ABOUT SOME GENERALIZATIONS OF (λ,μ)-COMPACTNESS

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In [MS] Makowsky and Shelah introduced the concept of $[\lambda,\mu]$-compactness and proved, among other things, that any $[\lambda^+,\lambda^+]$-compact logic allowing relativization is $[\lambda,\lambda]$-compact. Does an analogous result hold for (λ,λ)-compactness, too? We find that the answer is yes if either λ is regular, or $2^\lambda=\lambda^+$, while the question is open in the other cases.

The proof of [MS]'s result relies on the fact that every (λ^+,λ^+)-regular ultrafilter is (λ,λ)-regular: we can state a more general fact concerning the possibility of realizing a certain kind of types (in the sense of $L_{\omega\omega}$). At this point, however, the situation is not exactly as in [MS]: from (λ^+,λ^+)-compactness and arbitrary λ we can only obtain almost λ^+-(λ,λ)-compactness, for some rather natural notion of almost compactness. Now, if λ is regular, or $2^\lambda=\lambda^+$, then almost λ^+-(λ,λ)-compactness implies (λ,λ)-compactness, and we have the desired result.

DEFINITIONS. A class K of models is almost (λ,μ)-compact in the sense of the logic L iff the following holds: if $\Sigma\subset L$, $|\Sigma|=\lambda$ and every subset of Σ of cardinality $<\mu$ has a model in K, then there exists $\Sigma^*\subset\Sigma$ such that $|\Sigma^*|=\lambda$ and Σ^* has a model in K.

K is (λ,μ)-compact iff the above conclusion can be strengthened to: Σ has a model in K.

A logic L is [almost] κ-(λ,μ)-compact iff for every $\Gamma\subset L$ of cardinality $\le\kappa$ $\mathrm{Mod}_L(\Gamma)$ is [almost] (λ,μ)-compact in the sense of L (here, λ and μ are infinite cardinals, and κ can be any cardinal, or ∞; $\mathrm{Mod}_L(\Gamma)$ is the class of all models of Γ of any type).

$[\lambda,\mu]$-compactness is ∞-(λ,μ)-compactness.

For every cardinals $\mu\le\lambda$ let $S_\mu(\lambda)$ be the model $\langle S_\mu(\lambda),\subseteq,\{\alpha\}\rangle_{\alpha\in\lambda}$, and let $\tau=\tau(S_\mu(\lambda))$, $\tau^*=\tau(S_{\mu^*}(\lambda^*))$, and suppose w.l.o.g. $\tau\cap\tau^*=\emptyset$. We write $(\lambda,\mu)\overset{\kappa}{\Longrightarrow}(\lambda^*,\mu^*)$ iff there exists a model **A** with unary predicates U, U* such that $(\mathbf{A}\restriction\tau)|U\cong S_\mu(\lambda)$, $(\mathbf{A}\restriction\tau^*)|U^*\cong S_{\mu^*}(\lambda^*)$, and $|\tau\mathbf{A}-(\tau\cup\tau^*)|\le\kappa$ and, for every $\mathbf{B}\equiv\mathbf{A}$, if there is $b\in B$ such that **B** satisfies U(b) and $\{\alpha\}\subset b$, for every $\alpha\in\lambda$, then there is $b^*\in B$ such that **B** satisfies $U^*(b^*)$ and $\{\alpha\}^*\subset^* b^*$, for every $\alpha\in\lambda^*$.

If in the preceding definition we only ask that **B** satisfies U(b) and that $|\{\alpha\in\lambda \mid \mathbf{B} \text{ satisfies } \{\alpha\}\subset b\}|=\lambda$, we write "almost (λ,μ)" in place of (λ,μ); and similarly for b* and (λ^*,μ^*).

In all the preceding notations we omit κ, if it is finite.

PROPOSITION 1. If $\kappa\ge\sup\{2^\lambda,2^{\lambda^*}\}$, then $(\lambda,\mu)\overset{\kappa}{\Longrightarrow}(\lambda^*,\mu^*)$ iff every (λ,μ)-regular ultrafilter is (λ^*,μ^*)-regular.

See [MS] for the definition of regularity, and for some set theoretical consequences of the existence of (λ,λ)-regular non regular ultrafilters.

Proposition 1 follows from the fact that complete extensions are limits of ultrapowers, and that an ultrafilter D is (λ,μ)-regular iff in $\prod_D S_\mu(\lambda)$

there exists an element greater than all $\{\alpha\}$'s ($\alpha \in \lambda$).

The connection with compactness is given by the following proposition, whose proof uses a generalization for $S_\mu(\lambda)$ of [MS, Proposition 2.1].

PROPOSITION 2. *If $\kappa \geq \sup\{\lambda, \lambda^*\}$, then $(\lambda, \mu) \overset{\kappa}{\Longrightarrow} (\lambda^*, \mu^*)$ implies that every κ-(λ, μ)-compact logic is κ-(λ^*, μ^*)-compact. The same holds replacing everywhere (λ, μ) with almost (λ, μ), or (λ^*, μ^*) with almost (λ^*, μ^*), or both.*

In some particular cases we know [Lp] that the converse of Proposition 2 also holds: we conjecture that this is always the case, whenever $\lambda^* = \mu^*$.

LEMMA 1. (a) *If λ is regular, then almost $(\lambda, \lambda) \Longrightarrow (\lambda, \lambda)$.*

(b) *For every λ, $(\mathrm{cf}\,\lambda, \mathrm{cf}\,\lambda) \Longrightarrow (\lambda, \lambda)$.*

(c) *If $\lambda \geq \mu^* \geq \mu$, then almost $(\lambda, \mu) \Longrightarrow$ almost (λ, μ^*).*

(d) *Suppose that $\kappa \geq \lambda \geq \nu \geq \mu$ and that there are subsets $(X_\alpha)_{\alpha\in\kappa}$ of λ, each of cardinality ν, such that if $X \subset \lambda$ has cardinality λ, then $X_\alpha \subset X$ [$|X_\alpha \cap X| = \nu$, respectively], for some $\alpha \in \kappa$.*

Then almost $(\lambda, \mu) \overset{\kappa}{\Longrightarrow} (\nu, \mu)$ [almost $(\lambda, \mu) \overset{\kappa}{\Longrightarrow}$ almost (ν, μ), respectively].

(e) *For every $\lambda \geq \mu$, almost $(\lambda^+, \mu) \Longrightarrow$ almost (λ, μ).*

(f) *If $\kappa \geq \lambda^\nu$ and $\lambda \geq \nu \geq \mu$, then almost $(\lambda, \mu) \overset{\kappa}{\Longrightarrow} (\nu, \mu)$.*

(g) *If $\lambda \geq \lambda^*$ are regular, then $(\lambda, \lambda) \overset{\kappa}{\Longrightarrow} (\lambda^*, \lambda^*)$ iff there exists an expansion A of $\langle \lambda, \leq, \alpha \rangle_{\alpha\in\lambda}$ with at most κ new symbols such that whenever $B \equiv A$, $b \in B$ and B satisfies $\alpha \leq b$ for every $\alpha \in \lambda$, then there exists $b^* \in B$ such that B satisfies $\alpha \leq b^* < \lambda^*$ for every $\alpha \in \lambda^*$.*

THEOREM 1. *For every λ there exists a finite expansion A of the model $\langle S_{\lambda^+}(\lambda^+), \subset, \{\alpha\}_{\alpha\in\lambda^+}, U, U_\nu\rangle_{\nu\leq\lambda}$, where U is $S_{\mathrm{cf}\,\lambda}(\mathrm{cf}\,\lambda)$ and U_ν is $S_\nu(\lambda^+)$, for every $\nu \leq \lambda$, such that for every $B \equiv A$, if there is $b \in B$ such that B satisfies $\{\alpha\} \subset b$, for every $\alpha \in \lambda^+$, then there is $b^* \in B$ such that either B satisfies $U(b^*)$ and $\{\beta\} \subset b^*$, for every $\beta \in \mathrm{cf}\,\lambda$, or for some regular $\nu \leq \lambda$ B satisfies $U_\nu(b^*)$ and $|\{\beta \in \lambda^+ \mid$ B satisfies $\{\beta\} \subset b^*\}| = \lambda^+$.*

COROLLARY 1. (a) *For every λ, $(\lambda^+, \lambda^+) \Longrightarrow$ almost (λ, λ).*

(b) *If λ is regular, then $(\lambda^+, \lambda^+) \Longrightarrow (\lambda, \lambda)$.*

(c) *If $\kappa \geq 2^\lambda$, then $(\lambda^+, \lambda^+) \overset{\kappa}{\Longrightarrow} (\lambda, \lambda)$.*

In view of (a suitable variation of) Proposition 1, Theorem 1 is a generalization of [CN, Theorem 8.32]. A suggestive statement for Theorem 1 could be: $(\lambda^+, \lambda^+) \Longrightarrow \{(\mathrm{cf}\,\lambda, \mathrm{cf}\,\lambda)$ or almost (λ^+, ν) for some regular $\nu \leq \lambda\}$.

Proposition 2 and Corollary 1 immediately yield:

THEOREM 2. (a) *Every (λ^+, λ^+)-compact logic is almost λ^+-(λ, λ)-compact.*

(b) *If either λ is regular, or $2^\lambda = \lambda^+$, then every (λ^+, λ^+)-compact logic is (λ, λ)-compact.*

(c) *Every 2^λ-(λ^+, λ^+)-compact logic is (λ, λ)-compact.*

We mention also the following result that [MS] forgot to draw from [CN, Theorem 8.32]:

THEOREM 3. If $\mathrm{cf}(\lambda) \neq \lambda$ and L is a $[\lambda^+, \lambda^+]$-compact logic, then either L is $[\mathrm{cf}\,\lambda, \mathrm{cf}\,\lambda]$-compact, or L is $[\lambda^+, \nu]$-compact for some $\nu < \lambda$.

Let us now talk about some open problems: first, does (λ^+, λ^+)-compactness imply (λ, λ)-compactness, for every λ ? or, more generally, are there examples of almost (λ, μ)-compact non (λ, μ)-compact logics? We feel that such examples exist, but we know none (of course, even if almost compactness is always equivalent to compactness, this new concept is useful, since we need it in the proof of Theorem 2).

A more detailed study of the relation $(\lambda, \mu) \overset{\kappa}{\Longrightarrow} (\lambda^*, \mu^*)$ seems rather difficult, and involves set theoretical axioms with some flavour of large cardinals, as so does the more particular relation "every (λ, μ)-regular ultrafilter is (λ^*, μ^*)-regular". In some cases [Lp] a version of Proposition 1 holds for regularity of prime filters in Boolean algebras and smaller κ's .

Another problem is: try to generalize for (λ, μ)-compactness other results of [MS] for $[\lambda, \mu]$-compactness. We know that (λ, λ)-compactness for some λ implies (κ, κ)-compactness for some weakly compact cardinal κ (the analogue for $[\lambda, \lambda]$-compactness, implicit in [MS], yields a measurable κ). We conjecture that an analogue of [MS, Theorem 4.3] holds, that is: if L is (ω, ω)-compact and OC(L) is less than the first uncountable weakly compact cardinal, then $\mathrm{OC(L)} = \omega$.

A more general problem is to find a family playing the role for (λ, μ)--compactness the family **UF**(L) plays for $[\lambda, \mu]$-compactness.

References.

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