices in the graph associated to  $P$  which are on the right of  $X_j$ .

• Suppose that  $C_{ij}$  passes through a  $S_n$ -point  $P$  of  $X$ , for any  $n$ ; as before, we have two different possibilities. Indeed:

(a) let  $X_i$  corresponds to the vertex of valence  $n - 1$  in the associated graph to  $P$ , i.e.



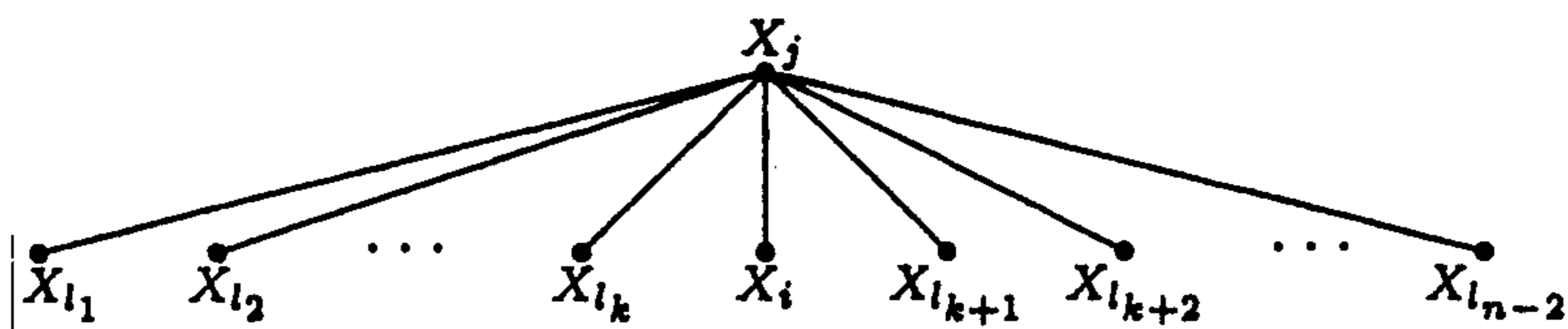
In this situation, the element in (3.19) lifts up to the  $n$ -tuple having components:

$$1 \in \mathcal{O}_{X_i},$$

$$0 \in \mathcal{O}_{X_j},$$

$$1 \in \mathcal{O}_{X_k}, \text{ for all } 1 \leq t \leq n - 2.$$

(b) let  $X_i$  corresponds to a vertex of valence 1 in the associated graph to  $P$ . Since  $C_{ij} \neq \emptyset$  by assumption, then  $X_j$  has to be the vertex of valence  $n - 1$ , i.e. we have the following picture:



Thus, the element in (3.19) lifts up to the  $n$ -tuple having components

$$1 \in \mathcal{O}_{X_i},$$

$$0 \in \mathcal{O}_{X_j},$$

$$0 \in \mathcal{O}_{X_k}, \text{ for all } 1 \leq t \leq n - 2.$$

• Suppose that  $C_{ij}$  passes through an  $E_n$ -point  $P$  for  $X$ . Then, each vertex of the associated graph to  $P$  has valence 2. Since such a graph is a cycle, it is clear that no lifting of (3.19) can be done.

To sum up, we see that  $\text{coker}(\lambda)$  is supported at the  $E_n$ -points of  $X$ . Furthermore, if we consider

$$\bigoplus_{1 \leq i < j \leq v} \mathcal{O}_{C_{ij}} \xrightarrow{ev_P} \mathcal{O}_P = \underline{C}_P, \quad \bigoplus f_{ij} \mapsto \sum f_{ij}(P)$$



it is clear that, if  $P$  is an  $E_n$ -point then

$$ev_P\left(\bigoplus_{1 \leq i < j \leq v} \mathcal{O}_{C_{ij}} / \text{Im}(\lambda)\right) \cong \underline{C}_P.$$

This means that

$$\text{coker}(\lambda) \cong \underline{C}^f.$$

By the exact sequences

$$0 \rightarrow \mathcal{O}_X \rightarrow \bigoplus_{1 \leq i \leq v} \mathcal{O}_{X_i} \rightarrow \text{Im}(\lambda) \rightarrow 0, \quad 0 \rightarrow \text{Im}(\lambda) \rightarrow \bigoplus_{1 \leq i < j \leq v} \mathcal{O}_{C_{ij}} \rightarrow \underline{C}^f \rightarrow 0,$$

we get (3.16). ■

In the next section we will see how the computation of the geometric genus is much more involved, even in the case of  $X$  having only  $E_3$ -points as Zappatic singularities.

#### 4. – The geometric genus of a Zappatic surface with only $E_3$ -points.

The main purpose of this section is to compute the *geometric genus* of a projective good Zappatic surface  $X = \bigcup_{i=1}^v X_i$  (cf. Remark 3.4), which is assumed to have only  $E_3$ -points, i.e. global normal crossing singularities. In terms of its associated graph, this means that  $G_X$  is a subgraph of the complete graph on  $v$  vertices, which has only 3-faces (i.e. triangles).

We first want to recall some definitions and results, which will be used in the sequel.

DEFINITION 4.1. – Let  $T$  be a smooth surface and  $C$  be a smooth, irreducible curve on  $T$ . Let  $\omega$  be a global meromorphic 2-form on  $T$  whose polar locus contains  $C$ . We may assume that  $x, y$  are local coordinates on  $T$  in an analytic neighbourhood of a point of  $C$  in such a way that  $y = 0$  is the local equation defining  $C$ . Then, the *Poincaré residue map* (or *adjunction map*)

$$\omega_T \otimes \mathcal{O}_T(C) \xrightarrow{\mathcal{R}_C} \omega_C$$

is locally defined by:

$$\omega = \frac{f(x, y)}{y} dx \wedge dy \mapsto -f(x, 0) dx$$

and  $-f(x, 0)dx$  is called the (*Poincaré residue*) of the 2-form  $\omega$  along  $C$  (see [10, p. 147]).

If, more generally,  $C$  is assumed to be a reduced (possibly reducible) divi-

tor with only normal crossing singularities, denote by  $\omega_C$  its dualizing sheaf and take local coordinates on  $T$  in an analytic neighbourhood of a node of  $C$  in such a way that  $xy = 0$  is its local defining equation. If

$$\omega = \frac{f(x, y)}{xy} dx \wedge dy,$$

then one defines the Poincaré residue map by considering

$$(4.2) \quad \mathcal{R}_C(\omega) \in H^0(C, \omega_C)$$

defined as the pair of forms:

$$\begin{aligned} \text{(i)} \quad \omega_x &= -\frac{f(x, 0)}{y} dx \text{ on the branch } y = 0, \\ \text{(ii)} \quad \omega_y &= \frac{f(0, y)}{x} dy \text{ on the branch } x = 0. \end{aligned}$$

REMARK 4.3. – If  $\mathcal{R}_0(\omega_x)$  denotes the usual Poincaré residue at the point  $x = 0$  of the meromorphic 1-form  $\omega_x$  on the (smooth) branch  $y = 0$ , observe that

$$(4.4) \quad \mathcal{R}_0(\omega_y) = -\mathcal{R}_0(\omega_x).$$

This is consistent with the definition of  $H^0(C, \omega_C)$ . Indeed, assume that  $C$  has only  $m$  nodes; then, if  $\nu: \tilde{C} \rightarrow C$  is the normalization morphism and if  $\{q_1, q'_1\}$ ,  $\{q_2, q'_2\}$ ,  $\dots$ ,  $\{q_m, q'_m\}$  are the pre-images in  $\tilde{C}$  of the  $m$  nodes of  $C$ , then  $\omega_C$  is the invertible subsheaf

$$\omega_C \subset \nu_* \left( \omega_{\tilde{C}} \left( \sum_{i=1}^m (q_i + q'_i) \right) \right)$$

such that a section  $\sigma$  of  $\nu_* \left( \omega_{\tilde{C}} \left( \sum_{i=1}^m (q_i + q'_i) \right) \right)$ , viewed as a section of  $\omega_{\tilde{C}} \left( \sum_{i=1}^m (q_i + q'_i) \right)$ , is a section of  $\omega_C$  if and only if

$$\mathcal{R}_{q_i}(\sigma) + \mathcal{R}_{q'_i}(\sigma) = 0, \quad 1 \leq i \leq m.$$

Unless otherwise stated, from now on  $X = \bigcup_{i=1}^v X_i$  will denote a projective, Zappatic surface with  $E_3$ -points only as Zappatic singularities and we use notation as in Definition 3.2 and in Notation 3.10.

It is well known that, for each  $i$ :

$$(4.5) \quad \omega_{X_i} \cong \omega_X \otimes \mathcal{O}_{X_i}(-C_i), \quad \text{with } C_i := X_i \cap \overline{(X \setminus X_i)} = \sum_{j \neq i} C_{ij},$$

where  $\omega_{X_i}$  is the canonical line bundle of  $X_i$ , whereas  $\omega_X$  denotes the dualizing sheaf of  $X$ . Note that  $\omega_X$  is an invertible sheaf by the hypotheses on  $X$ .

As in Remark 3.4, recall that the geometric genus of  $X$  is denoted

by  $p_g(X)$  and defined as  $p_g(X) = h^0(X, \omega_X)$ . In order to compute  $p_g(X)$ , we need some further remarks which will be fundamental in the sequel.

REMARK 4.6. — Observe that, if  $X = \bigcup_{i=1}^v X_i$  is as above, since the intersection  $X_i \cap X_j$  — when non-empty — is the double curve  $C_{ij} = C_{ji}$ , the index pair  $(i, j)$  with  $i < j$  uniquely determines the double curve  $C_{ij}$ . In the same way, when  $i < j$ , since the intersection  $X_i \cap X_j \cap X_k$  — when non-empty — is the triple point set  $\Sigma_{ijk} = \Sigma_{\tau(i)\tau(j)\tau(k)}$ , for  $k \neq i, j$  and for any  $\tau \in \text{Sym}(\{i, j, k\})$ , then the lexicographically ordered index triple uniquely determines the triple point set either  $\Sigma_{kij}$ , or  $\Sigma_{ikj}$ , or  $\Sigma_{ijk}$ , according to the fact that either  $k < i$ , or  $i < k < j$ , or  $k > j$ .

REMARK 4.7. — Since the only Zappatic singularities of  $X$  are assumed to be  $E_3$ -points, then  $G_X$  contains neither dashed edges, nor angles, nor open faces, nor  $n$ -faces, with  $n \geq 4$ . Furthermore, in such a case the graph  $G_X$  comes with a lexicographic orientation of the faces (see Remark 3.8). If  $X$  is, in particular, planar recall that we have strong constraints on the possible shape of the graph  $G_X$  — because of the geometry of planes (cf. Example 3.9) — and each non-zero  $m_{ijk}$  equals one.

Observe that by the connectedness hypothesis of  $X = \bigcup_{i=1}^v X_i$ , we get that  $C_i \neq \emptyset$ , for each  $1 \leq i \leq v$ . For simplicity of notation, in the sequel we shall always denote by  $C_{ij}$  the intersection of  $X_i$  and  $X_j$ , for any  $1 \leq i < j \leq v$ , with the obvious further condition that  $C_{ij} = C_{ji} = \emptyset$  when the index pair corresponds to two disjoint surfaces in  $X$ , i.e. when there is no edge  $(v_i, v_j)$  in the associated graph  $G_X$ .

We can define a natural map

$$(4.8) \quad \Phi : \bigoplus_{i=1}^v H^1(X_i, \mathcal{O}_{X_i}) \rightarrow \bigoplus_{1 \leq i < j \leq v} H^1(C_{ij}, \mathcal{O}_{C_{ij}})$$

in the following way: if

$$\pi_{ij} : \bigoplus_{1 \leq i < j \leq v} H^1(C_{ij}, \mathcal{O}_{C_{ij}}) \rightarrow H^1(C_{ij}, \mathcal{O}_{C_{ij}})$$

denotes the projection on the  $(ij)$ <sup>th</sup>-summand and if  $r_{C_{ij}}^{(i)} : H^1(X_i, \mathcal{O}_{X_i}) \rightarrow H^1(C_{ij}, \mathcal{O}_{C_{ij}})$  denotes the natural restriction map to  $C_{ij}$  as a divisor in  $X_i$ , where  $i < j$ , then

$$(4.9) \quad (\pi_{ij} \circ \Phi)((a_1, \dots, a_v)) := r_{C_{ij}}^{(i)}(a_i) - r_{C_{ij}}^{(j)}(a_j).$$

REMARK 4.10. — Observe that the above definition is consistent with the lexicographic order of the indices  $1 \leq i \leq v$ . In other words, (4.9) means that we consider the curve  $C_{ij}$  as a *positive curve* on the surface  $X_i$  and as a *negative*

curve on the surface  $X_j$ , when  $i < j$ . Furthermore, when the index pair is such that  $C_{ij} = \emptyset$ , we obviously consider  $H^1(C_{ij}, \mathcal{O}_{C_{ij}})$  as the zero-vector space and  $\pi_{ij} \circ \Phi$  as the zero-map.

Take an index pair  $i < j$  such that  $C_{ij} \neq \emptyset$ . By the adjunction sequence of  $C_{ij}$  on  $X_i$  and on  $X_j$ , we can consider the two obvious coboundary maps:

$$(4.11) \quad \begin{array}{ccc} & & H^1(X_i, \omega_{X_i}) \\ & \nearrow^{\delta_i} & \\ H^0(C_{ij}, \omega_{C_{ij}}) & & \\ & \searrow_{\delta_j} & \\ & & H^1(X_j, \omega_{X_j}). \end{array}$$

On the other hand, when the index pair  $i < j$  is such that  $C_{ij} = \emptyset$ , then  $H^0(C_{ij}, \omega_{C_{ij}})$  is considered as the zero-vector space and (4.11) are the zero-maps. Then, we can define the map

$$(4.12) \quad \Delta: \bigoplus_{1 \leq i < j \leq v} H^0(C_{ij}, \omega_{C_{ij}}) \rightarrow \bigoplus_{i=1}^v H^1(X_i, \omega_{X_i})$$

in the following way: if

$$\iota_{ij}: H^0(C_{ij}, \omega_{C_{ij}}) \hookrightarrow \bigoplus_{1 \leq i < j \leq v} H^0(C_{ij}, \omega_{C_{ij}})$$

denotes the natural inclusion of the  $(ij)^{\text{th}}$ -summand and if  $\gamma_{ij} \in H^0(C_{ij}, \omega_{C_{ij}})$ , then

$$(4.13) \quad (\Delta \circ \iota_{ij})(\gamma_{ij}) := (0, \dots, 0, \delta_i(\gamma_{ij}), 0, \dots, 0, -\delta_j(\gamma_{ij}), 0, \dots, 0),$$

where  $i < j$ .

Observe that the definition of  $\Delta$  is consistent with the lexicographic order of the indices  $1 \leq i \leq v$  and with our Remark 4.10.

The following preliminary result is an obvious consequence of our definitions.

PROPOSITION 4.14. – *With notation as above, we have*

$$\Delta = \Phi^v.$$

PROOF. – The proof directly follows from Serre's duality on each summand and from the fact that the matrix which represents  $\Delta$  is the transpose of the one representing  $\Phi$ . ■

We are now able to prove the main result of this section.

THEOREM 4.15. – Let  $X = \bigcup_{i=1}^v X_i$  be a projective, good Zappatic surface with only  $E_3$ -points as Zappatic singularities. Denote by  $\omega_X$  the dualizing sheaf of  $X$  and by  $G_X$  the associated graph of  $X$  (see Definition 3.7). Let  $\Phi$  be the map defined in (4.8), where  $C_{ij}$  irreducible for each  $1 \leq i < j \leq v$ , and let  $p_g(X)$  be the geometric genus of  $X$  as in Remark 3.4. Then the following inequality:

$$(4.16) \quad p_g(X) \leq b_2(G_X) + \sum_{i=1}^v p_g(X_i) + \dim(\text{coker}(\Phi))$$

holds, where as customary  $b_2(G_X)$  is the second Betti number of  $G_X$ .

Furthermore, a sufficient condition for the equality in (4.16) to hold is either that

(i) each irreducible component  $X_i$  is a regular surface, for  $1 \leq i \leq v$ , or that

(ii) for any irregular component  $X_j$  of  $X$ , the divisor  $C_j = X_j \cap \overline{(X \setminus X_j)}$  is ample on  $X_j$ .

PROOF. – To prove the first part of the statement, we construct a homomorphism

$$(4.17) \quad H^0(\omega_X) \xrightarrow{f} H_2(G_X, \mathbb{C})$$

and we show that

$$(4.18) \quad \ker(f) \cong \bigoplus_{i=1}^v H^0(X_i, \omega_{X_i}) \oplus (\text{coker}(\Phi)).$$

Then, for what concerns the second part, we prove that either hypothesis (i) or hypothesis (ii) implies the surjectivity of the map (4.17).

Recall that, from our hypotheses it follows that  $G_X$  is a subgraph of the complete graph on  $d$  vertices which contains only 3-faces and that there can be more than one 3-face incident on the same triple of vertices (equiv. edges). This occurs when (cf. Notation 3.3)  $m_{ijk} > 1$ , for a given triple  $v_i, v_j, v_k$  of vertices of  $G_X$ . It is obvious that, when  $\Sigma_{ijk} = \emptyset$ , then  $m_{ijk} = 0$ .

To construct  $f$ , by (4.5), we consider a global section  $\omega \in H^0(X, \omega_X)$  as a collection

$$\{\omega_i\}_{1 \leq i \leq v} \in \bigoplus_{1 \leq i \leq v} H^0(X_i, \omega_{X_i}(C_i)),$$

where each  $\omega_i$  is a global meromorphic 2-form on the corresponding irreducible component  $X_i$  having simple polar locus along the (possibly reducible) curve

$$C_i = X_i \cap \overline{(X \setminus X_i)} = \sum_{j \neq i} C_{ij},$$

for each  $1 \leq i \leq v$  (recall that  $C_{ij} = C_{ji}$  and that  $C_i \neq \emptyset$  for each  $1 \leq i \leq v$ , because of the connectedness hypothesis of  $X$ ).

Take an index pair  $i < j$  such that  $C_{ij} \neq \emptyset$  and consider  $C_{ij}$ , which is both an irreducible component of  $C_i$  and of  $C_j$ . As in (4.2), take  $\mathcal{R}_{C_{ij}}(\omega_i)$  the Poincaré residue of  $\omega_i$  on  $C_{ij}$  and denote it by  $\omega_{ij}$ . Then, we have

$$(4.19) \quad \omega_{ij} = -\omega_{ji},$$

for each  $1 \leq i < j \leq v$ . In the trivial case  $C_{ij} = \emptyset$ , we have  $\omega_{ij} = \omega_{ji} = 0$ ; so (4.19) holds.

Fix an index pair  $i < j$  such that  $C_{ij} \neq \emptyset$ . By recalling our Remark 4.6 and by our hypotheses, when  $X_i \cap X_j \cap X_k \neq \emptyset$  the lexicographically ordered index triple uniquely determine the triple point set either  $\Sigma_{kij}$ , or  $\Sigma_{ikj}$ , or  $\Sigma_{ijk}$  on  $C_{ij}$ , according to the case that either  $k < i$ , or  $i < k < j$ , or  $k > j$ . Now, given  $i < j$  as above, we can consider  $\omega_{ij}$  which is a meromorphic 1-form on the curve  $C_{ij} \subset X_i$  having simple poles at the points in  $\Sigma_{kij}, \Sigma_{ikj}, \Sigma_{ijk} \subset C_{ij}$  defined above and determined by those  $k \neq i, j$  such that  $X_i \cap X_j \cap X_k \neq \emptyset$ . Otherwise, when  $k$  is such that  $\Sigma_{ijk} = \emptyset$ , then  $\omega_{ij}$  must be considered as holomorphic at  $\Sigma_{ijk}$  and its residues at the empty point set are zero; such  $k$ 's are determined by those vertices  $v_k$  in  $G_X$  which do not form any 3-face with the edge  $e_{ij} = (v_i, v_j)$ .

On the other hand, if the index pair  $i < j$  is such that  $C_{ij} = \emptyset$ , one has that  $\Sigma_{kij} = \Sigma_{ikj} = \Sigma_{ijk} = \emptyset$  for each  $k \neq i, j$  and that each  $\omega_{ij}$  is zero.

Therefore, for simplicity of notation, for any index pair  $i < j$  we write

$$\omega_{ij} \in H^0\left(C_{ij}, \omega_{C_{ij}}\left(\sum_{k < i} \Sigma_{kij} + \sum_{k \in (i, j)} \Sigma_{ikj} + \sum_{k > j} \Sigma_{ijk}\right)\right),$$

recalling that, if  $C_{ij} = \emptyset$ , the above is the zero-vector space otherwise, if  $C_{ij} \neq \emptyset$  but some triple point set is the empty set, its points do not impose any pole to the meromorphic form  $\omega_{ij}$ .

In any case, one can compute the Poincaré residues at the given points, namely  $\mathcal{R}_{P_{kij}^r}(\omega_{ij})$ ,  $\mathcal{R}_{P_{ikj}^s}(\omega_{ij})$  and  $\mathcal{R}_{P_{ijk}^t}(\omega_{ij})$ , for any  $1 \leq r \leq m_{kij}$ ,  $1 \leq s \leq m_{ikj}$  and  $1 \leq t \leq m_{ijk}$ .

To simplify our notation, if e.g.  $k > j$ , we write  $(\omega_{ij}^t)_k$  (or directly  $\omega_{ijk}^t$ ) to denote the Poincaré residue  $\mathcal{R}_{P_{ijk}^t}(\omega_{ij})$  of  $\omega_{ij}$  at the  $t^{\text{th}}$  point  $P_{ijk}^t$  of the set  $\Sigma_{ijk}$ , for any  $1 \leq t \leq m_{ijk}$ . Similar notation for the other two cases.

As observed in Remark 4.6, if we fix  $i < j < k$  we focus on the triple point set  $\Sigma_{ijk}$  of  $X$ , which is given by

$$X_i \cap X_j \cap X_k = C_{ij} \cap C_{ik} \cap C_{jk},$$

where

$$C_{ij}, C_{ik} \subset X_i, \quad C_{ji} = C_{ij}, C_{jk} \subset X_j, \quad C_{ki} = C_{ik}, C_{kj} = C_{jk} \subset X_k.$$

Therefore, at any given triple point  $P_{ijk}^t$ , with  $i < j < k$  and  $1 \leq t \leq m_{ijk}$ , one can

compute six different residues. Indeed, once we choose one of the three surfaces as the ambient variety, we have two different possible choices of smooth, irreducible curves (and so of meromorphic 1-forms) to use for such a computation. By taking into account (4.19), Remark 4.3 and the lexicographic order of the indices, the residues at  $P_{ijk}^t$  satisfy

$$(4.20) \quad \omega_{\sigma(i)\sigma(j)\sigma(k)}^t = \operatorname{sgn}(\sigma) \omega_{ijk}^t,$$

where  $i < j < k$ ,  $1 \leq t \leq m_{ijk}$  and where  $\sigma \in \operatorname{Sym}(\{i, j, k\})$ . Therefore, for each section  $\omega \in H^0(X, \omega_X)$  there is, up to sign, a well determined value associated to each point  $P_{ijk}^t \in \Sigma_{ijk}$ . Recall that each such value is zero either if  $\Sigma_{ijk}$  is the empty point set or if some of the double curves is the empty set.

For  $i < j$  such that  $C_{ij} \neq \emptyset$ , the subsets of triple points of  $X$  lying on the curve  $C_{ij}$  are parametrized by those indices  $k \neq i, j$  such that the vertex  $v_k$  of  $G_X$  forms a number bigger than or equal to one of faces with the edge  $e_{ij} = (v_i, v_j)$ . By the Residue theorem on  $C_{ij}$  and by (4.20), we get

$$(4.21) \quad \sum_{k \neq i, j} \sum_{t=1}^{m_{\sigma_k(i)\sigma_k(j)\sigma_k(k)}} \operatorname{sgn}(\sigma_k) \omega_{\sigma_k(i)\sigma_k(j)\sigma_k(k)}^t = 0,$$

where  $\sigma_k \in \operatorname{Sym}(\{i, j, k\})$  is the permutation which lexicographically reorders the index triples  $(i, j, k)$ .

Otherwise, if  $C_{ij} = \emptyset$ , then (4.21) is trivially true, i.e. a sum of zeroes equals zero.

Choose, once and for all, the lexicographic orientation on the graph  $G_X$ . Therefore, if the edge  $\lambda_{ij} = (v_i, v_j)$  belongs to  $G_X$ , then it is an arrow from  $v_i$  to  $v_j$  if and only if  $i < j$ . Recall that a set  $\Sigma_{ijk}$ , for some  $i < j < k$ , corresponds to  $m_{ijk} = |\Sigma_{ijk}|$  faces of  $G_X$  insisting on the triple of vertices  $\{v_i, v_j, v_k\}$ . For any  $1 \leq t \leq m_{ijk}$ , we associate to the  $t^{\text{th}}$ -face the residue  $\omega_{ijk}^t$  computed as above.

From (4.20), (4.21) and from the fact that  $G_X$  is a 2-dimensional graph, it follows that the above computations determine a 2-cycle  $\{\omega_{ijk}^t\}$  of the graph  $G_X$ .

To sum up, the map  $f$  is defined as a composition of maps in the following way:

$$H^0(\omega_X) \xrightarrow{i} \bigoplus H^0(\omega_{X_i}(C_i)) \xrightarrow{a} \bigoplus H^0(\omega_{C_i}) \xrightarrow{b}$$

$$\bigoplus H^0\left(\omega_{C_{ij}}\left(\sum_{k,t} P_{ijk}^t\right)\right) \xrightarrow{c} \bigoplus H^0\left(\omega_{C_{ij}}\left(\sum_{k,t} P_{ijk}^t\right)\right)_{|P_{ijk}^t} \rightarrow H_2(G_X, \mathbb{C})$$

where  $i$  is the natural inclusion and  $a, b, c$  are given by the following exact sequences:

$$\begin{aligned} 0 &\rightarrow \bigoplus H^0(\omega_{X_i}) \rightarrow \bigoplus H^0(\omega_{X_i}(C_i)) \xrightarrow{a} \bigoplus H^0(\omega_{C_i}) \\ 0 &\rightarrow \bigoplus H^0(\omega_{C_i}(-C_{ij})) \rightarrow \bigoplus H^0(\omega_{C_i}) \xrightarrow{b} \bigoplus H^0(\omega_{C_i|C_{ij}}) \cong \bigoplus H^0\left(\omega_{C_{ij}}\left(\sum_{k,t} P_{ijk}^t\right)\right) \\ 0 &\rightarrow \bigoplus H^0\left(\omega_{C_{ij}}\left(\sum_{l \neq k, r \neq t} P_{ijl}^r\right)\right) \rightarrow \bigoplus H^0\left(\omega_{C_{ij}}\left(\sum_{l,r} P_{ijl}^r\right)\right) \xrightarrow{c} \bigoplus H^0\left(\omega_{C_{ij}}\left(\sum_{l,r} P_{ijl}^r\right)\right)_{|P_{ijk}^t}. \end{aligned}$$

To compute  $\ker(f)$ , take as above  $C_i = X_i \cap \overline{(X \setminus X_i)} = \sum_{j \neq i} C_{ij}$ , where we recall that  $C_{ij} = C_{ji}$ , for  $i \neq j$ , some — but not all — of them possibly empty. As in formulas (4.2) and (4.11), denote by  $\mathcal{R}_{C_i}$  the Poincaré residue map on the reducible, nodal curve  $C_i \subset X_i$  and by  $\delta_i$  the coboundary map on the surface  $X_i$ , for  $1 \leq i \leq v$ . Thus, the first row of the diagram:

$$(4.22) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{i=1}^v H^0(\omega_{X_i}) & \longrightarrow & \bigoplus_{i=1}^v H^0(\omega_{X_i}(C_i)) & \xrightarrow{\oplus \mathcal{R}_{C_i}} & \bigoplus_{i=1}^v H^0(\omega_{C_i}) & \xrightarrow{\oplus \delta_i} & \bigoplus_{i=1}^v H^1(\omega_{X_i}) \\ & & \searrow \zeta & & \uparrow \iota & & \uparrow \beta & & \nearrow \Delta \\ & & & & \ker(f) & \xrightarrow{\sigma_{1/2}} & \bigoplus_{i < j} H^0(\omega_{C_{ij}}) & & \end{array}$$

is naturally defined and exact. Apart from  $\Delta$ , introduced in (4.12), our aim is to define the maps  $\zeta, \iota, \sigma_{1/2}$  and  $\beta$  in such a way that the whole diagram commutes and that the subsequence determined by the maps  $\zeta, \sigma_{1/2}$  and  $\Delta$  is exact.

Obviously,  $\zeta$  and  $\iota$  are natural inclusions by the very definition of  $H^0(X, \omega_X)$ . For what concerns the map  $\beta$ , it suffices to define its image in one direct summand, i.e.

$$(\pi_h \circ \beta): \bigoplus_{i < j} H^0(\omega_{C_{ij}}) \rightarrow H^0(\omega_{C_h}),$$

where  $\pi_h$  is the projection on the  $h^{\text{th}}$ -summand, for a given  $h \in \{1, \dots, v\}$ . When  $h \neq i, j$ , the image is 0, therefore the relevant summands are the following:

$$(4.23) \quad \begin{aligned} &\bigoplus_{h < j} H^0(\omega_{C_{hj}}) \oplus \bigoplus_{i < h} H^0(\omega_{C_{ih}}) \rightarrow H^0(\omega_{C_h}), \\ &\left( \bigoplus_{h < j} \omega_{hj}, \bigoplus_{i < h} \omega_{ih} \right) \mapsto \sum_{h < j} \omega_{hj} - \sum_{i < h} \omega_{ih}, \end{aligned}$$

with the obvious condition that  $\omega_{lm} = 0$  when  $C_{lm} = \emptyset$ . First of all observe that  $\beta$  is well-defined by the definition of  $H^0(C_h, \omega_{C_h})$  (see Remark 4.3); moreover, the coefficients  $\pm 1$  are uniquely determined by the fact that  $C_h \subset X_h$  and by Remark 4.10.



To define  $\sigma_{1/2}$ , recall that  $\ker(f) \subseteq H^0(X, \omega_X) \subset \bigoplus_{i=1}^v H^0(\omega_{X_i}(C_i))$ ; thus, an element in  $\ker(f)$  is a collection of  $v$  meromorphic 2-forms  $(\gamma_1, \dots, \gamma_v) \in \bigoplus_{i=1}^v H^0(\omega_{X_i}(C_i))$  such that

$$\gamma_{ij} = -\gamma_{ji}, \quad \text{for } i < j,$$

and

$$\gamma_{\tau_k(i)\tau_k(j)\tau_k(k)}^t = 0 \text{ for each } k \neq i, j \text{ and for each } 1 \leq t \leq m_{\tau_k(i)\tau_k(j)\tau_k(k)},$$

where  $\tau_k \in \text{Sym}(\{i, j, k\})$  is the permutation which lexicographically reorders the index triple  $\{i, j, k\}$ . As before, we can limit ourselves to define its image on a given direct summand; therefore, if  $\pi_{ij}$  is the projection on the  $(ij)^{\text{th}}$ -summand, with  $i < j$ , then we have the following equivalent expressions

$$(4.24) \quad (\pi_{ij} \circ \sigma_{1/2})(\gamma_1, \dots, \gamma_v) = \frac{1}{2}(\mathcal{R}_{C_{ij}}(\gamma_i) - \mathcal{R}_{C_{ij}}(\gamma_j)) = \mathcal{R}_{C_{ij}}(\gamma_i) = -\mathcal{R}_{C_{ij}}(\gamma_j).$$

Observe that  $\sigma_{1/2}$  is well-defined since the  $\gamma_i$ 's are in the kernel of  $f$ ; furthermore, when  $C_{ij} = \emptyset$ , the image is obviously 0.

By using the definition of  $\Delta$  as in (4.13), it is straightforward to check that diagram (4.22) commutes. Furthermore, it is trivial to show that

$$\text{Im}(\zeta) = \ker(\sigma_{1/2}) \text{ and } \text{Im}(\sigma_{1/2}) \subseteq \ker(\Delta).$$

To show the converse, take  $\alpha \in \ker(\Delta)$ , thus  $(\bigoplus_i \delta_i(\beta(\alpha))) = 0$ , i.e.  $\beta(\alpha) \in \text{Im}(\bigoplus_i (\mathcal{R}_{C_i}))$ . This implies that  $\alpha \in \text{Im}(\sigma_{1/2})$ .

From the fact that the subsequence in (4.22) is exact, it follows that

$$\ker(f) \cong \ker(\sigma_{1/2}) \oplus \text{Im}(\sigma_{1/2}) \cong \text{Im}(\zeta) \oplus \ker(\Delta) \cong \bigoplus_{i=1}^d H^0(\omega_{X_i}) \oplus \ker(\Delta).$$

By Proposition 4.14, it follows that

$$(4.25) \quad \ker(f) \cong \bigoplus_{i=1}^d H^0(\omega_{X_i}) \oplus \text{coker}(\Phi).$$

This proves (4.16).

To show that the map  $f$  is surjective, we have to reconstruct a global section of  $\omega_X$  once we have a collection  $\{\omega_{ijk}^t\} \in H_2(G_X, \mathbb{C})$ .

Fix two indices  $l < m$  in  $\{1, \dots, d\}$ , such that  $C_{lm} \neq \emptyset$ ; this means that we consider the curve  $C_{lm}$  as the ambient variety to make our computations. From (4.20), we have three different possibilities:

• if  $k < l$ , then  $\mathcal{R}_{P_{klm}^t}(\omega_{lm}) = \omega_{lmk}^t = \text{Sgn}((k, l, m)) \omega_{klm}^t = \omega_{klm}^t$  for any  $P_{klm}^t \in \Sigma_{klm}$ , where  $(k, l, m)$  is a 3-cycle in  $\text{Sym}(\{k, l, m\})$ ;

- if  $l < k < m$ , then  $\mathcal{R}_{P_{lkm}^t}(\omega_{lm}) = \omega_{lkm}^t = \text{Sgn}((m, k)) \omega_{lkm}^t = -\omega_{lkm}^t$  for any  $P_{lkm}^t \in \Sigma_{lkm}$ , where  $(m, k)$  is a transposition in  $\text{Sym}(\{l, k, m\})$ ;
- if  $k > m$ , we directly have  $\mathcal{R}_{P_{klm}^t}(\omega_{lm}) = \omega_{lkm}^t$  for any  $P_{klm}^t \in \Sigma_{lkm}$ .

Therefore, by (4.21), on  $C_{lm}$  we have:

$$\sum_{k < l} \sum_{t=1}^{m_{klm}} \omega_{klm}^t - \sum_{k \in (l, m) \cap \mathbb{N}} \sum_{t=1}^{m_{lkm}} \omega_{lkm}^t + \sum_{k > m} \sum_{t=1}^{m_{lmk}} \omega_{lmk}^t = 0.$$

This means that the divisor

$$(4.26) \quad D = \sum_{k < l} \sum_{t=1}^{m_{klm}} \omega_{klm}^t P_{klm}^t - \sum_{k \in (l, m) \cap \mathbb{N}} \sum_{t=1}^{m_{lkm}} \omega_{lkm}^t P_{lkm}^t + \sum_{k > m} \sum_{t=1}^{m_{lmk}} \omega_{lmk}^t P_{lmk}^t \in \text{Div}(C_{lm})$$

is homologous to zero. By the Residue Theorem, (4.26) implies there exists a global meromorphic 1-form  $\omega_{lm} \in H^0(C_{lm}, \omega_{C_{lm}}(D))$  having the given residues at the points in  $\text{Supp}(D)$ , i.e. such that

$$\mathcal{R}_{P_{klm}^r}(\omega_{lm}) = \omega_{klm}^r, \quad \mathcal{R}_{P_{lkm}^s}(\omega_{lm}) = -\omega_{lkm}^s, \quad \mathcal{R}_{P_{lmk}^t}(\omega_{lm}) = \omega_{lmk}^t,$$

where  $1 \leq r \leq m_{klm}$ ,  $1 \leq s \leq m_{lkm}$  and  $1 \leq t \leq m_{lmk}$ .

The above discussion obviously holds for each choice of index pairs. Fix now an index  $1 \leq h \leq v$  and consider on the surface  $X_h$  the reducible nodal curve  $C_h = X_h \cap (X \setminus X_h)$ ; since we are on  $X_h$ , by Remark 4.10, we can write

$$C_h = C_h^+ + C_h^-,$$

where

$$C_h^- = \sum_{l < h} C_{lh}, \quad \text{and} \quad C_h^+ = \sum_{m > h} C_{hm}.$$

Thus, by the above discussion, on each  $C_{lh}$  (resp.  $C_{hm}$ ) we have a meromorphic 1-form  $-\omega_{lh}$  (resp.  $\omega_{hm}$ ) inducing the given residues at the given triple points. Fix three indices  $i, j, h$  and consider the set of triple points given by  $X_i \cap X_j \cap X_h \neq \emptyset$ . Since we are on the surface  $X_h$ , such set of points is determined by the intersection of  $C_{ih} = C_{hi}$  and  $C_{jh} = C_{hj}$ . We have the following possibilities:

- if  $i < j < h$ , the set of triple points is  $\Sigma_{ijh}$  and on  $C_{ih}$  (resp., on  $C_{jh}$ ) we have the meromorphic 1-form  $-\omega_{ih}$  (resp.,  $-\omega_{jh}$ ); therefore, by (4.20),

$$\mathcal{R}_{P_{ijh}^t}(-\omega_{ih}) + \mathcal{R}_{P_{ijh}^t}(-\omega_{jh}) = \omega_{ijh}^t - \omega_{ijh}^t = 0,$$

for any  $1 \leq t \leq m_{ijh}$ ;

• if  $h < i < j$ , the set of triple points is  $\Sigma_{hij}$  and on  $C_{hi}$  (resp., on  $C_{hj}$ ) we have the meromorphic 1-form  $\omega_{hi}$  (resp.,  $\omega_{hj}$ ); as before,

$$\mathcal{R}_{P_{hij}^t}(\omega_{hi}) + \mathcal{R}_{P_{hij}^t}(\omega_{hj}) = \omega_{hij}^t - \omega_{hij}^t = 0,$$

for any  $1 \leq t \leq m_{hij}$ ;

• if  $i < h < j$ , the set of triple points is  $\Sigma_{ihj}$  and on  $C_{ih}$  (resp., on  $C_{hj}$ ) we have the meromorphic 1-form  $-\omega_{ih}$  (resp.,  $\omega_{hj}$ ); thus,

$$\mathcal{R}_{P_{ihj}^t}(-\omega_{ih}) + \mathcal{R}_{P_{ihj}^t}(\omega_{jh}) = -\omega_{ihj} + \omega_{ihj} = 0,$$

for any  $1 \leq t \leq m_{ihj}$ .

In either case, by Remark 4.3, such forms glue together to determine an element in  $H^0(C_h, \omega_{C_h})$ . This can be done for each  $1 \leq h \leq d$ , determining a collection  $\{\bar{\omega}_h\} \in \bigoplus_{h=1}^v H^0(\omega_{C_h})$ .

Assume now to be in the case of hypothesis (i), so each  $X_h$  is regular; since  $h^1(X_h, \omega_{X_h}) = 0$ , for each  $1 \leq h \leq v$ , by the exact sequences

$$0 \rightarrow \omega_{X_h} \rightarrow \omega_{X_h}(C_h) \rightarrow \omega_{C_h} \rightarrow 0, \quad 1 \leq h \leq d,$$

the collection of forms  $\{\bar{\omega}_h\}$  lifts up to a collection of forms  $\{\omega_h\} \in \bigoplus_{h=1}^v H^0(\omega_{X_h}(C_h))$ . Take an index pair with  $h < k$ ; since  $C_{hk}$  is both a component of  $C_h^+$  on  $X_h$  and of  $C_k^-$  on  $X_k$ , then

$$\mathcal{R}_{C_{hk}}(\omega_k) = -\mathcal{R}_{C_{hk}}(\omega_h).$$

This means that the collection  $\{\omega_h\}$  is an element of  $H^0(X, \omega_X)$ , so the map  $f$  is surjective and formula (4.16) is proved.

Assume now to be in the case of hypothesis (ii); let  $X_j$  be an irreducible component of  $X$ , which is assumed to be an irregular surface. By Dolbeault cohomology, by the hypothesis on  $C_j$  and by the Kodaira vanishing theorem, we get the following diagram:

$$(4.27) \quad \begin{array}{ccccccc} H^0(\omega_{X_j}(C_j)) & \xrightarrow{\mathcal{R}_{C_j}} & H^0(\omega_{C_j}) & \xrightarrow{\delta_j} & H^1(\omega_{X_j}) & \longrightarrow & 0, \\ & & & & \uparrow \eta_j & & \\ & & & & H^0(X_j, \Omega_{X_j}^1) & & \end{array}$$

$\swarrow \text{tr}_{C_j}$

which can be seen to be commutative, where  $\text{tr}_{C_j}$  is the trace map of holomorphic 1-forms on  $X_j$  to holomorphic 1-forms on  $C_j$  whereas  $\eta_j$  is the map defined by the cup product with the first Chern class of  $C_j$ ,  $\gamma_{C_j} \in H^{1,1}(X_j)$ .

From the surjectivity of  $\delta_j$ , it follows that

$$H^0(\omega_{C_j}) \cong \text{Im}(\mathcal{R}_{C_j}) \oplus H^1(\omega_{X_j});$$

on the other hand, by the Hard Lefschetz theorem,  $\eta_j$  is an isomorphism. Since  $\delta_j$  is injective on  $\text{Im}(\text{tr}_{C_j})$ , then  $\text{Im}(\mathcal{R}_{C_j}) \cap \text{Im}(\text{tr}_{C_j}) = \{0\}$ . On the other hand,

$$\text{Im}(\text{tr}_{C_j}) \rightarrow H^1(\omega_{X_j})$$

is surjective. Then,

$$H^0(\omega_{C_j}) \cong \text{Im}(\mathcal{R}_{C_j}) \oplus \text{Im}(\text{tr}_{C_j}).$$

Observe that the elements of  $\text{Im}(\text{tr}_{C_j})$  give zero residues at the triple points of  $X$  lying on  $X_j$ , since such elements are restrictions to  $C_j$  of global holomorphic 1-forms of  $X_j$ . Therefore, the element  $\omega_j \in H^0(\omega_{C_j})$  of the collection  $\{\bar{\omega}_h\} \in \bigoplus_{h=1}^v H^0(\omega_{C_h})$ , which was constructed from the given non-zero collection of residues in  $H_2(G_X)$ , is determined via  $\mathcal{R}_{C_j}$  by an element in  $H^0(\omega_{X_j}(C_j))$ , which is necessarily not zero. Then we can conclude as above, proving also in this case the surjectivity of  $f$ . ■

In case  $X$  is a planar Zappatic surface, Theorem 4.15 implies the following:

COROLLARY 4.28. – *Let  $X = \bigcup_{i=1}^v \Pi_i$  be a planar Zappatic surface which has only  $E_3$  points as Zappatic singularities. Then,*

$$(4.29) \quad p_g(X) = b_2(G_X),$$

$$(4.30) \quad q(X) = b_1(G_X).$$

PROOF Formula (4.29) trivially follows from Theorem 4.15. Notice that, in such a case, the proof of Theorem 3.15 becomes simpler. Indeed, each  $\Sigma_{ijk}$  (cf. Notation 3.3) is either a singleton or empty, since the double curves are lines (cf. Remark 4.7).

Formula (4.30) follows from (3.17), (4.29) and from the fact that  $\chi(G_X) = 1 - b_1(G_X) + b_2(G_X)$ . ■

In § 5 we shall extend the above results to a good planar Zappatic surface  $X$ , assuming that  $X$  is smoothable, i.e. the central fibre of a degeneration.

5. - Zappatic degenerations.

In this section we will focus on degenerations of smooth surfaces to Zappatic ones.

DEFINITION 5.1. - Let  $\Delta$  be the spectrum of a DVR (equiv. the complex unit disk). Then, a *degeneration* (of relative dimension  $n$ ) is a proper and flat algebraic morphism

$$\begin{array}{c} \mathcal{X} \\ \downarrow \pi \\ \Delta \end{array}$$

such that  $\mathcal{X}_t = \pi^{-1}(t)$  is a smooth, irreducible,  $n$ -dimensional projective variety, for  $t \neq 0$ .

If  $Y$  is a smooth, projective variety, the degeneration

$$\begin{array}{c} \mathcal{X} \subseteq \Delta \times Y \\ \downarrow \pi \\ \Delta \end{array}$$

is said to be an *embedded degeneration* in  $Y$  of relative dimension  $n$ . When it is clear from the context, we will omit the term embedded.

A degeneration (equiv. an embedded degeneration) is said to be *semistable* if the total space  $\mathcal{X}$  is smooth and if the central fibre  $\mathcal{X}_0$  (where  $0$  is the closed point of  $\Delta$ ) is a divisor in  $\mathcal{X}$  with global normal crossings, i.e.  $\mathcal{X}_0 = \sum V_i$  is a sum of smooth, irreducible components  $V_i$ 's which meet transversally so that locally analytically the morphism  $\pi$  is defined by

$$(x_1, \dots, x_{n+1}) \xrightarrow{\pi} x_1 x_2 \dots x_k = t \in \Delta, \quad k \leq n + 1.$$

Given an arbitrary degeneration  $\pi : \mathcal{X} \rightarrow \Delta$ , the well-known Semistable Reduction Theorem (see [11]) states that there exists a base change  $\beta : \Delta \rightarrow \Delta$  (defined by  $\beta(t) = t^m$ , for some  $m$ ), a semistable degeneration  $\psi : \mathcal{Z} \rightarrow \Delta$  and a diagram

$$\begin{array}{ccccc} \mathcal{Z} & \xrightarrow{f} & \mathcal{X}_\beta & \longrightarrow & \mathcal{X} \\ & \searrow \psi & \downarrow & & \downarrow \\ & & \Delta & \xrightarrow{\beta} & \Delta \end{array}$$

such that  $f$  is a birational map obtained by blowing-up and blowing-down subvarieties of the central fibre. Therefore, statements about degenerations which are invariant under blowing-ups, blowing-downs and base-changes can

be proved by directly considering the special case of semistable degenerations.

From now on, we will be concerned with degenerations of relative dimension two, namely degenerations of smooth, projective surfaces.

DEFINITION 5.2. – Let  $\mathcal{X} \rightarrow \Delta$  be a degeneration (equiv. an embedded degeneration) of surfaces. Denote by  $\mathcal{X}_t$  the general fibre, which is by definition a smooth, irreducible and projective surface; let  $X = \mathcal{X}_0$  denote the central fibre. We will say that the degeneration is *Zappatic* if  $X$  is a Zappatic surface and  $\mathcal{X}$  is smooth except for:

- ordinary double points at points of the double locus of  $X$ , which are not the Zappatic singularities of  $X$ ;
- further singular points at the Zappatic singularities of  $X$  of type  $T_n$ , for  $n \geq 3$ , and  $Z_n$ , for  $n \geq 4$ .

A Zappatic degeneration will be called *good* if the central fibre is moreover a good Zappatic surface. Similarly, an embedded degeneration will be called a *planar Zappatic degeneration* if its central fibre is a planar Zappatic surface.

Notice that we require the total space  $\mathcal{X}$  to be smooth at  $E_3$ -points of  $X$ .

If  $\mathcal{X} \rightarrow \Delta$  is a good Zappatic degeneration, the singularities that  $\mathcal{X}$  has at the Zappatic singularities of the central fibre  $X$  are explicitly described in [4].

NOTATION 5.3. – Let  $\mathcal{X} \rightarrow \Delta$  be a degeneration of surfaces and let  $\mathcal{X}_t$  be the general fibre, which is by definition a smooth, irreducible and projective surface. Then, we consider some of the intrinsic invariants of  $\mathcal{X}_t$ :

- $\chi := \chi(\mathcal{O}_{\mathcal{X}_t})$ ;
- $K^2 := K_{\mathcal{X}_t}^2$ ;
- $p_g := p_g(\mathcal{X}_t)$ ;
- $\chi_{\text{top}} := \chi_{\text{top}}(\mathcal{X}_t)$ ;

If the degeneration is assumed to be embedded in  $\mathbb{P}^r$ , for some  $r$ , then we also have:

- $d := \text{deg}(\mathcal{X}_t)$ ;
- $g := (K + H)H/2 + 1$ , the sectional genus of  $\mathcal{X}_t$ .

We will be mainly interested in computing these invariants in terms of the central fibre  $X$ . For some of them, this is quite simple. For instance, when  $\mathcal{X} \rightarrow \Delta$  is an embedded degeneration in  $\mathbb{P}^r$ , for some  $r$ , and if the central fibre  $\mathcal{X}_0 = X = \bigcup_{i=1}^v X_i$ , where the  $X_i$ 's are smooth, irreducible surfaces of degree  $d_i$ ,

$1 \leq i \leq v$ , then by the flatness of the family we have

$$d = \sum_{i=1}^v d_i.$$

When  $\mathcal{X} \rightarrow \Delta$  is a good Zappatic degeneration (in particular a good, planar Zappatic degeneration), we can easily compute some of the above invariants by using our results of § 3. Indeed, by using our Notation 3.10 and Propositions 3.12, 3.15, we get the following results.

PROPOSITION 5.4. – *Let  $\mathcal{X} \rightarrow \Delta$  be a good Zappatic degeneration embedded in  $\mathbb{P}^r$ . Let  $\mathcal{X}_0 = X = \bigcup_{i=1}^v X_i \subset \mathbb{P}^r$  be the central fibre and let  $G = G_X$  be its associated graph. Let  $C$  be the double locus of  $X$ , i.e. the union of the double curves of  $X$ ,  $C_{ij} = C_{ji} = X_i \cap X_j$  and let  $c_{ij} = \deg(C_{ij})$ . Let  $D$  be a general hyperplane section of  $X$  and let  $D_i$  be the  $i^{\text{th}}$  smooth, irreducible component of  $D$ , which is a general hyperplane section of  $X_i$ , and denote by  $g_i$  its genus. Then:*

$$(5.5) \quad g = \sum_{i=1}^v g_i + \sum_{e_{ij} \in E} c_{ij} - v + 1.$$

When  $X$  is a good, planar Zappatic surface, if  $G^{(1)}$  denotes the 1-skeleton of  $G$ , then:

$$(5.6) \quad g = 1 - \chi(G^{(1)}) = e - v + 1.$$

PROOF. – It directly follows from our computations in Proposition 3.12 and from the flatness of the family of hyperplane sectional curves of the degeneration (cf. formula (2.4)). ■

PROPOSITION 5.7. – *Let  $\mathcal{X} \rightarrow \Delta$  be a good Zappatic degeneration and let  $\mathcal{X}_0 = X = \bigcup_{i=1}^v X_i$  be its central fibre. Let  $G = G_X$  be its associated graph and let  $E$  be the indexed set of edges of  $G$ . Let  $C$  be the double locus of  $X$ , which is the union of the double curves  $C_{ij} = C_{ji} = X_i \cap X_j$ . Denote by  $g_{ij}$  the genus of the smooth curve  $C_{ij}$ . Then*

$$(5.8) \quad \chi = \sum_{i=1}^v \chi(\mathcal{O}_{X_i}) - \sum_{e_{ij} \in E} \chi(\mathcal{O}_{C_{ij}}) + f.$$

Moreover, if  $\mathcal{X} \rightarrow \Delta$  is a good, planar Zappatic degeneration, then

$$(5.9) \quad \chi = \chi(G) = v - e + f.$$

PROOF. – It follows from Proposition 3.15 and from the invariance of  $\chi$  under flat degeneration. ■

In the particular case that  $\mathcal{X} \rightarrow \Delta$  is a semistable Zappatic degeneration, i.e. if  $X$  has only  $E_3$ -points as Zappatic singularities, then  $\chi$  can be computed also in a different way by topological methods (see formula (5.13) in Theorem 5.12).

The above results are indeed more general:  $X$  is allowed to have any good Zappatic singularity, namely  $R_n$ -,  $S_n$ - and  $E_n$ -points, for any  $n \geq 3$ , and moreover our computations do not depend on the fact that  $X$  is smoothable, i.e. that  $X$  is the central fibre of a degeneration. Notice also that a good Zappatic degeneration is not semistable in general.

For what concerns the geometric genus, assume now — unless otherwise stated — that the Zappatic surface  $X = \bigcup_{i=1}^v X_i$  is the central fibre of a semistable Zappatic degeneration  $\mathcal{X} \rightarrow \Delta$ , i.e.  $\mathcal{X}$  is smooth and  $X$  has only  $E_3$ -points as Zappatic singularities. In this case, Theorem 4.15 implies the following:

**PROPOSITION 5.10.** — *Let  $\mathcal{X} \rightarrow \Delta$  be a semistable Zappatic degeneration and  $\mathcal{X}_0 = X = \bigcup_{i=1}^v X_i$  be its central fibre. Let  $G_X$  be the associated graph to  $X$  and  $\Phi$  be the map defined in (4.8). Then, for any  $t \in \Delta$ ,*

$$(5.11) \quad p_g(\mathcal{X}_t) \leq b_2(G_X) + \sum_{i=1}^v p_g(X_i) + \dim(\text{coker}(\Phi)).$$

**PROOF.** — By semi-continuity, we have  $p_g(\mathcal{X}_t) \leq p_g(\mathcal{X}_0) = p_g(X)$ . One concludes by using formula (4.16). ■

On the other hand,  $p_g(\mathcal{X}_t)$  and  $\chi(\mathcal{X}_t)$  can be also computed by topological methods: indeed the Clemens-Schmid exact sequence relates the mixed Hodge theory of the central fibre  $X$  to that of  $\mathcal{X}_t$  by means of the monodromy of the total space  $\mathcal{X}$  (see [15] for definitions and statements). In our particular situation, the following result holds:

**THEOREM 5.12 (Clemens-Schmid).** — *Let  $\mathcal{X} \rightarrow \Delta$  be a semistable Zappatic degeneration and  $\mathcal{X}_0 = X = \bigcup_{i=1}^v X_i$  be its central fibre. Let  $G_X$  be the associated graph to  $X$  and  $\Phi$  be the map defined in (4.8). Then, for any  $t \neq 0$ ,*

$$(5.13) \quad \chi(\mathcal{O}_{\mathcal{X}_t}) = \chi(G_X),$$

$$(5.14) \quad p_g(\mathcal{X}_t) = b_2(G_X) + \sum_{i=1}^d p_g(X_i) + \dim(\text{coker}(\Phi)).$$

A proof of Theorem 5.12 can be found in [15, «Clemens-Schmid I and II»]. The above result, together with Theorem 4.15, implies the following:



COROLLARY 5.15. – Let  $X = \bigcup_{i=1}^v X_i$  be the central fibre of a semistable Zappatic degeneration  $\mathcal{X} \rightarrow \Delta$ . Then, for every  $t \in \Delta$ ,

$$(5.16) \quad p_g(\mathcal{X}_t) = p_g(X) = b_2(G_X) + \sum_{i=1}^v p_g(X_i) + \dim(\text{coker}(\Phi)).$$

In particular, the geometric genus of the fibres of  $\mathcal{X} \rightarrow \Delta$  is constant.

PROOF. – Formula 5.16 trivially follows from (4.16), (5.14) and from semicontinuity. ■

Recalling the proof of Theorem 4.15, we also have the following:

COROLLARY 5.17. – Let  $X = \bigcup_i X_i$  be a Zappatic surface with global normal crossings, i.e. with only  $E_3$ -points as Zappatic singularities. Let  $G_X$  be its associated graph. A necessary condition for the smoothability of  $X$  is the surjectivity of the homomorphism  $f: H^0(X, \omega_X) \rightarrow H_2(G_X)$  defined in (4.17).

PROOF. – If  $X$  is smoothable, then (5.16) implies that the equality in (4.16) holds. This implies that  $f$  is surjective by the proof of Theorem 4.15. ■

The above results naturally suggest the following:

QUESTION. – Is the homomorphism  $f: H^0(X, \omega_X) \rightarrow H_2(G_X)$  in (4.17) always surjective? Equivalently, does the equality in (4.16) always hold?

We believe that the answer to this question should be negative, but we have not been able to exhibit a counterexample so far.

In case the answer to the above question were negative, it should be interesting to compare the surjectivity of  $f$  with other smoothability conditions, like Friedman's one in [8].

We now generalize the computations for  $p_g$  to the case of good Zappatic degenerations, i.e. degenerations where the central fibre  $X$  is a union of surfaces having not only  $E_3$ -points, but also  $R_n$ -,  $S_n$ - and  $E_n$ -points for any  $n \geq 3$ .

DEFINITION 5.18. – Let  $\mathcal{X} \rightarrow \Delta$  be a good Zappatic degeneration and  $X = \mathcal{X}_0$  be its central fibre. Consider the semistable reduction  $\mathcal{X}' \rightarrow \Delta$  of  $\mathcal{X} \rightarrow \Delta$  together with its central fibre  $X' = \mathcal{X}'_0$ , which is a Zappatic surface with global normal crossings, i.e. with only  $E_3$ -points. We define the geometric genus as (cf. Remark 3.4):

$$(5.19) \quad p_g(X) := p_g(X') = h^0(X', \omega_{X'}).$$

As we will see in a moment, the definition is well-posed.

**THEOREM 5.20** *Let  $X = \mathcal{X}_0$  be a good Zappatic surface which is the central fibre of a degeneration  $\mathcal{X} \rightarrow \Delta$  and let  $G_X$  be its associated graph. Then*

$$(5.21) \quad p_g(X) = p_g(\mathcal{X}_t) = b_2(G_X) + \sum_{i=1}^v p_g(X_i) + \dim(\text{coker}(\Phi)).$$

**SKETCH OF THE PROOF.** – Complete details will appear in [4]. Here we give an outline of the proof.

The first step is to understand how to get the semistable reduction locally near  $R_n$ -,  $S_n$ - and  $E_m$ -points, for  $n \geq 3$  and  $m \geq 4$ .

Consider a  $R_n$ -point  $x \in X$ . Then  $x$  is an isolated singularity for the total space  $\mathcal{X}$  and it is a *minimal singularity* in the sense of Kollár ([12] and [13]). Let  $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$  be the blow-up at  $x$ . Its exceptional divisor  $F$  is a minimal degree surface (of degree  $n$ ) in  $\mathbb{P}^{n+1} = \mathbb{P}(\mathcal{T}_{\mathcal{X},x})$ , where  $\mathcal{T}_{\mathcal{X},x}$  is the tangent space of  $\mathcal{X}$  at  $x$ . Furthermore  $F$  is connected in codimension one and can be explicitly described. In particular one can show that either  $F$  is smooth or  $F$  has  $R_m$ -points, for  $m \leq n$ . Some points  $E_4$  can appear along the intersection of the exceptional divisor with the strict transform of  $X$ . In any event, after finitely many blow-ups of the total space at points, we resolve the singularities of the total space. It turns out that all the components of the exceptional divisor are rational and all the double curves involved in them are also rational.

The situation is completely similar for  $S_n$ -points and  $E_n$ -points,  $n \geq 4$ .

Therefore one can get the semistable reduction  $\mathcal{X}' \rightarrow \Delta$  of  $\mathcal{X} \rightarrow \Delta$  just by blowing-up  $\mathcal{X}$  at points which are good Zappatic singularities of the central fibre. Let  $\sigma: \mathcal{X}' \rightarrow \mathcal{X}$  be this blow-up and  $X' = \mathcal{X}'_0$  be the central fibre of  $\mathcal{X}' \rightarrow \Delta$ .

By (5.19)  $p_g(X)$  is defined to be  $p_g(X')$ ; Theorem 4.15 tells how to compute it.

Now, our second and last step is to prove that  $p_g(X')$  is given by (5.21) and that  $p_g(X') = p_g(\mathcal{X}'_t)$ , for  $t \neq 0$ . Since the semistable reduction of  $\mathcal{X} \rightarrow \Delta$  involves only the central fibre and since all the exceptional divisors as well as all the curves involved in them are rational, it suffices to prove that  $b_2(G_X) = b_2(G_{X'})$ .

Consider an open  $n$ -face (resp. a  $n$ -angle)  $G_x$  of  $G_X$ , namely  $G_x$  is the subgraph with  $n$  vertices corresponding to  $n$  surfaces forming a  $R_n$ -point (resp. a  $S_n$ -point)  $x$ . Let  $G'_x$  be the subgraph of  $G_{X'}$  containing the vertices corresponding to the proper transforms of the  $n$  surfaces and the exceptional divisors contained in  $\sigma^{-1}(x)$ . The above description of the infinitesimal neighbourhood of  $x$  shows that, as topological spaces, the subgraph  $G_x$  is a deformation retract of  $G_{X'}$ .

Similarly, if  $G_x$  is a closed  $n$ -face of  $G_X$  (i.e. a subgraph with  $n$  vertices corresponding to  $n$  surfaces forming an  $E_n$ -point  $x$ ), then the above description shows that the subgraph  $G'_x$  of  $G_{X'}$  containing the vertices corresponding to

the proper transforms of the planes and the exceptional divisors contained in  $\sigma^{-1}(x)$  is a triangulation of  $G_x$ .

It follows that the graphs  $G_X$  and  $G_{X'}$  have the same homological invariants, which is what we had to prove.

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Alberto Calabri: Dipartimento di Matematica, Università degli Studi di Roma  
 «Tor Vergata», Via della Ricerca Scientifica, 00133 Roma, Italy  
 e-mail: calabri@mat.uniroma2.it

Ciro Ciliberto: Dipartimento di Matematica, Università degli Studi di Roma  
 «Tor Vergata», Via della Ricerca Scientifica, 00133 Roma, Italy  
 e-mail: cilibert@mat.uniroma2.it

Flaminio Flamini: Dipartimento di Matematica Pura ed Applicata  
 Università degli Studi di L'Aquila, Via Vetoio, Loc. Coppito, 67100 L'Aquila, Italy  
 e-mail: flamini@mat.uniroma3.it

Rick Miranda: Department of Mathematics, 101 Weber Building  
 Colorado State University, Fort Collins, CO 80523-1874, U.S.A.  
 e-mail: miranda@math.colostate.edu

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