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Families of nodal curves on projective surfaces

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Table of Contents

Introduction

This thesis is devoted to the study of some fundamental properties of schemes parametrizing irreducible, nodal curves in a complete linear system on a smooth projective surface S. In particular, we focused on surfaces of general type since, for such surfaces, less is known than what is proven in other cases.

These schemes are classically called *Severi varieties* of irreducible, nodal curves on S since Severi, in [125] - Anhang F, was the first who studied the case $S = \mathbb{P}^2$.

We are mainly interested in dimensional and smoothness properties of such schemes as well as in geometrical properties and moduli behaviour of the elements they parametrize. More precisely, in our work we study conditions for a Severi variety on a surface of general type to be smooth of the expected dimension (*regular* for short). We are also interested in the moduli behaviour of the families of nodal curves parametrized by Severi varieties. In particular, we study the dimension of the locus they span in a moduli space of curves (the *number of moduli*).

These problems have complete answers in the case of plane curves (see [125] and [119], respectively) and the motivation for this work was to explore the possibility of extending such results to surfaces of general type.

The first chapter can be viewed as a brief summary of notation, fundamental definitions and technical results which are used for our analysis. The reader is frequently referred to a detailed bibliography in order to avoid repetitions of proofs of well-known results.

Since a natural approach to the dimension problem is to use deformation theory of nodal curves, Chapter 2 is devoted to recalling basic properties of equisingular deformations both from Horikawa's theory point of view and from the "Cartesian" approach. In Section 2.3 we also provide a chronological overview of the main known results concerning Severi varieties of nodal curves on a smooth, projective surface S.

Chapter 3 contains our main result on the smoothness and the dimension problem. It gives purely numerical sufficient conditions, on the divisor class $C \in Div(S)$ and on the number of nodes δ, guaranteeing that the corresponding Severi variety $V_{|C|,\delta}$ is everywhere regular. More precisely, we prove the following:

Theorem (see Theorem 3.2.3) *Let S be a smooth, projective algebraic sur-*

face and let C be a smooth, irreducible divisor on S. Suppose that:

1. $(C - 2K_S)^2 > 0$ *and* $C(C - 2K_S) > 0$*;*
2. *either*
$$(i) \quad K_S^2 > -4 \quad if \quad C(C - 2K_S) \geq 8,$$
or
$$(ii) \quad K_S^2 \geq 0 \quad if \quad 0 < C(C - 2K_S) < 8.$$
3. $CK_S \geq 0$*;*
4. $(CK_S)^2 - C^2K_S^2 < 4(C(C - 2K_S) - 4)$*;*
5. *either*
$$(i) \quad \delta \leq \frac{C(C-2K_S)}{4} - 1 \quad if \quad C(C - 2K_S) \geq 8,$$
or
$$(ii) \quad \delta < \frac{C(C-2K_S)+\sqrt{C^2(C-2K_S)^2}}{8} \quad if \quad 0 < C(C - 2K_S) < 8.$$

Then, if $[C'] \in\mid C \mid$ parametrizes a reduced, irreducible curve with only δ nodes as singular points, the Severi variety $V_{|C|,\,\delta}$ is smooth of codimension δ (i.e. regular) at the point $[C']$.

By a similar approach, in Chapter 4 we also determine numerical conditions implying that the nodal curves parametrized by $V_{|C|,\delta}$ are *geometrically linearly normal* (see Theorem 4.2.6). Further, this result is used to construct examples of obstructed curves in a given Severi variety.

Apart from deformation theory, our analysis is based on the study of some Bogomolov-unstable rank-two vector bundles on S which determine fundamental numerical criteria for the set of nodes of a nodal curve C' not to impose independent conditions to the linear system $|C|$. This approach was first considered by Chiantini and Sernesi (see [27]); our results generalize what is proven in [27] and in [53] (in the nodal case) as it clearly follows from the study of examples of blown-up surfaces, surfaces which are elements of a component of the Noether-Lefschetz locus of surfaces in $\mathbb{P}^3$ or smooth "canonical" complete intersection surfaces (see Sections 3.2, 3.3 and 4.3).

In Chapter 5, we consider the infinitesimal approach to the "moduli problem" of nodal curves of a Severi variety on a smooth, projective and regular surface S of general type. The problem is to find for which divisor classes D on S the number of moduli of the family $V_{|D|,\delta}$, $\delta \geq 0$, coincides with its dimension, as expected (see Section 5.1 for precise definitions). After having considered some examples which show that the "moduli problem" does not always have a positive answer, we give a cohomological condition ensuring that the morphism

$$\pi_{|D|,\delta} : V_{|D|,\delta} \to \mathcal{M}_g$$

($\mathcal{M}_g$ the moduli space of smooth curves of genus $g = p_a(D) - \delta$) has injective differential at a regular point $[X] \in V_{|D|,\delta}$. More precisely, we prove:

Theorem (see Theorem 5.2.1 and Remarks 5.1.2 and 5.1.4) *Let S be a smooth, projective algebraic surface, which is regular and of general type. Let D be a smooth, irreducible divisor on S and let $X \subset S$ be an irreducible, δ-nodal curve, $\delta \geq 0$, which is linearly equivalent to D and whose set of nodes is denoted by N. If*

$$h^1(S,\ \mathcal{I}_{N/S} \otimes \Omega^1_S(D + K_S)) = 0$$

and if $[X] \in V_{|D|,\delta}$ is assumed to be a regular point, then the component of $V_{|D|,\delta}$ containing $[X]$ parametrizes a family having the expected number of moduli (in the sense of Definition 5.1.1). In particular, when $\delta = 0$,

$$h^1(S,\ \Omega^1_S(D + K_S)) = 0$$

is a sufficient condition for the family $V_{|D|,0}$ to have the expected number of moduli.

We then prove the main theorem of this section, which determines linear equivalence classes of divisors for which the cohomological conditions above hold.

Theorem (see Theorems 5.2.2 and 5.2.5) *Let $S \subset \mathbb{P}^r$ be a smooth, algebraic and regular surface with hyperplane section H; let D and X be as in the previous theorem and let $g = p_a(X) - \delta$. Assume that:*

(i) *$\Omega^1_S(K_S)$ is globally generated;*

(ii) *$D \sim K_S + 6H + L$ on S, where L an ample divisor;*

(iii) *the Severi variety $V_{|D|,\delta}$ is regular at $[X]$.*

Then, the morphism

$$\pi_{|D|,\delta} : V_{|D|,\delta} \to \mathcal{M}_g$$

has injective differential at $[X]$. In particular, $\pi_{|D|,\delta}$ has finite fibres on each regular component of $V_{|D|,\delta}$, so each such component parametrizes a family having the expected number of moduli. In particular, when $\delta = 0$, hypotheses (i) and (iii) can be eliminated and the same conclusions are obtained for the family $V_{|D|,0}$.

We also give some improvements of this result by restricting our analysis to smooth complete intersection surfaces in $\mathbb{P}^r$ (see Theorem 5.3.1).

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Chapter 1

Preliminaries

This chapter is devoted to recalling some standard results, which will be frequently used in the sequel. We will review some fundamental definitions and, when it is not beyond the scope of our work, we shall explain, in detail, some useful techniques. This answers the sake of establishing a common language, fixing, once and for all, our notation. For terminology not explicitly introduced, we refer the reader to [64].

1.1 Notation and basic definitions

We work in the category of $\mathbb{C}$-schemes; all schemes will be *algebraic*, i.e noetherian and of finite type over $\mathbb{C}$. By a *point* of a scheme S we mean a closed point, unless otherwise specified (e.g. in the case of a *generic point* of S).

A *variety* X is an integral algebraic scheme; a *curve*, resp. a *surface*, is a variety of pure dimension 1, resp. 2 (exceptions to this terminology are given by reducible curves, which are considered in the sequel, or by Definition 2.2.28).

All sheaves will be coherent. Given a sheaf $\mathcal{F}$ on a scheme X, when there is no ambiguity, we denote by $H^i(\mathcal{F})$ the vector space $H^i(X, \mathcal{F})$ and by $h^i(\mathcal{F})$ its dimension, i.e. $h^i(\mathcal{F}) := dim_{\mathbb{C}}(H^i(X, \mathcal{F}))$.

$\underline{Hom}(\mathcal{E}, \mathcal{F})$ will denote the sheaf of homomorphisms from the sheaf $\mathcal{E}$ to the sheaf $\mathcal{F}$. The *dual sheaf* of $\mathcal{E}$ is simply denoted by $\mathcal{E}^{\vee}$. For a locally free sheaf L of rank one, i.e. an invertible sheaf or, equivalently, a line bundle, sometimes we shall use the notation L^{-1} instead of the previous one. The tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}$ will be simply denoted by $\mathcal{F} \otimes \mathcal{E}$. Furthermore, $\underline{Ext}^i(\mathcal{F}, \mathcal{E})$ will denote the sheaf of germs of extensions whereas $\underline{Tor}_i(\mathcal{F}, \mathcal{E})$ is the Tor sheaf of $\mathcal{F}$ and $\mathcal{E}$. In the same way, $\mathrm{Ext}^i(\mathcal{F}, \mathcal{E})$ and $\mathrm{Tor}_i(\mathcal{F}, \mathcal{E})$ will denote, respectively, the vector space of extensions and the Tor vector space, for $i \geq 0$.

Definition 1.1.1 *Let X be a projective scheme of dimension n. The symbol*

$p_a(X)$ *denotes the* arithmetic genus *of* X, *which is defined by*

$$p_a(X) := (-1)^n(\chi(\mathcal{O}_X) - 1),$$

where $\chi(-)$ *is the Euler-Poincare' characteristic of the sheaf* $-$.

If X is integral, then $H^0(\mathcal{O}_X) \cong \mathbb{C}$, so that $p_a(X) = \Sigma_{i=0}^{n-1}(-1)^i h^{n-i}(\mathcal{O}_X)$. In particular, when X is an integral curve, we find that $p_a(X) = h^1(X, \mathcal{O}_X)$.

If $Y \subset X$ is a closed subscheme, the ideal sheaf of Y in X is denoted by $\mathcal{I}_{Y/X}$ ($\mathcal{I}_Y$ if there is no ambiguity).

Given a scheme X, by a *divisor* on X we shall always mean a *Cartier divisor* on X. We denote by $\sim$ the *linear equivalence* of divisors on X.

If X is a scheme and $x \in X$ is a point, $T_x(X)$ will denote the *Zarisky tangent space* to X at x.

Remark 1.1.2 We write $dim_x(X)$ to denote the dimension of the scheme X at the point x.

A projective variety X is said to be *smooth* if it is non-singular at each point. For brevity, X is a projective *n-fold* if it is a smooth, projective variety of dimension n. Particular cases are when $n = 1$, so X is simply a smooth, projective curve, and $n = 2$, which is the case of a non-singular, projective surface.

$X \subset \mathbb{P}^r$ is said to be *non-degenerate* if it is not contained in a proper projective subspace of $\mathbb{P}^r$.

We recall that given $f : X \to Y$ a morphism of schemes, one can define the *sheaf of relative differentials* of X over Y which is denoted by $\Omega^1_{X/Y}$ (see [64], page 175). When $Y = Spec(\mathbb{C})$, then it is usual to write Ω^1_X and to call it the *cotangent sheaf* of X.

Proposition 1.1.3 *(see [64], Prop. 8.12, page 176) Let* $f : X \to Y$ *be a morphism of schemes and let* Z *be a closed subscheme of* X, *with ideal sheaf* $\mathcal{I}_Z$. *Then there is an exact sequence of sheaves on* Z:

$$\mathcal{I}_Z/\mathcal{I}_Z^2 \to \Omega^1_{X/Y} \otimes \mathcal{O}_Z \to \Omega^1_{Z/Y} \to 0.$$

Theorem 1.1.4 *(see [64], Theorem 8.15, page 177) Let* X *be a variety.* Ω^1_X *is a locally free sheaf of rank* $n = dim(X)$ *if and only if* X *is smooth. In this case,* Ω^1_X *is also said the* cotangent bundle *of* X.

Definition 1.1.5 *Let* X *be a* n*-fold, the* tangent sheaf *of* X *is defined as*

$$\mathcal{T}_X = \underline{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = {\Omega^1_X}^\vee.$$

It is a locally free sheaf of rank $n = dim(X)$, *therefore, it is more usually called the* tangent bundle *of* X. *On the other hand, one can define the* *canonical sheaf of X (or the canonical bundle, since X is smooth) to be the n^{th}-exterior power of the cotangent bundle, i.e.*

$$\Omega_X^n = \bigwedge^n \Omega_X^1.$$

Since for a smooth variety X the canonical bundle coincides with the dualizing sheaf ω_X (see [64], Section III.7), we shall also use the symbol ω_X for the canonical bundle whereas K_X will denote a canonical divisor *on X, which is defined by the condition $\mathcal{O}_X(K_X) \cong \omega_X$. If X is also projective, the* geometric genus *of X is defined as*

$$p_g(X) = h^0(\omega_X).$$

So, in the case of a smooth projective curve, its arithmetic genus and its geometric genus coincide by Serre duality.

We now recall some standard definitions of particular projective varieties.

Definition 1.1.6 *Let $X \subset \mathbb{P}^r$ be a smooth variety of codimension $c = r - n$. Then X is said to be a* complete intersection *if it is the transverse intersection of c hypersurfaces $F_1, \dots, F_c$. If $d_i = deg(F_i)$, this means that the sheaf morphism*

$$\bigoplus_{i=1}^{c} \mathcal{O}_X(-d_i) \stackrel{(F_1,\dots,F_c)}{\longrightarrow} \mathcal{I}_{X/\mathbb{P}^r}$$

is surjective.

It is a standard consequence of adjunction theory that, if $X \subset \mathbb{P}^r$ is a complete intersection, then

$$\omega_X = \mathcal{O}_X(-r - 1 + \Sigma_{i=1}^{c} d_i). \qquad (1.1)$$

Definition 1.1.7 *Let Y be a closed subscheme of a smooth variety X. We recall that Y is said to be a* local complete intersection *in X if the ideal sheaf $\mathcal{I}_{Y/X}$ can be locally generated by $m = codim_X(Y)$ elements at every point.*

Theorem 1.1.8 *(see [64], Theorem 8.17, page 178) Let X be a smooth variety and let $Y \subset X$ be an irreducible closed subscheme defined by the ideal sheaf $\mathcal{I}_Y$. Then,*

(i) Y is a local complete intersection in X if and only if the exact sequence, defined in Proposition 1.1.3, is also left-exact, i.e.

$$0 \to \mathcal{I}_Y/\mathcal{I}_Y^2 \to \Omega_X^1 \otimes \mathcal{O}_Y \to \Omega_Y^1 \to 0;$$

(ii) Y is smooth if and only if the above sequence holds and moreover Ω_Y^1 is locally free on Y.

The notion of being a local complete intersection is an intrinsic property of the scheme Y, i.e. it is independent of the smooth variety containing it. This can be proven by using the Lichtenbaum and Schlessinger cotangent complex of a morphism (see [81]); however, we shall not use this fact in the sequel. From the previous proposition, it immediately follows that if Y is itself non-singular, then it is a local complete intersection inside any non-singular X which contains it.

Suppose we have a projective scheme $X \subset \mathbb{P}^r$ and denote by H a general hyperplane in $\mathbb{P}^r$. With abuse of notation, we denote by $\mathcal{O}_X(H)$ (or by $\mathcal{O}_X(1)$) the sheaf $i^*(\mathcal{O}_{\mathbb{P}^r}(H))$, where $i : X \hookrightarrow \mathbb{P}^r$ is the natural embedding. In the same way, $\mathcal{O}_X(nH)$ (equiv. $\mathcal{O}_X(n)$) shall denote the locally free sheaf determined, on X, by a general hypersurface of degree n in $\mathbb{P}^r$.

If we have a sheaf $\mathcal{F}$ on a projective scheme $X \subset \mathbb{P}^r$, we shall always write $\mathcal{F}(n)$ to mean $\mathcal{F} \otimes \mathcal{O}_X(n)$.

Definition 1.1.9 *Let $X \subset \mathbb{P}^r$ be a normal scheme of pure dimension n. X is said to be* arithmetically Cohen-Macaulay *(or* projectively Cohen-Macaulay*) if*

(i) $H^i(X, \mathcal{O}_X(\rho)) = 0$, for each $\rho \in \mathbb{Z}$, $0 < i < n$;

(ii) the restriction morphism

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(\rho)) \to H^0(X, \mathcal{O}_X(\rho))$$

is surjective, for each $\rho \in \mathbb{Z}$; from the standard exact sequence defining $X \subset \mathbb{P}^r$,

$$0 \to \mathcal{I}_{X/\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_X \to 0,$$

this is equivalent to

$$H^1(\mathcal{I}_{X/\mathbb{P}^r}(\rho)) = 0, \ \forall\, \rho \in \mathbb{Z}.$$

It is a standard fact that smooth complete intersections $X \subset \mathbb{P}^r$ are arithmetically Cohen-Macaulay

Remark 1.1.10 If X is a normal scheme of pure dimension n and if we suppose that only condition (ii) in the previous definition holds, then X is said to be *projectively normal*. When X is projectively normal, the hypersurfaces of degree $\rho > 0$ cut out on X complete linear systems. Moreover, from (ii) with $\rho = 0$, if X is projectively normal then it is connected; in particular, if X is smooth and projectively normal, then X must be irreducible. X is said to be *linearly normal* if (ii) holds with $\rho = 1$, which means that X cannot be a projection in $\mathbb{P}^r$ of a smooth variety in $\mathbb{P}^N$ with $N > r$.

Obviously, when X is a curve, i.e. $n = 1$, the property of being projectively normal coincides with the one of being arithmetically Cohen-Macaulay. For brevity, in this chapter we shall write *a.C.M.* to mean arithmetically Cohen-Macaulay and *p.n.* to mean projectively normal.

Proposition 1.1.11 *Let $X \subset \mathbb{P}^r$ be a smooth variety of dimension n and let Y be a smooth hyperplane section of X. Then X is a.C.M. if and only if Y is a.C.M.*

Proof: It immediately descends from the following standard diagram

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathcal{I}_{X/\mathbf{P}^r}(\rho-1) & \to & \mathcal{O}_{\mathbf{P}^r}(\rho-1) & \to & \mathcal{O}_X(\rho-1) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathcal{I}_{X/\mathbf{P}^r}(\rho) & \to & \mathcal{O}_{\mathbf{P}^r}(\rho) & \to & \mathcal{O}_X(\rho) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathcal{I}_{Y/\mathbf{P}^{r-1}}(\rho) & \to & \mathcal{O}_{\mathbf{P}^{r-1}}(\rho) & \to & \mathcal{O}_Y(\rho) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

for each $\rho \in \mathbb{Z}$. □

We want to recall the definition of *liaison relation* for curves in $\mathbb{P}^3$, which will be used in the sequel (see Sections 3.3 and 4.3). It may be noted that many of the definitions below could be made in a greater generality by working in an arbitrary projective space, or even in a variety which is not a projective space. The decision to restrict to space curves has been made for simplicity and for the fact that all the examples considered in Chapters 3 and 4, which involve liaison relation, are determined by space curves which are residual in a complete intersection. For further details, the reader is referred, for example, to [91] or [107].

For any curve C in $\mathbb{P}^3$, $I(C)$ denotes its *homogeneous ideal*, which is defined by

$$I(C) := \bigoplus_{n \in \mathbb{Z}} H^0(\mathbb{P}^3, \mathcal{I}_{C/\mathbf{P}^3}(n)).$$

Definition 1.1.12 *(see [91], Def. (1.3), page 8) Let C and C' be curves in $\mathbb{P}^3$ and let $X = (F_1) \cap (F_2)$ be a complete intersection of the surfaces (F_i), determined by the vanishing of homogeneous polynomials F_i of degrees d_i, $1 \leq i \leq 2$, respectively. Then, C and C' are* (directly) geometrically linked *by X (written $C \bowtie_{X,g} C'$) if*

(a) $C \cup C' = X$ as schemes, i.e. $I(C) \cap I(C') = I(X)$;

(b) C contains no component of C' and conversely.

Observe that $deg(C) + deg(C') = deg(X)$. In a general setting this definition poses no problem. For example, if (F_1) is the union of two planes and (F_2) is a plane, then X has degree two and should represent the direct geometric linkage of two lines. However, one would like this to hold regardless of the positioning of the planes (provided they do not coincide). There are some positions for which the complete intersection of such planes links a line to itself. Yet, this violates both conditions in the definition. There is a need for a definition of linkage which is equivalent to Definition 1.1.12 when the curves have no component in common, but which does not require this condition.

Definition 1.1.13 *(see [91], Def. (1.4), page 9) Let C and C' be curves in $\mathbb{P}^3$ and let $X = (F_1) \cap (F_2)$ be a complete intersection of two surfaces in $\mathbb{P}^3$. Then, C and C' are* (directly) algebraically linked *by X (written $C \bowtie_{X,a} C'$) if*

(a) $C \cup C' \subseteq X$ as schemes, i.e. $I(X) \subseteq I(C) \cap I(C')$;

(b) $I(X) : I(C) = I(C')$;

(c) $I(X) : I(C') = I(C)$.

C' is said to be residual to C in the complete intersection X *and conversly.*

One can sheafify (b) and (c) of the previous definition to obtain equivalent conditions, where the ideals I are simply replaced by $\mathcal{I}$. The fact that the direct geometric linkage implies the direct algebraic linkage is a result of Peskine-Szpiro, [107]. Conversely, that direct algebraic linkage implies direct geometric linkage (in the event that C and C' have no common component) is proven in Rao's paper [113].

Definition 1.1.14 Liaison *is the equivalence relation among curves (in $\mathbb{P}^3$) generated by direct algebraic linkage. Two curves C and C' in the same liaison class are said to be* linked *and we write $C \bowtie C'$. If they are directly linked by a complete intersection X we write $C \bowtie_X C'$.*

Proposition 1.1.15 *Let $C \bowtie_X C'$ where $X = (F_1) \cap (F_2)$ and $deg(F_i) = d_i$. Then*

(a) $deg(C') = deg(X) - deg(C) = d_1 d_2 - deg(C)$;

(b) $p_a(C') = p_a(C) + p_a(X) - deg(C)(d_1 + d_2 - 4) - 1 = p_a(C) + \frac{1}{2}[(d_1 + d_2 - 4)(d_1 d_2 - 2deg(C))]$;

(c) If C is p.n., then C' is p.n.

Proof: These are standard consequences of Serre duality and Serre vanishing theorem applied to the exact sequence

$$0 \to \mathcal{I}_{C'}(\rho) \to \mathcal{O}_{\mathbf{P}^3}(\rho) \to \mathcal{O}_{C'}(\rho) \to 0,$$

$\rho \in \mathbb{Z}$. For details, see Proposition (1.12) in [91]. □

We now want to recall some useful exact sequences of sheaves on $\mathbb{P}^r$ and on smooth, projective varieties. We have already recalled the tangent sheaf of a smooth variety in Definition 1.1.5. Since $\mathbb{P}^r$ is obviously smooth, we have a tangent sheaf fitting in the following exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^r} \to \mathcal{O}_{\mathbf{P}^r}^{\oplus(r+1)}(1) \to \mathcal{T}_{\mathbf{P}^r} \to 0, \tag{1.2}$$

which is called the *Euler sequence* (see [64], II.8.20.1).

If $Y \subset X$ are schemes, the *normal sheaf* of Y in X is denoted by $\mathcal{N}_{Y/X}$ and defined as

$$\mathcal{N}_{Y/X} \cong \underline{Hom}_{\mathcal{O}_Y}(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y).$$

The sheaf $\mathcal{I}_Y/\mathcal{I}_Y^2$ is therefore called the *conormal sheaf* of X in Y.

Remark 1.1.16 If Y is a local complete intersection in a smooth variety X, then $\mathcal{I}_Y/\mathcal{I}_Y^2$ is locally free of rank $c = codim_X(Y)$, as it follows from Theorem 1.1.8 (i). Therefore, also $\mathcal{N}_{Y/X}$ is locally free, of the same rank. In such a case, we shall use the terminology of *conormal* and *normal bundles*, respectively. In this situation, some texts call the embedding $Y \subset X$ a *regular embedding* and Y is said to be *regularly embedded* in X (see for example [120], page 4-8).

In particular, when X is a smooth variety which is a complete intersection in $\mathbb{P}^r$, then its normal bundle completely splits; more precisely, with same notation as in Definition 1.1.6, if $c = codim_{\mathbb{P}^r}(X)$, then

$$\mathcal{N}_{X/\mathbb{P}^r} \cong \bigoplus_{i=1}^{c} \mathcal{O}_X(d_i). \tag{1.3}$$

If we have closed embeddings of schemes

$$Z \subset Y \subset X,$$

we determine other important exact sequences involving the fact that such schemes form a chain. More precisely, we have

$$0 \to \mathcal{I}_{Y/X} \to \mathcal{I}_{Z/X} \to \mathcal{I}_{Z/Y} \to 0, \tag{1.4}$$

which induces, after taking $\underline{Hom}(-, \mathcal{O}_Z)$, an exact sequence of normal sheaves

$$0 \to \mathcal{N}_{Z/Y} \to \mathcal{N}_{Z/X} \to \mathcal{N}_{Y/X} \otimes \mathcal{O}_Z,$$

where the cokernel has support contained in the locus where Z is not a local complete intersection in Y and/or Y is not a local complete intersection in X. If Z is regularly embedded in Y and Y is regularly embedded in X then the sequence is also exact on the right and becomes the following exact sequence of normal bundles

$$0 \to \mathcal{N}_{Z/Y} \to \mathcal{N}_{Z/X} \to \mathcal{N}_{Y/X} \otimes \mathcal{O}_Z \to 0. \tag{1.5}$$

Since this chapter is devoted to a general overview, singular varieties (in particular nodal curves embedded in smooth, projective surfaces) will be treated in more detail in Section 1.4. Therefore, here we focus on properties of such sheaves when they are defined on smooth subvarieties of smooth varieties, i.e. we shall consider here only vector bundles.

Let Y be a smooth subvariety of a non-singular variety X. If we dualize the *conormal sequence*

$$0 \to \mathcal{I}_Y/\mathcal{I}_Y^2 \to \Omega_X^1 \otimes \mathcal{O}_Y \to \Omega_Y^1 \to 0,$$

then we get the exact sequence

$$0 \to \mathcal{T}_Y \to \mathcal{T}_X \otimes \mathcal{O}_Y \to \mathcal{N}_{Y/X} \to 0, \tag{1.6}$$

called the *tangent sequence* of Y in X.

Proposition 1.1.17 *(see [64], Proposition 8.20, page 182) Let Y be a smooth subvariety of codimension c in a smooth variety X. Then, the canonical bundle of Y is such that*

$$\omega_Y \cong \omega_X \otimes \bigwedge^c(\mathcal{N}_{Y/X}).$$

In case $c = 1$, i.e. Y a smooth divisor of X, then

$$\omega_Y \cong \omega_X \otimes \mathcal{O}_X(Y) \otimes \mathcal{O}_Y.$$

Proof: These results trivially follow from taking the highest exterior powers of the bundles in the conormal exact sequence of Y in X and from recalling that formation of highest exterior power commutes with taking the dual of a given sheaf. □

From now on, each restriction of a sheaf $\mathcal{F}_X$, on a scheme X, to a proper subscheme $Y \subset X$, will be denoted by $\mathcal{F}_X \mid_Y$ instead of $\mathcal{F}_X \otimes \mathcal{O}_Y$.

When we consider nodal curves, it will be fundamental the use of some sheaves which are not locally free on such curves.

Definition 1.1.18 *A sheaf $\mathcal{F}$ on an integral scheme X is said to be a* torsion sheaf *if $dim(supp(\mathcal{F})) < dim(X)$, where $supp(\mathcal{F})$ denotes the support of $\mathcal{F}$. A sheaf is said to be* torsion-free *if it does not contain torsion subsheaves.*

If $\mathcal{F}$ and $\mathcal{E}$ are sheaves on a scheme X, we recall that an *extension* of $\mathcal{F}$ by $\mathcal{E}$ is an exact sequence of sheaves

$$0 \to \mathcal{E} \to \mathcal{H} \to \mathcal{F} \to 0.$$

Two such extensions are *equivalent* if there is a map between them, inducing the identity on the outer terms. The equivalence classes of extensions of $\mathcal{F}$ by $\mathcal{E}$ are parametrized by the vector space $\mathrm{Ext}^1(\mathcal{F}, \mathcal{E})$.

Theorem 1.1.19 *(Spectral sequence for Ext, see [47], page 265) Let X be a scheme and let $\mathcal{E}$ and $\mathcal{F}$ be sheaves on X. There exists a spectral sequence with $E_2^{p,q} = H^p(X, \underline{Ext}^q(\mathcal{F}, \mathcal{E}))$ which converges to $\mathrm{Ext}^{p+q}(\mathcal{F}, \mathcal{E})$, i.e. $E_\infty^{p,q} \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{F}, \mathcal{E})$. In particular, by the Leray spectral sequence, we have the following exact sequence:*

$$0 \to H^1(\underline{Hom}(\mathcal{F}, \mathcal{E})) \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{E}) \to H^0(\underline{Ext}^1(\mathcal{F}, \mathcal{E})) \to$$

$$\to H^2(\underline{Hom}(\mathcal{F}, \mathcal{E})) \to \mathrm{Ext}^2(\mathcal{F}, \mathcal{E}) \to \cdots.$$

Since the main subject of our thesis is the study of families of nodal curves on smooth projective surfaces, we want to end this section by briefly recalling some important definitions and results on (algebraic) surface theory, which will be frequently used in the sequel.

Notation: From now on, the symbol S will denote a smooth, irreducible projective surface, unless otherwise specified. $Div(S)$ and $Pic(S)$ denote its divisor group and its Picard group, respectively. The term *curve* is used for a reduced, irreducible divisor on S.

We recall that $D \in Div(S)$ is said to be *very ample* if there exists a closed immersion $S \overset{\phi}{\hookrightarrow} \mathbb{P}^r$ (for some r) such that

$$\mathcal{O}_S(D) = \phi^*(\mathcal{O}_{\mathbb{P}^r}(1)).$$

On the other hand, D is said to be *ample* if nD is very ample, for some $n \in \mathbb{N}$. There are other equivalent definitions of ample divisors on an arbitrary projective scheme X (see [64]).

A line bundle $L \in Pic(S)$ is said to be *globally generated* (or *free*) if, for each $p \in S$, there exists a global section in $H^0(S, L)$ which does not vanish on p; equivalently if the *evaluation morphism*

$$ev : H^0(S, L) \otimes \mathcal{O}_S \to L$$

is surjective.

If $D \in Div(S)$, we denote by $|D|$ (or, sometimes with the sheaf-notation, by $|\mathcal{O}_S(D)|$) the *complete linear system* consisting of all effective divisors on S which are linearly equivalent to D. A *linear system* Λ of divisors linearly equivalent to D is simply a projective subspace of $|D|$.

A point $p \in S$ is a *base point* for a linear system if it belongs to all divisors in the given linear system. The linear system is *base point free* if it has no base point. Otherwise, if the intersection of all the divisors in a linear system is not empty, it is called the *base locus* of the linear system.

Another important "ingredient" in (algebraic) surface theory is given by the Neron-Severi group of a surface S. In his GAGA paper, [123], Serre proves that the category of projective complex schemes is isomorphic to the category of projective analytic spaces. This means that one may use algebraic and analytical tools interchangeably. Therefore, on S we have the *exponential sequence*

$$0 \to \mathbb{Z} \to \mathcal{O}_S \to \mathcal{O}_S^* \to 0.$$

From the associated cohomology sequence, we get the *first Chern map*

$$H^1(S, \mathcal{O}_S^*) = Pic(S) \overset{c_1}{\to} H^2(S, \mathbb{Z}),$$

where $c_1(L)$ is called the *first Chern class* of the line bundle L.

Definition 1.1.20 *The image of c_1 is called the* Neron-Severi group *of S, denoted by $NS(S)$, which is a finitely generated abelian group. Its rank (i.e. the rank of its free-abelian part) is the* Picard number *of S, which is denoted by $\rho(S)$.*

One has the following exact sequence

$$0 \to Pic^0(S) \to Pic(S) \to NS(S) \to 0,$$

where $Pic^0(S) \cong H^1(\mathcal{O}_S)/H^1(\mathbb{Z})$ consists of line bundles L on S such that $c_1(L) = 0$. In the next section we shall discuss in more detail Chern classes of vector bundles on a smooth surface and their numerical properties.

By pulling-back to $Pic(S)$ the bilinear, non-degenerate form given by the intersection form on $H^2(S, \mathbb{Z})$, we have an itersection pairing on $Div(S)$ which depends only on the linear equivalence classes, i.e. if $C_1 \sim C_2$ on S then $C_1 \cdot D = C_2 \cdot D$, for each $D \in Div(S)$.

For simplicity, from now on we will omit the symbol $\cdot$ and we shall simply write CD to denote the intersection of divisors on S.

With this numerical tool, we can reinterpret the sheaf isomorphism

$$\omega_C \cong \omega_S \otimes \mathcal{O}_S(C) \otimes \mathcal{O}_C,$$

recalled in Proposition 1.1.17, when C is a smooth curve in S. Indeed, by taking the degrees of the line bundles in both sides of the previous isomorphism, we get the *adjunction formula*

$$2g(C) - 2 = C(C + K_S), \tag{1.7}$$

where $g(C) = p_g(C)$ is the geometric genus of the smooth, irreducible divisor C and K_S is a canonical divisor on S. We shall see in Section 1.4 that the adjunction formula can be generalized also to reduced, non-zero effective divisors X by using the dualizing sheaf instead of the canonical bundle; thus, the geometric genus in the formula above will be replaced by the arithmetic genus $p_a(X)$.

A curve C in S for which $C^2 < 0$ has a remarkable property: there are no other curves linearly equivalent to C, i.e. C cannot move in a linear system. Such curves are called *exceptional*. Important kind of exceptional curves are recalled in the following definition.

Definition 1.1.21 *Let S be a smooth, surface and let C be a curve on S. Then, C is an* exceptional curve of first kind *(or* (-1)-curve*) if it is a smooth, rational curve with self-intersection -1.*

Remark 1.1.22 From adjunction formula and from the fact that (-1)-curves are rational, we immediately get

$$C^2 = CK_S = -1.$$

A smooth surface S is called *minimal* if it does not contain (-1)-curves.

A fundamental subject which must be briefly recalled is Hodge theory on a smooth, projective surface S. In general, when X is a projective variety, then it is a Kählerian manifold with respect to the complex topology. From Hodge theory, (see, for example [8]), we write

$$H^{p,q}(X) := H^q(X, \Omega_X^p); \tag{1.8}$$

these complex vector spaces are such that

$$H^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X),\ 0 \leq i \leq n,$$

and

$$h^{p,q}(X) = h^{q,p}(X),$$

where $h^{p,q}(X) = dim_{\mathbb{C}}(H^{p,q}(X))$.

When $X = S$ is a smooth, projective surface, then

$$q(S) := h^{1,0}(S) = h^{0,1}(S) = h^1(\mathcal{O}_S) = h^0(\Omega_S^1)$$

is called the *irregularity* of S; therefore, if we denote by $b_i(S)$ the i^{th}-Betti number of S, $2b_1(S) = q(S)$. On the other hand,

$$p_g(S) := h^{2,0}(S) = h^{0,2}(S) = h^2(\mathcal{O}_S) = h^0(\omega_S)$$

is the geometric genus of S, as in Definition 1.1.5. Consider now the real vector space

$$H^{1,1}_{\mathbb{R}}(S) := H^{1,1}(S) \cap H^2(S, \mathbb{R}).$$

Theorem 1.1.23 *(Signature theorem, see [8], page 120) The cup-product on* $H^2(S, \mathbb{R})$*, induced by the intersection form, has signature* $(1, h^{1,1} - 1)$ *when restricted to* $H^{1,1}_{\mathbb{R}}(S)$ *.*

The Signature theorem here is mostly used in the algebraic form.

Theorem 1.1.24 *(Algebraic index theorem or Hodge index theorem, see [8], page 120) Let D and E be divisors on S. If $D^2 > 0$ and $DE = 0$, then $E^2 \leq 0$; moreover, $E^2 = 0$ if and only if E is homologous to 0, in particular $EB = 0$ for each divisor $B \in Div(S)$.*

Definition 1.1.25 *(see [43], page 17, or [100], page 85) A divisor D on S is* numerically equivalent to 0 *(in symbols $D \equiv 0$) if $DB = 0$ for all divisors $B \in Div(S)$. D_1 and D_2 are* numerically equivalent *($D_1 \equiv D_2$) if $D_1 - D_2$ is numerically equivalent to 0. Note that linear equivalence implies numerical equivalence. We let $Num(S)$ be the quotient of $Div(S)$ by the numerical equivalence relation, which is called* numerical divisor class group *of S.*

Combining the Algebraic index theorem and the Signature theorem we get the following:

Proposition 1.1.26 *(see [43], page 18) The induced map from $Num(S)$ to $H^2(S, \mathbb{Z}) \cap H^{1,1}(S) \subseteq H^2(S, \mathbb{C})$ is an isomorphism. Therefore, $Num(S)$ is the free-abelian part of $NS(S)$, i.e. $Num(S) = NS(S)/tors(NS(S))$, and its rank is the Picard number $\rho(S)$ of Definition 1.1.20.*

There is a fundamental numerical characterization of ample divisors on a smooth, projective surface S.

Theorem 1.1.27 *(Nakai-Moishezon criterion, see [64], page 365) A divisor H is ample on S if and only if $H^2 > 0$ and $HC > 0$, for each curve in S (i.e. for each reduced, irreducible divisor C on S).*

The ample divisors on S form an open cone in $Num(S) \otimes \mathbb{R} = NS(S) \otimes \mathbb{R}$, which is denoted by $N(S)^+$ (see, for example, [43], page 19, or [63], page 40). The next definition allows us to consider the elements in the closure of $N(S)^+$. These are the so called "nef" divisors. This terminology means *numerically effective divisors.* According to some authors (for example M. Reid) "nef" stands for *numerically eventually free.*

Definition 1.1.28 *Let S be a smooth, projective surface. An element $C \in Div(S)$ is said to be* nef*, if $CD \geq 0$ for each curve D in S. A nef divisor B is said to be* big *if $B^2 > 0$; this is equivalent (for nef divisors) to saying that $h^0(S, \mathcal{O}_S(mB))$ grows like m^2 (see [80]).*

Remark 1.1.29 To be more precise, an effective divisor D on a smooth surface S is called *big* if the projective map determined by the complete linear system $|nD|$, for $n >> 0$, maps S birationally onto its image. If D is nef, then the condition to be big is equivalent to saying that $D^2 > 0$.

Theorem 1.1.30 *(Kleiman's criterion, see [63], page 34) If D is a nef divisor, then $D^2 \geq 0$.*

Remark 1.1.31 By Kleiman's criterion, D is nef if and only if it is in the closure of the ample divisor cone of S (see [63] and [80]).

1.2 Vector bundles on projective varieties and some vanishing theorems.

In this section we want to recall some standard results on vector bundles on smooth, projective varieties, expecially on smooth, projective surfaces.

Let X denote a n-fold and E a rank-k vector bundle on X. $\mathbb{P}(E) := Proj(Sym(E^\vee))$ is a projective scheme of dimension $n+k-1$ and $\mathcal{O}_{\mathbb{P}(E)}(1)$ is its tautological line bundle.

We start by recalling some standard definitions on vector bundle theory (see, for example, [63], page 83). E is said to be *globally generated* if, as in the case of line bundles in Section 1.1, the evaluation morphism

$$ev : H^0(X, E) \otimes \mathcal{O}_X \to E$$

is surjective. A vector bundle E on X is *ample* if one of the following equivalent condition holds:

(i) For each coherent sheaf $\mathcal{F}$, there exists an integer $n_0 = n_0(\mathcal{F})$ such that, for each $n \geq n_0$, $\mathcal{F} \otimes Sym^n(E)$ is globally generated;

(ii) For each coherent sheaf $\mathcal{F}$, there exists an integer $n_1 = n_1(\mathcal{F})$ such that, for each $n \geq n_1$, $H^i(X, \mathcal{F} \otimes Sym^n(E)) = (0)$, for each $i > 0$;

(iii) The tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample on $\mathbb{P}(E)$.

Some standard properties of ampleness are the following.

Proposition 1.2.1 *Let X be a smooth, projective variety.*

(i) Let E be a vector bundle on X and let $Y \subset X$ be a closed subscheme; then E ample on X implies $E|_Y$ ample on Y.

(ii) Suppose to have a surjection of vector bundles $E \to E' \to 0$ on X, such that E is ample (resp. globally generated), then E' is ample (resp. globally generated).

(iii) Let $0 \to E' \to E \to E'' \to 0$ be an exact sequence of vector bundles on X. Then E' and E'' ample implies that E is ample.

(iv) $\bigoplus_{i=1}^n E_i$ is ample if and only if each E_i is ample.

(v) If E is ample, then $E^{\otimes m}$, $Sym^m(E)$ and $\bigwedge^j(E)$ are ample, for $m \in \mathbb{N}$ and $1 \leq j \leq k$.

(vi) If E_1 is ample and E_2 is ample or globally generated then $E_1 \otimes E_2$ is ample.

If X is a n-fold, denote by $A^r(X)$, $r \leq n$, the group of cycles of codimension r in X and by $A(X) = \bigoplus_{i=0}^n A^i(X)$ the Chow ring of X. Given E a vector bundle on X, then its *first Chern class* is defined by

$$c_1(E) := c_1(det(E)) \in A^1(X) \cong Pic(X)$$

(see for example [64], Appendix A.3). Observe that this definition is consistent with the one for line bundles given in Section 1.1 via the exponential sequence, since there is a homomorphism $A^i(X) \to H^{2i}(X, \mathbb{Z})$ which sends a codimenson i cycle to its fundamental class.

To define $c_i(E)$ in general, one observes that the definition is trivial for a vector bundle which is a direct sum of line bundles; indeed, if $E = L_1 \oplus \cdots \oplus L_n$, then

$$1 + c_1(E) + \cdots + c_n(E) = (1 + c_1(L_1)) \cdots (1 + c_1(L_n))$$

and the actual formula is obtained by equating the terms that lie in the correspondent group $A^i(X)$. From the *splitting principle*, one has a definition of *Chern classes* for an arbitrary vector bundle E.

Given a rank-k vector bundle E on X of dimension n, $c(E) = 1+c_1(E)+ \cdots + c_n(E)$ is called the *total Chern class*.

There are some standard, useful properties of Chern classes (see, for example, [43]).

Proposition 1.2.2 *Let X be a n-fold. Denote by E and L a rank-k vector bundle and a line bundle on X, respectively. Then,*

(i) $c_i(E^\vee) = (-1)^i c_i(E)$, $0 \leq i \leq n$,

(ii) $c_1(E \otimes L) = c_1(E) + kc_1(L)$,

(iii) $c_2(E \otimes L) = c_2(E) + (k-1)c_1(E)c_1(L) + \binom{k}{2} c_1(L)^2$.

There is another important formula for Chern classes of a rank-two vector bundle E on a smooth, projective surface S, which will be frequently used in the sequel.

Proposition 1.2.3 *Let S be a smooth, projective surface and let E be a rank-two vector bundle on S. Denote by Z a zero-dimensional subscheme of S and suppose there exists an exact sequence*

$$0 \to L \to E \to L' \otimes \mathcal{I}_{Z/S} \to 0,$$

where L and L' are line bundles on S. Then

(i) $c_1(E) = c_1(L) + c_1(L')$;

(ii) $c_2(E) = c_1(L)c_1(L') + l(Z)$, *where $l(Z)$ is the length of Z, which is defined as*

$$l(Z) := h^0(\mathcal{O}_Z) = h^0(\mathcal{O}_S/\mathcal{I}_Z).$$

Another important aspect of vector bundles on smooth varieties is given by the cohomological behaviour of such vector bundles. We shall frequently use several vanishing results, some of which are not so standard.

Theorem 1.2.4 *(Mumford's vanishing, [43], page 247) Let S be a smooth surface and let D be a big and nef divisor on S. Then $H^1(S, \mathcal{O}_S(-D)) = 0$.*

In 1982, a generalization of the previous result to all dimensions has been given:

Theorem 1.2.5 *(Kawamata-Viehweg vanishing, [80], page 25) Let X be a smooth, projective variety. If D is a big and nef divisor then*

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0,$$

for each $i > 0$.

There is also a generalization of the previous result, in the case $dim(X) = 2$, for $\mathbb{Q}$-divisors, but we shall not make use of this result in the sequel.

Particular cases to apply such cohomological results are given, for example, by blow-ups of smooth surfaces. Consider S a smooth, projective surface and let D be an ample (or nef and big) divisor on S. One could ask, for example, if the line bundle $\omega_S \otimes \mathcal{O}_S(D)$ is free (i.e. globally generated) at a point $x \in S$. Since, by the definition, this happens when the map

$$H^0(S, \omega_S \otimes \mathcal{O}_S(D)) \to H^0(S, \omega_S \otimes \mathcal{O}_S(D) \otimes \mathcal{O}_x)$$

is surjective, freeness in x is equivalent to proving that

$$(*)\ H^1(S, \omega_S \otimes \mathcal{O}_S(D) \otimes \mathcal{I}_x) = (0).$$

Vanishing theorems above cannot directly apply because the sheaf in question is not locally free on S. The traditional step around this problem is to blow-up at x, which at least reduces the question to one involving only invertible sheaf.

Lemma 1.2.6 *Let S be a smooth surface and let*

$$\mu : X = Bl_x(S) \to S$$

be the blow-up of S at a point $x \in S$. Denote by E the μ-exceptional divisor in X. Let L be a line bundle on S and $r > 0$ be any positive integer. Then, for all $i \geq 0$, there are isomorphisms

$$H^i(S, \omega_S \otimes L \otimes \mathcal{I}_x^r) \cong H^i(X, \omega_X \otimes \mu^* L \otimes \mathcal{O}_X(-(r+1)E)).$$

Proof: See [64] or [80]. □

Thus, with the previous result, one can apply some standard vanishing theorems, as Kodaira vanishing, if $L = \mathcal{O}_S(D)$ is ample, or Mumford's vanishing (Theorem 1.2.4), if D is big and nef, to prove $(*)$ and so to determine if $\omega_S \otimes \mathcal{O}_S(D)$ is globally generated at the given point.

There are also some vanishing results of vector bundles on smooth, projective varieties, or more generally, on compact, complex manifolds, known as *Griffiths vanishings* (see [128], pages 107, 109 and 110, respectively), which will be used in Chapter 5.

Theorem 1.2.7 *Let X be a complex, compact manifold and let E be a vector bundle on X. If E is globally generated and if L is a positive (ample, in the algebraic case) line bundle, then*

$$H^q(X,\ \mathcal{O}_X(K_X) \otimes Sym^l(E) \otimes det(E) \otimes L) = (0),\ \forall\, l,\ q > 0.$$

Similarly,

Theorem 1.2.8 *Let X be a complex, compact manifold and let E be a vector bundle on X. If E is ample and if L is a nef line bundle, then*

$$H^q(X,\ \mathcal{O}_X(K_X) \otimes Sym^l(E) \otimes det(E) \otimes L) = (0),\ \forall\, l,\ q > 0.$$

Remark 1.2.9 In the original statement L is taken to be a semi-positive line bundle; however, at page 114 of [128], it is remarked that "semi-positive" can be replaced by "nef".

Corollary 1.2.10 *Let X be a complex, compact manifold and let E be a vector bundle on X. If E is ample then*

$$H^q(X,\ \mathcal{O}_X(K_X) \otimes Sym^l(E) \otimes det(E)^{\otimes k}) = (0),\ \forall\ l,\ k,\ q > 0.$$

We end the section by recalling the notion of stability of vector bundles and some other tools related to it. There are several definitions of "stability" (see, for example [43]). We will be mainly concerned with Bogomolov-stability of rank-two vector bundles on a smooth, projective surface.

In 1960, Mumford introduced the notion of a stable or semistable vector bundle on a curve and used Geometric Invariant Theory ([99]) to construct moduli spaces for semistable vector bundles over a given curve. There were various attempts to generalize Mumford's definition of stability to surfaces or to higher-dimensional varieties in order to construct moduli spaces of vector bundles. In 1972, Takemoto gave the straightforward generalization to higher-dimensional (polarized) smooth, projective varieties, which is now called the *Mumford-Takemoto stability* or *μ-stability.*

Definition 1.2.11 *(Mumford-Takemoto stability) Let X be a n-fold with a fixed ample divisor H and let E a vector bundle on X. The* slope *of E (or its* normalized degree *with respect to H) is defined by*

$$\mu(E) := \frac{c_1(E)H^{n-1}}{rank(E)} \in \frac{1}{rank(E)}\mathbb{Z} \subset \mathbb{Q}.$$

*E is said to be μ-stable (resp. μ-semistable) if for all coherent subsheaves $\mathcal{F}$ of E, with $0 < rank(\mathcal{F}) < rank(E)$, we have $\mu(\mathcal{F}) < \mu(E)$ ($\mu(\mathcal{F}) \leq \mu(E)$ resp.). E is μ-*unstable *if it is not μ-semistable and μ-*strictly semistable *if it is μ-semistable but not μ-stable. Finally, a subsheaf $\mathcal{F}$ of E is* destabilizing *if $\mu(\mathcal{F}) \geq \mu(E)$.*

Since $\mathcal{F}$ is a torsion-free sheaf, then $\mu(\mathcal{F})$ is well defined because $\mathcal{F}$ is locally free in codimension one (see, for example, [95], page 61). For simplicity, here we shall simply write stability instead of μ-stability. There are equivalent definitions of stability; for example by using quotient sheaves of E instead of its subsheaves (see [43]). Moreover, the notion of (semi)stability is preserved under dualizing operation or tensoring operation by a line bundle. More precisely, E is (semi)stable if and only if $E^\vee$ is (semi)stable or if and only if $E \otimes L$ is (semi)stable, for each line bundle L.

Note that, if X is a curve, then $\mu(E) = deg(c_1(E))$ is independent from the choice of H and we obtain the original definition due to Mumford. However, for $dim(X) \geq 2$, if $rank(Num(X)) \geq 2$, then the definition of stability strongly depends on the choice of the numerical class of H.

On the other hand, in 1975, Bogomolov ([12] or [114]) gave an extension of Mumford's definition of stability on projective curves to varieties of higher-dimension and proved an instability criterion. The fundamental fact is that his criterion, at least in the case of a rank-two vector bundle E on $X = S$ a smooth, projective surface, translates into a numerical criterion only in terms of the Chern classes of E.

Definition 1.2.12 *(Bogomolov instability) Let S be a smooth projective surface. A rank-two vector bundle E on S is said to be* Bogomolov-unstable *if there exist $A,\ B \in Div(S)$ and a zero-dimensional scheme Z (possibly empty), fitting in the exact sequence*

$$0 \to \mathcal{O}_S(A) \to E \to \mathcal{I}_Z(B) \to 0 \qquad (1.9)$$

and moreover $c_1(\mathcal{O}_S(A)) - c_1(\mathcal{O}_S(B)) = A - B \in N(S)^+$, where $N(S)^+$ denotes the ample divisor cone of S (see Section 1.1). Otherwise, E is Bogomolov-stable.

Very concretely, we recall that the condition $A - B \in N(S)^+$ is equivalent to the conditions

(i) $(A - B)^2 > 0$, and

(ii) $(A - B)H > 0$ for each ample divisor H.

Bogomolov's statement is the following.

Theorem 1.2.13 *(Bogomolov's criterion) Let E be a rank-two vector bundle on a smooth, projective surface S. Then*

$$c_1(E)^2 - 4c_2(E) > 0$$

if and only if E is Bogomolov-unstable.

Proof: See the original paper [12] or [114]. □

From Proposition 1.2.3, since E fits in the exact sequence (1.9) we immediately get $c_1(E) = A + B$ and $c_2(E) = AB + l(Z)$.

Remark 1.2.14 Observe that, from Proposition 1.2.2, given a rank-two vector bundle E and a line bundle L on S, we have

$$c_1(E^\vee)^2 - 4c_2(E^\vee) = c_1(E)^2 - 4c_2(E)$$

and

$$c_1(E \otimes L)^2 - 4c_2(E \otimes L) = c_1(E)^2 - 4c_2(E);$$

thus, the polynomial

$$c_1(-)^2 - 4c_2(-)$$

in the Chern classes of a rank-two vector bundle E is invariant under the operations of tensoring E with a line bundle and of passing to the dual bundle $E^{\vee}$. Therefore, also the notion of Bogomolov's instability is preserved by such operations.

In the same paper, Bogomolov proved also that the cotangent bundle Ω^1_S of a smooth, projective surface of general type is Bogomolov-stable. As a consequence, he got the well known inequality for a surface S of general type

$$c_1^2(S) \leq 4c_2(S),$$

(see also Section 1.3).

We conclude by recalling an important technical tool in vector bundle theory on smooth, projective surfaces.

Definition 1.2.15 *Let E be a rank-k vector bundle on a smooth, projective surface S. The* discriminant *of E is denoted by $\delta(E)$ and defined by*

$$\delta(E) := (\frac{k-1}{2k})c_1(E)^2 - c_2(E).$$

By Theorem 1.2.13, if $k = 2$ and $\delta(E) > 0$ then E is Bogomolov-unstable. The definition of the discriminant of a vector bundle is a technical tool which is very useful for computations, expecially when we are dealing with elementary modifications of vector bundles on S or restrictions of stable bundles on S to its divisors (see [14]). These are fundamental techniques which will be considered in Chapter 5 to give some first positive answers to our "moduli problem" (see Section 5.1).

Definition 1.2.16 *Let C be an effective divisor on a smooth, projective surface S. If E and F are vector bundles on S and C, respectively, then a vector bundle $T_{C,F}(E)$ is obtained by an* elementary modification *of E by F along C if there exists an exact sequence*

$$0 \to T_{C,F}(E) \to E \to i_*F \to 0,$$

where i denotes the embedding $C \subset S$.

As usual, we just write F instead of i_*F meaning F with its natural $\mathcal{O}_S$-structure.

Proposition 1.2.17 *$T_{C,F}(E)$ is locally free. Moreover, we have the following numerical equalities.*

(i) $c_1(T_{C,F}(E)) = c_1(E) - rk(F)C$;

(ii) $c_2(T_{C,F}(E)) = c_2(E) + c_1(F) - rk(F)(c_1(E)C) + rk(F)(rk(F) - 1)C^2$;

(iii) $\delta(T_{C,F}(E)) = \delta(E) + \frac{rk(F)}{rk(E)}(c_1(E)C) + rk(F)(\frac{rk(E)-rk(F)}{2rk(E)})C^2 - c_1(F)$.

Proof: These are straightforward computations. The reader is referred to the original paper [14] or to [70], page 129. □

1.3 Fundamental properties of surfaces of general type

This section is devoted to recalling some of the fundamental properties of smooth, projective surfaces of general type. As it is well known, smooth surfaces can be basically divided into four "classes", according to the the behaviour of their pluricanonical divisors.

Definition 1.3.1 *Let X be a smooth, projective variety, K_X be a canonical divisor of X, φ_n be the rational map from X to the projective space associated to the complete linear system $\mid nK_X \mid$. The* Kodaira dimension of X, *written $\kappa(X)$, can be defined as the maximum dimension of the images $\varphi_n(X)$, for $n \geq 1$.*

If $\mid nK_X \mid = \emptyset$ then $\varphi_n(X) = \emptyset$ and one defines $\kappa(X) = -\infty$.

If $X = S$ is a surface, a first classification can be done in terms of the Kodaira dimension of S. Recall that

$$P_n(S) := h^0(S, \omega_S^{\otimes n})$$

is called the n^{th}-*plurigenus* of S, for $n \geq 1$; if $n = 1$, one simply denotes by $p_g(S)$ the first plurigenus of S, which coincides with the geometric genus of S. These, as well as $\kappa(S)$, are birational invariants of S.

If $\kappa(S) = 2$ then S is called a *surface of general type*. Therefore, S is of general type if and only if, for some N, $\varphi_N(S)$ is a surface.

Trivial examples of surfaces of general type are given by products of two smooth curves, $C_1 \times C_2$, where C_i of genus $g_i \geq 2$. Furthermore, if $S_{d_1,\ldots,d_{r-2}}$ denotes a surface in $\mathbb{P}^r$ which is the complete intersection of $r-2$ hypersurfaces of degrees $d_1, \ldots, d_{r-2}$ then, since $K_S = (\sum_{i=1}^{r-2} d_i - r - 1)H$, H a hyperplane section of S, we obtain that S_2, S_3, $S_{2,2}$ are rational, S_4, $S_{2,3}$, $S_{2,2,2}$ are $K3$ surfaces, whereas, all the other surfaces $S_{d_1,\ldots,d_{r-2}}$ are of general type. In particular, if $S \subset \mathbb{P}^3$ is smooth, of degree $d \geq 5$ then it is a regular surface (as all surfaces in $\mathbb{P}^3$) of general type.

Proposition 1.3.2 *Let S be a smooth, projective surface such that $\kappa(S) \geq 0$. Let $C \subset S$ be an effective, reduced and irreducible divisor such that $K_S C < 0$. Then C is a (-1)-curve (see Definition 1.1.21). Moreover, on such a S, there exists only a finite number of (-1)-curves. In particular, if S is minimal, then K_S is nef.*

Proof: Since $\kappa(S) \geq 0$, there exists an effective divisor $D \in \mid mK_S \mid$, for some $m > 0$. Suppose C irreducible such that $CK_S < 0$. Then $DC = mK_S C < 0$; this implies that C is a component of D and that $C^2 < 0$. Indeed, if $D = aC + D'$, then $DC = aC^2 + D'C$, with $D'C \geq 0$ and $a > 0$, which implies $C^2 < 0$. By adjunction,

$$-2 \leq 2p_a(C) - 2 = C^2 + CK_S \leq -2.$$

Therefore, $C^2 = CK_S = -1$ and $p_a(C) = p_g(C) = 0$, therefore C is rational and smooth. If $E_1 \neq E_2$ are (-1)-curves in S, then $E_1E_2 = 0$; so, these are independent elements of $NS(S)$. It follows that (-1)-curves in S are finitely many. $\square$

Remark 1.3.3 If S is a minimal surface of general type, then $K_S^2 > 0$ (see, for example, [8] or [10]). Since it is also nef, by Remark 1.1.29 K_S is big and nef.

There are fundamental inequalities involving Chern classes of S. Such inequalities will be used in some examples in Section 5.1, when we deal with some positive answers to the "moduli problem" for pluri-canonical linear sytems on a regular projective surface of general type.

To this aim, we recall that the Chern classes of S are defined as

$$c_i(S) := c_i(\mathcal{T}_S),\ 1 \leq i \leq 2.$$

Therefore, $c_1(S) = -K_S$ and $c_2(S) = e(S)$. As already mentioned in Section 1.2, Bogomolov proved, in [12] (see also [114]), that if S is a smooth, projective surface of general type, then Ω_S^1 (and so $\mathcal{T}_S$) is Bogomolov-stable, in the sense of Definition 1.2.12. By his numerical criterion (Theorem 1.2.13) we get

$$c_1(S)^2 \leq 4c_2(S). \tag{1.10}$$

A year later, Miyaoka and Yau independently improved the previous inequality by showing that for a surface S of general type

$$c_1(S)^2 \leq 3c_2(S) \tag{1.11}$$

holds (see the original papers [92] and [141] or [8], page 212).

We end this section by recalling a general result of Matsumura which is related to varieties of general type ([87]).

Proposition 1.3.4 *If V is a projective, non-singular variety and if the rational mapping of V determined by $\mid nK_V \mid$ is birational onto its image, for some $n > 0$, then $Bir(V)$, the group of all birational transformations of V onto itself, is finite.*

Proof: See Corollary 2, [87]. $\square$

In our case, if S is a smooth, projective surface of general type then $Aut(S)$ is a finite group. This gives a useful cohomological condition; indeed, since $H^0(\mathcal{T}_S)$ is the Lie algebra of the Lie group of automorphisms of S, then

$$H^0(\mathcal{T}_S) = 0, \tag{1.12}$$

for each smooth surface of general type. There are some papers, which determine a polynomial estimate, in terms of $c_2(S)$, for an upper-bound on the order of $Aut(S)$ (see, for example, [33]).

1.4 Embedded curves in a projective surface. Nodal and stable curves

In this section, we shall deal with singular (in particular nodal) curves on a smooth projective surface.

Let S be a smooth, projective surface and let $X \subset S$ be an effective, reduced divisor, which can be singular and even reducible. If $\varphi : C \to X \subset S$ is the normalization of X, then C is a smooth (possibly reducible) curve. For $x \in X$ singular, $\varphi^{-1}(x)$ consists of finitely many points, corresponding to the different branches of X through x. On X one has the *normalization sequence*

$$0 \to \mathcal{O}_X \to \varphi_* \mathcal{O}_C \to \underline{t} \to 0, \tag{1.13}$$

where $\underline{t}$ is a torsion sheaf, supported on the singularities of X.

In general, we can define the *local genus drop at* x (or the δ*-invariant at* x) to be the non-negative integer

$$\delta_x := dim_{\mathbb{C}}(\varphi_* \mathcal{O}_C / \mathcal{O}_X)_x = dim_{\mathbb{C}}(\underline{t}_x). \tag{1.14}$$

Thus, $\delta_x = 0$ if and only if x is a smooth point in X. For example, if x is an ordinary double point (a node) then $\delta_x = 1$.

For an arbitrary non-zero effective divisor X we can define the arithmetic genus of X by the formula

$$2p_a(X) - 2 = (K_S + X)X,$$

as in the adjunction formula (1.7) for the smooth case.

For simplicity, assume that X is reduced and irreducible. There exists on X the dualizing sheaf which is denoted by ω_X and defined as $\omega_X = \omega_S \otimes \mathcal{N}_{X/S}$; moreover, since X is a reduced divisor in a smooth, projective surface, ω_X is locally free (see [64]) and

$$2p_a(X) - 2 = deg(\omega_X).$$

The morphism φ is finite so, for any sheaf $\mathcal{F}$ on C, the higher direct images $R^i \varphi_* \mathcal{F}$ vanish. Hence we get the Leray isomorphisms

$$H^i(C, \mathcal{F}) \cong H^i(X, \varphi_* \mathcal{F}), \tag{1.15}$$

for each $i \geq 0$ (see [64]). From the fact that X is reduced and irreducible, it follows that

$$H^0(\mathcal{O}_X) \cong H^0(\mathcal{O}_C) \cong H^0(\varphi_* \mathcal{O}_C) \cong \mathbb{C}.$$

Thus, by (1.13), we get that

$$h^1(\mathcal{O}_X) = h^1(\varphi_* \mathcal{O}_C) + h^0(\underline{t}) = h^1(\mathcal{O}_C) + \delta,$$

where δ is the non-negative integer

$$\delta := \sum_{x \in X} \delta_x.$$

Furthermore, by Definition 1.1.1, $p_a(X) = 1 - \chi(\mathcal{O}_X)$; since $h^0(\mathcal{O}_X) = 1$, we have

$$p_a(X) = h^1(\mathcal{O}_X) = h^0(\omega_X),$$

by the definition of dualizing sheaf.

Therefore, the genera of C and X are related by

$$p_a(X) = p_a(C) + \delta = g(C) + \delta,$$

where $g(C)$ is the geometric genus of the smooth curve C and δ is called the *genus drop* of the curve X.

Observe that, if each singular point is a node, then δ coincides with the cardinality of the support of $Sing(X)$ (the singular locus of X), i.e. δ is the number of nodes of X.

Note that we can still define the global invariant δ even if X is assumed to be only reduced. As before, one obtains, more generally, that if C_i, $1 \leq i \leq n$, denote the (smooth) connected components of C, with $g(C_i) = g_i$, then

$$p_a(X) = \sum_{i=1}^{n} g_i + \delta + 1 - n. \tag{1.16}$$

In complete analogy with the smooth case, one can prove the Riemann-Roch theorem for any locally free sheaf F on a singular curve X. This theorem holds even if X is not reduced (see [8], Theorem 3.1, page 51). Since in the sequel we will mainly consider reduced, nodal curves, we prove this result only in the reduced case.

For every locally free sheaf F on X, one defines $deg(F)$ as $deg(det(F))$, as in the smooth case.

Theorem 1.4.1 *(Riemann-Roch theorem for reduced, embedded curves) Let $X \subset S$ be a reduced curve, embedded in a smooth, projective surface S and let F be a locally free sheaf of rank r on X. Then*

$$\chi(F) = deg(F) + r\chi(\mathcal{O}_X).$$

Proof: Since X is reduced, then we have the normalization map

$$\varphi : C \to X \subset S.$$

The fact that φ is birational implies that the degree of the map (defined as in [64], page 137) is such that $deg(\varphi) = 1$. Therefore,

$$deg(\varphi^* F) = deg(\varphi) deg(F) = deg(F).$$

Moreover, by the Leray isomorphisms in (1.15),

$$\chi(\varphi^* F) = \chi(\varphi_* \varphi^* F) \text{ and } \chi(\mathcal{O}_C) = \chi(\varphi_* \mathcal{O}_C).$$

So Riemann-Roch theorem on C implies

$$deg(F) = \chi(\varphi^* F) - r\chi(\mathcal{O}_C) = \chi(\varphi_* \varphi^* F) - r\chi(\varphi_* \mathcal{O}_C).$$

Since F is locally free, $(\varphi_* \varphi^* F/F) \cong (\varphi_* \mathcal{O}_C/\mathcal{O}_X)^{\oplus r} = \underline{t}^{\oplus r}$ and

$$\chi(F) = \chi(\varphi_* \varphi^* F) - \chi(\varphi_* \varphi^* F/F) =$$

$$= deg(F) + r\chi(\varphi_* \mathcal{O}_C) - r\chi(\varphi_* \mathcal{O}_C/\mathcal{O}_X) =$$

$$= deg(F) + r\chi(\mathcal{O}_X),$$

as stated. □

In 1.1.18 we have given the definition of a torsion-free sheaf. Equivalently, $\mathcal{F}$ is a *torsion-free sheaf of rank one* on a curve X (not necessarily smooth) if there exists a finite set N of points of X such that $\mathcal{F} |_{X \setminus N}$ is a line bundle and the map $\mathcal{F} \to i_* i^* \mathcal{F}$ is injective, where $i : X \setminus N \to X$ is the natural inclusion map (see [43], page 198). Thus, for example, $\varphi_* \mathcal{O}_C$ is a torsion-free sheaf of rank one on the singular curve X.

One can simply extend the notion of *degree* to torsion-free sheaves of rank one on singular curves.

Definition 1.4.2 *(see [43], page 198) Let $\mathcal{F}$ be a torsion-free sheaf of rank one on a reduced curve X. Then,*

$$deg(\mathcal{F}) := \chi(\mathcal{F}) + p_a(X) - 1.$$

As an immediate consequence of the previous definition and of the exact sequence (1.13), we get that $deg(\varphi_* \mathcal{O}_C) = \delta$.

We now want to study local properties of some fundamental invertible or torsion-free sheaves on a reduced, irreducible nodal curve embedded in a smooth, projective surface. Roughly speaking, a nodal curve X is a complete curve which is locally smooth or analytically isomorphic to $xy = 0$, where (x, y) are local coordinates of S. Denote by $N = \{p_1, \ldots, p_\delta\}$ the set of nodes of X; let $\varphi : C \to X \subset S$ be the normalization map and let $\{q_i,\ q_i'\}$ be the preimages in C of the node p_i in X, $1 \leq i \leq \delta$. Put $B = \sum_{i=1}^{\delta}(q_i + q_i')$ on C. Then, the dualizing sheaf ω_X is such that

$$\omega_X \subset \varphi_*(\omega_C(B))$$

(see [8] or [60]).

To deal more closely with the sheaf of differentials of X, by GAGA's results we can consider the analytic approach which is more appropriate for a local analysis (see [7]). In general, if Y is a complex, analytic space embedded in a complex, projective manifold V of dimension n, let $\mathcal{I}_Y$ be (as in the algebraic approach) the ideal sheaf of Y in V and Ω^1_V be the vector bundle of holomorphic 1-forms on V. The usual differentiation induces a map

$$\mathcal{I}_Y \to \Omega^1_V |_Y,$$

locally defined by

$$f \to d(f) := \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} dz_i,$$

where $(z_1, \ \ldots, \ z_n)$ are local coordinates for V. Since for each $f, g \in \mathcal{I}_Y$, $d(fg) = d(f)g + d(g)f$, we have an induced map

$$(\mathcal{I}_Y/\mathcal{I}_Y^2) \xrightarrow{d} \Omega_V^1 \mid_Y,$$

whose cokernel is called the sheaf of *Kähler differentials*. This is nothing but the cotangent sheaf Ω_Y^1 recalled in Section 1.1.

Remark 1.4.3 An intrinsic definition can be given by considering the $\mathcal{O}_Y$-module generated by the symbols $gd(f)$, where $g, f \in \mathcal{O}_Y$, modulo the usual relations imposed by $\mathbb{C}$-linearity and Leibnitz rule; thus, for each $\mathcal{O}_Y$-module $\mathcal{E}$,

$$\underline{Hom}(\Omega_Y^1, \mathcal{E}) = Der_{\mathbb{C}}(\mathcal{O}_Y, \mathcal{E}).$$

By Theorem 1.1.4, Ω_Y^1 is locally free if and only if Y is smooth and, by Theorem 1.1.8, if Y is reduced and a local complete intersection in V then $\mathcal{I}_Y/\mathcal{I}_Y^2$ is locally free of rank $c = codim_V(Y)$ and such that the exact sequence

$$0 \to \mathcal{I}_Y/\mathcal{I}_Y^2 \to \Omega_V^1 \otimes \mathcal{O}_Y \to \Omega_Y^1 \to 0$$

holds.

We restrict ourselves to the case $Y = X$ reduced curve, with only nodes as singularities, in $V = S$ a smooth, projective surface. The sheaf Ω_X^1 and its dual sheaf (denoted by Θ_X in the sequel) are fundamental tools in studying first-order equisingular deformations of nodal curves.

If $p \in X$ is a smooth point then, with a local coordinate z centered at p such that $z(p) = 0$,

$$\Omega_{X,p}^1 = \{f(z)dz \mid f(z) \ holomorphic \ around \ 0\}$$

so there is nothing else to say.

Let, therefore, p be a double point of X; thus, in a neighborhood of p the curve X is isomorphic to the curve $xy = 0$ in a certain open set $U \subset \mathbb{C}^2_{(x,y)}$, where (x, y) local coordinates for S around $p = (0, 0)$. Now,

$$\mathcal{O}_{X,p} = \{k + f_1(x) + f_2(y) \mid k \in \mathbb{C}, \ f_i \ holomorphic \ s.t. \ f_i(0) = 0\} = \frac{\mathbb{C}\{x, y\}}{(xy)}.$$

Therefore, by Remark 1.4.3,

$$\Omega_{X,p}^1 = \frac{\{(k + f_1(x) + f_2(y))(g_1(x)dx + g_2(y)dy)\}}{relations =< xy = 0, ydx + xdy = 0 >},$$

where g_i, f_i holomorphic which map 0 to 0. Thus, we get

$$(*) \ \ \Omega_{X,p}^1 = \{f_1(x)g_1(x)dx + f_2(y)g_2(y)dy + (k + f_1(x))g_2(y)dy +$$
$$+ (k + f_2(y))g_1(x)dx\}/relations < xy = 0, ydx + xdy = 0 > .$$

Now,

$$x^n dy = x^{n-1} x dy = -x^{n-1} y dx$$

and

$$y^n dx = -y^{n-1} x dy$$

over X, for $n \geq 2$; so only the linear terms of $f_1(x)$ and $f_2(y)$ give some contributions to the third and last summand of $(*)$. Therefore, the contribution of these two summands can be written as

$$(k + h_1(x)x)g_2(y)dy + (k + h_2(y)y)g_1(x)dx.$$

The contributions of $kg_2(y)dy$ and $kg_1(x)dx$ may be absorbed by $f_2(y)g_2(y)dy$ and $f_1(x)g_1(x)dx$ respectively, so we are left with

$$h_1(x)xg_2(y)dy + h_2(y)yg_1(x)dx;$$

of course, here the terms of g_1 and g_2 of degree greater than or equal to one do not give any contribution because $xy = 0$; so we are left with

$$xk_1 dy + yk_2 dx, \quad k_i \in \mathbb{C}.$$

By using $xdy = -ydx$, we get finally an expression of the form

$$\alpha y dx, \quad \alpha \in \mathbb{C}.$$

Thus,

$$\Omega^1_{X,p} \cong \{a(x)dx + b(y)dy + \alpha y dx\} \tag{1.17}$$

where $a(x) = f_1(x)g_1(x)$, $b(y) = f_2(y)g_2(y)$, respectively.

The terms of type $\alpha y dx$ are (somehow improperly) called *torsion differentials*, because $x(\alpha y dx) = 0$. From the above discussion we get that, if $\varphi : C \to X \subset S$ is the normalization of X, the following exact sequence holds

$$0 \to \mathcal{TD}_X \to \Omega^1_X \to \varphi_*\Omega^1_C = \varphi_*\omega_C \to 0. \tag{1.18}$$

$\mathcal{TD}_X$ is the sheaf of the torsion differentials, for which

a) $\mathcal{TD}_{X,p} = 0$, if p is a smooth point of X, or

b) $\mathcal{TD}_{X,p} \cong \mathbb{C}$, if p is a node of X.

Another important "ingredient" is the conductor ideal of a reduced, irreducible nodal curve X embedded in a smooth, projective surface S. This is strictly related to the first-order equisingular deformations of such a curve.

More generally, as we shall explain in more detail in Section 2.2, there are two ways of analyzing deformations of a reduced, irreducible singular curve in S satisfying certain geometric conditions.

In the "parametric" approach, one looks at deformations of the normalization map

$$\varphi : C \to X \subset S.$$

It is the fundamental result of Horikawa's theory that the tangent space to the space of such deformations is, a priori, a subspace of the space of sections of the normal sheaf of the map $\mathcal{N}_\varphi$ (see Section 2.2). This has the virtue of incorporating the condition that the geometric genus of X is preserved in the deformations. Moreover, $\mathcal{N}_\varphi$ is a sheaf on a smooth curve. On the other hand, it has the defect that, unless we know that φ has generically injective differential, the sheaf $\mathcal{N}_\varphi$ can have torsion. Fortunately, in the nodal case, $\mathcal{N}_\varphi$ is a line bundle on C (see Section 2.2).

In the other approach, which is usually called the "Cartesian" approach, we look instead at deformations of X as a subscheme of the surface S; so that the tangent space to the space of deformations is, a priori, a subspace of the space of sections of the normal bundle $\mathcal{N}_{X/S} \cong \mathcal{O}_X(X)$ of the divisor $X \subset S$. In some ways, this is more direct and it is in particular useful when we want to intersect our family of deformations with other subvarieties of the projective space determined by the complete linear system of curves linearly equivalent to X. But it has the drawback that we have to impose extra conditions to ensure that the geometric genus of X stays constant in deformations. These conditions, moreover, sometimes interact badly with conditions such as tangency with a fixed curve (see also [15] and [17]).

In some cases, there is a reasonably straightforward relationship between these two approaches. To start with, we make the following definition.

Definition 1.4.4 *Let S be a smooth, projective surface and let $\varphi : C \to X \subset S$ be a map from a smooth curve C into S with injective differential (for example when X is an irreducible, nodal curve and φ is the normalization map). We denote by*

$$\varsigma_X \subset \mathcal{O}_X$$

the conductor ideal *of the curve X, which may be characterized in several equivalent ways.*

(i) It is the annihilator of the sheaf $\varphi_*(\mathcal{O}_C)/\mathcal{O}_X$;

(ii) More concretely, it is the ideal in $\mathcal{O}_X$ whose restriction to each branch Δ_i of X, at each point $p \in X$, is equal to the restriction to that branch of the ideal of the union of all other branches of X through p. In other words, if $p_i \in C$ is the point lying over p in the branch Δ_i,

$$\varphi^* \varsigma_X = \mathcal{O}_C(-\sum_i (\sum_{j \neq i} mult_p(\Delta_i \Delta_j)) p_i).$$

Therefore, if X contains only a node p as singularity, such that $\varphi(p_1) = \varphi(p_2) = p$, then $\varphi^* \varsigma_X = \mathcal{O}_C(-p_1 - p_2)$.

Remark 1.4.5 From (i) above, we observe that ς_X is an ideal sheaf on X of $\mathcal{O}_X$-modules and of $\varphi_* \mathcal{O}_C$-modules, defined by

$$\varsigma_X = \{c \in \mathcal{O}_X \mid \ ca \in \mathcal{O}_X, \ \forall \, a \in \varphi_*(\mathcal{O}_C)\}.$$

If $p \in X$, we denote by $\delta(p)$ the dimension $dim_{\mathbb{C}}((\varphi_*(\mathcal{O}_C)/\mathcal{O}_X)_p)$; the point p is smooth if and only if $\delta(p) = 0$. Otherwise, if p is an ordinary double point, then $\delta(p) = 1$.

Remark 1.4.6 There is another standard definition for the conductor sheaf (see, for example, [71], page 365). More generally, let $f : V \to Y$ be a finite map from a n-fold into a $(n+1)$-fold Y such that the induced map $h : V \to f(V)$ is birational. Denote by Z the image $f(V)$, which is a closed subvariety of Y. Then, the conductor of the finite, birational map $h : V \to Z$ is given by the formula

$$\varsigma_Z = \underline{Hom}_{\mathcal{O}_Z}(h_*\mathcal{O}_V, \mathcal{O}_Z).$$

From this definition, if we consider our case $Y = S$, $V = C$, $Z = X$ an irreducible, nodal curve and $h = \varphi$ the normalization map, we can consider the normalization sequence (1.13)

$$0 \to \mathcal{O}_X \to \varphi_*\mathcal{O}_C \to \underline{t} \to 0$$

and we can apply the functor $\underline{Hom}(-, \mathcal{O}_X)$ to immediately get that the conductor ς_X is an ideal sheaf, since

$$0 \to \underline{Hom}_{\mathcal{O}_X}(\varphi_*\mathcal{O}_C, \mathcal{O}_X) \to \mathcal{O}_X \to \underline{Ext}^1(\underline{t}, \mathcal{O}_X) \to \dots$$

holds because $\underline{t}$ is a torsion sheaf on X.

No matter how we characterize the conductor, it is not hard to see that, since $\varphi : C \to X \subset S$ has injective differential, the normal bundle $\mathcal{N}_\varphi$ of the map and the normal bundle $\mathcal{N}_{X/S} \cong \mathcal{O}_X(X)$ of the curve X are related by

$$\mathcal{N}_\varphi = \varphi^*(\varsigma_X \otimes \mathcal{N}_{X/S}). \tag{1.19}$$

This can be most easily seen in terms of description (ii) in Definition 1.4.4. If the local defining equation $f(x,y)$ of X at a point $p \in X$ factors, in the completion of the local ring $\mathcal{O}_{X,p}$, as

$$f(x,y) = f_1(x,y) f_2(x,y) \cdots f_n(x,y),$$

then a general first-order deformation of the map φ will simply move each branch, resulting in a curve given by the equation

$$f_\epsilon(x,y) = (f_1(x,y) + \alpha_1\epsilon)(f_2(x,y) + \alpha_2\epsilon) \cdots (f_n(x,y) + \alpha_n\epsilon).$$

As a deformation of the map, that is, as a section of $\mathcal{N}_\varphi$, this will be non-zero at the point of C corresponding over the branch Δ_i given by $f_i(x,y) = 0$ if and only if the coefficient $\alpha_i \neq 0$. But the corresponding section of the normal bundle $\mathcal{N}_{X/S}$ (that is, the restriction to X of the coefficient of ϵ in $f_\epsilon(x,y)$) on this branch is $\alpha_i \prod_{i \neq j} f_j(x,y)$, which vanishes to order $\sum_{j \neq i} mult_p(\Delta_j \Delta_i)$.

In any event, the conclusion is that the sections of $\mathcal{N}_{X/S}$ coming from deformations of the map are simply those lying in the conductor ideal. We thus have a very useful dictionary between the two languages, at least as long as φ has an injective differential, otherwise the correspondence is more complicated.

To sum up, the conductor ideal determines the subsheaf of the normal bundle $\mathcal{N}_{X/S}$, whose global sections correspond to equisingular deformations of X in S. Even if more will be said in Section 2.2 on the subject of equisingular deformation theory and of the sheaves describing abstract and embedded equisingular deformations of the nodal curve X, here we shall make a brief overview of the principal exact sequences which must be considered to approach the problem.

Given S a smooth, projective surface, $X \subset S$ an irreducible, nodal curve and $\varphi : C \to X \subset S$ the normalization map, we can apply the contravariant, left-exact functor $\underline{Hom}(-, \mathcal{O}_X)$ to the exact sequence

$$0 \to \mathcal{I}_X/\mathcal{I}_X^2 \to \Omega_S^1 \mid_X \to \Omega_X^1 \to 0$$

(see Theorem 1.1.8). Thus, we get

$$0 \to \underline{Hom}(\Omega_X^1, \mathcal{O}_X) \to \underline{Hom}(\Omega_S^1, \mathcal{O}_X) \to$$

$$\to \underline{Hom}(Con_{X/S}, \mathcal{O}_X) \to \underline{Ext}^1(\Omega_X^1, \mathcal{O}_X) \to 0,$$

since $\Omega_S^1 \mid_X$ is locally free.

Observe that $\underline{Hom}(\Omega_S^1, \mathcal{O}_X) = \mathcal{T}_S \mid_X$, whereas Ω_X^1 is the cotangent sheaf (or the sheaf of Kählerian differentials) of X and its dual

$$\Theta_X := \underline{Hom}(\Omega_X^1, \mathcal{O}_X), \tag{1.20}$$

is called the *sheaf of derivations* of $\mathcal{O}_X$.

On the other hand,

$$T_X^1 := \underline{Ext}^1(\Omega_X^1, \mathcal{O}_X) \tag{1.21}$$

is called the *first cotangent sheaf of* X; this is a torsion sheaf supported on $Sing(X)$, with $T^1_{X,p} \cong \mathbb{C}$ for each $p \in Sing(X)$ ([81]). The resulting exact sequence

$$0 \to \Theta_X \to \mathcal{T}_S \mid_X \to N_{X/S} \to T_X^1 \to 0$$

splits into the shorter exact sequences

$$0 \to \Theta_X \to \mathcal{T}_S \mid_X \to N'_X \to 0$$

and

$$0 \to N'_X \to N_{X/S} \to T_X^1 \to 0.$$

The sheaf N'_X, defined as the kernel of the surjection $N_{X/S} \to T_X^1$, is called the *equisingular sheaf.* We shall see (Section 2.2 - Cartesian approach) that the $0-th$ and first cohomology groups of the sheaf $N'_{X/S}$ give the first-order equisingular embedded deformation space of X and its obstruction space, respectively. On the other hand, the first and the second cohomology groups of the sheaf Θ_X give the first-order abstract and locally trivial deformation space of X and its obstruction space, respectively (see Section 1.5 and [7]). Observe that N'_X and Θ_X are torsion-free sheaves on X, since $\Theta_X \subset \mathcal{T}_S \mid_X$ whereas $N'_X \subset N_{X/S}$.

Since X is nodal, it is well-known that

$$N'_X = \varsigma_X N_{X/S}, \tag{1.22}$$

where ς_X is the conductor sheaf defined above, and moreover that $N'_{X,p}$ coincides with the maximal ideal of the node $p \in X$, since $N_{X/S}$ is locally free whereas $\varsigma_{X,p} \cong m_p$ (see, for example, [131], page 111).

Using what we have introduced up to now, we can prove (1.22) directly (see [60]). Suppose that X is locally embedded in S, with ideal sheaf $\mathcal{I}_X$, as the analytic locus given by $xy = 0$ at the node $p = (0,0)$, in suitable local holomorphic coordinates on S. Locally, the first two terms of the exact sequence

$$0 \to \mathcal{I}_X/\mathcal{I}_X^2 \xrightarrow{\alpha} \Omega^1_S \mid_X \to \Omega^1_X \to 0$$

look like

$$\mathcal{O}_{X,p}\{xy\} \xrightarrow{\alpha_p} \mathcal{O}_{X,p}\{dx, dy\},$$

with the map α_p given by

$$\alpha_p(xy) = d(xy) = xdy + ydx,$$

in a neighborhood of $p = (0,0)$.

For what we have observed in (1.17), Ω^1_X is locally free of rank one except at the node p. Dualizing, we get

$$0 \to \Theta_X \to \mathcal{T}_S \mid_X \xrightarrow{\alpha^\vee} N_{X/S} \to T^1_X \to 0,$$

with $\alpha^\vee$ locally the map

$$\mathcal{O}_{X,p}\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\} \xrightarrow{\alpha_p^\vee} \mathcal{O}_{X,p}\{xy\}^\vee$$

defined by sending $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ to the natural functionals, i.e.

$$\frac{\partial}{\partial x} \longrightarrow \{xy \to y\}$$

and

$$\frac{\partial}{\partial y} \longrightarrow \{xy \to x\},$$

respectively. Since $N_{X/S}$ is $\underline{Hom}(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{O}_X) = \underline{Hom}(\mathcal{I}_X, \mathcal{O}_X)$, it is generated by the homomorphism

$$\{xy \to 1\}.$$

Therefore, the image of $\alpha^\vee$ is exactly $m_p N_{X,p} \subset N_{X,p}$. Hence, at a node p, the quotient $T^1_{X,p}$ is isomorphic to the fibre at p of $N_{X/S}$, i.e. $T^1_{X,p} \cong N_{X,p}/m_p N_{X,p}$. Since $N_{X/S}$ is locally free of rank one, $N_{X,p} \cong \mathcal{O}_{X,p}$ as modules, therefore $Im(\alpha_p^\vee) = N'_{X,p} \cong m_p$, as stated in Tannenbaum's paper [131].

Remark 1.4.7 From what observed, we obtain that N'_X is not locally free on X.

For what concerns Θ_X, at a node $p \in X$ we have

$$f_1(x)\frac{\partial}{\partial x} + f_2(y)\frac{\partial}{\partial y} \xrightarrow{\alpha_p^\vee} \{xy \to f_1(x)y + f_2(y)x = 0\},$$

by the condition $xy = 0$. This means that

$$ker(\alpha_p^\vee) = \{f_1(x)\frac{\partial}{\partial x} + f_2(y)\frac{\partial}{\partial y} \mid \ f_i \ holomorphic, \ f_i(0) = 0, \ 1 \leq i \leq 2\}.$$

So, $\Theta_{X,p}$ is of rank two at a node p.

Remark 1.4.8 By summarizing, Θ_X is not locally free on X; moreover, if $\underline{t}$ is supported on δ nodes, then

$$\chi(\Theta_X \otimes \underline{t}) = 2\delta. \tag{1.23}$$

We end this section by considering particular cases of nodal (possibly reducible) curves, i.e. *stable curves*, which play a major role in the compactification of the moduli space of smooth curves of given genus.

Definition 1.4.9 *([35]) Let T be any scheme and let $p_a \geq 2$ be a positive integer. A* stable curve *of (arithmetic) genus p_a over T is a proper, flat morphism*

$$\pi : \mathcal{C} \to T,$$

whose geometric fibres are reduced, connected, one-dimensional schemes $\mathcal{C}_t$ such that:

(i) $\mathcal{C}_t$ has only ordinary double points as singularities;

(ii) if E is a non-singular, rational component of $\mathcal{C}_t$, then E meets the other components of $\mathcal{C}_t$ in at least 3 points;

(iii) $dim_{\mathbb{C}}(H^1(\mathcal{O}_{\mathcal{C}_t})) = p_a$.

If $T = Spec(\mathbb{C})$, then $\mathcal{C}$ will be simply denoted by C.

Suppose that $\pi : \mathcal{C} \to T$ is a stable curve. Since π is flat and its geometric fibres are local complete intersections, the morphism π is locally a complete intersection morphism, i.e. locally $\mathcal{C}$ is isomorphic as T-scheme to $V(f_1, \dots, f_{n-1}) \subset \mathbf{A}^n \times U$, where $\mathbf{A}^n$ is the n-dimensional affine space, $U \subset T$ is an open subset and $f_1, \dots, f_{n-1} \in H^0(\mathcal{O}_{\mathbf{A}^n \times U})$ form a regular sequence (see, also Definition 2.2.13, Chapter 2). Therefore, there exists the relative dualizing sheaf, $\omega_{\mathcal{C}/T}$, which is an invertible sheaf on $\mathcal{C}$. Thus, if for example $T = Spec(\mathbb{C})$ and if $\mathcal{F}$ is a sheaf on C, then

$$H^1(C, \mathcal{F})^\vee \cong \mathrm{Hom}(\mathcal{F}, \omega_C).$$

Observe that, in view of the connectedness of $\mathcal{C}_t$, for each $t \in T$, and of condition (ii) of the definition, the automorphism group of $\mathcal{C}_t$ is a finite group, for each $t \in T$.

If we weaken condition (ii) by replacing the number 3 by 2, the resulting curves are called *semistable*. For simplicity, if we limit ourselves to the case $T = Spec(\mathbb{C})$, the semistability of C geometrically amounts to allowing chains $C_1, \ldots, C_k$ of smooth, rational curves as subcurves of C. More precisely, saying that we have a chain means that : C_1 and C_k each meets the complement of the chain in C in a single node; the other C_i are disjoint from this complement, and each C_i, $2 \leq i \leq k-1$, meets each of C_{i-1} and C_{i+1} in a single node and meets no other components of the chain.

Stable curves with marked points are defined analogously. In this case, the finiteness condition can be equivalently reformulated as saying that every rational component of the normalization of C has, at least, 3 points lying over singular and/or marked points of C. Also, as before, if we weaken either of these conditions by replacing the number 3 by 2, the resulting pointed curves are called semistable.

The fundamental differences between stable and semistable curves are explained in the following:

Example. Let C be a semistable curve and let E be a smooth rational component of C, meeting the rest of the curve in exactly 2 points $p_1, p_2 \in E$. Then,

$$\omega_C \otimes \mathcal{O}_E = \omega_E(p_1 + p_2) = \mathcal{O}_E,$$

because, since E is rational, $\omega_E \cong \mathcal{O}_E(-p_1 - p_2)$. Thus, $\omega_C \mid_E = \mathcal{O}_E$ cannot be ample on E and, therefore, ω_C is not ample on C. This depends on the fact that C is supposed to be semistable.

Stable curves are nodal curves for which the linear systems associated to multiples of the dualizing sheaf behave as the pluricanonical linear systems on smooth curves of genus $g \geq 2$. Indeed, if C is a stable curve of arithmetic genus $p_a(C) \geq 2$, then: ω_C is ample, $H^1(C, \omega_C^{\otimes n}) = 0$, for each $n \geq 2$ and $\omega_C^{\otimes n}$ is very ample for $n \geq 3$ (see [7] or [35]).

1.5 Basic results on families of projective varieties and deformation theory

For the reminder of this chapter we will be considering families of schemes. Even if in the next chapters we will be concerned with families of schemes over $\mathbb{C}$, the first definitions of this section are given in a more generality for families defined over a field $\mathbf{k}$, which is not necessarily algebraically closed or of characteristic 0. For terminology not explicitely recalled, we refer the reader to [64] and to [120].

Let $f : X \to Y$ be a morphism of schemes, defined over a field $\mathbf{k}$, and let $y \in Y$ be a point. We denote by

$$X_y = X \times_Y Spec(\mathbf{k}(y))$$

the fibre of f over y, where $\mathbf{k}(y)$ is the residue field of y.

Definition 1.5.1 *Given a scheme X_0 over a field $\mathbf{k}$, a* family of deformations *of X_0 is defined as a flat morphism $f : \mathcal{X} \to Y$, with Y connected, together with a point $y_0 \in Y$, such that $\mathbf{k}(y_0) = \mathbf{k}$ and $\mathcal{X}_{y_0} \cong X_0$. The other fibres $\mathcal{X}_y$ of f are called* deformations *of X_0.*

Definition 1.5.2 *Let Y and S be schemes. A* flat family of closed subschemes of Y parametrized by S *is a closed subscheme $\mathcal{X} \subset Y \times S$, such that the morphism $\mathcal{X} \to S$, induced by the projection $Y \times S \to S$, is flat. The family is called* trivial *if $\mathcal{X} = X \times S$, for some closed subscheme $X \subseteq Y$. When $S = Spec(A)$, with A a noetherian $\mathbf{k}$-algebra with residue field $\mathbf{k}$, then $\mathcal{X}$ is a* local family of closed subschemes of Y*; moreover, if A is also artinian, then $\mathcal{X}$ is an* infinitesimal deformation *of $\mathcal{X}_o$ in Y (or* first-order embedded deformations *of $\mathcal{X}_o$ in Y), where $o \in S$ is the closed point of S.*

Our next aim is to briefly recall some basic properties of first-order abstract and first-order embedded deformations of smooth abstract curves and of smooth or nodal curves embedded in a non-singular, projective surface S; these subjects are related to the infinitesimal study of the moduli space $\mathcal{M}_g$ of curves of genus g and of the Hilbert scheme of subschemes in a given scheme, respectively.

For simplicity, in the sequel we shall consider all schemes defined over $\mathbb{C}$, even if the treatement can be done in a more generality.

As notation, we will frequently consider the affine scheme

$$\Delta_\epsilon := Spec(\mathbb{C}[\epsilon]/(\epsilon^2)), \tag{1.24}$$

where $\mathbb{C}[\epsilon]/(\epsilon^2)$ is the ring of dual numbers; Δ_ϵ has only a closed point o, corresponding to the unique prime ideal (ϵ). It is a standard result that a $\mathbb{C}[\epsilon]/(\epsilon^2)$-valued point of a scheme X is the same as a closed point p of X together with an element of the Zariski tangent space to X at p, in symbols

$$T_p(X) = (m_p/m_p^2)^\vee \cong Morph((\Delta_\epsilon, o),\ (X,p);\ o \to p). \tag{1.25}$$

This is a fundamental point relationing first-order deformations and tangent space to fine moduli (or parameter) spaces of suitable functors (for terminology of coarse and fine moduli space and parameter space, we follow [60]).

Definition 1.5.3 *Let r be a positive integer and let $P(t) \in \mathbb{Q}[t]$. The* Hilbert scheme *of $\mathbb{P}^r$ relative to $P(t)$, $Hilb^r_{P(t)}$, is a scheme parametrizing a flat family of closed subschemes of $\mathbb{P}^r$*

$$\begin{array}{ccc} \mathcal{W} & & \subset \mathbb{P}^r \times Hilb^r_{P(t)} \\ \downarrow & \pi & \\ Hilb^r_{P(t)} & & \end{array}$$

all whose fibres have Hilbert polynomial $P(t)$ and having the following universal property:

- *for every flat family of closed subschemes of* $\mathbb{P}^r$

$$\begin{array}{ll} \mathcal{X} & \subset \mathbb{P}^r \times \Sigma \\ \downarrow f & \\ \Sigma & \end{array}$$

  with Hilbert polynomial $P(t)$, there is a unique morphism

$$g : \Sigma \to Hilb^r_{P(t)},$$

  called the classifying map *for the family f, such that π induces f by base change, i.e.*

$$\mathcal{X} = \Sigma \times_{Hilb^r_{P(t)}} \mathcal{W} \subset \Sigma \times \mathbb{P}^r.$$

The Hilbert scheme is a generalization of complete linear systems of hypersurfaces and of Grasmannians. Once proven that it exists, $Hilb^r_{P(t)}$ is unique so that π is the *universal family.*

Theorem 1.5.4 *For every $P(t)$, $Hilb^r_{P(t)}$ exists and is a projective scheme.*

Proof: For a proof, see [60] or [120]. □

From the universal property it immediately follows that if $X \subset \mathbb{P}^r$ has Hilbert polynomial equal to $P(t)$, there is a unique point of $Hilb^r_{P(t)}$ which parametrizes X (i.e. whose fibre in the universal family is X).

Notation. From now on, when X is a scheme which belongs to a family of schemes parametrized by a base-scheme $\mathcal{X}$, we will denote by $[X]$ the point in $\mathcal{X}$ parametrizing X.

The existence of the universal family ensures us that $Hilb^r_{P(t)}$ is a fine moduli (or a parameter) space for the geometric objects determined by closed subschemes of $\mathbb{P}^r$ with given Hilbert polynomial $P(t)$. The functor posing such a moduli problem is called the *Hilbert functor* of $\mathbb{P}^r$, relative to the given polynomial $P(t)$. For every scheme Σ, let

$$\underline{\mathcal{H}ilb}^r_{P(t)}(\Sigma) := \{flat\ families\ of\ closed\ subschemes\ of\ \mathbb{P}^r \\ with\ Hilbert\ polynomial\ P(t)\ parametrized\ by\ \Sigma\}.$$

Since flatness is preserved under base change, this defines a contravariant functor from the category of schemes to the category of sets. Therefore, the universal property of the Hilbert scheme means that the couple $(Hilb^r_{P(t)}, \pi)$ represents the functor $\underline{\mathcal{H}ilb}^r_{P(t)}$.

Among families of closed subschemes of $\mathbb{P}^r$, complete families play a central role.

Definition 1.5.5 *Let*

$$\begin{array}{ll} \mathcal{X} & \subset \mathbb{P}^r \times \Sigma \\ \downarrow^{f} & \\ \Sigma & \end{array}$$

be a flat family of closed subschemes of $\mathbb{P}^r$ *with Hilbert polynomial* $P(t)$ *and let* $g : \Sigma \to Hilb^r_{P(t)}$ *be the classifying map. The family* f *is called* complete *(resp.* effectively parametrized*) at a closed point* $s \in \Sigma$ *if the morphism* g *is smooth (resp. ètale) at* s*.*

Remark 1.5.6 The completeness in s implies that $g(\Sigma)$ contains a neighborhood of the point $g(s) \in Hilb^r_{P(t)}$; in other words, f parametrizes all "sufficiently small" deformations of $\mathcal{X}_s$ in $\mathbb{P}^r$. On the other hand, "effectively parametrized" implies that the classifying map g is finite-to-one nearby the point s.

The functoriality of the Hilbert scheme makes the study of their local properties particularly natural. If we consider $z \in Hilb^r_{P(t)}$ a closed point, then let $X = \mathcal{W}_z \subset \mathbb{P}^r$ be the closed subscheme parametrized by z, where $\mathcal{W}$ is such as in Definition 1.5.3. Let $\mathcal{N}_X$ be the normal sheaf of X in $\mathbb{P}^r$, which is locally free if X is regularly embedded in $\mathbb{P}^r$ (see Remark 1.1.16). The local properties of $Hilb^r_{P(t)}$ at $z = [X]$ are given by the properties of the local ring $\mathcal{O}_{Hilb^r_{P(t)},[X]}$.

Definition 1.5.7 X *is said to be* unobstructed *(resp.* obstructed*) if* $Hilb^r_{P(t)}$ *is non-singular (resp. singular) at* $[X]$*.*

As in (1.25) above, the Zariski tangent space to $Hilb^r_{P(t)}$ at $[X]$ corresponds to the vector space of scheme morphisms

$$\mathrm{Morph}((\Delta_\epsilon, o),\ (Hilb^r_{P(t)}, [X]);\ o \to [X]),$$

where (Δ_ϵ, o) is as in (1.24). In view of the functoriality of the Hilbert scheme, to give a morphism $f : \Delta_\epsilon \to Hilb^r_{P(t)}$ is equivalent to giving a flat family $(\mathcal{X}_\epsilon, \Delta_\epsilon, p_\epsilon)$, with central fibre X, i.e. a first-order embedded deformation of X in $\mathbb{P}^r$ (see Definition 1.5.2). Thus, the Zariski tangent space $T_{[X]}(Hilb^r_{P(t)})$ to $Hilb^r_{P(t)}$ at $[X]$ can be easily identified.

Theorem 1.5.8

$$T_{[X]}(Hilb^r_{P(t)}) \cong H^0(X, \mathcal{N}_X). \qquad (1.26)$$

Proof: See [60], pages 13-14, or [120], Proposition (8.1). □

We recall that, if X is any scheme and $x \in X$, then

$$dim(\mathcal{O}_{X,x}) \leq dim_{\mathbb{C}}(T_x(X)),$$

where the dimension on the left is the Krull dimension of the local ring. The equality holds if and only if X is smooth at x. Applying all this to the Hilbert scheme, we find the following

Corollary 1.5.9

$$dim_{[X]}(Hilb^r_{P(t)}) \leq h^0(X, \mathcal{N}_X),$$

with the further information that the equality holds if and only if $Hilb^r_{P(t)}$ is smooth at $[X]$.

Let

$$\begin{array}{ll} \mathcal{X} & \subset \mathbb{P}^r \times \Sigma \\ \downarrow^{\ f} & \\ \Sigma & \end{array}$$

be a flat family of closed subschemes of $\mathbb{P}^r$ with Hilbert polynomial $P(t)$ and let $g : \Sigma \to Hilb^r_{P(t)}$ be the classifying map. Let $s \in \Sigma$ be a closed point and $X = \mathcal{X}_s \subset \mathbb{P}^r$.

Definition 1.5.10 *The differential*

$$dg_s : T_s(\Sigma) \to H^0(X, \mathcal{N}_X) \tag{1.27}$$

is called the characteristic map *of the family f.*

The following is a useful criterion.

Proposition 1.5.11 *In the above situation, the following statements are true:*

(i) If f is complete (resp. effectively parametrized) at s, then the characteristic map is surjective (resp. bijective);

(ii) If Σ is smooth at s and the characteristic map is surjective (resp. bijective), then f is complete (resp. effectively parametrized) at s and $Hilb^r_{P(t)}$ is smooth at $g(s)$.

Proof: These are direct consequences of Definition 1.5.5 and of properties of smooth and ètale morphisms (see [120]). □

Remark 1.5.12 With this set up, one can easily show, for example, that a complete intersection $X \subset \mathbb{P}^r$ is always unobstructed; if $dim(X) > 0$, this can be proven by constructing a complete family of complete intersections (see [120], page 8-4).

In general, other information on the dimension estimate can be deduced from the *theory of obstructions*. The main result is that there exists a vector space $o(X, \mathbb{P}^r)$, called the *space of obstructions* of X in $\mathbb{P}^r$, such that for any first-order embedded deformation α of X in $\mathbb{P}^r$, one has a $\mathbb{C}$-linear map $o(\alpha)$ (the *obstruction map*)

$$\mathrm{Ex}^1(\mathbb{C}[\epsilon]/(\epsilon^2), \mathbb{C}) \stackrel{o(\alpha)}{\to} o(X, \mathbb{P}^r),$$

where $\mathrm{Ex}^1(\mathbb{C}[\epsilon]/(\epsilon^2), \mathbb{C})$ denotes the space of algebra extensions of $\mathbb{C}[\epsilon]/(\epsilon^2)$ by $\mathbb{C}$ (see [120]). By definition, $o(\alpha)$ maps an extension B to 0 if and only if α can be extended to an infinitesimal deformation parametrized by $Spec(B)$.

The main feature of the obstruction theory is the following result.

Proposition 1.5.13 *Let $X \subset \mathbb{P}^r$ be a closed subscheme with Hilbert polynomial $P(t)$. Then:*

(i) $h^0(\mathcal{N}_X) - dim(o(X, \mathbb{P}^r)) \leq dim_{[X]}(Hilb^r_{P(t)})$;

(ii) If, moreover, X is a local complete intersection (i.e. it is regularly embedded in $\mathbb{P}^r$) then $o(X, \mathbb{P}^r)$ is a subspace of $H^1(\mathcal{N}_X)$.

Proof: See [77], Theorem 2.8 and Proposition 2.14. □

Remark 1.5.14 A nice consequence of the above result is that if $h^1(\mathcal{N}_X) = 0$ and X is a local complete intersection, then $[X]$ is a smooth point for $Hilb^r_{P(t)}$. The converse does not hold. In fact, we have already recalled in Remark 1.5.12 that a complete intersection X is always unobstructed, but if $dim(X) > 0$ in general $h^1(\mathcal{N}_X)) \neq 0$, as one can immediately find, for example, with a complete intersection of type $(2, m)$, $m \geq 4$, in $\mathbb{P}^3$.

When X is a local complete intersection curve in $\mathbb{P}^r$, we shall denote by $Hilb_{d,g,r}$ the Hilbert scheme of curves in $\mathbb{P}^r$ of degree d and (arithmetic) genus g, since in such a case the Hilbert polynomial is $P(t) = dt + (1 - g)$. We can give a simpler formula for the lower-bound of the dimension of each component of this Hilbert scheme.

Corollary 1.5.15 *If X is a local complete intersection curve in $\mathbb{P}^r$,*

$$\chi(\mathcal{N}_X) \leq dim_{[X]}(Hilb_{d,g,r}) \leq h^0(\mathcal{N}_X), \quad (1.28)$$

and the second inequality is an equality if and only if X is unobstructed. Moreover,

$$\chi(\mathcal{N}_X) = (r + 1)d + (r - 3)(1 - g)$$

is called the Hilbert number*, denoted by $h(d, g, r)$.*

Proof: (1.28) is nothing but Proposition 1.5.13. To compute $h(d, g, r)$ it suffices to consider the normal sequence of X in $\mathbb{P}^r$ and the Euler sequence of $\mathbb{P}^r$ restricted to X. □

The number $h(d, g, r)$ is sometimes called also the *expected dimension* (or the *estimated dimension*) of the Hilbert scheme. Non-special curves or canonical curves, of genus $g \geq 3$, are examples of smooth points of their corresponding Hilbert scheme components with the expected dimensions.

Definition 1.5.16 *A component W of the Hilbert scheme $Hilb_{d,g,r}$ whose general point corresponds to an irreducible, non-singular and non-degenerate curve, is called* regular *if it is smooth of the expected dimension $h(d,g,r)$ at its general point. If it is not regular, W is said to be* superabundant.

Notice that a superabundant component may very well have the expected dimension in which case it is not smooth at its general point, i.e. it is not reduced. There is a famous example, due to Mumford ([98]), of a non-reduced component of the Hilbert scheme of curves of degree 14 and genus 24 in $\mathbb{P}^3$. On the other hand, we already know that one may also have superabundant components which are generically smooth, but of dimension larger than $h(d,g,r)$. Examples are given by complete intersection curves with special normal bundle; in this case, the expected dimension is strictly less than the effective one.

There are a number of useful variants of the Hilbert scheme (see [60] or [120]); one is the Hilbert scheme parametrizing subschemes of a given subscheme of $\mathbb{P}^r$.

Fix $r \geq 1$, a polynomial $P(t) \in \mathbb{Q}[t]$ and a closed subscheme $Y \subset \mathbb{P}^r$. For every scheme Σ let

$$\underline{\mathcal{H}ilb}^Y_{P(t)}(\Sigma) := \begin{array}{l}\{flat\ families\ \mathcal{X} \subset Y \times \Sigma\ of\ closed\ subschemes\ of\ Y \\ parametrized\ by\ \Sigma\ with\ Hilbert\ polynomial\ P(t)\}.\end{array}$$

We obtain a contravariant functor

$$\underline{\mathcal{H}ilb}^Y_{P(t)} : (schemes) \to (sets)$$

called the *Hilbert functor of Y relative to the polynomial $P(t)$*, which is a subfunctor of $\underline{\mathcal{H}ilb}^r_{P(t)}$.

Theorem 1.5.17 *For every Y and $P(t)$ as above $\underline{\mathcal{H}ilb}^Y_{P(t)}$ is represented by a closed subscheme $Hilb^Y_{P(t)}$ of $Hilb^r_{P(t)}$, called the* Hilbert scheme of Y relative to $P(t)$, *and by a universal family*

$$\begin{array}{ccccc} \mathcal{U}_Y \subset & Y \times Hilb^Y_{P(t)} \subset & \mathbb{P}^r \times Hilb^Y_{P(t)} \\ \downarrow & & \\ Hilb^Y_{P(t)} & & & . \end{array}$$

Proof: See [120], page 10-5. □

The infinitesimal study of the schemes $Hilb^Y_{P(t)}$ can be carried out without changes as in the case of the ordinary Hilbert schemes. As for the general case, one finds that for a given $X \subset Y \subset \mathbb{P}^r$ the tangent space to $Hilb^Y_{P(t)}$ at $[X]$ is such that

$$T_{[X]}(Hilb^Y_{P(t)}) \cong H^0(X, \mathcal{N}_{X/Y}), \qquad (1.29)$$

where $\mathcal{N}_{X/Y}$ is the normal sheaf of X in Y (see [41], page 137). As before, this gives an "a priori" estimate for the dimension of the componets of

$Hilb^Y_{P(t)}$. There is also a definition of the *obstruction space of* X *in* Y, denoted by $o(X,Y)$, analogous to that of $o(X,\mathbb{P}^r)$. If X is regularly embedded in Y, then $o(X,Y) \subset H^1(X,\mathcal{N}_{X/Y})$, and the previous results on $Hilb^r_{P(t)}$ generalize with no changes (see [77], page 34, and [120], page 10-8).

With complete analogy of what we have considered before, given a flat family

$$\mathcal{X} \subset Y \times B \to B$$

of subschemes of Y parametrized by B, such that $X = \mathcal{X}_{b_0}$, one can define the *characteristic map* of the family,

$$\rho : T_{b_0}(B) \to H^0(\mathcal{N}_{X/Y}) = T_{[X]}(Hilb^Y_{P(t)}).$$

Indeed, as before, an element v in $T_{b_0}(B)$ can be interpreted as a morphism of pointed schemes

$$v : (\Delta_\epsilon, o) \to (B, b_0).$$

Pulling back the family $\mathcal{X}$ via v yields a first-order embedded deformation of X in Y, so that an element of $H^0(\mathcal{N}_{X/Y})$.

On the other hand, we shall consider the scheme $\mathcal{M}_g$, which is the coarse moduli space for smooth, complete, connected curves of genus g. There are several approaches to construct $\mathcal{M}_g$: Teichmüller approach, the Hodge Theory approach and the Geometric Invariant Theory (GIT) approach, due to Mumford (see [60] for a brief overview). The main result is the following:

Theorem 1.5.18 *Given a positive integer* $g \geq 2$, $\mathcal{M}_g$ *is a coarse moduli space. It is a quasi-projective variety of dimension* $3g-3$. *Indeed, if we denote by* $\mathcal{M}^0_g$ *the open dense subscheme of* $\mathcal{M}_g$ *consisting of isomorphism classes of smooth, non-trivial-automorphism-free curves of genus* g*, then for* $[C] \in \mathcal{M}^0_g$ *(which is a smooth point of* $\mathcal{M}_g$*)*

$$T_{[C]}(\mathcal{M}^0_g) \cong H^1(C, \mathcal{T}_C). \tag{1.30}$$

Proof: See [60], Chapters 2 and 4. □

The compactification of $\mathcal{M}_g$ is a fundamental result of Deligne and Mumford ([35]), which shows that stable curves of (arithmetic) genus g are the right class of curves to consider for moduli problems.

Notation. $\overline{\mathcal{M}}_g$ *and* $\overline{\mathcal{M}}_{g,n}$ *denote the coarse moduli spaces of stable and* n*-pointed stable curves. These spaces exist as projective varieties and they are* stable compactifications *of* $\mathcal{M}_g$ *and* $\mathcal{M}_{g,n}$*, respectively (see [60]).*

An immediate remark on the geometry of such compactifications is the following. Fix a stable curve X with δ nodes and ν irreducible components X_i, each of geometric genus g_i. To specify such a stable curve, we have to consider the normalizations C_i's of the X_i's and the points, on each of these curves, that will be identified to form the nodes of X; there will be 2δ such points in all. The family of such curves has dimension

$$(*) \ \sum_{i=1}^{\nu} (3g_i - 3) + 2\delta$$

(see [60]); by using Leray's isomorphisms, we can compute the arithmetic genus g of X, which is

$$g = (\sum_{i=1}^{\nu} g_i) + \delta - \nu + 1,$$

then $(*)$ equals

$$3g - 3 - \delta. \tag{1.31}$$

From (1.31), we deduce the following important property ([60]).

Proposition 1.5.19 *The locus $\mathcal{S}_\delta$ in $\overline{\mathcal{M}}_g$, determined by stable curves with exactly δ nodes, has pure codimension δ in $\overline{\mathcal{M}}_g$. Moreover, the locus of curves with more than δ nodes lies in the closure of the locus of those with exactly δ nodes.*

We can relate what we have discussed up to now to infinitesimal deformation theory. In the last part of this chapter, we shall limit ourselves to consider only deformations related to curves.

There are several variations of the basic first-order deformation theory plan. In general, the application of deformation theory involves three steps: (i) pose the appropriate deformation theoretic problem, (ii) calculating the space of first-order deformations and (iii) constructing, if it exists, a versal deformation space. For what concerns point (i), denote by C a smooth, complete, connected curve. Then:

(a) A deformation of C is nothing but a flat family $\pi : \mathcal{X} \to B$ together with an isomorphism $C \cong \pi^{-1}(o)$.

(b) A deformation of a pointed curve $(C, p_1, \dots, p_k)$ is a flat family $\pi : \mathcal{X} \to B$ together with an isomorphism $\psi : C \cong \pi^{-1}(o)$ and disjoint sections $\sigma_i : B \to \mathcal{X}$ such that $\sigma_i(o) = \psi(p_i)$.

(c) A deformation of a map $\varphi : C \to Y$, with C and Y fixed, is a map $\tilde{\varphi} : C \times B \to Y \times B$, whose restriction to $C \times \{o\}$ is φ.

(d) A deformation of a map $\varphi : C \to Y$, with Y fixed, is a deformation $\pi : \mathcal{X} \to B$ of C together with a map $\tilde{\varphi} : \mathcal{X} \to Y \times B$ fitting into the commutative diagram:

$$\begin{array}{ccccccccc} C & \stackrel{\cong}{\to} & \pi^{-1}(o) & \to & \mathcal{X} & \stackrel{\tilde{\varphi}}{\to} & Y \times B & \leftarrow & Y \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ o & = & o & \in & B & = & B & \ni & o \end{array}.$$

(Particular cases of (d) are *equisingular deformations* which will be the core of the next chapter).

Step (ii) reduces to compute first-order deformations.

The space of first-order deformations of a smooth curve of genus g (case (a)) is the vector space $H^1(C, \mathcal{T}_C)$, related to the infinitesimal study of the moduli space $\mathcal{M}_g$ at the smooth point $[C]$.

For case (c), the space of first-order deformations of φ, with C and Y assumed to be fixed, is $H^0(\varphi^*(\mathcal{T}_Y))$. In particular, if $Y = C$ and φ is the identity map, then the first-order deformation space is just the space $H^0(C, \mathcal{T}_C)$ of global vector fields on C.

Case (d) is related to Horikawa's theory; as already mentioned, if φ has everywhere injective differential, the first-order deformation space of the map $\varphi : C \to Y$, with Y fixed, is nothing but $H^0(\mathcal{N}_\varphi)$ (see Section 2.2 for more details). In particular, the space of the first-order deformations of the identity $C \overset{id}{\hookrightarrow} Y \subset \mathbb{P}^r$ is the space $H^0(\mathcal{N}_{C/Y})$.

Finally, in case (b), the space of the first-order deformations of the pointed curve is the vector space $H^1(\mathcal{T}_C(-p_1 - \ldots - p_k))$. Note that in this example, there is a natural map to the space of first-order deformations of C alone; indeed, if we put $D = p_1 + \ldots + p_k \in Div(C)$, it is the map on H^1's associated to the exact sequence

$$0 \to \mathcal{T}_C(-D) \to \mathcal{T}_C \to \mathcal{O}_D \to 0.$$

In particular, if C has no global vector fields, then the space of the first-order deformations of the points on the fixed variety is just $H^0(\mathcal{O}_D)$, i.e. the direct sum of tangent spaces to C at the points p_i.

Another important example is the space of first-order deformations of a singular curve (see [7] and [60], Chapter 3). We shall consider only the case of nodal curves in a smooth, projective surface S.

Denote by X a reduced, irreducible, δ-nodal curve in S and let $\varphi : C \to X \subset S$ be its normalization map. To introduce first-order deformation theory of nodal (in particular semistable) curves we need an object which replaces, in this situation, the tangent bundle on a smooth curve. Such a key tool is the sheaf of derivations of X in (1.20), Section 1.4, which was denoted by Θ_X. Recall that, by applying the contravariant, left-exact functor $\underline{Hom}(-, \mathcal{O}_X)$ to the exact sequence

$$0 \to \mathcal{I}_X/\mathcal{I}_X^2 \to \Omega_S^1 \mid_X \to \Omega_X^1 \to 0,$$

we obtained

$$0 \to \Theta_X \to \mathcal{T}_S \mid_X \to \mathcal{N}_{X/S} \to \underline{Ext}^1(\Omega_X^1, \mathcal{O}_X) \to 0;$$

moreover, we posed

$$T_X^1 = \underline{Ext}^1(\Omega_X^1, \mathcal{O}_X),$$

which is the first cotangent sheaf of X, supported on $Sing(X)$ (see Sect. 1.4, (1.21)). As in [7], if we denote by $\mathbf{T}_X^1$ the set of isomorphism classes of all first-order deformations of X, then

$$\mathbf{T}_X^1 \cong \mathrm{Ext}^1(\Omega_X^1, \mathcal{O}_X)$$

(formula 4.2.13 in [7]). Thus the study of first-order deformations of X nodal reduces to compute the dimension of the vector space of the equivalence class extensions of Ω_X^1 by $\mathcal{O}_X$.

By applying the local-to-global spectral sequence for Ext of Theorem 1.1.19 to

$$0 \to \Theta_X \to \mathcal{T}_S \mid_X \to \mathcal{N}_{X/S} \to T^1_X \to 0,$$

we get

$$0 \to H^1(\Theta_X) \to \mathbf{T}^1_X \cong \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \xrightarrow{v} H^0(T^1_X) \to 0. \tag{1.32}$$

This exact sequence has the following geometric interpretation.

Proposition 1.5.20 *Let S be a smooth, projective surface, let $X \subset S$ be a reduced, irreducible nodal curve and let $\varphi : C \to X \subset S$ be the normalization map. Let $p_a(X)$ be the arithmetic genus of X and g its geometric genus.*

- *(i) The vector space $H^1(X, \Theta_X)$ parametrizes the isomorphism classes of locally trivial first-order deformations of X; in particular, all the singularities of X persist by deforming X along these directions of deformation.*
- *(ii) T^1_X is a skyscraper sheaf supported at the singular points of X. For each node p of X, $T^1_{X,p} \cong \mathbb{C}$. In particular $h^0(X, T^1_X) = \delta$, the number of nodes of X. Any first-order deformation of X whose isomorphism class in $\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)$ is mapped by v into a non-zero element of $T^1_{X,p}$ represents a smoothing directions of the singularity p of X.*
- *(iii) $dim(\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)) = 3p_a(X) - 3$, if $p_a(X) \geq 2$ and X is stable.*

Proof: See [7], Proposition 4.2.18. □

Corollary 1.5.21 *For a stable curve X, with δ nodes and arithmetic genus $p_a(X) \geq 2$, we have that*

$$h^1(\Theta_X) = 3p_a(X) - 3 - \delta$$

and

$$H^0(X, T^1_X) = \oplus_{p \in Sing(X)} T^1_{X,p};$$

in particular, one can independently smooth each singular point of X (up to the first-order).

Remark 1.5.22 If $p_1, \ldots, p_\delta, q_1, \ldots, q_\delta$ are points in C mapping to the nodes of X via $\varphi : C \to X \subset S$, i.e. $\varphi(p_i) = \varphi(q_i)$, then one has

$$\underline{Hom}(\Omega^1_X, \mathcal{O}_X) = \varphi_* T_C(-B),$$

where $B = \sum_{i=1}^{\delta}(p_i + q_i)$. So, by Leray isomorphism (1.15), deformations of X preserving the singularities correspond to deformations of the 2δ- pointed smooth curve $(C; p_1, \ldots, p_\delta, q_1, \ldots, q_\delta)$ (see [60]).

Chapter 2

Families of Nodal Curves on Algebraic Surfaces and Severi Varieties

This chapter is devoted to the subject of singular curves in projective varieties; in particular we shall focus on nodal curves on smooth projective surfaces. We shall begin (see Section 2.1) with a brief introduction to the problem, by recalling some of the most important motivations which have classically and recently stimulated researches in these topics. Since this is not intended as an exhaustive survey, the reader is referred to a sufficiently detailed bibliography.

In Section 2.2, we recall the basic definitions and properties of the schemes which parametrize nodal curves on smooth, projective surfaces. These are classically known as *Severi varieties*, even if Severi considered only plane curves. We shall give the precise definition of what we intend for *Severi variety*, since it is slightly different from what is classically meant. Indeed, we shall consider families of irreducible, nodal curves in given linear systems, which determine only locally closed subschemes in these projective spaces. Moreover, we shall "specialize" the first-order deformation theory, recalled in the previous chapter, to the case of *equisingular deformations*. This allows us to give a cohomological interpretation of the tangent space at a smooth point of a Severi variety and then to easily compute the dimension of such a scheme. This will be fundamental in the sequel for our analysis (see Chapters 3 and 4).

In the last section, we shall give an overview of what is known, up to now, about Severi variety theory. Indeed, in recent times, there have been many results on this subject and in many directions. In fact, one may study several problems concerning Severi varieties: existence (i.e. non-emptyness), dimensional and smoothness problems, enumerative problems, irreducibility and the behaviour, from a moduli point of view, of the curves parametrized by such varieties. We have tried to recall some of the most important results on these subjects and, when it was not too far from the scope of our work, to briefly examine the techniques used to prove such results. For a deeper analysis, the reader is referred to the original articles.

2.1 Singular curves on projective varieties: motivations, related problems and historical background

The classification of (algebraic) varieties is based on the study of invariants that one can discover on them. One way to determine important invariants consists in observing the possible subvarieties that a given variety can contain; this leads, for example, to the classical definition of the group of divisors existing on a variety X (see Section 1.1); this is strictly related to its analytic structure, i.e. to the meromorphic functions defined on X.

In order to control the invariants of a variety, it is fundamental to know their behaviour in a family of varieties. For example, in the case of S a smooth surface in $\mathbb{P}^3$, of given degree $d \geq 4$, there is a well-known result, the *Noether-Lefschetz theorem*, which states that the Picard group of the generic surface is a rank-one free $\mathbb{Z}$-module, generated by the plane section of S, i.e. $Pic(S) \cong \mathbb{Z}[H]$ (see [103]). Then, one naturally arrives to the definition of the Noether-Lefschetz locus.

Definition 2.1.1 *Let d be a positive integer bigger than 3 and let $N = h^0(\mathcal{O}_{\mathbf{P}^3}(d)) - 1$. The* Noether-Lefschetz locus*, $NL(d)$, is the set of points $s \in \mathbb{P}^N$ corresponding to smooth surfaces S in $\mathbb{P}^3$ such that $Pic(S)$ is not generated, over $\mathbb{Z}$, by $\mathcal{O}_S(H)$.*

It is well-known that this locus is a countable union of proper subvarieties of $\mathbb{P}^N$ (see, for example. [83]).

Recent researches have given further refinements for the classification problem; one way is to study what kind of singular curves can exist in a given variety or, in the case of a surface S, in a given Picard class. The study of this problem has been recently stimulated not only by research interests in Algebraic Geometry, but also by several applications to other fields, as Particle Physics and Hyperbolic Geometry. Indeed, on the one hand the theory of strings of nuclear physicists deals with the enumerative geometry of rational curves contained in some projective threefolds, which has given "life" to the *quantum cohomology* machinery (see, for example, [45], [78], [79] and [109]); on the other hand, the study of singular curves is naturally related to the hyperbolic geometry of complex projective varieties. We recall that a compact, complex manifold M is said to be *hyperbolic* (in the sense of Kobayashi, [74]) if there are no non-constant, entire holomorphic maps $f : \mathbb{C} \to M$. An important problem is to characterize which projective varieties X, over the complex field, are hyperbolic. The most optimistic lower bound for the degree of hyperbolic n-dimensional hypersurfaces would be $2n + 1$, for $n \geq 2$ (see [37]). So, in the case of a smooth surface $S \subset \mathbb{P}^3$, the bound is strongly expected to be equal to 5, which is precisely the lowest possible degree for S to be of general type. In [37] there is a partial answer to Kobayashi's conjecture on the hyperbolicity of generic surfaces of general type in $\mathbb{P}^3$; indeed, it is proven that the general surface of degree at least 42 is hyperbolic in the sense of Kobayashi. So the problem still remains open for low degrees.

An approach, for an intermediate step, to give an affirmative answer to this problem was given by Demailly ([36]), since he proved that hyperbolicity on a smooth, projective variety X implies that it is *algebraically hyperbolic*, i.e. there exists a real positive number ϵ such that every algebraic curve $C \subset X$, of geometric genus g and degree d, satisfies $2g - 2 \geq \epsilon d$. Clemens ([31]) proved, in the following result, that a general surface of degree at least 6 in $\mathbb{P}^3$ is algebraically hyperbolic.

Theorem 2.1.2 *Let C be a curve on a surface S of degree $d \geq 5$ in $\mathbb{P}^3$. By the Noether-Lefschetz theorem, if S is general, C must be a complete intersection of type (d, k) in $\mathbb{P}^3$, for some $k \in \mathbb{N}$. Then, there is no (d, k)-type curve on S with geometric genus $g \leq \frac{1}{2}dk(d-5)$. In particular, there is no curve with geometric genus $g \leq \frac{1}{2}d(d-5)$ on a general surface of degree $d \geq 5$ in $\mathbb{P}^3$.*

Clemens argument was extended by Ein ([39], [40]), to the case of complete intersections in higher dimensional varieties. Other generalizations of Clemens result in higher dimensions are given by Voisin ([137]). There are also some recent results of Chiantini, Lopez and Ran ([26]) concerning the study of the desingularization of subvarieties of generic hypersurfaces in any variety.

For what concerns singular curves in projective surfaces, we mention that in 1994 Geng Xu ([143]) improved Clemens result for surfaces in $\mathbb{P}^3$ and gave an affirmative answer to a conjecture of Harris, which stated that on a generic surface of degree $d \geq 5$ in $\mathbb{P}^3$ there are neither rational nor elliptic curves.

Theorem 2.1.3 *On a generic surface S of degree $d \geq 5$ in $\mathbb{P}^3$, there is no curve with geometric genus $g \leq \frac{1}{2}d(d-3) - 3$ and this bound is sharp. Moreover, this sharp bound can be achieved only by tri-tangent hyperplane sections if $d \geq 6$. In particular, S is algebraically hyperbolic.*

In [25], Chiantini and Lopez translate Xu's local analysis approach with a global study of the *focal locus* of a family in order to consider algebraic hyperbolicity problem for surfaces which are general in a given component of the Noether-Lefschetz locus of surfaces in $\mathbb{P}^3$, and for surfaces in $\mathbb{P}^4$ which are projectively Cohen-Macaulay.

For what concerns families of curves on surfaces, expecially surfaces of general type, we must also recall the paper of Bogomolov [13], where he considers the problem of bounding curves in surfaces of general type by their geometric genus. His technique relies on the idea of lifting curves to the tangent bundle of the surface and uses foliation arguments. In [85], Lu and Miyaoka determine some effective bounds for the degree of a curve in a surface of general type, with the only constraints on the number of ordinary nodes and ordinary triple points of the curve. Moreover, they prove that in a surface of general type, there are only a finite number of rational curves and elliptic curves with a fixed bound on the number of ordinary nodes and ordinary triple points of each curve. They generalize these results to higher dimensions in [86].

From this brief overview, it is then clear how important is the problem of studying families of smooth and singular curves on varieties and how much must still be proven. From now on, we shall focus on the problem of studying some properties of some families of singular curves on a smooth, projective surface.

Among singular curves, nodal curves play a central role in this subject. In [125], Anhang F, Severi studied families of plane curves of given degree and given geometric genus. The varieties, which parametrize such plane curves, can be viewed as particular cases of Hilbert schemes. As recalled in Section 1.5, we know that in general the structure of the projective variety $Hilb_{d,g,r}$ is very hard to understand; it may be reducible, with components of arbitrarily many dimensions, except that $dim(Hilb_{d,g,r}) \geq 3g - 3 + \rho + (r+1)^2 - 1$ when $Hilb_{d,g,r} \neq \emptyset$. Moreover, recall that there are examples of non-reduced components of some Hilbert schemes; in fact, in [98], Mumford describes a family of non-singular space curves whose component of the Hilbert scheme is generically non-reduced.

In the case $r = 2$, i.e. what is classically known as *Severi variety* of plane curves, the situation is more favourable. To make clearer the exposition, we have to recall some definitions and some notation from [59] and [60].

In Definition 1.1.5, we have recalled the definition of geometric genus for a smooth and irreducible variety.

Definition 2.1.4 *If C is any reduced (but possibly reducible) curve, by* geometric genus *of C we mean the arithmetic genus of its normalization. Thus, if C is the union of ν irreducible components C_i of geometric genera g_i, respectively, the geometric genus of C will be*

$$g(C) = (\sum_{i=1}^{\nu} g_i) - \nu + 1. \tag{2.1}$$

Let now $\mathbb{P}^N$, $N = \frac{d(d+3)}{2}$, be the projective space parametrizing all plane curves of degree d. There are some locally closed subschemes of $\mathbb{P}^N$, fitting in the following diagram

$$\begin{array}{ccccc} U_{d,\,\delta} & \subset & U_{d,\,g} & \subset & \mathbb{P}^N \\ \cup & & \cup & & \\ V_{d,\,\delta} & \subset & V_{d,\,g} & & \end{array}$$

where

· $U_{d,\,g}$ is the subset of $\mathbb{P}^N$ consisting of reduced (but not necessarily irreducible) curves of degree d and geometric genus g (as in Definition 2.1.4);

· $V_{d,\,g}$ is the subset of $U_{d,\,g}$ consisting of irreducible curves of degree d and geometric genus g;

· $U_{d,\,\delta}$ is the subset of $U_{d,\,g}$ consisting of reduced curves of degree d, with only $\delta = \frac{1}{2}d(d-1) - g$ nodes as singularities.

· $V_{d,\,\delta}$ is the intersection of $V_{d,\,g}$ with $U_{d,\,\delta}$, i.e. it consists of reduced and irreducible curves of degree d, with only $\delta = \frac{1}{2}(d-1)(d-2) - g$ nodes as singularities.

It is well-known that these are locally-closed subschemes of $\mathbb{P}^N$; their closures are classically named *Severi varieties* of plane curves of degree d, with the further conditions on the geometric genus or on the number of nodes, respectively.

Observe that, a priori, for nodal curves in $U_{d,\,\delta}$, the number of nodes δ is such that $0 \leq \delta \leq \frac{d(d-1)}{2}$. There always exist curves of degree d with δ nodes, for δ varying in this range of values, but the bigger is the value of δ, the higher is the possibility to find only reducible nodal curves. Indeed, the maximum $\delta = \frac{d(d-1)}{2}$ is reached only by d-gons, i.e. by curves consisting of d distinct lines, no three of which have a common point. We shall prove the theorem, due to Severi, of the existence of irreducible, nodal plane curves with $\delta \leq \frac{(d-1)(d-2)}{2}$ (see Theorem 3.1.1). Indeed, for irreducible curves of degree d with δ nodes to exist it is necessary and sufficient that δ satisfies this stronger inequality.

Severi ([125]) focused on this subject and proved the following result.

Theorem 2.1.5 *(Theorem of Severi, see [44], [60] (pag.30) and [145]) For d, g and $\delta \leq \frac{d(d-1)}{2}$ non-negative integers, one has:*

(i) $U_{d,\,\delta}$ is non-empty;

(ii) $U_{d,\,\delta}$ is everywhere smooth of dimension $\frac{d(d+3)}{2} - \delta = 3d + g - 1$;

(iii) If $\delta > \sigma$, then $U_{d,\,\delta}$ is contained in the closure of $U_{d,\,\sigma}$. The branches of $\overline{U}_{d,\,\sigma}$ through a point $[C]$ in $U_{d,\,\delta}$ are non-singular and correspond to the choices of σ assigned points among the δ nodes of C. The other $\delta - \sigma$ virtual nodes disappear when one deforms $[C]$ to the corresponding branch of $\overline{U}_{d,\,\sigma}$;

(iv) If $[C] \in U_{d,\,\delta}$ and if $\mathcal{A}$ is a set of $\sigma < \delta$ nodes of C, the curves C', with $[C']$ in the corresponding branch of $\overline{U}_{d,\,\sigma}$, have k irreducible components, where k is the number of connected components of the complement of $\mathcal{A}$ in C.

Corollary 2.1.6 *If $\delta \leq \frac{(d-1)(d-2)}{2}$, $V_{d,\,\delta}$ is non-empty and everywhere smooth of dimension $\frac{d(d+3)}{2} - \delta = 3d + g - 1$.*

Observe that $3d+g-1 = N-\delta$, which means that the nodes of an irreducible, nodal curve C impose independent conditions to the complete linear system $\mid dL \mid$ (L a line in $\mathbb{P}^2$) to which C belongs. As Zariski wrote, *this is in* *agreement with the intuitive expectation that the requirement that a curve of degree d possesses δ nodes (in non-assigned position) imposes δ independent algebraic (non-linear) conditions on the curve* ([145], pag. 209). As we shall see in Theorem 3.1.1, this can be easily proven by using equisingular deformation theory.

The application of deformation theory to the geometry of these schemes was carried out first by Arbarello and Cornalba [2] and then, independently and via a different approach, by Zariski [145].

Zariski proved a general result on algebraic systems of plane curves. Before stating his result, we recall some terminology.

Definition 2.1.7 *Let S be a smooth surface and let V be a variety. Denote by Σ a subvariety of the product $S \times V$. (Σ, V) (with abuse of language only Σ) is said to be an* algebraic system of curves *parametrized by V if, for each $y \in V$, $\Sigma_y = \Sigma \cdot (S \times \{y\})$ is a curve in S. (Σ, V) is said to be* irreducible *if V is; the dimension of the algebraic system is, by definition, $dim(V)$. The algebraic system of curves is called* complete (or maximal) *if there exists no irreducible algebraic system Σ' containing Σ as a proper subsystem and such that the general curve C of Σ is a general curve of Σ'. In the case of $S = \mathbb{P}^2$, (Σ, V) is an algebraic system of plane curves of degree d and genus g if, for a general $y \in V$, the curve Σ_y is of degree d and genus g.*

Theorem 2.1.8 *(Zariski, [145]) Let Σ denote a complete, irreducible algebraic system of plane curves of degree d, in which the general curve C is reduced and has arbitrary singularities. Let g denote the geometric genus of C, as in Definition 2.1.4. Then,*

$$dim(\Sigma) \leq 3d + g - 1,$$

with equality if and only if C has only nodes as singularities.

From this, it follows that the general element of $\overline{V}_{d,g}$ is an irreducible nodal curve of geometric genus $g = p_a(dL) - \delta$. Therefore, we have the following

Corollary 2.1.9 *With notation as in Theorem 2.1.5, $V_{d,\delta}$ is dense in $V_{d,g}$.*

Another important point, which we didn't mention in Severi's theorem above (Theorem 2.1.5), is the fact that $\overline{V}_{d,\, g}$ is irreducible. Indeed, this was not proven by Severi, since his approach was wrong. With terminology as in Definition 2.1.7, he stated that:

Each complete, irreducible system, whose general point is an irreducible plane curve of degree d, with only δ nodes as singularities, contains all the d-gons of the plane (i.e. all the curves consisting of d lines in general position).

To prove this, he asserted that each complete, algebraic system contains curves C_0 consisting of d distinct lines through a point O (and this was correct). Moreover, he also stated that, if a general curve C of such a complete system has only δ nodes as singularities and if it specializes to the above curve $C_0 = L_1 + \cdots + L_d$, then the δ nodes of C specialize to δ nodes of C_0 centered at O, i.e. in the specialization each node of C specializes to a node formed by two of the d lines L_i. This second assertion is false. There are some counterexamples treated in [145]. This has been an important open problem for a long time. By using Severi's correct statement on the smoothness of $\overline{V}_{d,\,g}$ and Corollary 2.1.9, to prove its irreducibility is equivalent to showing that the dense open subset $V_{d,\delta}$ is connected.

One of Severi's reasons for proving that $V_{d,\delta}$ is connected was to give a proof of the (connectedness) irreducibility of the moduli space $\mathcal{M}_g$, using only methods of Algebraic Geometry, and to find a proof for his conjecture on the unirationality of $\mathcal{M}_g$, for all g. Indeed, there is a well-defined map

$$\pi_{d,\delta} : V_{d,\delta} \to \mathcal{M}_g,$$

where $g = \frac{(d-1)(d-2)}{2} - \delta$, by considering the simultaneous desingularization of all the nodal curves parametrized by $V_{d,\delta}$. Since every curve can be represented by a nodal plane curve of sufficiently large degree d, then $\pi_{d,\delta}$ is dominant on $\mathcal{M}_g$, for d large with respect to g. Moreover, Severi already knew, in 1921, that $\mathcal{M}_g$ is unirational for $g \leq 10$ ([125]; see also [3] for a modern proof).

After the Deligne-Mumford compactification of $\mathcal{M}_g$ (see Section 1.5), whose boundary points correspond to stable curves of arithmetic genus g, to show the (connectedness) irreducibility of $\mathcal{M}_g$ is equivalent to the assertion that any non-singular curve of genus g can be degenerated to a stable curve. This was proven later by Fulton in the Appendix of [61]; in the same article, Harris and Mumford gave a negative answer to Severi's conjecture, by proving that $\mathcal{M}_g$ is of general type for $g \geq 24$ and that the Kodaira dimension of $\mathcal{M}_{23}$ is bigger then 0.

However, after the article [61] Severi's assertion on the irreducibility of $\overline{V}_{d,\,g}$ was still an open problem. Only after more than 60 years from Severi's statement, Harris completed, in [58], the proof of the irreducibility of the Severi variety $\overline{V}_{d,\,g}$ of the projective plane, by showing that the dense open subset $V_{d,\,\delta}$ is connected (cf. also Arbarello-Cornalba, [4], for a proof in case $3d - 2g - 6 > 0$). Harris proof is along lines suggested by Severi; he argues that there is a unique component of $\overline{V}_{d,\,g}$ containing the locus of rational nodal curves and, then, that every component of $\overline{V}_{d,\,g}$ contains $\overline{V}_{d,\,0}$. The first of these two statements has been known for some time before and we will recall it later on. To establish the second, Harris shows that any component W of $\overline{V}_{d,\,g}$ contains suitable degenerations (said *good degenerations*) to curves of lower geometric genus. This is done by forcing the curves in W to have a higher and higher contact with a line $L \subset \mathbb{P}^2$. For details, the reader is referred to the original paper. Here we only want to mention the following result.

Proposition 2.1.10 *(see [58])*

There is a unique component of $\overline{V}_{d,\,g}$ which contains the variety $\overline{V}_{d,\,0}$.

Proof: Note first that the variety $\overline{V}_{d,\,0}$ of rational curves of degree d is irreducible. Indeed, the previous variety contains the subscheme of irreducible, maximal nodal curves $V_{d,\,\frac{(d-1)(d-2)}{2}}$ as an open dense subset. This open set is connected, since each rational, nodal curve in $\mathbb{P}^2$ is simply the projection of the rational normal curve $X \subset \mathbb{P}^d$ from a subspace $\Lambda \cong \mathbb{P}^{d-3} \subset \mathbb{P}^d$ to a plane $\Gamma \cong \mathbb{P}^2$. This gives a dominant rational map from the product of a Grassmannian $\mathbb{G}(d-3, d)$ with the variety $PGL(3, \mathbb{C})$, of isomorphisms of Γ with $\mathbb{P}^2$, to the Severi variety $\overline{V}_{d,\,0}$, showing that $\overline{V}_{d,\,0}$ is irreducible. Then, we can explicitly describe the variety $\overline{V}_{d,\,g}$ in an analytic neighborhood of a general point of $\overline{V}_{d,\,0}$. In fact, since a general point in $\overline{V}_{d,\,0}$ corresponds to a curve E with $\frac{(d-1)(d-2)}{2}$ nodes and since there exist deformations of E smoothing independently any subset of these nodes[1], there are exactly $\binom{\frac{(d-1)(d-2)}{2}}{g}$ sheets of $\overline{V}_{d,\,g}$ through E, corresponding to the subsets of cardinality g in the whole set of nodes. The statement of the proposition now ammounts to the assertion that, as E varies in $\overline{V}_{d,\,0}$, these $\binom{\frac{(d-1)(d-2)}{2}}{g}$ sheets are interchanged transitively. This will in turn follow from the fact that :

as E varies in $\overline{V}_{d,\,0}$, the monodromy group acts as the full symmetric group of nodes of E.

Since, as we have observed before, E can be realized as a projection of a rational, smooth curve X from a linear subspace $\Lambda \subset \mathbb{P}^d$, the nodes of E correspond to the points of intersection of Λ with the the chordal variety of X. By the Uniform Position theorem (see [5]), the monodromy action on the points of intersection of any irreducible variety with a general plane of complementary dimension is the full symmetric group. □

As an immediate consequence of the previous proposition, we see that to prove that $\overline{V}_{d,\,g}$ is irreducible reduces to prove that every component of $\overline{V}_{d,\,g}$ contains $\overline{V}_{d,\,0}$ in its closure. By induction, there must be proven that any component of $\overline{V}_{d,\,g}$ contains in its closure a nodal curve of geometric genus $g' < g$. This is the analysis of *nice deformations* of Harris (see [58] or [59]).

2.2 Equisingular deformations and Severi varieties of irreducible, nodal curves on smooth, projective surfaces

As mentioned in the introduction of the present chapter, here we shall deal with *equisingular deformation theory* of nodal curves on a smooth, projective surface S. We will describe the Cartesian and the parametric approach to the problem, by analyzing in detail the exact sequences of sheaves and the

[1] We shall see in the next section that this is an important consequence of the fact that the nodes impose independent conditions to the complete linear system to which E belongs.

vector spaces which describe the problem. Then, we shall give the definition of what we intend, from now on, for *Severi variety*. This will enable us to give a natural interpretation of the vector spaces which represent the tangent space and the space of obstructions at a point $[X]$ of such a variety, in terms of the ideal sheaf of the 0-dimensional scheme of nodes of the singular curve X.

As in (1.24), Section 1.5, let $\Delta_\epsilon = Spec(\mathbb{C}[\epsilon]/(\epsilon^2))$ and let o denote its unique closed point.

We shall focus on equisingular deformations of an irreducible, nodal curve $X \subset S$. So, let

$$\varphi : C \to X \subset S$$

be the normalization map of X, let $p_a = p_a(X)$ denote the arithmetic genus of X, whereas g its geometric genus, so that $p_a = g + \delta$, where δ denotes the number of nodes of X. As already mentioned in Section 1.4, one can approach the problem of studying equisingular deformations of the nodal curve X in two different ways.

The first one can be called the *Cartesian approach*, which deals with the cohomology of the equisingular sheaf. As computed in Section 1.4, from the conormal sequence of X in S one determines the following exact sequence on X:

$$0 \to \Theta_X \to \mathcal{T}_S\,|_X \to N_{X/S} \to T^1_X \to 0 \tag{2.2}$$

(see Sect. 1.4, (1.20) and (1.21)), which splits into the shorter exact sequences

$$0 \to \Theta_X \to \mathcal{T}_S\,|_X \to N'_X \to 0 \tag{2.3}$$

and

$$0 \to N'_X \to N_{X/S} \to T^1_X \to 0, \tag{2.4}$$

where N'_X, the kernel of the surjection $N_{X/S} \to T^1_X$, is the equisingular sheaf.

If X is such a curve, one can consider the functor of Artin rings

$$\mathcal{H}_X(A) = \{divisors\ X_A \subset S_A,\ A - flat,\ inducing\ X\ on\ the\ closed\ fibre\}$$

for each Artin local $\mathbb{C}$-algebra A and the subfunctor

$$\mathcal{H}'_X(A) = \{X_A \subset S_A\ locally\ trivial\ over\ A\}.$$

As we shall see in the sequel, the $0-th$ and first cohomology groups of the sheaf N'_X give the tangent and obstruction spaces of $\mathcal{H}'_X$, respectively.

On the other hand, by Horikawa's works ([67], [68] and [69]), we can consider deformations of non-degenerate morphisms which are related to the *parametric approach* (see also [2] and [15] for an Algebraic Geometry approach). More generally, these are concerned with families of maps from a possibly variable smooth domain to a fixed smooth target space. In other words, one considers a flat, smooth, proper family $f : \mathcal{Y} \to B$ over a smooth, connected base B, a smooth variety W and a morphism $\Phi : \mathcal{Y} \to B \times W$ of B-schemes. For each $b \in B$, we let $\phi_b : \mathcal{Y}_b \to W$ be the restriction of Φ to the fibre $\mathcal{Y}_b$ of $\mathcal{Y}$ over B and

$$d\phi_b : T\mathcal{Y}_b \to \phi_b^* TW$$

be its differential.

In this way, we get a *family of deformations of morphisms into* W, which is the quadruplet $(\mathcal{Y}, \Phi, f, B)$ of schemes $\mathcal{Y}$, B and morphisms Φ and f, with f proper and surjective.

From now on, we will assume that each ϕ_b has injective differential $(d\phi_b)_y$ at each point y. Thus, for each $b \in B$, we have an injective sheaf morphism,

$$0 \to \mathcal{T}_{\mathcal{Y}_b} \stackrel{(\phi_b)_*}{\to} \phi_b^*(\mathcal{T}_W).$$

In particular, for $b = o$, the central point of B (for which $\mathcal{Y}_o \cong Y$), we have an exact sequence

$$0 \to \mathcal{T}_Y \stackrel{(\phi_o)_*}{\to} \phi_o^*(\mathcal{T}_W) \to \mathcal{N}_{\phi_o} \to 0.$$

Remark 2.2.1 Observe that, when $\phi_o = i$ is the identity and $Y \subset W$, then $\mathcal{N}_{\phi_o} = \mathcal{N}_{Y/W}$.

Definition 2.2.2 *The cokernel* $\mathcal{N}_{\phi_o}$ *of the morphism* $(\phi_o)_*$ *is the* normal sheaf of the map ϕ_o *and* $H^0(\mathcal{N}_{\phi_o})$ *is called the* characteristic system of the map.

Equivalently, if we let

$$d\Phi : T\mathcal{Y} \to \Phi^* T(B \times W)$$

be the differential of Φ and $\mathcal{N} = Coker(d\Phi)$ the normal sheaf of Φ, then the normal sheaf $\mathcal{N}_{\phi_b}$ is the restriction of $\mathcal{N}$ to the fibre $\mathcal{Y}_b$, that is $\mathcal{N}_{\phi_b} = \mathcal{N} \otimes \mathcal{O}_{\mathcal{Y}_b}$. Note that, since ϕ_b has injective differential, $\mathcal{N}_{\phi_b}$ will be locally free.

The standard applications of Horikawa's theory are based on a fundamental fact. If the family Φ of morphisms is nowhere isotrivial (i.e. the restriction of Φ to the subfamily $\mathcal{Y}_{B_0} = f^{-1}(B_0) \subset \mathcal{Y}$ is not isotrivial for any analytic arc $B_0 \subset B$), then at a general point $b \in B$ there is an "a priori" bound on the dimension of the family:

$$dim(B) \le h^0(\mathcal{Y}_b, \mathcal{N}_{\phi_b})$$

(see Proposition 2.2.3 and Theorem 2.2.5). The Chern classes of the normal sheaf $\mathcal{N}_{\phi_b}$ are in general readily calculated, so that in many cases it may be possible to exactly estimate $h^0(\mathcal{Y}_b, \mathcal{N}_{\phi_b})$. The difficulty arises when ϕ_b

has not an injective differential; indeed, if ϕ_b is assumed to be only equidimensional onto its image, then the sheaf $\mathcal{N}_{\phi_b}$ will have a torsion subsheaf supported exactly on the locus where $d\phi_b$ fails to be an injective bundle map. Thus, if for example $S = \mathbb{P}^2$ and if the differential $d\phi_b$ vanishes at points of $\mathcal{Y}_b$, the sheaf $\mathcal{N}_{\phi_b}$ will have torsion there so that the quotient $\mathcal{N}_{\phi_b}/(\mathcal{N}_{\phi_b})_{tors}$ (and hence $\mathcal{N}_{\phi_b}$ itself) may well be special, i.e. $h^1(\mathcal{N}_{\phi_b}) \neq 0$. In such a case, the dimension $h^0(\mathcal{Y}_b, \mathcal{N}_{\phi_b})$ will indeed be larger than the expected estimate for the dimension of our family. As already mentioned in section 1.4, there is a standard result, due to Arbarello and Cornalba [2], which states that first-order deformations of the map ϕ_b corresponding to a torsion section of $\mathcal{N}_{\phi_b}$ can never be equisingular.

We shall briefly recall the basic results of Horikawa's theory.

Proposition 2.2.3 *The* characteristic map *of the family B is the vector space homomorphism*

$$\rho_\Phi : T_o(B) \to H^0(\mathcal{N}_{\phi_o}),$$

which is the differential at o of the map

$$B \to Hilb^W_{P_{\phi_o(Y)}(t)}$$

(see Definition 1.5.10). If ρ denotes the Kodaira-Spencer map *of the given deformation*[2] *and ∂ is the coboundary map*

$$\partial : H^0(\mathcal{N}_{\phi_o}) \to H^1(\mathcal{T}_Y)$$

of the cohomology sequence associated to

$$0 \to \mathcal{T}_Y \stackrel{(\phi_o)_*}{\to} (\phi_o)^*(\mathcal{T}_W) \to \mathcal{N}_{\phi_o} \to 0,$$

then one has a factorization $\rho = \partial \circ \rho_\Phi$, i.e. a commutative diagram

$$\begin{array}{ccc} T_o(B) & \stackrel{\rho_\Phi}{\longrightarrow} & H^0(\mathcal{N}_{\phi_o}) \\ & \searrow \rho & \downarrow \partial \\ & & H^1(\mathcal{T}_Y) \end{array}.$$

Proof: For a proof, see also [18], Proposition 8.4, or directly [67], Proposition 1.4. □

In a complete analogy, one can extend Definition 1.5.5 of complete families to this situation.

Definition 2.2.4 *A family $(\mathcal{Y}, \Phi, f, B)$ is* complete *at $o \in B$ if for any family $(\mathcal{Y}', \Phi', f', B')$, such that $\phi'_{o'} : \mathcal{Y}'_{o'} \to W$ is equivalent to $\phi_o : \mathcal{Y}_o \to W$ for a point $o' \in B'$, there exists a morphism h from $o' \in U \subset B'$ into B with $h(o') = o$ such that the restriction of $(\mathcal{Y}', \Phi', f', B')$ on U is equivalent to the family induced by h from $(\mathcal{Y}, \Phi, f, B)$.*

[2] ρ is the infinitesimal deformation map of the family $(\mathcal{Y}, f, B)$ (see [76] or [124])

Horikawa ([67]) proved in this context a generalization of Proposition 1.5.11.

Theorem 2.2.5 *Let $\varphi : Y \to W$ be a morphism of smooth varieties with injective differential at each $y \in Y$.*

(1) Let $(\mathcal{Y}, \Phi, f, B)$ be a family of deformations of morphisms into W such that $\phi_o = \varphi$, $o \in B$ such that $\mathcal{Y}_o \cong Y$. If the characteristic map

$$T_o(B) \xrightarrow{\rho_\Phi} H^0(\mathcal{N}_\varphi)$$

is surjective, then the family is complete at o (in the sense of Definition 2.2.4);

(2) If $H^1(Y, \mathcal{N}_\varphi) = (0)$, then there exists a family $(\mathcal{Y}, \Phi, f, B)$ of morphisms into W with injective differentials and a point $o \in B$ such that:

(i) $\phi_o : \mathcal{Y}_o \to W$ is equivalent to $\varphi : Y \to W$;

(ii) $T_o(B) \xrightarrow{\rho_\Phi} H^0(\mathcal{N}_\varphi)$ is bijective.

In case (1), if B is smooth, with classical terminology one can also say that the characteristic system $H^0(\mathcal{N}_\varphi)$ of the map $\varphi : Y \to W$ is *complete.*

Remark 2.2.6 From this brief overview, it is clear that $H^0(\mathcal{N}_\varphi)$ is the exact analouge of $H^1(\mathcal{T}_Y)$ (in the case of deformations of the smooth variety Y) or of $H^0(\mathcal{N}_{Y/W})$ (in the case $\varphi = i$ the identity embedding), discussed in Section 1.5. Recall that a sufficient condition in order that the *Kuranishi family* is smooth is the sharp assumption $h^2(\mathcal{T}_Y) = 0$ (see, for example, [124]). On the other hand, by Remark 1.5.14, if Y is assumed to be l.c.i. in W, then the condition $h^1(\mathcal{N}_{Y/W}) = 0$ implies that Y is unobstructed, i.e. it corresponds to a smooth point of the Hilbert scheme $Hilb^W_{P(t)}$. In a similar fashion, Horikawa proves the generalization for $\mathcal{N}_\varphi$, as stated in in Theorem 2.2.5 (2); moreover, in this case, the Kodaira-Spencer map coincides with the coboundary map

$$\partial : H^0(\mathcal{N}_\varphi) \to H^1(\mathcal{T}_\mathcal{Y}),$$

as it immediately follows from the diagram in Proposition 2.2.3. We emphasize the deformation-theoretic interpretation of the cohomology sequence of

$$0 \to \mathcal{T}_Y \to \varphi^*\mathcal{T}_W \to \mathcal{N}_\varphi \to 0.$$

The coboundary

$$\partial : H^0(\mathcal{N}_\varphi) \to H^1(\mathcal{T}_Y),$$

takes a deformation of the map φ to the corresponding deformation of Y, forgetting the map. The kernel consists of deformations of the map φ fixing both Y and W, modulo automorphisms of Y.

When Y is a smooth curve of genus g (with no non-trivial automorphism) and $\phi_0 = \varphi = i$ (so that $Y \subset W$) this gives a relationship between the tangent space at $[Y]$ to $Hilb^W_{P(t)}$ and the tangent space at $[Y]$ to $\mathcal{M}_g$.

Remark 2.2.7 Assume that Y is a smooth curve. If the characteristic system is complete, then $H^0(\mathcal{N}_\varphi)$ determines the exact number of parameters necessary to describe the first-order deformation family of morphisms parametrized by B, as in Definition 2.2.4. Indeed, a priori, B could be described by redundant parameters. $h^0(\mathcal{N}_\varphi)$ is the number of independent parameters to entirely describe B, whereas

$$dim(Im(\partial)) \leq h^1(\mathcal{T}_Y)$$

determines the *number of moduli* of the first-order deformed curves, i.e. the number of parameters to describe the local moduli space for the elements of the family.

Sometimes, the characteristic map of the family of Proposition 2.2.3 is called the *Horikawa map* of the family (of morphisms) Φ at a point $b \in B$ (see [2] and [15]).

As already mentioned, in our case we have a map $\varphi : C \to X \subset S$ which is the normalization of an irreducible, δ-nodal curve $X \subset S$, where S is a smooth, projective surface. Thus φ has injective differential at each point of C, so $\mathcal{N}_\varphi$ is a line bundle on C fitting in the exact sequence

$$0 \to \mathcal{T}_C \to \varphi^*\mathcal{T}_S \to \mathcal{N}_\varphi \to 0. \tag{2.5}$$

The coboundary map

$$\cdots H^0(C, \mathcal{N}_\varphi) \xrightarrow{\partial} H^1(C, \mathcal{T}_C) \to \cdots \tag{2.6}$$

of the cohomology sequence of (2.5) applies the Horikawa class to the Kodaira-Spencer class of the family $p_\delta : \mathcal{C}_\delta \to \Delta_\epsilon$, where $\mathcal{C}_o \cong C$, o the closed point of Δ_ϵ (see Theorem 2.2.5 and Remark 2.2.6).

In this case, we will show that the Cartesian approach and the parametric approach coincide by proving that $\mathcal{N}'_X \cong \varphi_*\mathcal{N}_\varphi$ as sheaves on X (see Remark 2.2.11). This depends on the following more general fact:

Definition 2.2.8 *(see [18], Def. 9.15 and 11.4) Let Y and W be smooth varieties. Let $\varphi : Y \to W$ be a non degenerate morphism (i.e. it has generically injective differential) and assume that $\Sigma := \varphi(Y)$ is birational to Y. The map φ is said to be* stable *if the direct image sheaf $\varphi_*(\mathcal{N}_\varphi)$ is isomorphic to the* equisingular sheaf *$\mathcal{N}'_\Sigma$, defined as the cokernel of the sheaf injection*

$$\mathcal{T}_W(-log(\Sigma)) \hookrightarrow \mathcal{T}_W,$$

where $\mathcal{T}_W(-log(\Sigma))$ denotes the sheaf of tangent vectors on W which are also tangent to Σ.

Proposition 2.2.9 *(see [18], Prop. 11.6) Let $\psi : Y \to W$ be a non degenerate morphism and let $\Sigma = \psi(Y)$, with Y and W smooth varieties. Assume $dim(Y) = 1$ and that $p \in \Sigma$ is a singular point of Σ. If ψ is stable, then*

(i) Σ is birational to Y;

(ii) p is an ordinary double point (i.e. a node);

(iii) $dim(W) = 2$.

Proof: We do not mention here the proof of this result and we refer the reader directly to [18]. □

Example: The morphism $t \to (t^2, t^3)$, giving an ordinary cusp in the affine plane over $\mathbb{C}$, is not stable since in fact the deformation

$$t \to (x_1(t) = a_0 + t^2, x_2(t) = b_0 + b_1 t + t^3)$$

gives a node; in fact, $\frac{\partial(x_1)}{\partial t} = 2t$, $\frac{\partial(x_2)}{\partial t} = b_1 + t^2$. The points given by $t = \pm\sqrt{-b_1}$ are two points on the affine line mapping to the same point. Therefore, the morphism cannot be equisingular, otherwise all the deformed curves would have only cusps.

Turning back to our case, denote by N the set of nodes of $X \subset S$, i.e.

$$N := Sing(X).$$

For each non-empty subset $I \subset N$, denote by $T^1_{X|I}$ the restriction of the first cotangent sheaf T^1_X to I, extended by zero on X, and by

$$\mathcal{N}^I_X = ker(\mathcal{N}_{X/S} \to T^1_{X|I}).$$

We will shortly denote $\mathcal{N}^N_X$ by $\mathcal{N}'_X$, and we may view $\mathcal{N}_{X/S}$ as being $\mathcal{N}^{\emptyset}_X$ (see, also, [119]). For every set inclusions

$$\emptyset \neq J \subset I \subset N,$$

we have sheaf inclusions

$$\mathcal{N}'_X \subset \mathcal{N}^I_X \subset \mathcal{N}^J_X \subset \mathcal{N}_{X/S}.$$

Denote by $\mid I \mid$ the number of elements of the set I. The following proposition is an obvious generalization of what is proven in [119] in the case of $X \subset \mathbb{P}^r$.

Proposition 2.2.10 *With notation as above, the following properties hold.*

(i) For each $I \subseteq N$,

$$\chi(\mathcal{N}^I_X) = \chi(\mathcal{N}_{X/S}) - \mid I \mid .$$

(ii) We have

$$\chi(\mathcal{N}_{X/S}) = \chi(\mathcal{N}_{\varphi}) + \delta,$$

where $\delta = \mid N \mid$.

Proof:

(i) The statement directly follows from the exact sequence

$$0 \to \mathcal{N}_X^I \to \mathcal{N}_{X/S} \to T^1_{X|I} \to 0.$$

(ii) On C we have the exact sequence

$$0 \to \mathcal{N}_\varphi \to \varphi^*(N_{X/S}) \to \varphi^*(T^1_X) \to 0; \tag{2.7}$$

indeed, by combining the pull-back of (2.4) and (2.5), we get:

$$\begin{array}{cccccccccc} 0 \to & \mathcal{T}_C & \to & \varphi^*\mathcal{T}_S & \to & \mathcal{N}_\varphi & \to 0 & & \\ & \downarrow & & \| & & \downarrow & & & \\ 0 \to & \varphi^*\Theta_X & \to & \varphi^*\mathcal{T}_S & \to & \varphi^*\mathcal{N}_{X/S} & \to & \varphi^* T^1_X & \to 0, \end{array}$$

which gives (2.7). By considering the exact sequence

$$0 \to \mathcal{O}_X \to \varphi_*(\mathcal{O}_C) \to \underline{t} \to 0, \tag{2.8}$$

where $\underline{t}$ is a sky-scraper sheaf supported on N, and by the projection formula, we deduce

$$0 \to \varphi_*\mathcal{N}_\varphi \to \mathcal{N}_{X/S} \otimes \varphi_*(\mathcal{O}_C) \to T^1_X \otimes \varphi_*(\mathcal{O}_C) \to 0. \tag{2.9}$$

By tensoring (2.8) with $\mathcal{N}_{X/S}$ we obtain the exact sequence:

$$0 \to \mathcal{N}_{X/S} \to \mathcal{N}_{X/S} \otimes \varphi_*(\mathcal{O}_C) \to \mathcal{N}_{X/S} \otimes \underline{t} \to 0. \tag{2.10}$$

Now compare (2.9) and (2.10) and use that $\chi(X, \varphi_*(\mathcal{N}_\varphi)) = \chi(C, \mathcal{N}_\varphi)$. We get

$$\chi(C, \mathcal{N}_\varphi) = \chi(X, \mathcal{N}_{X/S}) + \chi(X, \mathcal{N}_{X/S} \otimes \underline{t}) - \chi(X, T^1_X \otimes \varphi_*(\mathcal{O}_C)) =$$

$$\chi(X, \mathcal{N}_{X/S}) + (dim(S) - 1)\delta - 2\delta = \chi(X, \mathcal{N}_{X/S}) - \delta.$$

Observe that

$$\chi(X, T^1_X \otimes \varphi_*(\mathcal{O}_C)) = h^0(X, T^1_X \otimes \varphi_*(\mathcal{O}_C)) = 2\delta$$

since, by projection formula, $T^1_X \otimes \varphi_*(\mathcal{O}_C) \cong \varphi_*\varphi^* T^1_X$ and $h^0(X, \varphi_*\varphi^* T^1_X) = h^0(C, \varphi^* T^1_X)$. □

Remark 2.2.11 Combining (2.9) and (2.10) one can easily deduce the following exact sequence on X:

$$0 \to \mathcal{N}'_X \to \varphi_*\mathcal{N}_\varphi \to \mathcal{N}_{X/S} \otimes \underline{t} \to T^1_X \otimes \underline{t} \to 0. \tag{2.11}$$

From (2.11), one obtains

$$h^0(\mathcal{N}'_X) \leq h^0(\mathcal{N}_\varphi).$$

Since S is a surface, $\mathcal{N}_{X/S}$ is locally free of rank one on X; thus

$$\mathcal{N}_{X/S} \otimes \underline{t} \cong T^1_X \otimes \underline{t}.$$

From this fundamental fact, we directly find that

$$\varphi_* \mathcal{N}_\varphi \cong \mathcal{N}'_X, \tag{2.12}$$

i.e. φ is stable in the sense of Definition 2.2.8; hence,

$$h^i(X, \mathcal{N}'_X) = h^i(C, \mathcal{N}_\varphi), \quad 0 \le i \le 1.$$

Moreover, by using the pull-back of (2.4) and the exact sequence (2.7), we also find

$$\mathcal{N}_\varphi \cong \varphi^* \mathcal{N}'_X, \tag{2.13}$$

being the kernels of the same surjection of sheaves.

By Theorem 2.2.5, we know that if $H^1(C, \mathcal{N}_\varphi) = 0$ then a family of morphisms into S, $f : \mathcal{C} \to B$ and $\Phi : \mathcal{C} \to B \times S$, always exists such that, for some closed point $o \in B$, $\phi_o : \mathcal{C}_o \to S$ is equivalent to $\varphi : C \to X \subset S$ and the characteristic map is bijective (i.e. the vector space $H^0(C, \mathcal{N}_\varphi)$ represents the tangent space to B at the smooth point o, corresponding to $[\varphi : C \to X \subset S]$). In such a case, the map φ is called *non-obstructed.* We can directly obtain these results by considering the Cartesian approach. More precisely, we can see that the $0 - th$ and first cohomology groups of the sheaf $N'_{X/S}$ give the first-order equisingular deformation space and its obstruction space, respectively. For this approach we shall use [119], [120] and [121].

We start by recalling some useful definitions of infinitesimal algebra and locally complete intersection schemes (see, [57]).

Definition 2.2.12 *A local ring (R, m) is called a* complete intersection *(c.i.)* ring *if the m-adic completion $\hat{R}$ is isomorphic to a ring of the form B/I, where B is a complete regular local ring and I is an ideal generated by a regular sequence. A scheme X is a* local complete intersection *(l.c.i.)* at the point $x \in X$ *if the local ring $\mathcal{O}_{X,x}$ is a complete intersection ring.*

Thus, X is a local complete intersection scheme if it is so at every point.

Definition 2.2.13 *Let Y and T be schemes and let $g : Y \to T$ be flat morphism. g is said to be a* complete intersection morphism at $y \in Y$ *if the fibre $g^{-1}(g(y))$ is a l.c.i. at the point y. If g is a c.i. morphism at every $y \in Y$, then it is said to be a* a complete intersection morphism.

Proposition 2.2.14 *Let $g : Y \to T$ be flat morphism of finite type.*

(i) If g is a c.i. morphism, then for every scheme morphism $T' \to T$, the induced morphism $g' : Y' = Y \times_T T' \to T'$ is a c.i. morphism.

(ii) If g is proper, then the set of $t \in T$ such that $g^{-1}(t)$ is a l.c.i. is an open subset $U \subset T$.

Proof: See [57], (19.3.9) (ii) and (19.3.8). □

If $X \subset \mathbb{P}^r$ is a projective curve, a point $x \in X$ is an ordinary double point (node) if the complete local ring $\hat{\mathcal{O}}_{X,x}$ of X at x is isomorphic to the ring $\mathbb{C}[[T_1, T_2]]/(T_1^2 + T_2^2)$; thus X is a l.c.i. at a node x.

Let now $S \subset \mathbb{P}^r$ be a smooth, projective surface. As in Section 1.5, given a polynomial $p(t) \in \mathbb{Q}[t]$, we denote by $Hilb^S_{p(t)}$ the Hilbert scheme which parametrizes closed subschemes of S having Hilbert polynomial $p(t)$, and by $\underline{\mathcal{H}ilb}^S_{p(t)}$ the corresponding Hilbert functor. Fix integers $d > 0$ and $p_a \geq 0$, and let $p(t) = dt + 1 - p_a$. Let B be a scheme and $\mathcal{Y} \subset S \times B$ a flat family of closed subschemes of S parametrized by B, having Hilbert polynomial $p(t)$ (it follows that $\mathcal{Y}$ has relative dimension 1 over B). One can consider the first relative cotangent sheaf of $\mathcal{Y}$ over B, $T^1(\mathcal{Y}/B, \mathcal{O}_\mathcal{Y})$, with respect to $\mathcal{O}_\mathcal{Y}$, in such a way that $T^1(\mathcal{Y}/B, \mathcal{O}_\mathcal{Y})$ is independent of the embedding $\mathcal{Y} \subset \mathbb{P}^r \times B$, its definition is consistent with base change and $Supp(T^1(\mathcal{Y}/B, \mathcal{O}_\mathcal{Y}))$ is contained in the locus where $\mathcal{Y}$ is not smooth over B. Moreover, when $B = Spec(\mathbb{C})$,

$$T^1(\mathcal{Y}/B, \mathcal{O}_\mathcal{Y})_y \cong \mathcal{O}_{\mathcal{Y},y}/m_{\mathcal{Y},y} = \mathbb{C} \Leftrightarrow \mathcal{Y} \textit{ has a node at } y, \tag{2.14}$$

(see [81]).

Proposition 2.2.15 *Let $\mathcal{Y} \to B$ be a flat family of closed subschemes of a smooth surface S, parametrized by B, having Hilbert polynomial $p(t)$. Suppose that the following properties are satisfied:*

(a) The projection $\mathcal{Y} \to B$ is a complete intersection morphism;

(b) $T^1(\mathcal{Y}/B, \mathcal{O}_\mathcal{Y})$ is the structure sheaf of a reduced closed subscheme $\Delta \subset \mathcal{Y}$ (i.e. $dim(T^1(\mathcal{Y}/B, \mathcal{O}_\mathcal{Y})_y) \leq 1$ at every point $y \in \mathcal{Y}$) and Δ is ètale over B, of degree δ.

Then, for every geometric point $b \in B$, the fibre $\mathcal{Y}_b$ is a reduced curve having δ ordinary double points and no other singularities; in particular, $\mathcal{Y}_b$ is a smooth curve if $\delta = 0$.

Proof: Since $T^1(\mathcal{Y}/B, \mathcal{O}_\mathcal{Y})$ satisfies the base change property, we have

$$T^1(\mathcal{Y}_b, \mathcal{O}_{\mathcal{Y}_b}) \cong T^1(\mathcal{Y}/B, \mathcal{O}_\mathcal{Y}) \otimes \mathcal{O}_{\mathcal{Y}_b}.$$

From this and property (b) it follows that $T^1(\mathcal{Y}_b, \mathcal{O}_{\mathcal{Y}_b})$ is the structure sheaf of δ distinct reduced closed points $\{p_1, \ldots, p_\delta\}$. Therefore, $\mathcal{Y}_b$ is smooth outside $\{p_1, \ldots, p_\delta\}$ and at each $p = p_i$ we have

$$T^1(\mathcal{Y}_b, \mathcal{O}_{\mathcal{Y}_b}))_p \cong \mathbb{C}.$$

It follows that $\mathcal{Y}_b$ has an ordinary double point at p, from the fact that $\mathcal{Y}_b$ is a local complete intersection and from property (2.14). □

For each intenger $\delta \geq 0$, we define a functor

$$\underline{\mathcal{F}}^S_{\delta,p(t)} : (Schemes) \to (Sets)$$

letting:

$$\underline{\mathcal{F}}^S_{\delta,p(t)}(B) = \begin{array}{l}\{(\mathcal{Y} \subset S \times B) \in \underline{\mathcal{H}ilb}^S_{p(t)}(B) \ \mid \ T^1(\mathcal{Y}/B, \mathcal{O}_{\mathcal{Y}}) \ satisfies \\ properties\ (a)\ and\ (b)\ and\ \mathcal{Y}_b\ is\ connected\ for\ each\ b \in B\}.\end{array}$$

This is a functor because properties (a) and (b) are obviously functorial. We have an injective functorial morphism

$$\underline{\mathcal{F}}^S_{\delta,p(t)} \to \underline{\mathcal{H}ilb}^S_{p(t)}.$$

It can be proven (see [100]) that $\underline{\mathcal{F}}^S_{\delta,p(t)}$ is representable by a locally closed subscheme $\mathcal{V}^S_{\delta,p(t)}$ of $Hilb^S_{p(t)}$ and by the restriction to it of the universal family $\mathcal{X}$ over $Hilb^S_{p(t)}$. Denote such a restriction by $\mathcal{X}_\delta \subset S \times \mathcal{V}^S_{\delta,p(t)}$. This is called the *universal family of curves* of S of degree d, arithmetic genus p_a, having only δ nodes and no other singularity.

We will most often write $Hilb^S_{d,p_a}$, $\mathcal{V}^S_{\delta,d,p_a}$ and $\underline{\mathcal{F}}^S_{\delta,d,p_a}$ instead of $Hilb^S_{d,p(t)}$, $\mathcal{V}^S_{\delta,p(t)}$ and $\underline{\mathcal{F}}^S_{\delta,p(t)}$. For any scheme B, an element $(\mathcal{Y} \subset S \times B) \in \underline{\mathcal{F}}^S_{\delta,d,p_a}(B)$ will be called a *family of curves of* S of degree d, arithmetic genus p_a having δ nodes and no other singularity. This terminology is justified by the following result:

Corollary 2.2.16 *Suppose that B is a reduced scheme and that $\mathcal{Y} \subset S \times B$ is a c.i. flat family. Assume that for every closed point $b \in B$ the fibre $\mathcal{Y}_b$ is a reduced curve of degree d and arithmetic genus p_a, having only δ nodes and no other singularity. Then $(\mathcal{Y} \subset S \times B) \in \underline{\mathcal{F}}^S_{\delta,d,p_a}(B)$.*

Proof: Let $f : S \to Hilb^S_{d,p_a}$ be the functorial morphism induced by the family $\mathcal{Y}$. By hypothesis, $f(S)$ is contained in the support of $\mathcal{V}^S_{\delta,d,p_a}$. Since S is reduced, by the universal property of $\mathcal{V}^S_{\delta,d,p_a}$, S factors through it. Therefore $\mathcal{Y}$ is the pullback of the universal family over $\mathcal{V}^S_{\delta,d,p_a}$ and this proves the statement. $\square$

The families $\mathcal{X}_\delta \subset S \times \mathcal{V}^S_{\delta,d,p_a}$ are generalizations of the families of Severi varieties in the plane. We will now study the local geometrical properties of the schemes $\mathcal{V}^S_{\delta,d,p_a}$ by investigating the infinitesimal deformations of a reduced projective curve $X \subset S$, of degree d and arithmetic genus p_a, having only nodes as singularities.

For simplicity, if A is a local $\mathbb{C}$-algebra and if Y is a scheme, we will denote by Y_A the fibre-product $Y \otimes_{Spec(\mathbb{C})} Spec(A)$. Let $\mathcal{A}$ be the category of local artinian $\mathbb{C}$-algebras. Consider the *local Hilbert functor*

$$\mathcal{H}_{X/S} : \mathcal{A} \to (Sets)$$

defined, for every $A \in \mathcal{A}$, by

$$\mathcal{H}_{X/S}(A) := \begin{array}{l}\{flat\ families\ of\ deformations\ of \\ X \subset S\ parametrized\ by\ Spec(A)\}.\end{array}$$

Note that every such family $\mathcal{Y} \subset S \times Spec(A)$ satisfies condition (a) of Proposition 2.2.15, because the fibre over the central point $Spec(\mathbb{C})$ does, and that $T^1(\mathcal{Y}/Spec(A), \mathcal{O}_{\mathcal{Y}}) = \mathcal{O}_\Delta$, where Δ is a closed subscheme of $\mathcal{Y}$.

For every subset $I \subseteq N$ we define a subfunctor of $\mathcal{H}_{X/S}$:

$$\mathcal{H}^I_{X/S} : \mathcal{A} \to (Sets);$$

this is determined by considering, for each $A \in \mathcal{A}$, only family of deformations $\mathcal{Y} \subset S \times Spec(A)$ such that Δ is ètale over $Spec(A)$, at every $p \in I$. In particular,

$$\mathcal{H}^{\emptyset}_{X/S} = \mathcal{H}_{X/S}$$

and $\mathcal{H}^N_{X/S}$ will be shortly denoted $\mathcal{H}'_{X/S}$; this last functor is the restriction to $\mathcal{A}$ of $\underline{\mathcal{F}}^S_{\delta,p(t)}$. Using the theorem of Schlessinger [116], it is straightforward to check that the functors $\mathcal{H}^I_{X/S}$ are prorepresentable for each subset $I \subseteq N$.

We now denote by $\hat{\mathcal{A}}$ the category of complete, local $\mathbb{C}$-algebras with residue field $\mathbb{C}$, and local homomorphisms, and let (R, m) be in $\hat{\mathcal{A}}$. Let $t_R := (m/m^2)^\vee$ be the *Zarisky tangent space* of R. By the structure theorem for complete local $\mathbb{C}$-algebras, we have an isomorphism

$$R \cong \mathbb{C}[[T_1, \dots, T_n]]/a, \tag{2.15}$$

where $n = dim_{\mathbb{C}}(t_R)$ and where a is an ideal contained in $(T_1, \dots, T_n)^2 = (\underline{T})^2$. The $\mathbb{C}$-vector space

$$o(R) := (a/(\underline{T})a)^\vee$$

is called the *obstruction space* of R. Its dimension equals the minimal number of generators of a; in particular, $o(R) = (0)$ if and only if $R \cong \mathbb{C}[[T_1, \dots, T_n]]$. If $o(R) \neq (0)$, then R will be called *obstructed*. It is possible to show that, modulo isomorphism, $o(R)$ does not depend on the presentation of (2.15) (see [120]). If we denote by $dim(R)$ the Krull dimension of R, we have the following obvious inequalities:

$$dim_{\mathbb{C}}(t_R) - dim_{\mathbb{C}}(o(R)) \leq dim(R) \leq dim_{\mathbb{C}}(t_R), \tag{2.16}$$

and the second is an equality if and only if $o(R) = (0)$ if and only if both inequalities are equalities.

If we translate in this context what we have observed in Section 1.5 about Hilbert schemes, we get that $H^0(X, \mathcal{N}_{X/S})$ is the Zarisky tangent space of $\mathcal{O}_{Hilb^S_{d,p_a},[X]}$. Moreover, it is a standard verification to check that $o(\mathcal{O}_{Hilb^S_{d,p_a},[X]})$ is a subspace of $H^1(X, \mathcal{N}_{X/S})$; this amounts to check that $H^1(X, \mathcal{N}_{X/S})$ is an obstruction space for the functor $\mathcal{H}_{X/S}$ (see [100] or [120]). From (2.16), applied to $R = \hat{\mathcal{O}}_{Hilb^S_{d,p_a},[X]}$, we deduce the following well known inequalities, which generalize Corollary 1.5.15:

$$\chi(\mathcal{N}_{X/S}) \leq dim_{[X]}(Hilb^S_{d,p_a}) \leq h^0(\mathcal{N}_{X/S}); \tag{2.17}$$

indeed, $dim_{[X]}(Hilb^S_{d,p_a}) = dim(\hat{\mathcal{O}}_{Hilb^S_{d,p_a},[X]})$.

It is similarly easy to show that, for each $I \subseteq N = Sing(X)$, $H^0(\mathcal{N}^I_X)$ and $H^1(\mathcal{N}^I_X)$ are respectively the tangent space and an obstruction space for the functor $\mathcal{H}^I_{X/S}$. Therefore, in view of the Proposition 2.2.10, we also have the following inequalities:

$$\chi(\mathcal{N}_{X/S}) - \mid I \mid \leq dim(R^I) \leq h^0(\mathcal{N}^I_X). \tag{2.18}$$

In particular, for $I = N$, noting that $dim_{[X]}(\mathcal{V}^S_{\delta,d,p_a}) = dim(\hat{\mathcal{O}}_{\mathcal{V}^S_{\delta,d,p_a},[X]})$, we have

$$\chi(\mathcal{N}_{X/S}) - \delta \leq dim_{[X]}(\mathcal{V}^S_{\delta,d,p_a}) \leq h^0(\mathcal{N}'_X), \tag{2.19}$$

where $\delta =\mid N \mid$, the number of nodes of X.

To every morphism $h : A \to B$ in $\hat{\mathcal{A}}$, there are associated homomorphisms of vector spaces $dh : t_A \to t_B$ and $o(h) : o(A) \to o(B)$. We will need the following result.

Lemma 2.2.17 *Let $h : A \to B$ be a surjective local homomorphism of complete local $\mathbb{C}$- algebras with residue field $\mathbb{C}$. Then, for every irreducible component U of Spec(A) and V of Spec(B) such that $V \subset U$, we have*

$$dim(U) \geq dim(V) \geq dim(U) - dim_{\mathbb{C}}(t_A) + dim_{\mathbb{C}}(t_B) - dim_{\mathbb{C}}(ker(o(h))).$$

Proof: The first inequality is obvious. We may assume that

$$A = \mathbb{C}[[T_1, \ldots, T_n, Z_1, \ldots, Z_m]]/a,$$

where $n = dim_{\mathbb{C}}(t_B)$ and $m = dim_{\mathbb{C}}(t_A) - dim_{\mathbb{C}}(t_B)$, $a \subset (\underline{T}, \underline{Z})^2$ and

$$B = \mathbb{C}[[T_1, \ldots, T_n]]/b,$$

$b \subset (\underline{T})^2$, and that h is induced by the projection

$$H : \mathbb{C}[[T_1, \ldots, T_n, Z_1, \ldots, Z_m]] \to \mathbb{C}[[T_1, \ldots, T_n]],$$

such that $T_j \to T_j$, $Z_k \to 0$. We have $b = (H(a), \beta_1, \ldots, \beta_t)$, and, by definition of $o(h)$, $t = dim_{\mathbb{C}}(ker(o(h)))$. Therefore, $B = \mathbb{C}[[T_1, \ldots, T_n, Z_1, \ldots, Z_m]]/c$, where

$$c = (a, Z_1, \ldots, Z_m, H^{-1}(\beta_1), \ldots, H^{-1}(\beta_t)).$$

Since U and V correspond to minimal primes of the set $Ass(a)$ (determined by the associated prime ideals of a) and of $Ass(c)$, respectively, the conclusion follows from the generalized Krull "Hauptidealsatz". □

We will now apply the above lemma to prove a refinement of inequalities (2.17), (2.18) and (2.19).

Proposition 2.2.18 *Let S be a smooth, projective surface, $X \subset S$ be a nodal curve, N the set of nodes of X and $R = \hat{\mathcal{O}}_{Hilb^S_{d,p_a},[X]}$. Then:*

(i) Given $J \subset I \subset N$, and irreducible components Φ of Spec(R^I) and Ψ of Spec(R^J), such that $\Phi \subset \Psi$, we have

$$dim(\Psi)- \mid I \mid + \mid J \mid \leq dim(\Phi) \leq dim(\Psi).$$

(ii) If in (i) the first inequality is an equality then, for every $J \subset G \subset I$, and for every irreducible component Σ of Spec(R^G), such that $\Phi \subset \Sigma \subset \Psi$, we have

$$dim(\Sigma) = dim(\Psi)+ \mid G \mid - \mid I \mid (= dim(\Phi)+ \mid G \mid - \mid J \mid).$$

Proof:

(i) From the exact sequence of sheaves on X

$$0 \to \mathcal{N}_X^I \to \mathcal{N}_X^J \to T_X^1 \mid_{I \setminus J} \to 0,$$

we deduce an exact sequence of vector spaces, such that the injection

$$H^0(\mathcal{N}_X^I) \xrightarrow{t} H^0(\mathcal{N}_X^J)$$

coincides with the differential map df^{JI}, where $f^{JI} : Spec(R^J) \hookrightarrow Spec(R^I)$, and the surjection

$$H^1(\mathcal{N}_X^I) \xrightarrow{o} H^1(\mathcal{N}_X^J)$$

fits into the commutative diagram:

$$\begin{array}{ccc} H^1(\mathcal{N}_X^I) & \xrightarrow{o} & H^1(\mathcal{N}_X^J) \\ \cup & & \cup \\ o(R^I) & \xrightarrow{o(f^{JI})} & o(R^J) \end{array}$$

It follows that $ker(o(f^{JI})) \subset ker(o)$, and we have

$$\mid I \mid - \mid J \mid = h^0(T_X^1 \mid_{I \setminus J}) = h^0(\mathcal{N}_X^J) - h^0(\mathcal{N}_X^I) + dim(ker(o)) \geq$$

$$h^0(\mathcal{N}_X^J) - h^0(\mathcal{N}_X^I) + dim(ker(o(f^{JI}))).$$

The conclusion follows from Lemma 2.2.17 applied to $h = f^{JI}$.

(ii) If $dim(\Psi) - \mid I \mid + \mid J \mid = dim(\Phi)$, then both

$$dim(\Sigma) < dim(\Psi) - \mid I \mid + \mid G \mid$$

and

$$dim(\Sigma) > dim(\Psi) - \mid I \mid + \mid G \mid = dim(\Phi) + \mid G \mid - \mid J \mid$$

contradict (i). □

The next result provides a sufficient cohomological condition for the hypothesis (ii) of the previous proposition to be satisfied.

Proposition 2.2.19 *Assume that for some $I \subseteq N$ we have $H^1(\mathcal{N}_X^I) = (0)$. Then, for all $J \subseteq I$, $Spec(R^J)$ is smooth of dimension $\chi(\mathcal{N}_{X/S}) - \mid J \mid$ and, in particular, $Hilb_{d,p_a}^S$ is smooth of dimension $\chi(\mathcal{N}_{X/S})$ at $[X]$.*

Proof: Since $H^1(\mathcal{N}_X^I)$ is an obstruction space for the functor $\mathcal{H}_{X/S}^I$, the hypothesis implies that $H_{X/S}^I$ is smooth, hence that $Spec(R^I)$ is smooth of dimension $h^0(\mathcal{N}_X^I) = \chi(\mathcal{N}_{X/S}) - \mid I \mid$. From the exact sequence

$$0 \to \mathcal{N}_X^I \to \mathcal{N}_X^J \to T_X^1 \mid_{I \setminus J} \to 0$$

we deduce that we also have $H^1(\mathcal{N}_X^J) = (0)$ for all $J \subseteq I$. This concludes the proof. □

Corollary 2.2.20 *If $H^1(\mathcal{N}_X') = (0)$, then $Spec(R^J)$ is smooth of dimension $\chi(\mathcal{N}_{X/S}) - \mid J \mid$, for all $J \subseteq N$; in particular, $Hilb_{d,p_a}^S$ and $\mathcal{V}_{\delta,d,p_a}^S$ are both smooth at $[X]$, of dimension $\chi(\mathcal{N}_{X/S})$ and $\chi(\mathcal{N}_{X/S}) - \delta$, respectively.*

Proof: It is an immediate consequence of the previous proposition and of the fact that $H^1(\mathcal{N}'_X)$ is an obstruction space for the functor $\mathcal{H}'_{X/S}$, which is prorepresented by $\hat{\mathcal{O}}_{\mathcal{V}^S_{\delta,d,p_a},[X]}$. □

By generalizing Definition 1.5.7 for Hilbert schemes, we can make the following:

Definition 2.2.21 *The curve X will be called* obstructed *(resp.* unobstructed*) in $\mathcal{V}^S_{\delta,d,p_a}$ or in $Hilb^S_{d,p_a}$ if $[X]$ is a singular (resp. a non-singular) point of $\mathcal{V}^S_{\delta,d,p_a}$ or of $Hilb^S_{d,p_a}$.*

We want to briefly consider another important aspect which can be called the *smoothability property.*

Definition 2.2.22 *Let S be a smooth, projective surface. A closed subscheme $X \subset S$ having Hilbert polynomial $P(t)$ is called* smoothable in S *if $[X] \in Hilb^S_{p(t)}$ belongs to an irreducible component H of $Hilb^S_{p(t)}$ whose general point parametrizes a non-singular subvariety of S. If $X \subset S$ is a curve of degree d and arithmetic genus p_a having δ nodes and no other singularity, X is* smoothable in S *if and only if $[X] \in \overline{\mathcal{V}^S}_{0,d,p_a}$ (where $\overline{\mathcal{V}^S}_{0,d,p_a}$ denotes the closure in $Hilb^S_{d,p_a}$). The curve $X \subset S$ is said to be* i-smoothable *in S ("i" stands for "independently"), if there are irreducible componets Z_j of $\mathcal{V}^S_{j,d,p_a}$, $0 \leq j \leq \delta$, such that*

$$[X] \in Z_\delta \subset \overline{Z}_{\delta-1} \subset \cdots \subset \overline{Z}_1 \subset \overline{Z}_0.$$

The property of i-smoothability in S means that X is the flat specialization of curves contained in S and having any number of nodes between δ and 0. Of course, the fact that X is i-smoothable in S implies that X is smoothable in S, but the converse, in general, is not true (see [21]). Note that, in the previous definition, we have $dim(Z_j) \leq dim(Z_{j-1})-1$ because Z_j is properly contained in $\overline{Z}_{j-1}$. Therefore, we also have $dim(Z_\delta) \leq dim(Z_0) - \delta$. From Proposition 2.2.18, it follows that we must have equality, hence that

$$dim(Z_j) = dim(Z_0) - j,\ 0 \leq j \leq \delta,$$

and Z_0 must coincide with a component of $Hilb^S_{d,p_a}$.

It can be proven ([121]) that, conversely, in order for X to be i-smoothable in S, it is sufficient that $[X]$ is contained in a component of $\mathcal{V}^S_{\delta,d,p_a}$ which has codimension δ in a component of $Hilb^S_{d,p_a}$. We state, without proving it, the following:

Theorem 2.2.23 *(i) Let Φ and Ψ be irreducible components of $\mathcal{V}^S_{\delta,d,p_a}$ and of $Hilb^S_{d,p_a}$, respectively, such that $\Phi \subset \Psi$. Then we have*

$$dim(\Psi) - \delta \leq dim(\Phi) \leq dim(\Psi).$$

(ii) Assume that the irreducible component Φ *of* $\mathcal{V}^S_{\delta,d,p_a}$ *is contained in the component* Ψ *of* $Hilb^S_{d,p_a}$ *and that* $dim(\Phi) = dim(\Psi) - \delta$*. Then, for every* $0 \leq j \leq \delta$*, there is an irreducible component* Z_j *of* $\mathcal{V}^S_{j,d,p_a}$*, such that*

$$\Phi = Z_\delta \subset \overline{Z}_{\delta-1} \subset \cdots \subset \overline{Z}_1 \subset \overline{Z}_0 = \Psi.$$

Equivalently, every closed point of Φ *parametrizes a nodal curve which is i-smoothable in* S*.*

The criterion of i-smoothability can often be applied when the curve X is given as a member of a family of nodal curves whose dimension l is known. The difficulty in applying it is to ensure that the curves so constructed vary in an open subset of $\mathcal{V}^S_{\delta,d,p_a}$, so to conclude that $\mathcal{V}^S_{\delta,d,p_a}$ has dimension l and not larger. A typical case is when we have a l-dimensional family of reducible curves having δ nodes, whose general member has $k \geq 2$ irreducible components, and we need to exclude that it is contained in a $(l+1)$-dimensional family of curves having the same number δ of nodes and whose general member has $h < k$ irreducible components. This can be done by means of a classical result, which we restate here.

Theorem 2.2.24 *Let* $\pi : \mathcal{Y} \to T$ *be a proper flat family of curves (i.e.* π *is a flat projective morphism of relative dimension one), parametrized by an irreducible scheme* T*. Suppose that for all closed points* $t \in T$*, the fibre* $\mathcal{Y}_t$ *has* δ *nodes and no other singularity. Then all fibres have the same number of irreducible components.*

The proof of the previous theorem can be given by reducing to the following other classical result, whose modern proof can be found in [102].

Theorem 2.2.25 *Let* $\mathcal{Y} \to T$ *be a proper flat family of curves, such that* T *is integral. Let* $A = \mathcal{Y}_\eta$ *be the generic geometric fibre,* $t_0 \in T$ *be a closed point,* $B = \pi^{-1}(t_0)$ *such that the associated cycle* $z(B) = \sum_{i=1}^s m_i B_i$*, where* $m_i = length(\mathcal{O}_{B_i,z_i})$*,* z_i *is the generic point of* B_i *and the indices have been chosen so that* $g(B_i) > 0$ *if and only if* $1 \leq i \leq r$*, for a suitable integer* r*,* $0 \leq r \leq s$*. Then, the following inequality holds:*

$$g(A) \geq \sum_{i=1}^{r} (m_i\, g(B_i) - m_i + 1).$$

In the case of plane curves there is also a converse "existence theorem". Since we will be mainly interested in irreducible curves, we do not prove here these results and refer the reader to the original paper.

As a direct corollary of Theorem 2.2.24 we have the following result.

Corollary 2.2.26 *Let* $\mathcal{X}_\delta \subset S \times \mathcal{V}^S_{\delta,d,p_a}$ *be the universal family. The number of irreducible components of* $(\mathcal{X}_\delta)_w$*,* w *a closed point of* $\mathcal{V}^S_{\delta,d,p_a}$*, is constant when* w *varies in a connected component of* $\mathcal{V}^S_{\delta,d,p_a}$*.*

Theorem 2.2.23 gives a criterion of i-smoothability of purely geometric nature. Using Proposition 2.2.19, we can obtain a cohomological criterion of i-smoothability, already given in [119] when $X \subset \mathbb{P}^r$.

Theorem 2.2.27 *If $H^1(X, \mathcal{N}'_X) = (0)$, then both $\mathcal{V}^S_{\delta,d,p_a}$ and $Hilb^S_{d,p_a}$ are smooth at $[X]$ of dimension $\chi(\mathcal{N}_{X/S}) - \delta = h^0(\mathcal{N}_{X/S}) - \delta$ and $\chi(\mathcal{N}_{X/S}) = h^0(\mathcal{N}_{X/S})$, respectively, and X is i-smoothable in S.*

Proof: It immediately follows from Proposition 2.2.19, Corollary 2.2.20 and Theorem 2.2.23. □

With such a cohomological criterion, we shall see in Theorem 3.1.1 how it becomes easy to prove that Severi varieties of irreducible plane curves, with only nodes as singularities, are everywhere smooth of the expected dimension and, so, that each element of such a variety is an i-smoothable nodal curve in $\mathbb{P}^2$.

We now want to focus our attention on equisingular first-order deformations of a nodal curve $X \subset S$ in the linear system to which X belongs. These deformations sometimes do not coincide with all the first-order equisingular deformations, but are parametrized by a smaller subscheme. As we shall immediately see, this depends on the fact that S could or could not be a regular surface (i.e. $h^1(S, \mathcal{O}_S) = 0$ or $\neq 0$ respectively). So there is a need to find a vector space which could "take care" of this fact.

To this aim, let $D \in Div(S)$ be an effective divisor on S. Denote by $\mid D \mid$ the complete linear system associated to D. Suppose that the general element of $\mid D \mid$ is a smooth, irreducible curve. As usual, $p_a(D)$ denotes the (arithmetic) genus of D, i.e.

$$p_a(D) = \frac{D(D + K_S)}{2} + 1.$$

By the hypothesis on its general element, it makes sense to consider the subscheme of $|D|$ which parametrizes all curves $X \sim D$ that are irreducible and have only δ nodes as singular points. As before, such a subscheme is functorially defined and locally closed.

Definition 2.2.28 *We denote by*

$$V_{|D|,\,\delta}$$

the locally closed subscheme of $\mid D \mid$, which parametrizes a universal family of reduced and irreducible curves belonging to $\mid D \mid$ and having exactly δ nodes and no other singularities. The scheme $V_{|D|,\,\delta}$ will be called Severi variety *of irreducible, δ-nodal curves in the given linear system. If $V_{|D|,\,\delta} \neq \emptyset$ and if $[X] \in V_{|D|,\,\delta}$, then N will denote the scheme of nodes of X; it is a closed zero-dimensional subscheme of S of degree δ. As usual, the geometric genus of X will be denoted by $g = p_a(D) - \delta$.*

If we want to consider first-order equisingular deformations of $[X] \in V_{|D|,\,\delta}$ in the given linear system $|\,D\,|$, we can use the fact that $V_{|D|,\,\delta} \subset |\,D\,|$.

Since $|\,D\,|$ is a projective space, from the theory of Zariski tangent spaces to Grasmannians, we have

$$T_{[X]}(|\,D\,|) \cong H^0(\mathcal{O}_S(D))/ < X >, \tag{2.20}$$

where $< X >$ denotes the one-dimensional subspace generated by X. This space coincides with the vector space of first-order deformations of X in $|\,D\,|$. First-order equisingular deformations of X in $|\,D\,|$ are parametrized by its subspace

$$T_{[X]}(V_{|D|,\,\delta}).$$

In what follows, we describe how to give a cohomological interpretation of $T_{[X]}(V_{|D|,\,\delta})$, as in (2.20) for $T_{[X]}(|\,D\,|)$.

Denote by $\varphi : C \to X \subset S$ the normalization map of X. From the exact sequence (2.5),

$$0 \to \mathcal{T}_C \to \varphi^*\mathcal{T}_S \to \mathcal{N}_\varphi \to 0,$$

it follows that

$$\mathcal{T}_C \otimes \mathcal{N}_\varphi \cong \bigwedge^2(\varphi^*\mathcal{T}_S).$$

Denoting by K_S and K_C a canonical divisor of S and C respectively, we have therefore

$$\mathcal{O}_C(-K_C) \otimes \mathcal{N}_\varphi \cong \mathcal{O}_C(\varphi^*(-K_S))$$

which implies

$$\mathcal{N}_\varphi \cong \mathcal{O}_C(\varphi^*(-K_S)) \otimes \mathcal{O}_C(K_C).$$

Let $\mu : \tilde{S} \to S$ be the blow-up of S at the set of nodes N, such that $B = \sum_{i=1}^{\delta} E_i$ is the μ-exceptional divisor. We have, $\mu^*(K_S) = K_{\tilde{S}} - B$ and $\mu^*(X) = C + 2B$. From adjunction theory on $\tilde{S}$ and from the fact that $\mu\,|_C = \varphi$ it follows that $\mathcal{O}_C(K_C) \cong \mathcal{O}_C(K_{\tilde{S}} + C) = \mathcal{O}_C(\varphi^*(K_S) + B + \varphi^*(X) - 2B) = \mathcal{O}_C(\varphi^*(K_S + X) - B)$. Therefore,

$$\mathcal{N}_\varphi \cong \mathcal{O}_C(\varphi^*(X) - B). \tag{2.21}$$

Since $X \sim D$ on S, we have

$$H^i(\mathcal{N}_\varphi) \cong H^i(\mathcal{O}_C(\varphi^*(X) - B)),\ i \geq 0.$$

From what we have recalled about Horikawa's theory, $H^0(\mathcal{N}_\varphi)$ parametrizes all first-order equisingular deformations of X in S. Now, the scheme inclusions $N \subset X \subset S$ determine the following exact sequence

$$0 \to \mathcal{I}_{X/S} \cong \mathcal{O}_S(-X) \to \mathcal{I}_{N/S} \to \mathcal{I}_{N/X} \to 0. \tag{2.22}$$

Tensoring (2.22) with $\mathcal{O}_S(D)$, we get

$$0 \to \mathcal{O}_S \to \mathcal{I}_{N/S}(D) \to \mathcal{I}_{N/X}(D) \to 0, \tag{2.23}$$

since $X \sim D$ on S. Therefore, we have the following diagram of sheaves:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \rightarrow & & \mathcal{O}_S & \xrightarrow{\cdot X} & \mathcal{I}_{N/S}(D) & \rightarrow & \mathcal{I}_{N/X}(D) & \rightarrow 0 & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \rightarrow & & \mathcal{O}_S & \rightarrow & \mathcal{O}_S(D) & \rightarrow & \mathcal{O}_X(D) & \rightarrow 0 & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \rightarrow & & 0 & \rightarrow & \mathcal{O}_N(D) & \rightarrow & \mathcal{O}_N(D) & \rightarrow 0 & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & . \end{array}$$

From the first line, we observe that

$$T_{[X]}(V_{|D|,\,\delta}) \cong H^0(\mathcal{I}_{N/S}(D))/ < X >, \tag{2.24}$$

which is the subspace of

$$H^0(\mathcal{N}_\varphi) \cong H^0(\mathcal{O}_C(\varphi^*(X) - B))$$

contained in

$$T_{[X]}(|\ D\ |) \cong H^0(\mathcal{O}_S(D))/ < X > .$$

In other words, a first-order deformation $X + \epsilon X'$, $\epsilon^2 = 0$, is in $V_{|D|,\,\delta}$ if and only if it is in $|\ D\ |$ and $N \subset X'$.

Remark 2.2.29 From Lemma 1.2.6 and the diagram above, we also observe that if S is assumed to be a regular surface, then

$$H^0(\mathcal{N}_\varphi) \cong H^0(\mathcal{I}_{N/S}(D))/H^0(\mathcal{O}_S),$$

i.e.

$$H^0(\mathcal{N}_\varphi) \cong H^0(\mathcal{I}_{N/S}(D))/ < X >, \tag{2.25}$$

which means that all first-order equisingular deformations of X in S are in $|D|$.

On the contrary, if S is an irregular surface, the left-hand space in (2.25) properly contains the right-hand one; so first-order equisingular deformations of X in $|D|$ are a subclass of those of X in S.

To conclude the section, observe that from the exact sequence

$$0 \rightarrow \mathcal{I}_{N/S}(D) \rightarrow \mathcal{O}_S(D) \rightarrow \mathcal{O}_N(D) \rightarrow 0, \tag{2.26}$$

we get

$$h^0(\mathcal{I}_{N/S}(D)) = h^0(\mathcal{O}_S(D)) - \delta + \sigma,$$

where $\sigma = h^1(\mathcal{I}_{N/S}(D)) - h^1(\mathcal{O}_S(D)) \geq 0$, since

$$H^1(\mathcal{I}_{N/S}(D)) \rightarrow H^1(\mathcal{O}_S(D))$$

is surjective. Therefore, $h^0(\mathcal{I}_{N/S}(D)) \geq h^0(\mathcal{O}_S(D)) - \delta$, so

$$dim(T_{[X]}(V_{|D|,\,\delta})) \geq h^0(\mathcal{O}_S(D)) - (\delta + 1) =$$

$$= dim(T_{[X]}(\mid D \mid)) - \delta = dim(\mid D \mid) - \delta,$$

where the last equality holds since $\mid D \mid$ is smooth at every point. The first inequality is an equality if and only if the surjection

$$H^1(\mathcal{I}_{N/S}(D)) \to H^1(\mathcal{O}_S(D))$$

is an isomorphism; this happens if and only if

$$H^0(\mathcal{O}_S(D)) \to H^0(\mathcal{O}_N(D))$$

is surjective, i.e. if and only if the set N imposes independent conditions to the linear system $\mid D \mid$. In such a case,

$$dim(T_{[X]}(V_{|D|,\,\delta})) = dim(\mid D \mid) - \delta \geq dim_{[X]}(V_{|D|,\,\delta}).$$

For what stated in Theorem 2.2.23 (i), in general

$$dim_{[X]}(V_{|D|,\,\delta}) \geq dim(\mid D \mid) - \delta.$$

Thus, the obstruction space for such deformations is a subspace of $H^1(S, \mathcal{I}_N(D))$, so the set of nodes N imposes independent conditions to $\mid D \mid$ if and only if $V_{|D|,\,\delta}$ is smooth at $[X]$ of codimension δ in $\mid D \mid$.

Definition 2.2.30 *The Severi variety* $V_{|D|,\,\delta}$ *is said to be* regular *at the point* $[X]$*, if it is smooth at* $[X]$ *of dimension* $dim(\mid D \mid) - \delta$*,* δ *the number of nodes of* X*. Otherwise, the component of* $V_{|D|,\,\delta}$ *containing* $[X]$ *is said to be a* superabundant component*.*

In the sequel, we shall focus on vanishing conditions for the vector space $H^1(S, \mathcal{I}_{N/S}(D))$ in order to find sufficient conditions for regularity of Severi varieties on smooth surfaces.

Remark 2.2.31 The regularity property is very strong, since, for what stated in Theorem 2.2.27, it implies that the nodes of X can be independently smoothed, i.e. X is i-smoothable. In this intepretation this fact is more evident; indeed, let $[X] \in V_{|D|,\,\delta}$ and let $p \in N$ be one of the nodes of X. Denote by M the complement of p in N. Suppose that N imposes independent conditions to $\mid D \mid$; therefore, M does the same, so that $h^1(S, \mathcal{I}_{N/S}(D)) = h^1(S, \mathcal{I}_{M/S}(D))$. Hence, we have $h^0(S, \mathcal{I}_{N/S}(D)) + 1 = h^0(S, \mathcal{I}_{M/S}(D))$. Any element of the vector space $H^0(S, \mathcal{I}_{N/S}(D))$, not in $H^0(S, \mathcal{I}_{M/S}(D))$, defines an infinitesimal deformation of X which smooths the node p and leaves unsmoothed all the other nodes. This means that $[X] \in \overline{V}_{|D|,\,\delta-1}$.

2.3 Several problems and known results concerning Severi varieties on projective surfaces

This section is devoted to an overview of some of the most important results concerning Severi varieties, or more generally parameter spaces, of singular curves on projective surfaces. Since there is a huge number of articles concerning such topics, in order to avoid confusion, we treat separately some of the known results in the case of $\mathbb{P}^2$, in the case of other type of rational surfaces, in the case of $K3$ surfaces and, finally, we shall consider some known results which are mainly related to surfaces of general type. When it is not too far from the scope of this work, we shall try to recall, at least in some aspects, the basic techniques used to prove some of these results. For what concerns the regularity problem of Severi varieties on surfaces of general type and the moduli problem of smooth and nodal curves on such surfaces, here we only briefly mention the known results. Indeed, since these subjects are the core of our research, we have devoted Chapters 3, 4 and 5 to such topics, where these results will be recalled and our new results will be proven.

As for the plane case, one may ask several questions concerning a Severi variety $V_{|D|,\delta}$ of irreducible, δ-nodal curves in a given linear system $|D|$, on a smooth, projective surface S.

(i) the existence problem: when is such a $V_{|D|,\delta}$ not empty?

(ii) the dimension and smoothness problem: what are the dimensions of its components? Are they everywhere smooth?

(iii) the irreducibility problem: when $V_{|D|,\delta}$ is irreducible?

(iv) the enumerative problem: what is the degree of $V_{|D|,\delta}$ as a subscheme of $| D |$?

(v) the moduli problem: how is the behaviour of the moduli of the elements of $| D |$ and of $V_{|D|,\delta}$?

(vi) the configuration problem: which is the configuration of points that the nodes of the elements of $V_{|D|,\delta}$ form in the scheme $Hilb^{\delta}(S)$?

One can easily understand why there is so much production of papers related to Severi varieties. However, in some cases, there are still only partial answers. In the next chapters, we will mainly focus on points (ii) and (v), expecially in the case of S a smooth surface of general type, case in which less is known than what is proven in other cases of smooth, projective surfaces.

From Section 2.2, we know that the natural approach to the *dimension problem* is to use equisingular deformations of nodal curves. It is easy to see that if S is a rational surface and if $[X] \in V_{|D|,\delta}$ then its set of nodes N imposes independent conditions to $| D |$, hence all components are regular. A similar argument also works for other regular surfaces like $K3$ and Enriques surfaces. For surfaces of general type this approach completely fails; one way is to use a rank-two vector bundle approach to get some vanishing criteria involving the ideal sheaf $\mathcal{I}_{N/S}$.

Turning to the *existence problem*, in the plane case one takes the advantage from the fact that there are rational nodal curves of any degree d, namely the general projections to a plane of rational normal curves of degree d in $\mathbb{P}^d$. By smoothing some of their nodes one proves the non-emptyness of $V_{|D|,\delta}$, for all $0 \leq \delta \leq \frac{(d-1)(d-2)}{2}$. There are also some results about the non-emptyness of $V_{|D|,\delta}$ in the case of Del Pezzo surfaces. The existence problem becomes more difficult for other surfaces, like $K3$'s, Enriques surfaces or surfaces of general type.

For what concerns the *irreducibility*, apart from the classical case of the plane, there are some recent results which state that, for example, on a general surface $S \subset \mathbb{P}^3$ of degree large enough some Severi varieties are reducible. We shall treat this aspect in the sequel.

In recent times there has also been a lot of interest about *enumerative problem*. Impulse to the subject has come from recent ideas from quantum field theory leading to the definition of quantum cohomology. As a by product, formulas enumerating rational curves on certain varieties were derived from the properties of certain generating functions representing the free-energy of certain topological field theory. A mathematically formal construction of quantum cohomology came soon afterwards ([78], [79] and [115]). Different proofs of some of the classical enumerative formulas were provided later in [16] and [17] using different methods that could be generalized to cases (such as Hirzebruch surfaces) for which the quantum cohomology theory did not give enumerative results. Moreover, a recursive formula enumerating plane curves of any genus has been recently proven using purely algebro-geometric techniques ([15]) and a generating function exists together with a differential equation implying such a recursion ([46]).

Another important problem concerning curves in Severi varieties is the following; given a complete linear system $\mid D \mid$ or a family of nodal curves $V_{|D|,\delta}$ on S, one can be interested in studying its properties from the point of view of *moduli*. This reduces to understand how is the behaviour of the natural functorial morphism

$$\pi_{|D|,\,\delta} : V_{|D|,\delta} \longrightarrow \mathcal{M}_g,$$

where $g = p_a(D) - \delta$, for each $\delta \geq 0$; the problem is to determine the dimension of its image in the moduli space. In [119], Sernesi completely solved the case $S = \mathbb{P}^2$ and also found some results on smooth irreducible components of Hilbert schemes of smooth curves in $\mathbb{P}^r$. These results were followed by other researches of Pareschi ([106]) in $\mathbb{P}^3$ and Lopez ([84]) in $\mathbb{P}^r$. We shall treat in detail this problem in the sequel.

The *configuration problem* deals with the study of the map

$$V_{|D|,\delta} \longrightarrow Hilb^{\delta}(S).$$

If the map is dominant, then the nodes of the elements parametrized by $V_{|D|,\delta}$ can be taken in general position on S. There are some interesting results of Treger in the plane case, which we shall mention in the sequel.

We can make an overview of some results, which are known up to now, on these subjects.

The plane case:

We have already treated the classical case of Severi in Theorem 2.1.5; here we only want to add that there are also some classical results concerning plane curves with nodes and cusps. For the classical approach of B. Segre (in [118]), we refer the reader to [138] and [144], VIII.5, where there is a beautiful description of his results and some other results and problems concerning necessary and sufficient conditions for existence and smoothness of the variety parametrizing plane curves of degree n with exactly δ nodes and κ cusps.

Among the "non-classical" results, the first which must be recalled is the one of Wahl, [138] (1972). Even if families of plane nodal and cuspidal curves have been classical studied by Severi and Segre, they never considered the problem of the existence of these families as varieties. By using flattening stratification methods, Wahl is the first who proves the existence of a scheme $\mathcal{X}$ (in his notation) and a universal family of plane curves $\mathcal{E} \to \mathcal{X}$ whose fibres are curves of a given degree n with δ nodes and κ cusps, and $\mathcal{X}$ is a union of locally closed subschemes of $\mathbb{P}^N$, where $N = \frac{n(n+3)}{2}$. He also considers locally trivial embedded deformations of plane curves with arbitrary singularities, identifies infinitesimal equisingular deformations and obstructions as vector spaces of a generalized equisingular sheaf $\mathcal{N}'_{X/\mathbb{P}^2}$ on such a curve X and shows the existence of a formal versal deformation space in the sense of Schlessinger.

In 1975, Arbarello, [1], generalizes some known results of B. Segre by showing a further impossibility of extending for high values of the genus g, precisely $g > 36$, the process given by Severi in studying the moduli of curves (see Section 2.1). He proves that, in such a range of values of g and on a rational surface, no algebraic system of general moduli curves exist, supposing that their singularities are nodes which vary in a rational variety. This was a proof of the impossibility to extend the procedure used by Severi in proving the unirationality of $\mathcal{M}_g$, for $g \leq 10$.

In 1981, Arbarello e Cornalba, [2], gave a partial answer to Petri's conjecture, which the authors restated as:

Given a divisor D on a general moduli curve C, the cup product

$$\mu_0 : H^0(C, \mathcal{O}_C(D)) \otimes H^0(C, \mathcal{O}_C(-D) \otimes \omega_C) \to H^0(C, \omega_C),$$

is injective.

The geometric interpretation they give to Petri's conjecture implies that what has to be proven is that if $\varphi : C \to \mathbb{P}^r$ is a morphism from a general moduli curve C, then each first-order deformation of φ is non-obstructed, that is $H^1(C, \mathcal{N}_\varphi) = 0$. By using this version of the conjecture, they prove the case $r = 2$; as already mentioned in Section 2.2, this was the occasion in which they translated the beautiful results of Horikawa's theory in a purely Algebraic Geometry approach.

In the same year ([3]), they consider the classical problem to determine under which conditions there exists an irreducible curve of degree n in $\mathbb{P}^2$ with nodes at δ preassigned general points and no other singularity. Necessary obvious conditions are

$$\frac{(n+1)(n+2)}{2} - 1 \geq 3\delta$$

and
$$\frac{(n-1)(n-2)}{2} \geq \delta,$$
the first coming from the fact that it takes three linearly independent conditions to impose a nodal singularity at a preassigned general point to curves varying in a linear system, which has no fixed component and is not composed with a pencil, whereas the second condition is just the statement that a plane curve of degree n with more than that number of nodes is reducible. They prove that these conditions are also sufficient, with one exception. The fundamental tool for their analysis is always Horikawa's theory.

Theorem. ([3], Theorem (3.2)) *Let $p_1, \dots, p_\delta$ be general points in $\mathbb{P}^2$. Let n be a positive integer such that the above inequalities hold. Then there exists an irreducible plane curve of degree n having nodes at $p_1, \dots, p_\delta$ and no other singularity, unless*
$$n = 6, \ \delta = 9.$$
In this case the only plane curve of degree n passing doubly through $p_1, \dots, p_\delta$ is a smooth cubic counted twice.

They also generalize this result to rational surfaces (see [3], Proposition (3.7) and Corollary (4.6)).

A year later, Zarisky proves in [145] his generalization of Severi's theorem. We have already discussed his results, so we refer the reader to Theorem 2.1.8, in Section 2.1. Here we want just to recall that his result implies that the general member of the locus $V_{n,g}$, of reduced irreducible curves of degree n in $\mathbb{P}^2$ and of genus g, has only $\delta = \frac{(n-1)(n-2)}{2} - g$ nodes and no other singularities (see Corollary 2.1.9).

In 1983, Arbarello and Cornalba ([4]) prove the irreducibility of $\overline{V}_{n,g}$, whenever $3n \geq 2g + 7$. To understand what this bound means, we recall that the *Brill-Noether number* is defined as
$$\rho(g, r, n) = g - (r + 1)(g - n + r),$$
which, when non-negative, determines the dimension of the variety of linear series of degree n and dimension r on a general curve of genus g.

Theorem. ([4], Theorem (1.1)) *The variety $\overline{V}_{n,g}$ is irreducible whenever $\rho(2, n, g) \geq 1$ or, which is the same, when $3n \geq 2g + 7$.*

Denote by
$$\pi_{n,\delta} : V_{n,\delta} \to \mathcal{M}_g$$
the morphism from the Severi variety of irreducible, δ-nodal plane curves of degree n to the moduli space of curves of genus $g = \frac{(n-1)(n-2)}{2} - \delta$. This morphism is defined, since the family of irreducible, plane nodal curves
$$\begin{array}{ll} \mathcal{X} & \subset \mathbb{P}^2 \times V_{n,\delta} \\ \downarrow & \\ V_{n,\delta} & \end{array}$$

parametrized by $V_{n,\delta}$ can be simultaneously desingularized, i.e. there exists a diagram of proper morphisms

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Phi} & \mathcal{X} \quad \subset \mathbb{P}^2 \times V_{n,\delta} \\ & \searrow & \downarrow \\ & & V_{n,\delta} \end{array}$$

where the diagonal map is a proper morphism of curves and Φ is fibrewise the normalization map. The map Φ is the blow-up of $\mathcal{X}$ along its singular locus and the morphism $\pi_{n,\,\delta}$ is functorially defined by the family $f : \mathcal{C} \to V_{n,\delta}$.

With this set-up, the theorem above states that $\overline{V}_{n,g}$ is irreducible as soon as the general fibre of $\pi_{n,\,\delta}$ has dimension at least $9 = 1 + dim(PGL(3, \mathbb{C}))$ (provided $g > 1$).

In 1984, Sernesi studied families of projective curves with good properties from the moduli point of view (see [119]). In his paper, he investigates, for given $r,\ n,\ g$ the existence of a smooth irreducible open subset U of the Hilbert scheme of $\mathbb{P}^r$ parametrizing irreducible (and non singular if $r \geq 3$) curves of degree n and genus g having the expected number of moduli.

Definition 2.3.1 *Given the functorial morphism*

$$\pi : U \to \mathcal{M}_g,$$

from a family of smooth curves in $\mathbb{P}^r$ *(*δ*-nodal,* $\delta \geq 0$*, when* $r = 2$*) of geometric genus* g*, the* number of moduli of the family U *is the dimension of the image, i.e.* $dim(\pi(U))$*.* U *is said to have the* expected number of moduli *if* $\pi(U)$ *has dimension equal to*

$$min(3g - 3,\ 3g - 3 + \rho(g, r, n)).$$

Of course, when $\rho(g, r, n) \geq 0$, a family U has the expected number of moduli $3g - 3 = dim(\mathcal{M}_g)$ when every sufficiently general curve of genus g belongs to it; in this case the family is said to have *general moduli.* When $\rho(g, r, n) < 0$, every family U does not have general moduli, i.e. it has *special moduli* and the number $-\rho(g, r, n)$ determine the expected codimension of $\pi(U)$ in $\mathcal{M}_g$.

In the case of $\mathbb{P}^2$, by using a detailed analysis of the parametric version of equisingular deformation theory (i.e. the equisingular sheaves $\mathcal{N}^I_X$ of Section 2.2) and the *Brill-Noether map*

$$\mu_0(D) : H^0(C, \mathcal{O}_C(D)) \otimes H^0(C, \mathcal{O}_C(-D) \otimes \omega_C) \to H^0(C, \omega_C),$$

Sernesi proves the following result:

Theorem 2.3.2 *For all* $n,\ g$ *such that*

$$n - 2 \leq g \leq \frac{(n-1)(n-2)}{2}, \quad n \geq 5,$$

the general point of $U = V_{n,\delta}$ *parametrizes a nodal curve* Γ *such that*

(1) the lines of $\mathbb{P}^2$ *determine a complete linear system on the normalization* $C \to \Gamma$*;*

(2) $\mu_0(D)$ *has maximal rank.*

Moreover, the family $V_{n,\delta}$ *has the expected number of moduli.*

Observe that,

$$\begin{aligned} &number\ of\ moduli\ of\ V_{n,\delta} \le dim(V_{n,\delta}) - dim(Aut(\mathbb{P}^2)) = \\ &= 3n + g - 9 = 3g - 3 + \rho(g, 2, n). \end{aligned}$$

Thus, if $\rho(g, 2, n) < 0$, $V_{n,\delta}$ has at most the expected number of moduli. In order for $V_{n,\delta}$ to exactly have the expected number of moduli, it is sufficient that a general point of $V_{n,\delta}$ parametrizes a curve Γ which is birationally (but not projectively) equivalent to only finitely many curves of the family, i.e. the normalization C of Γ has only finitely many linear systems of degree n and dimension 2.

In the same article, by using techniques of smoothability, he proves the existence of a component of the Hilbert scheme, parametrizing smooth curves in $\mathbb{P}^r$, $r \geq 3$, of given degree and genus, which has the expected number of moduli. These results were improved later by Pareschi ([106]) in the case of $\mathbb{P}^3$ and by Lopez ([84]) for $r \geq 4$.

In 1986, we have contemporairly two different proofs of Severi's statement of the irreducibility of $\overline{V}_{n,g}$, whose proof was incorrect (we have already mentioned these facts in Section 2.1). Ran, in [108], proves it by degenerating $\mathbb{P}^2$ to $S = \tilde{\mathbb{P}}^2{}_1 \cup \ldots \cup \tilde{\mathbb{P}}^2{}_{n-1} \cup \tilde{\mathbb{P}}^2{}_n$, where $\tilde{\mathbb{P}}^2{}_i$ is the blow-up of $\mathbb{P}^2$ at a point and $\tilde{\mathbb{P}}^2{}_i \cap \tilde{\mathbb{P}}^2{}_{i+1}$ is exceptional on $\tilde{\mathbb{P}}^2{}_i$ and a general line in $\tilde{\mathbb{P}}^2{}_{i+1}$, for $1 \leq i \leq n-1$. Considering the resulting degeneration of an arbitrary component V of $V_{n,g}$, he shows, by an inductive argument, that the limit of V contains a curve of the form $\sum_{1 \leq i \leq j \leq n} M_{ij}$, where $M_{ij} \subset \tilde{\mathbb{P}}^2{}_i$; M_{ii} is a general line and M_{ij} for $i < j$ is a ruling. By analyzing the local structure of the limit of V along $\sum_{1 \leq i \leq j \leq n} M_{ij}$, he proves the irreducibility of $\overline{V}_{n,g}$.

On the other hand we have the procedure of *good degenerations* of Harris, [58], which we have already discussed in Section 2.1; the reader is referred there and to the original article.

A year later, Nobile took in exam the problem of determining the possible variation of the geometric genus of a projective curve varying in a family (see [102]). He gives a numerical characterization of the boundary points of Severi varieties of irreducible plane curves having degree n and δ nodes. We refer the reader to Theorem 2.2.25, where we have already stated his result.

In the same year, Treger ([133]) described the configuration of nodes of the nodal curves in $V_{n,\delta}$. In this paper, by using Harris result of the irreducibility of Severi varieties in the plane, he proves that the map

$$p_\delta : V_{n,\delta} \to Sym^\delta(\mathbb{P}^2),$$

which maps a nodal curve to the set of its nodes, is a birational morphism onto its image; more precisely, he proves:

Theorem. *(i) If* $n(n+3)/6 \leq \delta \leq (n-1)(n-2)/2$ *and* $(n,d) \neq (6,9)$*, then* $p_d : V_{n,\delta} \to Sym^{\delta}(\mathbb{P}^2)$ *is a birational morphism onto its image.*

(ii) If $\delta \leq min(n(n+3)/6, (n-1)(n-2)/2)$ *and* $(n,d) \neq (6,9)$ *then for a general element in* $Sym^{\delta}(\mathbb{P}^2)$*,* $(p_1, \ldots, p_{\delta})$*, there exists a curve in* $V_{n,\delta}$ *having nodes in* $(p_1, \ldots, p_{\delta})$*.*

A year later, he reproved the irreducibility of $\overline{V}_{n,g}$ via a different approach from the ones of Harris and Ran (see [134]). Moreover, in [135], he gives another proof of the theorem above without using Harris result but with the machinery of stratification of $Hilb^{\delta}(\mathbb{P}^2)$ with respect to the Hilbert function.

In [38], Diaz and Harris proved that $V_{n,\delta}$ are affine varieties, for each n and δ. So, for example, each family $\mathcal{F}$ of curves of degree n, which is parametrized by a projective curve, cannot contain only δ-nodal curves of the given degree; thus, if the general element of $\mathcal{F}$ belongs to $V_{n,\delta}$, this family must contain some curves with further and/or more complicated singularities.

In 1989, Ran was concerned with a range of enumerative problems in the plane ([109]). He developed a recursive procedure for counting the number of plane curves of degree d in $\mathbb{P}^2$, with δ nodes, passing through $\frac{d^2+2d-1}{2} - \delta$ general points. The recursion is based on suitable degenerations of the blow-up of $\mathbb{P}^2$ in a point, say S, to a surface $S_0 = S_1 \cup S_2$ (called *fan*), where $S_i \cong S$, $i = 1, 2$. Unfortunatly, his formula was not correct. He solved the problem in a later paper ([110]).

In the same year, Greuel and Karras ([50]) generalized the existence results of Wahl in the plane to other fixed type of analytic singularities. Their analysis makes use of some vanishing theorems they prove for arbitrary rank-one sheaves on reduced curves.

In [129] Shustin studies the asymptotical behaviour of the variety $V(n,\delta,\kappa)$ parametrizing irreducible plane curves of degree n with δ nodes and κ cusps as their only singularities. Hirzebruch ([66]) showed that $V(n,\delta,\kappa) = \emptyset$ if

$$\frac{9}{8}\delta + 2\kappa > \frac{5}{8}n^2;$$

Shustin proves that, on the other hand, $V(n,\delta,\kappa) \neq \emptyset$, when

$$\delta + 2\kappa \leq \frac{1}{2}n^2 + O(n).$$

Moreover,

(i) if $\delta + 2\kappa \leq \alpha_0 n^2$, where

$$\alpha_0 = \frac{7-\sqrt{13}}{81},$$

then $V(n,\delta,\kappa)$ is non-empty, non singular and regular of dimension $\frac{1}{2}n(n+3) - \delta - 2\kappa$;

(ii) if $\delta + 2\kappa \leq \alpha_1 n^2$, where

$$\alpha_1 = \frac{2}{225},$$

then $V(n, \delta, \kappa)$ is irreducible.

In 1995, Greuel and Lossen ([51]) generalized the results in [50], extended their analysis also to topological type of singularities and not only analytic type. Moreover, they defined the equisingular sheaf which takes care of such kind of singularities and proved some vanishing results, in terms of topological or analytic tools as Tjurina number, Milnor number, etc., in order to find sufficient conditions for the smoothness and the dimension problem of families of such curves in $\mathbb{P}^2$.

In [62], Harris and Pandharipande compute, with a new approach, the degree of $\overline{V}_{d,\delta}$, as a subvariety of $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(d))$, in cases $\delta = 1, 2$, and 3. The formulas they find are classical; what is new is the method, which involves the Bott residue formula. Another method to obtain formulas in higher cogenus, i.e. $\delta = 4$, 5 and 6, is the one of Vainsencher [136].

As we have already mentioned, interest and enthusiasm for these enumerative problems are revived by ideas from quantum field theory which lead to various enumerative predictions of rational curves on varieties. It was proposed by Gromov and Witten (see [140]) to study a new series of invariants on a variety V by using intersection theory on the moduli space $\overline{\mathcal{M}}_{g,n}$ (see Section 1.5 for notation). These invariants, called *Gromov-Witten invariants*, depend on the geometry of curves lying on V. For certain varieties (such as projective spaces) a subclass of these invariants corresponds to enumerative invariants. In the case of curves of genus 0, it is proven that the Gromov-Witten invariants satisfy a series of properties; the most important of them is the so called *splitting principle* or *composition law*, which gives a way of computing these invariants recursively. In [78], Kontsevich derived a formula for rational curves in the plane, assuming the associativity of the quantum product for $\mathbb{P}^2$ (not yet proven at that time). This allows one to compute degrees of all Severi varieties of rational curves in the plane.

Between 1994 and 1995, complete proofs of Kontsevich formulas are given indipendently by Kontsevich and Manin ([78] and [79]) and by Ruan and Tian ([115]). In both cases the goal is to give a rigorous definition of the Gromov-Witten invariants, so that they satisfy the necessary properties. In [115], this is done using symplectic topology and the Gromov theory of pseudo-holomorphic curves. In [79] the authors follow an algebro-geometric approach and use the existence of a good compactification of the moduli space of maps from $\mathbb{P}^1$ to V (constructed in the later paper [78]).

Finally, in 1996, Caporaso and Harris computed the degrees of Severi varieties in the plane of curves of any genus (see [15]). By studying the geometry of Severi varieties $\overline{V}_{d,\delta}$, for any d and any $g = \frac{(d-1)(d-2)}{2} - \delta$, they derive a recursive formula for their degrees. The approach is a purely algebro-geometric one. Indeed, for each $p \in \mathbb{P}^2$, they consider the hyperplane $H_p \subset \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(d)))$ of curves containing the point and then they study the geometry of intersections of the variety $\overline{V}_{d,\delta}$ with a succession of hyperplanes of the form H_{p_i}, where the points p_i are general points on a fixed line $L \subset \mathbb{P}^2$. At each stage they describe the irreducible components of the intersection and these all belong to specific collection of varieties, which are called *generalized Severi varieties*. They express the intersection of a generalized Severi variety with a hyperplane H_p corresponding to a general point $p \in L$ as a union of generalized Severi varieties of dimension one less. This is the key point in order to derive the recursive formula for their degrees. Their computations agree, in a certain sense, with a result of Getzler ([46]), who describes a generating function having the degrees of the generalized Severi varieties as coefficients.

Recently, Greuel, Lossen and Shustin ([54]) have derived new sufficient conditions in order that a variety V, parametrizing curves in $\mathbb{P}^2$, of degree d, having exactly r singular points of given topological or analytic singularities, is smooth of the expected dimension. The technical tool is to study Castelnuovo function of suitable zero-dimensional schemes associated to the r-tuples of singular points (*clusters*).

Among papers concerned with plane singular curves (not necessarily only nodal), we have to mention the papers of Gradolato and Mezzetti ([48]) and of Lindner ([82]), in which the cases of singular curves with nodes and cusps and also triple points are treated. The case of higher singularities are treated in [101]. There are also recent papers of Ciliberto and Miranda, [29] and [30], where they pose the problem of studying the dimension of the space of plane curves of degree d, having multiplicities m_i in n general points p_i. In some cases they compute the dimensions of such linear systems and they make a list of those systems which have unexpected dimension.

<u>Rational surfaces</u>:

For what concerns Severi varieties of nodal curves in smooth, rational surfaces, the first papers which must be recalled are those of Tannenbaum, [130] and [131] (1979-1980). In the first paper, he treats the problem of determining the possible geometric genera of curves lying on smooth rational surfaces; in the second, he generalizes Severi's proof in the plane to the case of smooth rational surfaces S in order to determine existence conditions of Severi varieties on S (see Corollary (2.14) in [131]).

In 1984, Nobile ([101]) studied families of reduced, singular curves on smooth rational surfaces, such that the singularities vary in an equisingular fashion. He generalizes the results of Zariski in the plane case ([145]) and of Tannenbaum on nodal curves on rational surfaces ([131]). Indeed, given a reduced curve C in a smooth surface S, he studies the functor of the first infinitesimal deformations of C, inside S, which preserves the singularities, in the sense of the theory developed by Wahl and Zarisky ([138] and [145]). His main result is the analogous of what Zarisky proves in the case of $S = \mathbb{P}^2$. Indeed, in the case where all the singularities are nodes, he shows that this functor is smooth and that its Zariski tangent space has dimension $h^0(\mathcal{N}_{C/S}) - \delta$; moreover, under slightly stronger conditions (see Theorem (3.2), [101]) the converse is also true, i.e. the equality sign forces the singularities to be nodes.

In more generality, Gradolato and Mezzetti ([49]) study the varieties of curves with ordinary singular points, even of high order, on regular (in particular rational) surfaces. They analyze the process of weakening singularities, in order to construct irreducible curves starting from reducible ones.

In 1995 Caporaso and Harris, ([17]), focused on enumeration problems of families of rational curves on smooth, rational surfaces. Given S a smooth, rational surface and D an effective divisor on it, they denote by $V(D) \subset\mid D\mid$ the closure of the locus of irreducible, rational curves in $\mid D \mid$. Let $r(D)$ be the dimension of $V(D)$; if D has non-negative self-intersection and if $V(D)$ is non-empty, then $r(D) = -K_S D - 1$ (see [77]). The particular aspect of the geometry of $V(D)$ is that it gives information on its degree, denoted by $N(D)$. This degree can be directly characterized as the number of irreducible, rational curves which are linearly equivalent to D and pass through $r(D)$ general points of S. They compute this $N(D)$ in some cases of rational surfaces via the *cross-ratio method*, which is based on the analysis of a one-parameter family of rational, irreducible curves in $\mid D \mid$ passing through $r(D) - 1$ general points of S. In some cases, as $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, blow-ups of $\mathbb{P}^2$ at general points and $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(n)$ (Hirzebruch surfaces), $n \leq 2$, this approach is succesful. For $\mathbb{F}_n$, $n \geq 3$, the situation is more complicated.

For what concerns smoothness and irreducibility problems in the case of rational surfaces, we must also recall a recent paper of Greuel, Lossen and Shustin. In [52], the case of nodal curves on the projective plane blown-up at r generic point is studied. More precisely, denoting by $\mathbb{P}^2_r$ such a blow-up, let $E_0, \ldots, E_r$ be the strict trasform of a generic line in $\mathbb{P}^2$ and the exceptional divisors of the blown-up points, respectively. Then, the authors give asymptotical sufficient conditions for the smoothness, irreducibility and non-emptyness of the variety $V_{irr}(d; d_1, d_2, \ldots, d_r; \delta)$ parametrizing irreducible curves $C \in \mid dE_0 - \sum_{i=1}^{r} d_i E_i \mid$ with δ nodes as the only singularities. Moreover, they extend their conditions for the smoothness and for the irreducibility to families of reducible curves on $\mathbb{P}^2_r$ and, for $r \leq 9$, they give a complete answer concerning the existence of nodal curves in $V_{irr}(d; d_1, d_2, \ldots, d_r; \delta)$.

Another paper of Caporaso and Harris, devoted to enumerative problems in the case of rational surfaces, is [16] where the authors introduced another method for such computations: *the rational fibration method*. This has been done to give answers to some cases which could not be treated in [17] with the cross-ratio method (mentioned above). In fact, in cases $\mathbb{F}_n$, $n \geq 3$, the number $N(D)$ deals with other terms of the degrees of some generalized Severi varieties parametrizing curves with a point of k-fold tangency with a fixed curve $E \subset S$. This method determines a computational technique which always involves an analysis of one-parameter families of rational curves passing through $r(D) - 1$ general points of S and their limits, but extracts more information from it. The main advantage is that it allows the authors to compute the degrees of the tangential loci involved in the cases they study: $\mathbb{P}^2$, $\mathbb{F}_2$ and $\mathbb{F}_3$.

Similar results, but with different methods, are obtained by Ran in [111] and [112]. In the first article, he uses Mori's bend-and-break technique to solve enumerative problem of rational curves, on a rational surface S, which are not assumed to form the family of "all" rational curves of a given homology class in S. It is required that they vary at least in a three-dimensional family. Mori's techniques is based on the fact that, once a rational curve 'bends' sufficiently (in a surface this means moving in a 3-parameter family) it will 'break', i.e. it admit a reducible limit. In [112], he uses the same considerations to obtain enumerative formulas for rational and elliptic curves with some nodes and cusps.

$K3$ surfaces:

The classification theory of (algebraic) surfaces shows that there are at most countably many rational curves on a $K3$ surface. Asking whether there are any rational curves at all on a general $K3$ surface has an affirmative answer for what Mori and Mukai have proven in [97]. This can be viewed as a first step for non-emptyness results of some Severi varieties on a $K3$ surface.

Tannenbaum generalized Severi's results to the case of a $K3$ surface ([132]). For what concerns reducible nodal curves on a $K3$-surface, he also showed that this case is completely different from what Nobile ([101]) proved in the case of a rational surface (see also in this section the part devoted to rational surfaces). Indeed, he proves that if S is a smooth $K3$-surface and if $[Y] \in | C |$ corresponds to a reduced, reducible curve with exactly δ nodes as singularities and with r irreducible components, then the closure of the scheme parametrizing such curves is smooth at $[Y]$ of dimension $dim(| C |) - \delta + (r-1)$. So the expected dimension is never reached at points corresponding to reducible curves, nevertheless, these points are smooth for such a closed scheme.

Yau and Zaslow, [142], compute the number of rational curves in $|\mathcal{O}_S(1)|$, when S is a general $K3$ surface in $\mathbb{P}^g$. Since their computations involve compactified Jacobians of the rational curves (see [11] for a detailed exposition), this Jacobians are not very well-understood if the singularities are other than nodes. Hence, the only gap left in this enumerative problem is the hypothesis that all rational curves in $| \mathcal{O}_S(1) |$ on a general $K3$ surface are nodal. In this way, it has been posed the following:

Conjecture: *For $g \geq 3$, all rational curves in the linear system $| \mathcal{O}_S(1) |$ on a general $K3$ surface in $\mathbb{P}^g$ are nodal.*

A partial affirmative answer to this conjecture is given by Chen in [22]. He uses techniques of degeneration applied to a general $K3$, degenerating it to a trigonal $K3$ surface, formed by the union of scrolls; such surface lies on the boundary of a complete family of genus g $K3$ surfaces. As a corollary of his main theorem, which we will state below, he proves that for $g \leq 9$ and $g = 11$ the conjecture is true and so he justifies Yau-Zaslow's computations, at least, for these cases. Moreover, the author extends the existence of irreducible rational curves to every complete linear system on a general $K3$ surface; indeed, he proves the following result.

Theorem. *([22]) For any integer $g \geq 3$ and $d > 0$, the linear system $|\mathcal{O}_S(d)|$, on a general $K3$ surface S in $\mathbb{P}^g$ contains an irreducible, rational and nodal* *curve.*

Even if it is already considered known, no complete proof of this result has appeared in the literature before the one of Chen. This implies also that Severi varieties on a general $K3$ of curves in $\mid \mathcal{O}_S(d) \mid$ are always non-empty. Moreover, since a rational nodal curve on a $K3$ surface is a regular component of the reducible Severi variety $V_{|\mathcal{O}_S(d)|,\, dim(|\mathcal{O}_S(d)|)}$, by using the i-smoothability property one has also the following fundamental result.

Theorem. *If $S \subset \mathbb{P}^g$ is a general $K3$ surface, $g \geq 3$, then, for each $d > 0$ and for each $0 \leq \delta \leq dim(\mid \mathcal{O}_S(d) \mid)$, the Severi variety $V_{|\mathcal{O}_S(d)|,\, \delta}$ contains regular irreducible components.*

In [23], the same author completely solved the above conjecture by showing that, for all $g \geq 3$, all rational curves in $\mid \mathcal{O}_S(1) \mid$, on a general primitive $K3$ surface S in $\mathbb{P}^g$, are nodal.

In [24], Chiantini and Ciliberto use Chen's method to study Severi varieties of nodal curves on a general quartic surface in $\mathbb{P}^3$. Put $N(4,n) := dim \mid \mathcal{O}_S(n) \mid$, where S is such a general quartic. Then, $N(4,n) = 2n^2+1 = p_a(nH)$. Since the behaviour of $K3$ surfaces, with respect to regularity of Severi varieties, is very similar to the one of the plane or of smooth, rational surfaces (see Theorem 3.1.1 and Remark 3.1.2), from Chen's result (for which $V_{|nH|, p_a(nH)} \neq \emptyset$) and from the regularity and the i-smoothability property, we get $V_{|nH|,\delta}$ regular and non-empty, for each $\delta \leq N(4,n)$. Moreover, if one looks at the universal Severi variety $\mathcal{V}_{n,\delta}$, Chen's result also implies that this universal variety, for $\delta = N(4,n)$, has a component which dominates the projective space $\mid \mathcal{O}_{\mathbf{P}^3}(4) \mid$. By degenerating a general quartic to a surface S_0 which is union of two quadrics, they prove the following result (see Theorem 2.2., [24]).

Theorem. *On a general quartic in $\mathbb{P}^3$, for all $\delta \leq N(4,n)$, there is an irreducible component of the Severi variety $V_{|nH|,\delta}$ such that, for C general in this component, $G(C,S) = Sym(4n)$, i.e. the monodromy group of the covering coincides with the full symmetric group.*

A part from the importance of this result by itself, it is fundamental for further analysis that Ciliberto and Chiantini do on Severi varieties of smooth, projective surfaces of degree $d \geq 5$ in $\mathbb{P}^3$ in order to prove reducibility of some Severi varieties of surfaces of general type. We shall give more details when we consider the part devoted to such surfaces.

We conclude by recalling a recent result of Knutsen, [73], where he studies necessary and sufficient conditions for the existence of pairs (S,C), where S is a $K3$ surface of degree $2n$ in $\mathbb{P}^{n+1}$ and C is a smooth (reducible or irreducible) curve of degree d and genus $g \geq 0$.

Surfaces of general type:

For what concerns surfaces of general type, in Section 2.1 we have already recalled Clemens result on the non-existence of curves of genus $g \leq \frac{1}{2}d(d-5)$ on a general surface of degree $d \geq 5$ in $\mathbb{P}^3$ (see Theorem 2.1.2).

Clemens argument was extended by Ein ([39], [40]) to the case of complete intersections in higher dimensional varieties. In fact, in general, if M

denotes a (smooth) projective n-fold and $X \subset M$ is a generic complete intersection of type $(m_1, \dots, m_k)$, then put $m = m_1 + \cdots + m_k$. Ein proves that if $m \geq 2n - k + 1$ (resp. $m \geq 2n - k$), then all subvarieties of X are of general type (resp. every subvariety of X is non-rational). If X is, for example, a complete intersection of type $(2, 2, 2)$ in $\mathbb{P}^5$, then X is a $K3$ surface and X contains a rational curve. So this result is sharp at least for complete intersection surfaces.

The above results imply the emptyness of some Severi varieties on complete intersection surfaces of general type.

In 1993, Chang and Ran, [20], consider generic hypersurfaces of degree at least 5 in $\mathbb{P}^3$ and $\mathbb{P}^4$ and reduced, irreducible but arbitrarily singular divisors upon them. They prove that such divisors cannot admit desingularizations having numerically effective anticanonical class. This gives, in some way, an assertion of numerical positivity of the canonical bundle on these desingularizations. This was the first step to prove the conjectures of Clemens and Harris, which stated that the generic quintic 3-fold in $\mathbb{P}^4$ cannot be birationally ruled and that the generic surface of degree $d \geq 5$ in $\mathbb{P}^3$ can contain neither rational nor elliptic curves.

We have already mentioned that, in 1994, Geng Xu ([143]) improved Clemens result on surfaces in $\mathbb{P}^3$ and completely proved Harris conjecture (see Theorem 2.1.3, in Section 2.1). Indeed, he shows that on a generic surface of degree $d \geq 5$ in $\mathbb{P}^3$, there is no curve with geometric genus $g \leq \frac{1}{2}d(d-3) - 3$ and that this sharp bound can be achieved only by 3-tangent hyperplane sections if $d \geq 6$.

In 1997 Chiantini and Sernesi ([27]) considered regularity properties for some Severi varieties on surfaces of general type; the methods they used involve Bogomolov's theory of unstable vector bundles on surfaces. Since this will be one of the main subject of the next chapter, we shall precisely mention their results in Section 3.1, before proving the main theorem of Chapter 3. Chiantini and Sernesi's approach was later generalized, in [53], to curves on surfaces of general type with arbitrary fixed analytic or topological singularities. Since, in the case of nodes, our result generalizes also Greuel-Lossen-Shustin's statement, we shall give details in Section 3.1.

Chiantini and Ciliberto, [24], mainly consider Severi varieties of surfaces of degree $d \geq 5$ in $\mathbb{P}^3$ by focusing on existence and dimensional problems. They give examples of superabundant components of Severi varieties $V_{|nH|,\delta}$ on surfaces of sufficiently high degree. One can easily determine first examples of regular components of some Severi varieties. Indeed, results in classical Enumerative Geometry ensure that there exists a finite number (bigger than one) of 3-tangent planes to a general smooth surface $S_d \subset \mathbb{P}^3$, $d = deg(S)$ (see, for example, [136]). So, given a general $S_d \subset \mathbb{P}^3$, each 3-nodal curve, which is a section of S_d by a 3-tangent plane, is an element of $V_{|H|,\,3}$ and also an irreducible, regular component of such a Severi variety. More generally, Chiantini and Ciliberto prove that, for any $d \geq 5$ and for any $0 \leq \delta \leq dim \mid D \mid$, there exist regular components of $V_{|nH|,\delta}$; the proof is by induction, and the first step is Chen's result for $d = 4$, a $K3$ surface.

Firstly, they use degeneration techniques to prove the existence of surfaces of degree d which split into a general surface of degree $d-1$ plus a general tangent plane of it (*suitable good surfaces*). Their next step consists in showing that on such reducible surfaces there exist limit nodal curves with good properties with respect to monodromy point of view (we have already mentioned to the aspect of monodromy, in the part devoted to $K3$ surfaces). By partially smoothing these curves they finally prove the following result, which generalizes the trivial case of 0-dimensional Severi variety $V_{|H|,3}$.

Theorem 2.3.3 *(see Theorem 3.1., [24]) Let S be a general surface in $\mathbb{P}^3$, of degree $d \geq 4$. For all $n \geq d$ and $\delta \leq N(d,n)$ there is an irreducible, regular component of $V_{|nH|,\delta}$.*

On the other hand, one can consider some examples of superabundant components of such Severi varieties. To this aim, put $N(d,n) = dim(| \mathcal{O}_S(n) |)$, where S is a smooth surface of degree $d \geq 5$ in $\mathbb{P}^3$. We can construct two different types of superabundant components of Severi varieties. The first occur when we consider, for example $d \geq 20$, $n = 3$ and $\delta \geq 20$. Since $N(d,3) = 19$, the expected dimension of $V_{|3H|,\delta}$ is negative, so we expect an empty Severi variety. However, if we denote by C the intersection of S with a general cone over a singular plane cubic and by L a line in the cone, then $C \sim dH$ on the cone so $CL = dHL = d \geq 20$. Since the generatrix of the cone passing through the node of the plane cubic is a double line, we have that $V_{|3H|,\delta}$ is non-empty. One can also give examples of superabundant components even in the range of values $\delta \leq N(d,n)$. Indeed, take a smooth surface of degree $d \geq 8$ and take n an even, positive integer such that $n >> d$. On a general plane, embedded in $\mathbb{P}^3$, by Severi's result (Theorem 2.1.5), there is a family of dimension $(nd-n-2)/2$ of irreducible nodal curves of degree n with

$$\alpha = \frac{n^2 - 5n + 2 - nd}{2}$$

nodes. Fixing a general point $p \in \mathbb{P}^3$, we have a family of cones with vertex at p over these nodal plane curves. Such cones have α double lines; intersecting these cones with S, we get a family of nodal curves, which is contained in $V_{|nH|,\delta}$, with $\delta = d\alpha$. Therefore, the dimension of $V_{|nH|,\delta}$ is, at least, $(nd-n-2)/2$, since two cones cannot intersect S in the same curve C because C and p uniquely determine the cone. If we compute the expected dimension of $V_{|nH|,\delta}$, we get

$$N(d,n) - d\alpha = \frac{d^3 - 12d^2 + 11d - 6}{6}.$$

This becomes strictly smaller that $(nd-n-2)/2$ as soon as $n < \frac{d(d-11)}{3}$.

Putting together Theorem 2.3.3 and the previous examples of superabundant components, one immediately deduces the following:

Proposition. *On a general surface in $\mathbb{P}^3$ of general type some Severi varieties are reducible.*

Apart from the trivial case of 0-dimensional Severi varieties, they conjectured that for S general surface of degree $d \geq 5$ in $\mathbb{P}^3$ and for all $0 \leq \delta \leq N(d,n)$, there is only one regular, irreducible component of the corresponding Severi variety.

In [25], Chiantini and Lopez translate Xu's local analysis approach with a global study of the *focal locus* of a family of curves to consider the problem of bounding geometric genera of curves on surfaces in $\mathbb{P}^3$, which are general elements in a given component of the Noether-Lefschetz locus of surfaces in $\mathbb{P}^3$. They prove the following result.

Theorem 2.3.4 *(see Theorem (1.3) in [25]) Let D be a reduced curve in $\mathbb{P}^3$ and s, d be two integers such that $d \geq s+4$. Moreover, suppose that:*

(i) there exists a surface $Y \subset \mathbb{P}^3$ of degree s which contains D;

(ii) the general element of the linear system $\mid \mathcal{O}_Y(dH-D) \mid$ is smooth and irreducible.

Denote by S a general surface of $\mathbb{P}^3$, of degree d, containing D. Thus, S does not contain reduced, irreducible curves $C \neq D$ of geometric genus $g < 1 + deg(C)\frac{(d-s-5)}{2}$.

With the same approach, they consider also surfaces in $\mathbb{P}^4$ which are projectively Cohen-Macaulay (see Theorem (1.4) in [25])

Enumerative problems are recently considered by Kleiman and Piene ([72]). They enumerate the singular curves in a complete linear system on a smooth projective surface S. The divisor must be suitably ample and the curves may have up to 8 nodes, or a triple point and up to 3 nodes. The curves must obviously pass through appropriately many general points on S. The number of curves is given by a universal polynomial in some basic Chern numbers. More precisely, limiting ourselves to the nodal case, consider a n-dimensional linear system $\mid D \mid$ on S, where D is an effective divisor on S. Assume that $V_{|D|,\delta}$ is non-empty. The goal is to compute how many δ-nodal curves pass through $n-\delta$ general points on S. At least for $\delta \leq 8$, they compute the number N_δ of such δ-nodal curves, in terms of a polynomial P_δ in the following Chern numbers:

$$d = D^2, \; k = DK_S, \; s = K_S^2, \; x = c_2(S).$$

Denote by $\mathcal{L} = \mathcal{O}_S(D)$ the line bundle associated to D and assume that $\mathcal{L} \cong \mathcal{M}^{\otimes m} \otimes \mathcal{N}$, where $\mathcal{M}$ is a very ample line bundle and $\mathcal{N}$ is globally generated. If $\delta \leq 8$ and if $m \geq 3\delta$, then

$$N_\delta = P_\delta(d,k,s,x)/(\delta!).$$

Such a formula coincides with classical results as, for example, the *Zeuthen-Segre formula*, $N_1 = 3d + 2k + x$, which gives the number of 1-nodal curves in a general pencil and extends some results of Vainsencher [136].

Finally, for what concerns families of curves on surfaces of general type, we have already recalled in Section 2.1 the paper of Bogomolov ([13]) and its generalizations by Lu and Miyaoka ([85] and [86]).

To conclude this overview, we only want to mention the papers of Chang and Ran, [21], and of Ballico and Chiantini, [6]. Both are concerned with nodal curves in projective spaces. The first consider families of nodal, reducible curves in $\mathbb{P}^r$, whose general element is a curve $C = C_1 \cup C_2$ which is the union of two smooth curves meeting transversely. They study the problem of relating the smoothability property of C with the ones of C_1 and C_2. This is a generalization of what Sernesi studied in [119]. On the other hand, Ballico and Chiantini show some non-emptyness results of Severi varieties of nodal curves in $\mathbb{P}^r$, $r > 2$, with fixed geometric genus. For $r = 3$, they also consider the variety $V_\delta(E)$, where E is a rank two vector bundle on $\mathbb{P}^3$, parametrizing sections of E whose zero-locus is nodal, with fixed geometric genus. They determine non-obstructedness results for these varieties.

Chapter 3

Regularity for Severi Varieties on Projective Surfaces

In this chapter we consider the problem of regularity of Severi varieties on a smooth, projective surface. In our analysis, we shall be mainly interested in smooth, projective surfaces of general type, which are the main subject of our thesis since, for such surfaces, less is proven than for other classes of surfaces (see Section 2.3).

We have already stated in Definition 2.2.28 that, for a given linear system $\mid D \mid$ on a smooth, projective surface S, whose general member is a smooth, irreducible curve, we denote by $V_{|D|,\delta}$ the locally closed subscheme of $\mid D \mid$ which parametrizes irreducible curves with only δ nodes as singularities and, with abuse of language, we use the word *Severi variety* to name such a locally closed subscheme.

Section 3.1 is devoted to the explanation of the technical tools which inspired us and that are used to prove the main theorem of the chapter; therefore, we will briefly recall the approach to the problem of regularity of Severi varieties on a surface S of general type, which was first considered by Chiantini and Sernesi ([27]). It is based on the study of a rank-two vector bundle on S, associated with the set of nodes of an irreducible, nodal curve X on S, which is numerically equivalent to pK_S on S, where $p \in \mathbb{Q}^+,\ p \geq 2$. They find an upper-bound on δ, ensuring that the Severi variety $V_{|D|,\,\delta}$ is smooth of codimension δ in $\mid D \mid$. This approach was slightly generalized in [53], by enlarging the class of considered divisors to some ample divisors on S.

In Section 3.2, we shall state and prove the main new result of the chapter, which gives purely numerical conditions on the class of divisors and upper-bounds on the number of nodes, ensuring that the corresponding Severi variety is regular (i.e. smooth of the expected dimension, as in Definition 2.2.30). Our result generalizes what is proven in [27] and [53] as it will be clear from the study of some examples of blown-up surfaces or surfaces which belong to a component of the Noether-Lefschetz locus of surfaces in $\mathbb{P}^3$ (see Definition 2.1.1).

To conclude the chapter, in Section 3.3 we shall also consider the problem to determine the sufficient conditions for the regularity of Severi varieties on surfaces of general type in $\mathbb{P}^3$ which contain a fixed divisor as, for example, a line. This is related to some results contained in [25], where the question of algebraic hyperbolicity for surfaces S in $\mathbb{P}^3$ and in $\mathbb{P}^4$ is treated.

3.1 Some known results on the regularity problem of Severi varieties on surfaces of general type

As we already stated in Section 2.1, in [125] - Anhang F, Severi studied some properties of the variety $\overline{V}_{d,\, g}$, defined as the closure of the locus $V_{d,\, g}$ consisting of reduced and irreducible plane curves, of geometric genus g, in the projective space parametrizing plane curves of degree d. $V_{d,\, g}$ contains, as an open dense subscheme, the locus $V_{d,\, \delta}$ parametrizing irreducible curves having only δ nodes as singularities. We can restate his result in the case of irreducible curves. He proved that, for every $d \geq 3$ and $0 \leq \delta \leq \binom{d-1}{2}$, $V_{|dL|,\, \delta}$ is non-empty and everywhere smooth of codimension δ in $\mid dL \mid$, where L denotes a line in $\mathbb{P}^2$.

In a modern language, a natural approach to dimension and regularity problems is to use deformation theory of nodal curves. Severi's classical result can be easily extended to rational or ruled surfaces, to $K3$ surfaces as well as to Enriques surfaces. We shall briefly recall the procedure for these cases since it contains the key point for which we cannot expect to extend the same procedure to smooth surfaces of general type.

Let S be a smooth, projective surface and let $|D|$ be a complete linear system on S whose general member is a smooth, irreducible curve. Let $p_a(D) = \frac{D(D+K_S)}{2} + 1$ be its genus. For $\delta \geq 1$, suppose that the Severi variety $V_{|D|,\, \delta}$ is non-empty. Let $[X] \in V_{|D|,\, \delta}$ and let N be the set of δ nodes of X. Thus, the geometric genus of X is $g = p_g(X) = p_a(X) - \delta$.

From (2.20) and (2.24) in Section 2.2, we know that the Zariski tangent space to $\mid D \mid$ at $[X]$ is isomorphic to

$$H^0(S, \mathcal{O}_S(D))/ < X >,$$

whereas the Zariski tangent space to $V_{|D|,\, \delta}$ at $[X]$ is

$$T_{[X]}(V_{|D|,\, \delta}) \cong H^0(S, \mathcal{I}_N(D))/ < X >,$$

where $\mathcal{I}_N \subset \mathcal{O}_S$ denotes the ideal sheaf of the 0-dimensional subscheme N of S. We observed that N imposes independent conditions to $\mid D \mid$ if and only if

$$dim(V_{|D|,\, \delta}) = dim\; T_{[X]}(V_{|D|,\, \delta}) = dim \mid D \mid -\delta$$

at $[X]$. In this case, $V_{|D|,\, \delta}$ is regular at $[X]$ (see Definition 2.2.30).

Recall that the regularity property is very strong since it implies that the nodes of X can be independently smoothed (see Remark 2.2.31).

We now consider, via modern techniques, the proof of Severi's theorem for irreducible, plane nodal curves (see, for example, [27] or [131]).

Theorem 3.1.1 *(Severi) Let $S = \mathbb{P}^2$, $d \geq 3$ and D be any irreducible divisor of degree d. Let $\delta \geq 1$ be such that $\delta \leq p_a(D) = (d-1)(d-2)/2$. Then the Severi variety $V_{|D|,\,\delta}$ is non-empty and smooth of pure dimension*

$$dim(|\, D\, |) - \delta = d(d+3)/2 - \delta.$$

Proof: Suppose that $[X] \in V_{|D|,\,\delta}$ and let N be the scheme of nodes of X. In view of the exact sequence

$$0 \to \mathcal{I}_N(d) \to \mathcal{O}_{\mathbb{P}^2}(d) \to \mathcal{O}_N(d) \to 0$$

and of the fact that $H^1(\mathbb{P}^2,\ \mathcal{O}_{\mathbb{P}^2}(d)) = 0$, to prove the regularity of $V_{|D|,\,\delta}$ at $[X]$ it is necessary and sufficient to prove that $h^1(\mathbb{P}^2,\ \mathcal{I}_N(d)) = 0$.

Let $\sigma := h^1(\mathbb{P}^2,\ \mathcal{I}_N(d))$. Since $h^0(N,\ \mathcal{O}_N(d)) = \delta$, from the above sequence we deduce that

$$h^0(\mathcal{I}_N(d)) = h^0(\mathcal{O}_{\mathbb{P}^2}(d)) - \delta + \sigma = \binom{d+2}{2} - \delta + \sigma.$$

Let $\varphi : C \to X \subset \mathbb{P}^2$ be the normalization map of X and let $\tilde{N}$ be the pullback of N to C, such that $deg(\tilde{N}) = 2\delta$. If $\mu : \tilde{S} \to \mathbb{P}^2$ denotes the blow-up of $\mathbb{P}^2$ in N and $B = \sum_{i=1}^{\delta} E_i$ the μ-exceptional divisor in $\tilde{S}$, $C = \mu^*(X) - 2B$ is irreducible in $\tilde{S}$. Tensoring the exact sequence defining C with $\mathcal{O}_{\tilde{S}}(\mu^*(X) - B)$ gives us

$$0 \to \mathcal{O}_{\tilde{S}}(B) \to \mathcal{O}_{\tilde{S}}(\mu^*(X) - B) \to \mathcal{O}_C(\mu^*(X) - B) \cong \varphi^*(\mathcal{O}(d))(-\tilde{N}) \to 0,$$

with $CB = 2\delta$. Since B is μ-exceptional, $h^0(\mathcal{O}_{\tilde{S}}(B)) = 1$. We therefore have an injective map

$$\frac{H^0(\mathcal{O}_{\tilde{S}}(\mu^*(X) - B))}{H^0(\mathcal{O}_{\tilde{S}}(B))} \cong \frac{H^0(\mathbb{P}^2,\ \mathcal{I}_N(d))}{< X >} \to H^0(C,\ \varphi^*(\mathcal{O}(d))(-\tilde{N})).$$

Since $\varphi^*(\mathcal{O}(d))(-\tilde{N})$ has degree $d^2 - 2\delta = 2g - 2 + 3d = deg(K_C) + 3d$, it is a non-special divisor on the smooth curve C. By Riemann-Roch on C, it follows that

$$\begin{aligned} h^0(S, \mathcal{I}_N(d)) - 1 \leq \ & h^0(C,\ \varphi^*(\mathcal{O}(d))(-\tilde{N})) & = d^2 - 2\delta + 1 - g \\ & = (d^2 + 3d)/2 - \delta & = \binom{d+2}{2} - \delta - 1. \end{aligned}$$

In conclusion $h^0(\mathcal{I}_N(d)) \leq \binom{d+2}{2} - \delta$, hence $\sigma = 0$.

To prove that $V_{|D|,\,\delta} \neq \emptyset$ for all $\delta \geq 1$ we start from the case $\delta = p_a(D)$, i.e. $g = 0$. The family $V_{|D|,\,p_a(D)}$ is not empty because it contains any general projection of a rational normal curve in $\mathbb{P}^d$. Let $[X] \in V_{|D|,\, p_a(D)}$ and let N denote the scheme of nodes of X. Fix $p \in N$ and denote by M the complement of $\{p\}$ in N. Since $h^1(\mathcal{I}_N(d)) = h^1(\mathcal{I}_M(d))$, we have

$$h^0(\mathcal{I}_M(d)) = h^0(\mathcal{I}_N(d)) + 1.$$

Any element of the vector space $H^0(\mathcal{I}_M(d))$ not in $H^0(\mathcal{I}_N(d))$ defines a first-order deformation of X which smooths the node p and leaves unsmoothed all the other nodes. This means that X belongs to the closure of $V_{|D|,\, p_a(D)-1}$; therefore $V_{|D|,\, p_a(D)-1} \neq \emptyset$. By descending induction on δ one proves that $V_{|D|,\, \delta} \neq \emptyset$, for all $1 \leq \delta \leq p_a(D)$. □

Remark 3.1.2 For what stated in Theorem 2.2.27, the regularity of Severi varieties of irreducible plane curves implies that such curves are *i*-smoothable in $\mathbb{P}^2$ (in the sense of Definition 2.2.22). We emphasize that the reason why the proof of Theorem 3.1.1 works is because

$$\varphi^*(\mathcal{O}(d))(-\tilde{N}) = \varphi^*(\mathcal{O}(d-3))(-\tilde{N}) \otimes \varphi^*\mathcal{O}(3) \cong \mathcal{O}_C(K_C) \otimes \varphi^*\mathcal{O}(3),$$

so it is a non-special line bundle on C. It is then clear that if we consider any smooth rational or ruled surface S and any smooth, irreducible curve X on S, such that $\mid D \mid$ is base point free (where D smooth and $D \sim X$ on S) and $K_S X < 0$, then the first part of the proof can be repeated for these cases with no change. This holds in particular for any Del Pezzo surface[1]. Therefore we get the following more general statement:

Theorem. *Let S be a rational or ruled surface and let $D \subset S$ be a smooth, irreducible curve such that $\mid D \mid$ is base point free and $DK_S < 0$. If for some $\delta \leq p_a(D)$ we have that $V_{|D|,\, \delta} \neq \emptyset$, then $V_{|D|,\, \delta}$ is smooth of codimension δ in $\mid D \mid$.*

Remark 3.1.3 Another important case is when S is a K3 surface (see, also, [132]) and D a smooth, irreducible curve such that $p_a(D) \geq 2$. Then $\mid D \mid$ is base point free (see [88]). Since $\mathcal{O}_S(K_S) \cong \mathcal{O}_S$, from Serre duality we get $h^1(\mathcal{O}_S(D)) = h^1(\mathcal{O}_S(-D))$; in addition $h^0(\mathcal{O}_S(-D)) = 0$ because $-D$ is not effective. Furthermore, since $h^1(\mathcal{O}_S) = 0$ and $h^0(\mathcal{O}_S) = h^0(\mathcal{O}_D)$, then $h^1(\mathcal{O}_S(-D)) = 0$; so that

$$h^0(\mathcal{O}_S(D)) = h^0(\mathcal{O}_D(D)) + 1.$$

From adjunction formula and the fact that S is a $K3$ surface, we get $\omega_D = \mathcal{O}_D(D)$, i.e. $h^0(\mathcal{O}_D(D)) = p_a(D) = p_g(D)$, which means that $dim(\mid D \mid) = p_a(D)$. In this case, for each $1 \leq \delta \leq p_a(D)$ and for any $[X] \in V_{|D|,\, \delta}$ we have

$$h^0(S,\, \mathcal{I}_N(X)) - 1 = p_a(X) - \delta + h^1(S,\, \mathcal{I}_N(X)) \leq h^0(C,\, \varphi^*(\mathcal{O}(X))(-\tilde{N}))$$

[1] For this result, we also refer the reader to [101] and [131].

$$= h^0(C,\ \mathcal{O}(K_C)) = p_a(X) - \delta.$$

It follows that $H^1(S,\ \mathcal{I}_N(X)) = (0)$ and therefore $V_{|D|,\ \delta}$ is smooth of codimension δ in $\mid D \mid$ at $[X]$. Note that, in particular, there are only finitely many nodal rational curves in $\mid D \mid$.

As recalled in Section 2.3, existence problems in the case of a $K3$ surface are treated, for example, in [22] or in [97], where conditions which imply $V_{|D|,\ \delta} \neq \emptyset$ are studied.

Remark 3.1.4 The last case we want to treat is the one with S an Enriques surface; this means $h^1(\mathcal{O}_S) = h^2(\mathcal{O}_S) = 0$ and $\omega_S^{\otimes 2} = \mathcal{O}_S$. Let D be a smooth, irreducible curve on S. From the standard exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(D) \to \mathcal{O}_D(D) \to 0,$$

we get $dim(\mid D \mid) = h^0(\mathcal{O}_D(D))$. From adjunction formula, $4p_g(D) - 4 = 2D(D + K_S) = 2D^2$; thus $deg(\mathcal{O}_D(D)) = deg(K_D)$. On the other hand $\omega_D = \mathcal{O}_D(D + K_S)$ is not isomorphic to $\mathcal{O}_D(D)$. Therefore, from Riemann-Roch and Serre duality on D, $h^0(\mathcal{O}_D(D)) = p_g(D) - 1 + h^0(\omega_D \otimes \mathcal{O}_D(-D)) = p_g(D) - 1$, since $\omega_D \otimes \mathcal{O}_D(-D)$ is a line bundle in $Pic^0(D)$ which is not isomorphic to the trivial bundle. We deduce that $dim \mid D \mid = p_g(D) - 1$, so we shall consider $1 \leq \delta \leq p_g(D) - 1$. Now, suppose that $V_{|D|,\ \delta}$ is not empty and let $[X] \in V_{|D|,\ \delta}$. By using the same procedure of Theorem 3.1.1 and Serre duality on C, we obtain

$$h^1(C,\ \varphi^*(\mathcal{O}(X))(-\tilde{N})) = h^0(C, \mathcal{O}_C(\varphi^*(K_S))).$$

Since $\mathcal{O}_C(\varphi^*(K_S))$ is a non-trivial line bundle in $Pic^0(C)$, $h^0(\mathcal{O}_C(\varphi^*(K_S))) = 0$ so that $\mid \varphi^*(\mathcal{O}(X))(-\tilde{N}) \mid$ is a non-special linear system on C. Therefore, $h^0(C, \varphi^*(\mathcal{O}_X(X))(-\tilde{N})) = X^2 - 2\delta - p_g(C) + 1 = X^2 - \delta - p_a(X) + 1$, so $h^0(S,\ \mathcal{I}_N(X)) \leq X^2 - \delta - p_a(X) + 2$. Since $h^0(S,\ \mathcal{I}_N(X)) = p_a(X) - \delta + h^1(S,\ \mathcal{I}_N(X))$, it follows that $h^1(S,\ \mathcal{I}_N(X)) \leq X^2 - 2p_a(X) + 2 = X^2 - (X^2 + XK_S) = -XK_S$. In conclusion, if D is a smooth, irreducible divisor such that $DK_S = XK_S \geq 0$, then $V_{|D|,\ \delta}$ is regular.

If S is a smooth surface of general type, we cannot expect that Theorem 3.1.1 extends without changes to linear systems on S. Indeed, consider for example a very ample divisor D on a smooth, minimal surface S of general type. Thus $K_SD > 0$, since D is very ample and K_S is big and nef (see Proposition 1.3.2 and Remark 1.3.3). We get that $D^2 < deg(K_D)$, which means that the characteristic linear system $|\mathcal{O}_D(D)|$ is special. Since $K_SD > 0$ on S, then $h^0(\mathcal{O}_D(D)) = h^1(\omega_D \otimes \mathcal{O}_D(-D)) < g = p_a(D) = h^1(\mathcal{O}_D)$. Thus, from the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(D) \to \mathcal{O}_D(D) \to 0,$$

we deduce that $dim(\mid D \mid) \leq p_a(D) - 1$; therefore $V_{|D|,\ p_a(D)}$, for example, cannot have the expected codimension and we should in fact expect that

$V_{|D|,\, p_a(D)} = \emptyset$ (except the fact that there could exist superabundant components). The next step is to understand for which values of δ, $V_{|D|,\,\delta}$ is smooth of the expected dimension (having supposed its non-emptyness).

Information about regularity of a Severi variety $V_{|D|,\,\delta}$ on a smooth surface S of general type can be obtained by studying suitable rank-two vector bundles on S. The first who used such approach were Chiantini and Sernesi.

Remark 3.1.5 Let N be a set of δ points in S. If N does not impose independent conditions to a linear system $\mid D \mid$ on S, then, from the exact sequence

$$0 \to \mathcal{I}_N(D) \to \mathcal{O}_S(D) \to \mathcal{O}_N(D) \to 0,$$

we have that the restriction map $H^0(\mathcal{O}_S(D)) \to H^0(\mathcal{O}_N(D))$ is not surjective. Let $N_0 \subset N$ be a minimal subset for which the composition $H^0(\mathcal{O}_S(D)) \to H^0(\mathcal{O}_N(D)) \to H^0(\mathcal{O}_{N_0}(D))$ does not surject; this means that $H^1(S,\, \mathcal{I}_{N_0}(D)) \neq 0$ and that N_0 satisfies the *Cayley-Bacharach condition* (see [56] and [80]). From Serre duality, we have

$$H^1(S,\, \mathcal{I}_{N_0}(D)) \cong \mathrm{Ext}^{2-1}(\mathcal{O}_S(D)\otimes\mathcal{I}_{N_0}, \omega_S)^\vee \cong \mathrm{Ext}^1(\mathcal{O}_S(D-K_S)\otimes\mathcal{I}_{N_0}, \mathcal{O}_S)^\vee.$$

Thus, a non-zero element of $H^1(\mathcal{I}_{N_0}(D))$ corresponds to a non-trivial rank-two vector bundle fitting in the following exact sequence

$$0 \to \mathcal{O}_S \to E \to \mathcal{I}_{N_0}(D - K_S) \to 0. \qquad (3.1)$$

For what stated in Proposition 1.2.3, we get

$$\begin{aligned} c_1(E) &= c_1(\mathcal{O}_S) + c_1(\mathcal{O}_S(D - K_S)) = D - K_S, \\ c_2(E) &= c_1(\mathcal{O}_S)c_1(\mathcal{O}_S(D - K_S)) + deg(N_0) = deg(N_0) := \delta_0, \end{aligned}$$

with $\delta_0 > 0$ since N_0 cannot be empty.

Before stating the main result of Chiantini and Sernesi, we recall that $NS(S)$ denotes the Neron-Severi group of S whereas the symbol $\equiv$ denotes the numerical equivalence of divisors on S (see Definitions 1.1.20 and 1.1.25).

Theorem 3.1.6 *(see Theorem 2.2. in [27]) Let S be a smooth surface with K_S an ample divisor and let C be an irreducible curve on S such that $\mid C \mid$ contains smooth elements and such that*

$$C \equiv pK_S, \quad p \geq 2,\ p \in \mathbb{Q}^+.$$

Suppose that C has $\delta \geq 1$ nodes and no other singularities and assume either

$$\delta < \frac{p(p-2)}{4}K_S^2$$

or

$$\delta < \frac{(p-1)^2}{4}K_S^2 \ \ p \in \mathbb{Z} \textit{ odd and } NS(S) \cong \mathbb{Z}[K_S].$$

Then the nodes of C impose independent conditions to $\mid C \mid$. In particular the Severi variety $V_{|C|,\,\delta}$ is smooth of codimension δ at $[C]$.

Proof: For details, the reader is referred to the original paper. □

This general result gives, for example, information on regularity of Severi varieties on smooth surfaces in $\mathbb{P}^3$ of degree $d \geq 5$, i.e. surfaces of general type. Indeed, since $K_S = (d-4)H$, where H is a plane section of S, then K_S is very ample; so the previous result applies to any nodal curve $C \sim nH$, for $n \geq 2(d-4)$, getting:

Proposition 3.1.7 *(see Proposition 2.4. in [27]) Let S be a smooth surface of degree $d \geq 5$ in $\mathbb{P}^3$ with plane section H. If $C \sim nH$, $n \geq 2d-8$, has δ nodes and no other singularities, and if $\delta < nd(n-2d+8)/4$, then C corresponds to a regular point of a component of the Severi variety $V_{|nH|,\,\delta}$ (in the sense of Definition 2.2.28).*

When S is a general quintic surface in $\mathbb{P}^3$ and $C \sim pK_S = pH$, p an odd integer, Theorem 3.1.6 gives:

Proposition 3.1.8 *(see Proposition 2.5. in [27]) Let S be a smooth surface of degree 5 in $\mathbb{P}^3$ with plane section H and Picard group isomorphic to $\mathbb{Z}$. If $C \sim pH$ ($p \geq 3$ odd) has δ nodes and no other singularities and $\delta < \frac{5(p-1)^2}{4}$, then the Severi variety $V_{|pH|,\,\delta}$ is regular at $[C]$.*

Remark 3.1.9 One can apply Bogomolov-unstable vector bundle procedure also to $K3$ or rational surfaces and get estimates on δ which imply that $V_{|C|,\,\delta}$ is smooth of the expected dimension. However, for these surfaces one obtains statements which are weaker than Theorem 3.1.1 or Remarks 3.1.2 and 3.1.3. On the other hand, Chiantini and Sernesi provided examples which show that their numerical bounds on δ are sharp at least for a general quintic surface in $\mathbb{P}^3$. We shall see, in Section 4.3, that our result (Theorem 3.2.3), which generalizes Theorem 3.1.6, is sharp for a larger class of projective surfaces: at least for general "canonical" complete intersections in $\mathbb{P}^r$.

An improvement of Theorem 3.1.6 is given in [53]. The authors generalized Chiantini and Sernesi's approach in two directions. In fact, they allowed arbitrary singularities and they weakened the assumption of K_S being ample, so that $S = \mathbb{P}^2$ is included. They consider V as the variety of irreducible curves, with fixed (topological or analytic) singularity types, on a smooth projective surface S. They give sufficient conditions for the smoothness of V and, in the case of simple singularities, they prove that these conditions are optimal with respect to the asymptotical behaviour. Before stating their result, we have to recall some of their notation.

Let D be a non-singular curve on a smooth, projective surface S and denote by $V = V(S_1, \ldots, S_r)$ the variety of irreducible (reduced) curves $C \in | D |$ having exactly r singularities of topological (or analytic) types $S_1, \ldots, S_r$. V has the $T-$*property* at $[C] \in V$ if the conditions imposed by the individual singularities of C are independent, i.e. V is smooth at $[C]$ with the expected codimension.

The authors denote by

$$I^{ea}(C,\ z) = (f,\ f_x,\ f_y)\mathcal{O}_{S,z}$$

the *Tjurina ideal* of $(C,\ z)$, where f is a local equation for the germ (C, z), and

$$I^{es}(C,\ z) := \{g \in \mathcal{O}_{S,z} \mid\ f + \epsilon g \text{ equisingular over } Spec(\mathbb{C}[\epsilon]/(\epsilon^2))\},$$

which is the *equisingularity ideal* of $(C,\ z)$ (see [50] and [139]). The schemes $X^{es}(C)$ and $X^{ea}(C)$, concentrated at the singular points $z_1,\ \ldots,\ z_r$ of C, are defined by the ideal $I^{es}(C,\ z)$ and $I^{ea}(C,\ z)$, respectively. The corresponding ideal sheaves on S are denoted by $\mathcal{I}_{X^{es}(C)}$ and $\mathcal{I}_{X^{ea}(C)}$. The topological case is the "es"-case, whereas the "ea"-case reflects the analytic singularities. Since they treat simultaneously the analytic and the topological case, for brevity they denote by $X'(C)$ both cases.

For a singular type S_i, $\tau'(S_i)$ denotes the codimension of the corresponding equisingularity (resp. Tjurina) ideal, whereas $k(S_i)$ is used for the intersection multiplicity of type S_i with a corresponding generic polar, that is $k(S_i) = \mu(S_i) + mt(S_i) - 1$, where μ denotes the *Milnor number* of S_i (see [8], pag. 62) and mt its multiplicity. Since we are interested only in the case of ordinary double points, we do not go deeper into detail and refer the reader to the original article. We only want to point out that, in the case we are interested in, each singular point z_i is an ordinary double point, so $S_i = S_j$ for $1 \leq i \neq j \leq r = \delta$; moreover, if p is a node, $\tau'(p) = 1$ and $k(p) = 1 + 2 - 1 = 2$. In [50] and [51] it is proven that the variety $V(S_1,\ \ldots,\ S_r)$ has the T-property at $[C]$ if and only if

$$h^1(S,\ \mathcal{I}_{X^{es}(C)}(C)) = 0 \quad (\textit{resp. } h^1(S,\ \mathcal{I}_{X^{ea}(C)}(C)) = 0.$$

Theorem 3.1.10 *(see Theorem 1 in [53]) Let S be a smooth projective surface and let $C \subset S$ be an irreducible curve with precisely r singularities at $z_1,\ \ldots,\ z_r$ of topological (resp. analytical) types $S_1,\ \ldots,\ S_r$, such that:*

1. *C is ample;*
2. *$C - K_S$ is ample;*
3. *$C^2 \geq K_S^2$.*

If

(i) $\sum_{i=1}^r \tau'(S_i) < \frac{(C-K_S)^2}{4},\ \ \sum_{i=1}^r k(S_i) < \frac{C(C-K_S)}{2}$;

(ii) $\frac{(\sum_{i=1}^r(\tau'(S_i)+1))^2}{C^2} < (\sum_{i=1}^r(1 - \frac{CK_S}{C^2}(\tau'(S_i) + 1))) - \frac{(CK_S)^2 - C^2K_S^2}{4C^2}$;

(iii) $\frac{(\sum_{i=1}^r k(S_i))^2}{C^2} < (\sum_{i=1}^r(k(S_i)(1 - \frac{CK_S}{C^2}) - \tau'(S_i))) - \frac{(CK_S)^2 - C^2K_S^2}{4C^2}$.

Then, $V(S_1, \ldots, S_r)$ has the T-property at $[C]$, i.e. it is smooth of codimension $\sum_{i=1}^{r} \tau'(S_i)$ at $[C]$.

Proof: The methods of the proof are based on the Chiantini and Sernesi idea, involving Bogomolov's theory of unstable vector bundles on surfaces, on local analysis of singularities and on suitable H^1-vanishing criteria. For details, the reader is referred to the original paper. $\square$

We only want to remark that, in the case of $r = \delta$ nodes, then formulas (i), (ii) and (iii) above become

(i) $\delta < \frac{(C-K_S)^2}{4}$, $2\delta < \frac{C(C-K_S)}{2}$, i.e. $\delta < min(\frac{(C-K_S)^2}{4}, \frac{C(C-K_S)}{4})$;

(ii) $= (iii)$ $16\delta^2 - 4\delta C(C - 2K_S) + ((CK_S)^2 - C^2K_S^2) < 0$.

In the case of nodes, condition $(ii) = (iii)$ is fundamental to the aim of finding a positive real upper-bound for δ; thus, there is an implicit request that the discriminant of the inequality is positive. Since such a discriminant is $16C^2(C-2K_S)^2$, these numerical hypotheses imply also that $(C-2K_S)^2 > 0$, otherwise condition $(ii) = (iii)$ would not hold.

As corollaries of the previous results, the authors find suitable sufficient conditions to deduce that $V(S_1, \ldots, S_r)$ has the T-property when S is: $\mathbb{P}^2$, a surface with trivial canonical divisor (e.g. a $K3$-surface, an abelian surface), a smooth surface of degree $d \geq 5$ in $\mathbb{P}^3$. For details, we refer the reader to the original paper.

3.2 The main theorem

In the previous section we have recalled what is already proven with respect to the problem of finding sufficient conditions which imply the regularity of a Severi variety of nodal curves on S, expecially in the case of S a smooth surface of general type. Before stating and proving our main result, we briefly recall and make further necessary definitions.

In Definition 1.1.28 we stated that, if S is a smooth, projective surface, C is a nef divisor on S if $CF \geq 0$, for each effective divisor F on S; by Kleiman's criterion (see Remark 1.1.31), this corresponds to the fact that C is in the closure of the ample divisor cone of S.

Definition 3.2.1 Let S be a smooth, projective surface and $C \in Div(S)$. We shall denote by $H(C, K_S)$ the *Hodge number* of C and S, defined by

$$H(C, K_S) := (CK_S)^2 - C^2K_S^2.$$

The Algebraic index theorem (Theorem 1.1.24) ensures us that this number is non-negative when C (or K_S) is a nef divisor. Indeed, by Kleiman's criterion, if C is nef then $C^2 \geq 0$. Thus,

(a) if $C^2 = 0$, then $H(C, K_S) = (CK_S)^2 \geq 0$;

(b) if $K_S \sim 0$, then one trivially has $H(C, K_S) = 0$;

(c) if $C^2 > 0$ and K_S a non-zero divisor then, either $CK_S = 0$, so $K_S^2 < 0$ (by the Algebraic index theorem), or $CK_S \neq 0$, then $H(C, K_S) \geq 0$.

Since it will be used in the proof of our main theorem, we recall the following simple result.

Lemma 3.2.2 *Let S be a smooth surface and let C be an effective, reduced and irreducible divisor on S. Assume that $C^2 > 0$. Then C is a nef divisor.*

Proof: Suppose there exists a divisor D on S such that $CD < 0$. Since C is irreducible, the only possibility is that C is a component of D. Let $\alpha \in \mathbb{N}$ be the multiplicity of C in D, so that

$$D = D' + \alpha C,$$

with D' an effective divisor which does not contain C. Then, $0 > CD = \alpha C^2 + D'C$. Since $\alpha > 0$ and $D'C \geq 0$, this would imply $C^2 < 0$. Thus, for each effective divisor D, $CD \geq 0$, i.e. C is nef. $\square$

We are now able to prove our result.

Theorem 3.2.3 *Let S be a smooth, projective surface and let C be a smooth, irreducible divisor on S. Suppose that:*

1. $(C - 2K_S)^2 > 0$ *and* $C(C - 2K_S) > 0$;
2. *either*
$$(i) \quad K_S^2 > -4 \quad if \quad C(C - 2K_S) \geq 8,$$
or
$$(ii) \quad K_S^2 \geq 0 \quad if \quad 0 < C(C - 2K_S) < 8.$$
3. $CK_S \geq 0$;
4. $H(C, K_S) < 4(C(C-2K_S)-4)$, *where $H(C, K_S)$ is the Hodge number of C and S (Definition 3.2.1);*
5. *either*
$$(i) \quad \delta \leq \tfrac{C(C-2K_S)}{4} - 1 \quad if \quad C(C - 2K_S) \geq 8,$$
or
$$(ii) \quad \delta < \tfrac{C(C-2K_S)+\sqrt{C^2(C-2K_S)^2}}{8} \quad if \quad 0 < C(C - 2K_S) < 8.$$

Then, if $[C'] \in |\, C \,|$ parametrizes a reduced, irreducible curve with only δ nodes as singular points and if N denotes the 0-dimensional scheme of nodes in C', in the above hypotheses N imposes independent conditions to $|\, C \,|$, i.e. the Severi variety $V_{|C|,\,\delta}$ is smooth of codimension δ (i.e. regular) at $[C']$.

Proof: For simplicity we will write K, instead of K_S, to denote a canonical divisor of S. By contradiction, assume that N does not impose independent conditions to $\mid C \mid$. Let $N_0 \subset N$ be a minimal 0-dimensional subscheme of N for which this property holds and let $\delta_0 =\mid N_0 \mid$. As already mentioned in Remark 3.1.5, this means that $H^1(S,\ \mathcal{I}_{N_0}(C)) \neq (0)$ and that N_0 satisfies the Cayley-Bacharach condition. Therefore, a non-zero element of $H^1(\mathcal{I}_{N_0}(C))$ corresponds to a non-trivial rank-two vector bundle $E \in \mathrm{Ext}^1(\mathcal{I}_{N_0}(C - K), \mathcal{O}_S)$; so, one can consider the obvious exact sequence

$$0 \to \mathcal{O}_S \to E \to \mathcal{I}_{N_0}(C-K) \to 0. \tag{3.2}$$

This implies that

$$c_1(E) = C - K,\ c_2(E) = \delta_0 \leq \delta,$$

so

$$c_1(E)^2 - 4c_2(E) = (C-K)^2 - 4\delta_0. \tag{3.3}$$

Observe also that, from hypotheses 1. and 3., it immediately follows that $C(C-K) > 0$ and $C^2 > 0$. Indeed, $C(C-K) = C(C-2K) + CK > 0$ and $C^2 > CK \geq 0$. Since C is irreducible, from Lemma 3.2.2, then C is a nef divisor.

We now want to compute (3.3) in cases 5.(i) and 5.(ii).
In the first one,

$$(C-K)^2 - 4\delta_0 \geq (C-K)^2 - 4\delta = C^2 - 2CK - 4 + 4 + K^2 - 4\delta \geq K^2 + 4 > 0,$$

by 2(i).
In the other case, using 5.(ii) and the Index Theorem,

$$(C-K)^2 - 4\delta_0 \geq (C-K)^2 - 4\delta = C^2 - 2CK + K^2 - 4\delta > K^2 \geq 0,$$

since we supposed 2(ii).

In both cases, the vector bundle E is Bogomolov-unstable (see Theorem 1.2.13), so there exist $M,\ B \in Div(S)$ and a 0-dimensional scheme Z (possibly empty) such that

$$0 \to \mathcal{O}_S(M) \to E \to \mathcal{I}_Z(B) \to 0 \tag{3.4}$$

holds and $(M-B)$ in the ample divisor cone of S, $N(S)^+$. This means that

$$\begin{aligned} (M-B)^2 &> 0, \\ (M-B)H &> 0, \quad \forall\, H \text{ ample divisor.} \end{aligned} \tag{3.5}$$

The exact sequence (3.4) ensures us that $H^0(E(-M)) \neq 0$. If we consider the tensor product of the exact sequence (3.2) by $\mathcal{O}_S(-M)$, we get

$$0 \to \mathcal{O}_S(-M) \to E(-M) \to \mathcal{I}_{N_0}(C-K-M) \to 0. \tag{3.6}$$

We state that $H^0(\mathcal{O}_S(-M)) = 0$; otherwise, $-M$ would be an effective divisor, therefore $-MH > 0$ for each ample divisor H. But, from (3.4), it follows that $c_1(E) = M + B$, so, by (3.2) and (3.4),

$$M - B = 2M - C + K \in N(S)^+. \tag{3.7}$$

Thus, for every ample divisor H,

$$MH > \frac{(C-K)H}{2}; \tag{3.8}$$

from (3.8) and from Kleiman's criterion, we get

$$MC \geq \frac{(C-K)C}{2}. \tag{3.9}$$

It follows that $-MC < 0$ so, since C is nef, $-M$ cannot be effective.

If we consider the cohomological exact sequence associated to (3.6), we deduce that there exists a divisor Δ in $\mid C - K - M \mid$ such that $N_0 \subset \Delta$. If the irreducible nodal curve $C' \sim C$, whose set of nodes is N, were component of Δ, then $-M - K$ would be an effective divisor. By applying (3.9) and by using the fact that $C(C-K) > 0$ and hypothesis 3., one determines

$$C'(-M-K) = C(-M-K) = -CK - CM \leq -CK - \frac{(C-K)C}{2} =$$

$$= -\frac{(C+K)C}{2} = -\frac{KC}{2} - \frac{C^2}{2} < -CK \leq 0,$$

which contradicts the effectiveness of $-M-K$, since C is nef.

Bezout's theorem implies that

$$C'\Delta = C'(C - K - M) \geq 2\delta_0. \tag{3.10}$$

On the other hand, taking M maximal, we may further assume that the general section of $E(-M)$ vanishes in a 2-codimensional locus Z of S. Thus, $c_2(E(-M)) = deg(Z) \geq 0$. From computations in Proposition 1.2.2, we obtain

$$c_2(E(-M)) = c_2(E) + M^2 + c_1(E)(-M) = \delta_0 + M^2 - M(C-K),$$

which implies

$$\delta_0 \geq M(C - K - M). \tag{3.11}$$

By applying the Algebraic index theorem to the divisor pair $(C,\ 2M - C + K)$, we get

$$C^2(2M - C + K)^2 \leq (C(C-K) - 2C(C-K-M))^2. \tag{3.12}$$

From (3.10) and from the fact that $C(C-K)$ is positive, it follows that

$$C(C-K) - 2C(C-K-M) \leq C(C-K) - 4\delta_0. \tag{3.13}$$

We observe that the left side member of (3.13) is non-negative, since $C(C-K)-2C(C-K-M)=C(2M-C+K)$, where C is effective and, by (3.7), $2M-C+K\in N(S)^+$. Thus, (3.13) still holds when we square both its members and, together with (3.12), this gives

$$C^2(2M-C+K)^2\le (C(C-K)-4\delta_0)^2. \tag{3.14}$$

On the other hand, by using (3.11), we get

$$(2M-C+K)^2=4(M-\frac{(C-K)}{2})^2=$$
$$=(C-K)^2-4(C-K-M)M\ge (C-K)^2-4\delta_0,$$

i.e.

$$(2M-C+K)^2\ge (C-K)^2-4\delta_0. \tag{3.15}$$

Putting together (3.14) and (3.15), we get

$$F(\delta_0):=16\delta_0^2-4C(C-2K)\delta_0+(CK)^2-C^2K^2\ge 0. \tag{3.16}$$

Summarizing, the assumption on N, stated at the beginning, implies (3.16)[2]. We want to show that our numerical hypotheses hold if and only if the opposite inequality is satisfied. To this aim, observe that the discriminant of the equation $F(\delta_0)=0$ is $16C^2(C-2K)^2$, so, by hypotheses 1. and 3., it is positive. The inequality $F(\delta_0)<0$ is verified iff $\delta_0\in(\alpha(C,K),\ \beta(C,\ K))$, where

$$\alpha(C,K)=\frac{C(C-2K)-\sqrt{C^2(C-2K)^2}}{8}\in \mathbb{R}\ and$$
$$\beta(C,K)=\frac{C(C-2K)+\sqrt{C^2(C-2K)^2}}{8}\in \mathbb{R}\ ;$$

we have to show that, with our numerical hypotheses, $\delta_0\in(\alpha(C,K),\ \beta(C,\ K))$.

From 5., it immediately follows that $\delta_0<\beta(C,K)$, since, as we shall see in the sequel, the bound in 5.(i) is smaller than $\beta(C,K)$. Note also that $\alpha(C,K)\ge 0$. Indeed, if $\alpha(C,K)<0$, then $C(C-2K)<\sqrt{C^2(C-2K)^2}$, which contradicts the Algebraic index theorem, since $C(C-2K)>0$.

Observe that $\alpha(C,K)<1$ if and only if

$$C(C-2K)-8<\sqrt{C^2(C-2K)^2} \tag{3.17}$$

To simplify the notation, we put $t=C(C-2K)$ so that (3.17) becomes

$$t-8<\sqrt{t^2-4H(C,K)}. \tag{3.18}$$

Two cases can occur.

[2]We remark that, in the case of nodes, this condition is the same of [53]; moreover, their hypotheses (ii) and (iii) (see Theorem 3.1.10) coincide in the case of nodes and become $F(\delta_0)<0$.

If $t - 8 < 0$, there is nothing to prove since the right side member of (3.18) is always positive.

Note, before proceeding to consider the other case, that in this situation we want that $\beta(C, K) > 1$ in order to have at least a positive integral value for the number of nodes; but $\beta(C, K) > 1$ if and only if $(0 <)8 - t < \sqrt{t^2 - 4H(C, K)}$. By squaring both members of the previous inequality, we get $4H(C, K) < 16t - 64$, which is our hypothesis 4.; so the upper-bound for δ is surely greater than 1. Moreover, the expression for such bound is the one in 5.(ii) and it can not be written in a better non-trivial form.

On the other hand, if $t - 8 \geq 0$, by squaring both members of (3.18), we get $H(C, K) < 4(C(C - 2K) - 4)$, which is our hypothesis 4.. Therefore, $\alpha(C, K) < 1$; moreover, the condition $\beta(C, K) > 1$ is trivially satisfied, since it is equivalent to $t - 8 > -\sqrt{C^2(C - 2K)^2}$. From (3.17), we can write

$$\frac{C(C - 2K) + C(C - 2K) - 8}{8} < \frac{C(C - 2K) + \sqrt{C^2(C - 2K)^2}}{8},$$

so we can replace the bound $\delta < \beta(C, K)$ with the more "accessible" one $\delta \leq \frac{C(C-2K)}{4} - 1$, which is the bound in 5.(i).

Observe that

$$\frac{C(C - 2K) + C(C - 2K) - 8}{8} < \frac{C(C - 2K) + \sqrt{C^2(C - 2K)^2}}{8} \leq \frac{C(C - 2K)}{4},$$

so, it is not correct to directly write $\delta < \frac{C(C-2K)}{4}$. Therefore, 5.$(i)$ is the right approximation.

In conclusion, our numerical hypotheses contradict (3.16), therefore the assumption $h^1(\mathcal{I}_N(C)) \neq 0$ leads to a contradiction. □

Remark 3.2.4 (1) The previous theorem gives purely numerical conditions to deduce some information about Severi varieties of smooth projective surfaces. In the next section, we shall discuss a class of interesting examples of projective surfaces to which our theorem easily applies. Indeed, we will consider smooth surfaces in $\mathbb{P}^3$ which are elements of a component of the Noether-Lefschetz locus; more precisely, surfaces of general type, of degree $d \geq 5$, which contain a line.

(2) Our result obviously generalizes the one of Chiantini and Sernesi. In their case, since $C \equiv pK_S$, $p \in \mathbb{Q}$ and $p \geq 2$, we always have $\alpha(C, K_S) = 0$ and $\beta(C, K_S) = \frac{p(p-2)}{4}K_S^2$; this depends on the fact that $H(pK_S, K_S) = 0$, for every p. We recall that, with the further hypotheses that $p \in \mathbb{Z}^+$, p odd, and that the Neron-Severi group of S is $NS(S) \cong \mathbb{Z}[K_S]$, they proved that one can take $\delta < \frac{(p-1)^2}{4}K_S^2$. These bounds are sharp, at least for a general quintic surface in $\mathbb{P}^3$.

Furthermore, as recalled in Remark 1.2 in [27], in the case of rational or ruled surfaces (for which $CK_S < 0$) or $K3$ surfaces (for which $CK_S =$

0) if $\mid C \mid$ is base-point-free the argument for $S = \mathbb{P}^2$ can be repeated without changes, since the normal bundle to the normalization map $\varphi : \tilde{C}' \to C' \subset S$, $\mathcal{N}_\varphi$, is non-special on $\tilde{C}'$. Observe that, such a line bundle equals $\varphi^*(\mathcal{O}_{C'}(C'))(-\tilde{N})$, where $\tilde{N}$ is the pullback of the set of nodes N to $\tilde{C}'$. Our result focuses on cases in which $CK_S \geq 0$ (see hypothesis 3.), where the previous approach fails.

(3) One can immediately deduce that when the Hodge number is zero, i.e. when we are considering a divisor pair such that $(CK)^2 = C^2K^2$ then, in the previous proof, we find $\alpha(C,K) = 0$ and $\beta(C,K) = \frac{C(C-2K)}{4}$. So, for example, we obtain once again what stated for the cases of $K3$ surfaces or abelian surfaces (see [27] and [53]).

(4) Theorem 3.2.3 also generalizes, in the case of nodes, the result in [53] (i.e. Theorem 3.1.10). This will be clear after having considered the following examples.

Examples:

1) Let $S \subset \mathbb{P}^3$ be a smooth general quartic. We have, $\mathcal{O}_S(K_S) \cong \mathcal{O}_S$. Let H denote the plane section of S and let D be a general element of $\mid 2H \mid$. From Bertini's theorems, it follows that D is smooth and irreducible. If $\pi : \tilde{S} \to S$ denotes the blow-up of S at a smooth point $p \in S$ and E the associated π-exceptional divisor, then $K_{\tilde{S}} \sim E$, i.e. the canonical divisor of the blown-up surface is linearly equivalent to the exceptional divisor. Thus, the results in [27] cannot be applied, since K_S is not ample. Moreover, also $C \sim 2\pi^*(H)$ is not ample, since $CK_{\tilde{S}} = 0$; so, the first hypothesis in 3.1.10 does not hold.

Nevertheless, observe that the generic element of $\mid C \mid$ is smooth and irreducible. Moreover, $C - 2K_{\tilde{S}} \sim 2\pi^*(H) - 2E$ so that $(C - 2K_{\tilde{S}})^2 = 16 - 4 = 12$, $C(C - 2K_{\tilde{S}}) = 16$, $CK_{\tilde{S}} = 0$, $K^2_{\tilde{S}} = -1 > -4$, $H(C, K_{\tilde{S}}) = 0 - 16(-1) = 16$ and $4(C(C - 2K_{\tilde{S}}) - 4) = 4(12) = 48$. Since we are in situation 5.(i), i.e. $C(C - 2K_{\tilde{S}}) \geq 8$, we get $\delta \leq \frac{16}{4} - 1 = 3$. Therefore, if $V_{|2\pi^*(H)|,\delta} \neq \emptyset$ on $\tilde{S}$ and if $\delta \leq 3$, then it is everywhere regular.

2) Let S be a smooth quintic surface in $\mathbb{P}^3$ which contains a line L. Denote by $\Gamma \subset S$ a plane quartic which is coplanar to L, so that $\Gamma \sim H - L$ (H denotes the plane section of S). Thus,

$$H^2 = 5,\ HL = 1,\ L^2 = -3,\ H\Gamma = 4,\ \Gamma^2 = 0 \text{ and } \Gamma L = 4.$$

Choose $C \sim 3H + L$, so that $\mid C \mid$ contains curves which are residue to Γ in the complete intersection of S with the smooth quartic surfaces in $\mathbb{P}^3$ containing Γ. $\mid 3H + L \mid$ is base-point-free and not composed with a pencil. Indeed, if we consider the minimal linear series $\Lambda = \mid 3H \mid +L$, such that $\Lambda \subset \mid 3H + L \mid$, each element of Λ has L as a fixed divisor. This means that the possibly fixed locus of $\mid 3H + L \mid$ should be contained in L, but $(3H + L)L = 0$. Moreover, if $\mid 3H + L \mid$ had been composed with a pencil, then each curve $C \sim 3H + L$ would be. This cannot happen, since $3H$ is an ample divisor. Once again, by Bertini's theorems, its general member is smooth and irreducible; but C and $C - K_S$ cannot be both either ample or, even, nef divisors. In fact, $CL = 3 - 3 = 0$ and $(C - K_S)L = (2H + L)L = 2 - 3 = -1$. Therefore, the result in 3.1.10 cannot be applied.

Neverthless, $CK_S = C(C - 2K_S) = H(C, K_S) = 16$, $(C - 2K_S)^2 = 4$, $K_S^2 = 5$, $4(C(C-2K_S)-4) = 48$ and, since $C(C-2K_S) > 8$, $\delta \leq \frac{16}{4} - 1 = 3$. Thus, if $\mid 3H + L \mid$ contains some nodal, irreducible curves, then, if $\delta \leq 3$, $V_{|3H+L|,\delta}$ is everywhere regular.

3.3 Regularity results for Severi varieties on smooth surfaces in $\mathbb{P}^3$, of degree $d \geq 5$, which contain a line

We now consider a class of examples to which our result easily applies. We shall focus on surfaces of $\mathbb{P}^3$ containing a line. Such approach can be generalized to surfaces belonging to other components of the Noether-Lefschetz locus.

To start with, let $S \subset \mathbb{P}^3$ be a smooth quintic and $L \subset S$ be a fixed line. Since $p_a(L) = p_g(L) = 0$, by the adjunction formula and by the fact that $K_S \sim H$ we get $L^2 = -3$. As before,

$$K_S^2 = 5, \; LH = 1, \; L^2 = -3.$$

Examples: (1) Consider $C \sim 7H + 2L$ on S, so C is not numerically equivalent to a rational multiple of K_S. Moreover, we immediately find that $C - K_S$ cannot be ample. Indeed, $C - K_S \sim 7H + 2L$ so $(C - K_S)L = 7LH + 2L^2 = 0$, with L an effective divisor. This contradicts the Nakai-Moishezon criterion for the ampleness of an effective divisor (see Theorem 1.1.27). Once again, Theorems 3.1.6 and 3.1.10 cannot be applied.

By simple computations, we get $(C - 2K_S)^2 = 133$, $C(C - 2K_S) = 187$, $K_S^2 = 5$, $CK_S = 37$, $H(C, K_S) = 1369 - 1305 = 64$ and $4(C(C-2K_S)-4) = 732$. We have to prove that the general element in $\mid 7H + 2L \mid$ is smooth and irreducible. This directly follows from Bertini's theorems:

For the smoothness, we have to show that $\mid 7H+2L \mid$ is base point free. It is sufficient to prove that there are no base points on L. To this aim, consider the exact sequence

$$0 \to \mathcal{O}_S(7H + L) \to \mathcal{O}_S(7H + 2L) \to \mathcal{O}_L(7H + 2L) \cong \mathcal{O}_{\mathbf{P}^1}(1) \to 0.$$

$\mid \mathcal{O}_{\mathbf{P}^1}(1) \mid$ is base point free, since $deg(\mathcal{O}_{\mathbf{P}^1}(1)) = 1 > 0$. It is sufficient to show that the restriction map

$$H^0(\mathcal{O}_S(7H + 2L)) \to H^0(\mathcal{O}_{\mathbf{P}^1}(1))$$

is surjective, equivalently that $H^1(\mathcal{O}_S(7H + L)) = 0$. Thus, take the exact sequence

$$0 \to \mathcal{O}_S(7H - L) \to \mathcal{O}_S(7H) \to \mathcal{O}_L(7H) \cong \mathcal{O}_{\mathbf{P}^1}(7) \to 0$$

and consider its tensor product with $\mathcal{O}_S(L)$. We get:

$$\cdots \to H^1(\mathcal{O}_S(7H)) \to H^1(\mathcal{O}_S(7H+L)) \to H^1(\mathcal{O}_{\mathbf{P}^1}(4)) \to \cdots.$$

We immediately conclude by observing that $H^1(\mathcal{O}_S(7H)) = 0$, since S is a smooth surface in $\mathbb{P}^3$, and $H^1(\mathcal{O}_{\mathbf{P}^1}(4)) = 0$, by Serre duality and by the fact that $\omega_{\mathbf{P}^1} \cong \mathcal{O}_{\mathbf{P}^1}(-2)$.

For the irreducibility, we have only to show that $\mid 7H+2L \mid$ is not composed with a pencil. This immediately follows from the fact that $(7H+2L)L = 1$, i.e. each element in $\mid 7H+2L \mid$ has only one point on L, and that $deg(7H+2L) = 37$.

In conclusion, if $\delta \leq 45$ and if $V_{|7H+2L|,\,\delta} \neq \emptyset$, then $V_{|7H+2L|,\,\delta}$ is everywhere regular.

(2) The previous example deals with a particular equivalence class of divisor on a smooth quintic in $\mathbb{P}^3$. We are now interested in some more general results of regularity for Severi varieties of curves, on a smooth quintic $S \subset \mathbb{P}^3$, which are residue to the line L in the complete intersection of S with a general surface of degree a passing through the line. Thus $C \sim aH - L$ on S. By straightforward computations, we get

$$deg(C) = (aH - L)H = 5a - 1,$$

$$p_a(C) = \frac{5a^2 + 3a}{2} - 1.$$

We want to find conditions on a in order to apply our main result.

(i) $\mid C \mid$ has a smooth and irreducible general member.

For the smoothness, we have to prove that $\mid aH - L \mid$ is base point free and not composed with a pencil. Since $a \geq 1$, $aH - L = (a-1)H + H - L$. If $a \geq 2$, the linear system $\mid (a-1)H \mid$ can not have fixed intersection points on L. We can restrict ourselves to consider the behaviour of $\mid H - L \mid$ on L. If $\mid H - L \mid$ admitted fixed points on L, each of those points should be a tangent point between S and the general plane of $\mathbb{P}^3$ passing through the line. This would imply that S is a singular surface in such points, which contradict the hypothesis. Moreover, $\mid H - L \mid$ can not be composed with a pencil, since $\mid H - L \mid + L \subset \mid H \mid$.

For the irreducibility, we can use the fact that C and L are directly geometrically linked in $\mathbb{P}^3$, i.e. $C \bowtie_g L$ as in Definition 1.1.12. For what proven in Propositions 1.1.11 and 1.1.15, this implies that C is projectively normal in $\mathbb{P}^3$, so $H^1(\mathcal{I}_{C/\mathbf{P}^3}(\rho)) = 0$ for each $\rho \in \mathbb{Z}$. By choosing $\rho = 0$, from the exact sequence

$$0 \to \mathcal{I}_{C/\mathbf{P}^3} \to \mathcal{O}_{\mathbf{P}^3} \to \mathcal{O}_C \to 0,$$

we get $H^0(\mathcal{I}_{C/\mathbf{P}^3}) = H^1(\mathcal{I}_{C/\mathbf{P}^3}) = 0$ so $H^0(\mathcal{O}_C) \cong H^0(\mathcal{O}_{\mathbf{P}^3})$. This proves that C is a connected curve; since we have already proven its smoothness, then the general member is also irreducible.

(ii) Numerical hypotheses.

As before, $K_S^2 = H^2 = 5$. Then, $(C - 2K_S)^2 = 5a^2 - 22a + 21 > 0$ if $a \leq 1$ and $a \geq 3$. On the other hand, $C(C - 2K_S) = 5a^2 - 12a - 1$, which is positive if $a \geq 3$. $CK_S = 5a - 1$, which is positive if $a \geq 1$. The last condition to verify is that

$$(*) \;\; H(C, K_S) < 4(C(C - 2K_S) - 4).$$

On the one hand, we have $H(C, K_S) = 16$; on the other one, $C(C - 2K_S) - 4 = 5a^2 - 12a - 5$. Therefore, condition $(*)$ becomes $20a^2 - 48a - 36 > 0$, which is satisfied if $a > 3$. Thus, one observes that all the numerical conditions in Theorem 3.2.3 simultaneously hold if $a \geq 4$. Moreover, $C(C - 2K_S) = 5a^2 - 12a - 1 \geq 8$ if and only if $5a^2 - 12a - 9 \geq 0$ which is satisfied when $a \geq 3$. This means that, in this case, if we look for an upper-bound for the number of nodes of an irreducible curve in $\mid aH - L \mid$, $a \geq 4$, ensuring that the corresponding point is a regular point of $V_{|aH-L|,\delta}$, we have to use the formula 5.(i) in Theorem 3.2.3.

Before generalizing example (2) to surfaces in $\mathbb{P}^3$, of degree $d \geq 6$, which contain a line, we want to observe what happens in the case $a = 1$, where our sufficient numerical conditions do not hold. Thus, if $a = 1$, we are considering a pencil of planes through the chosen line L; since a general element in $H - L$ is a plane quartic on S, then $p_a(H - L) = 3$. A priori, we could impose at most 3 nodes to the linear system $\mid H - L \mid$. Since $\mid H - L \mid$ is a pencil and since a single node obviously imposes one independent condition to the pencil, then $V_{|H-L|,\,1}$ is smooth and 0-dimensional, as expected. If $\delta = 2,\ 3$, a dimensional computation gives that $V_{|H-L|;\,\delta}$ should be empty, but, a priori, there could be superabundant components. Since S is a smooth surface of general type with $Pic(S) \cong \mathbb{Z}^n$, $n \geq 2$, it is not projectively ruled (we recall that a projectively ruled surface must contain an infinite number of lines). As a consequence of a result of Kronecher and Castelnuovo (see [32], pag. 270), the tangent plane to a smooth point of S is not tangent to S in any other point, so the Severi varieties for $\delta = 2,\ 3$ are empty as expected.

As we have already mentioned above, we can completely generalize our numerical procedure in Example (2) to the case of a smooth surface of degree $d \geq 6$ which contains a line L. Let $S \subset \mathbb{P}^3$ be such a surface and $C \sim aH - L$, so that

$$deg(C) = ad - 1,$$
$$p_a(C) = \frac{ad(a + d) - 2a - d(4a + 1) + 3}{2}.$$

Moreover, $L^2 = 2 - d$, since $K_S \sim (d - 4)H$ and $LH = 1$.

For the smoothness and the irreducibility of the general member of $\mid C \mid$ one can use the same argument for $d = 5$.

Now, $K_S^2 = (d - 4)^2 d \geq 24$, since $d \geq 6$, and $C - 2K_S = (a - 2d + 8)H - L$.

Numerical hypotheses.

1. $(C - 2K_S)^2 = (a - 2d + 8)^2 d - 2(a - 2d + 8) + 2 - d = a^2 d - 2a(2d^2 - 8d + 1) + 4d^3 - 32d^2 + 67d - 14 > 0$ when $a > 2d - 7$ and $a < 2d - 9 - \frac{2}{d}$; thus we will consider $a \geq 2d - 6$ and $a \leq 2d - 9$, since $0 < \frac{2}{d} \leq \frac{1}{3}$.

2. $C(C-2K_S) = a^2d-2a(d^2-4d+1)+d-6 > 0$; the associated equation has solutions given by

$$a = d-4+\frac{1}{d} \pm \sqrt{(d-4+\frac{1}{d})^2-1+\frac{6}{d}}.$$

We have to consider two different cases:

(i) $-1+\frac{6}{d} < 0$, which implies $d > 6$. In these cases the inequality gives us

$$a < d-4+\frac{1}{d} - \sqrt{(d-4+\frac{1}{d})^2-1+\frac{6}{d}} \in \mathbb{R}^+$$

and

$$a > d-4+\frac{1}{d} + \sqrt{(d-4+\frac{1}{d})^2-1+\frac{6}{d}} \in \mathbb{R}^+,$$

since $-1 < -1+\frac{6}{d} < 0$. Therefore, the first of these two inequalities determines an upper-bound for a which is a real number in the open interval $(0\ ,\ 1)$. The second one can be approximated by $a \geq 2d-8+\frac{2}{d}$, since $-1+\frac{6}{d} < 0$. From point 1., we already have that $a \geq 2d-6$; so with this bound both these first conditions are satisfied in cases $d > 6$.

(ii) $-1+\frac{6}{d} \geq 0$, which gives $d \leq 6$. Since we are considering the cases $d \geq 6$, this happens only when $d = 6$. Thus, the inequality gives $a < 0$ or $a > 4+\frac{1}{3} = 2d-8+\frac{2}{d}$. As before, we conclude by using the fact that in the previous item we have $a \geq 2d-6 = 6$.

3. $CK_S = (d-4)(ad-1) \geq 0$, which means $a \geq \frac{1}{d}$, since $d-4 \geq 2$.

4. $(CK_S)^2 = (d-4)^2(a^2d^2-2ad+1)$, $C^2K_S^2 = (d-4)^2(a^2d^2-2ad+2d-d^2)$, so $H(C, K_S) = (d-4)^2(d-1)^2$. On the other hand, $4(C(C-2K_S)-4) = 4a^2d-8a(d^2-4d+1)+4d-40$. Therefore, $H(C, K_S) < 4(C(C-2K_S)-4)$ if and only if $4a^2d-8a(d^2-4d+1)-(d^4-10d^3+33d^2-44d+56) > 0$. The associated equation gives us

$$a = \frac{1}{2d}(2(d^2-4d+1) \pm \sqrt{d^5-6d^4+d^3+28d^2+4}).$$

For simplicity, put

$$\Delta := d^3-6d^2+d+28+\frac{24}{d}+\frac{4}{d^2},$$

so that the previous equality becomes

$$a = (d-4+\frac{1}{d}) \pm \frac{1}{2}\sqrt{\Delta}.$$

Since we want positive bounds on a, the inequality is satisfied when

$$a > (d-4+\frac{1}{d}) + \frac{1}{2}\sqrt{\Delta};$$

we have to control if this upper-bound is bigger then $2d-6$. This happens if and only if

$$\sqrt{\Delta} > 2d-4-\frac{2}{d},$$

i.e. $d^3-10d^2+17d+20+\frac{8}{d} > 0$. Dividing by d^2, this happens if

$$(*)\ \ d > 10-(\frac{17}{d}+\frac{20}{d^2}+\frac{8}{d^3}).$$

Since $d \geq 6$, condition $(*)$ is satisfied only if $d = 6,\ 7$.

To summarize, for $6 \leq d \leq 7$, all the numerical hypotheses in Theorem 3.2.3 simultaneously hold if $a \geq 2d - 6$ (note that, for $d = 5$ we obtained $a \geq 4$ so that $d = 5,\ 6,\ 7$ behave in the same way). On the other hand, for $d \geq 8$, the condition on the Hodge number (i.e. hypothesis 4. in Theorem 3.2.3) determines a bound on a which is bigger than the one determined by the other conditions, i.e. $2d - 6$. Indeed, condition 4. holds if and only if

$$4a^2d - 8a(d^2 - 4d + 1) - (d^4 - 10d^3 + 33d^2 - 44d + 56) > 0.$$

This gives

$$a > d - 4 + \frac{1}{d} + \frac{1}{2}\sqrt{d^3 - 6d^2 + d + 28 + \frac{24}{d} + \frac{4}{d^2}}. \tag{3.19}$$

For what we have computed up to now, the right side member of (3.19) is bigger than $2d - 6$ when $d \geq 8$. Therefore, in this case, all the numerical conditions of Theorem 3.2.3 simultaneously hold if (3.19) holds.

In order to find a better expression for such a lower-bound on a, we first observe that, since $d \geq 8$,

$$\sqrt{d^3 - 6d^2 + d + 28 + \frac{24}{d} + \frac{4}{d^2}} \leq \sqrt{d^3 - 6d^2 + d + 28 + \frac{24}{8} + \frac{4}{64}} =$$

$$= \sqrt{d^3 - 6d^2 + d + 31 + \frac{1}{16}} < \sqrt{d^3 - 6d^2 + d + 32}.$$

We are looking for a real number b such that $\sqrt{d^3 - 6d^2 + d + 32} \leq \sqrt{(d\sqrt{d} - b)^2}$. For such a value, we have

$$2b\sqrt{d} \leq 6d - 1 + \frac{b^2 - 32}{d}. \tag{3.20}$$

Moreover, (3.19) becomes

$$a \geq d - 3 + \frac{d}{2}\sqrt{d} - \frac{b}{2}. \tag{3.21}$$

Obviously, the right side member of (3.21) must be greater than $2d - 6$ for $d \geq 8$. Observe that this happens if and only if

$$d\sqrt{d} > 2d + b - 6. \tag{3.22}$$

Therefore, putting $\varphi(d) := d\sqrt{d} - 2d + 6$, from (3.22) we have $b < \varphi(d)$. The function $\varphi(d)$ is monotone increasing for $d \geq 2$ so, to find a uniform bound on b for all $d \geq 8$, it is sufficient to consider $b < \varphi(8)$, i.e. $b \leq 12$. By taking into account (3.20), we find that in all cases a good choice is $b = 9$. Indeed, for $d = 8,\ 9$ and $b = 9$, (3.20) trivially holds; for $d \geq 10$ and $b = 9$, (3.20) becomes

$$(**)\ \ 18\sqrt{d} \leq 6d - 1 + \frac{49}{d},$$

$0 < \frac{49}{d} < 5$. Since for $d \geq 10$, we have that $18\sqrt{d} \leq 6d - 1$, a fortiori $(**)$ is satisfied. So $b = 9$ is a uniform choice for all cases $d \geq 8$. Thus, (3.19) can be replaced by $a \geq d - 3 + \frac{d\sqrt{d} - 9}{2}$.

Analogous computations show that, when $d \geq 5$, only condition 2.(i) can occur, i.e. $C(C - 2K_S) \geq 8$. Therefore the expression for the bound on the number of nodes is the one in 5.(i), which is

$$\delta \leq \frac{a^2d - 2a(d^2 - 4d + 1) + 2d - 15}{4}.$$

We have therefore proven the following:

Proposition 3.3.1 *Let S be a smooth surface in $\mathbb{P}^3$ of degree $d \geq 5$, which contains a line L. Consider on S the linear system $\mid aH - L \mid$, with*

1. $a \geq 2d - 6$, *if* $5 \leq d \leq 7$;
2. $a \geq \lceil d - 3 + \frac{d\sqrt{d}-9}{2} \rceil$, *if* $d \geq 8$.

(We denote by $\lceil x \rceil$ the round-up *of the real number x, i.e. the smallest integer which is bigger than or equal to x). Suppose, also, that the Severi variety $V_{|aH-L|,\delta}$ is not empty. Then, if*

$$\delta \leq \frac{a^2d - 2a(d^2 - 4d + 1) + 2d - 15}{4},$$

the Severi variety is everywhere regular.

Remark 3.3.2 We want to point out that the previous proposition, in a certain sense, agrees with what is proven by Chiantini and Lopez in [25] (see Theorem 2.3.4 in Chapter 2). Recall, in fact, that if D is a reduced curve in $\mathbb{P}^3$, s and d integers, such that $d \geq s + 4$, and if one assumes that:

(i) there exists a surface $Y \subset \mathbb{P}^3$ of degree s which contains D, and

(ii) the general element of the linear system $\mid \mathcal{O}_Y(dH - D) \mid$ is smooth and irreducible,

then they prove that $S \subset \mathbb{P}^3$, general smooth surface of degree d containing D, does not contain reduced, irreducible curves $C \neq D$ of geometric genus $g < 1 + deg(C)\frac{(d-s-5)}{2}$.

In the case of our proposition, S is a surface of degree $d \geq 5$ and $D = L$, such that $L^2 = 2 - d$. Thus, we can consider $s = 1$, i.e. Y is a plane containing the line L and $\mid \mathcal{O}_Y(dH - L) \mid = \mid \mathcal{O}_{\mathbb{P}^2}(d-1) \mid$ which has a smooth and irreducible general element. Therefore, if there exists a curve C of a given degree, then

$$p_g(C) \geq 2 + \frac{(d-6)}{2}deg(C) = 2 + \frac{(d-6)}{2}CH.$$

If, moreover, C is a nodal curve, then

$$\delta = p_a(C) - p_g(C) \leq \frac{C^2 + CK_S}{2} + 1 - 2 - \frac{(d-6)}{2}CH =$$

$$\frac{C^2}{2} + \frac{(d-4)}{2}CH - \frac{(d-6)}{2}CH - 1 = \frac{C^2}{2} + CH - 1.$$

On the other hand, since in such cases, when all our numerical hypotheses hold, we have $C(C - 2K_S) \geq 8$, then $5(i)$ determines

$$\delta \leq \frac{C(C - 2dH + 8H)}{4} - 1 = \frac{C^2}{4} - \frac{(d-4)}{2}CH - 1.$$

Observe that $\frac{C^2}{4} - \frac{(d-4)}{2}CH - 1 \leq \frac{C^2}{2} + CH - 1$ if and only if $\frac{C^2}{4} + CH(\frac{d}{2} - 1) \geq 0$. Since $d \geq 5$ and since C is big and nef (consequence of condition 1. and 3. in Theorem 3.2.3), this latter inequality is always strictly verified. This means that our bounds on δ are in the range of values, for the number of nodes, that are necessary for the existence of such a curve.

Chapter 4

Geometric Linear Normality for Nodal Curves on Smooth Projective Surfaces

In this chapter, we generalize some results of [27] on the geometric linear normality property of nodal curves on smooth surfaces in $\mathbb{P}^r$. As it will be clear from the main result of this chapter, this is a problem which is strictly related to the regularity of some Severi varieties.

In Section 4.1, we recall some terminology and notation which are useful for our analysis. Section 4.2 contains our main result, which determines numerical conditions establishing when a nodal curve on a smooth projective surface is *geometrically linearly normal* (roughly speaking, when the pull-back of the hyperplane linear system to the normalization of such a curve is complete). More precisely, for a nodal curve C on a smooth, projective, non-degenerate and linearly normal surface S we have proven that there exists a sharp upper-bound $\delta = \delta(C, S)$ such that C is geometrically linearly normal if the number of its nodes is less than δ.

Section 4.3 is devoted to interesting examples; in particular, we treat an example of a quintic surface which belong to a component of the Noether-Lefschetz locus of surfaces in $\mathbb{P}^3$ to which our numerical criterion easily applies. To show the sharpness of our bound, smooth "canonical" complete intersection surfaces are considered. In these cases, we also determine examples of obstructed curves of some Severi varieties which, however, we prove that are generically smooth of the expected dimension.

4.1 Notation and preliminaries

As stated in Section 1.1, projective, non-singular complete intersection varieties are linearly normal, i.e. they are not isomorphic projection of non-degenerate varieties in higher dimensional projective spaces. From the cohomological point of view, a projective variety $X \subset \mathbb{P}^r$ is linearly normal if and only if $h^1(X, \mathcal{I}_X(H)) = 0$, i.e. the linear system $\mid \mathcal{O}_X(H) \mid$ cut out on X by the hyperplanes in $\mathbb{P}^r$ is complete (see Remark 1.1.10). This definition makes sense even if X is singular, but sometimes this is no longer true. In fact, if $X = C$ is a singular curve, there are singular complete intersection curves $C \subset \mathbb{P}^n$ whose normalization $\tilde{C} \to C \subset S$ factors through a birational, non-degenerate map $\tilde{C} \to \mathbb{P}^r$, for some $r > n \geq 3$ (see [27] and Section 4.3).

On the other hand, when the geometric genus of C is close enough to the arithmetic genus of C, this factorization is impossible. Indeed, let C be, for example, a plane curve of a given degree d and let $\varphi : \tilde{C} \to C \subset \mathbb{P}^2$ be its normalization, which is supposed to be contained in $\mathbb{P}^3$. If we suppose that $p_g(C) = g(\tilde{C}) = \tilde{g}$ and $p_a(C)$ are such that $p_a(C) - p_g(C) = \epsilon$ small enough, i.e. if the number of nodes is small with respect to the arithmetic genus, such a projection cannot exist. Indeed, let $\tilde{C}$ be an extremal curve in $\mathbb{P}^3$ of degree $d = 2k$; from Castelnuovo's bounds (see, for example, [55]) it follows that $g(\tilde{C}) = (k-1)^2$ and $d \geq 2 \cdot 3 + 1 = 7$, so $\tilde{C}$ lies on a smooth quadric. Therefore, $\tilde{C} \sim mL_1 + nL_2$, where L_1 and L_2 denote two lines in the two different rulings of the quadric. $\tilde{C}$ is an extremal curve if and only if $m = n = k$. Now, if $C \subset \mathbb{P}^2$ is birational to $\tilde{C}$, this means that $p_a(C) = \frac{1}{2}((2k-1)(2k-2)) = 2k^2 - 3k + 1$. Thus, $\delta = p_a(C) - g(\tilde{C}) = 2k^2 - 3k + 1 - k^2 + 2k - 1 = k^2 - k$ and $d = 2k \geq 7$, i.e. $k \geq 4$. This implies that δ must be bigger than 11.

From the previous example, one may look for bounds, for the number $p_a(C) - g(\tilde{C})$, which exclude that C can be obtained as a birational projection of a smooth curve lying in some higher dimensional projective space. Therefore, one can extend the notion of linear normality by considering the geometric linear normality property of singular varieties $X \subset \mathbb{P}^r$, having some restricted type of singularities which can arise from projections.

To this aim, we recall the following:

Definition 4.1.1 *(see [27], Definition 3.1) Let C be any reduced curve in $\mathbb{P}^r$. C is said to be* geometrically linearly normal *if the normalization map $\varphi : \tilde{C} \to C \subset \mathbb{P}^r$ cannot be factored into a non-degenerate map $\tilde{C} \to \mathbb{P}^N$, with $N > r$, followed by a projection.*

Observe that, if $S \subset \mathbb{P}^r$ is a smooth, non-degenerate linearly normal surface and if H denotes the hyperplane section on S,

$$h^0(S, \mathcal{O}_S(H)) = h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(H)) = r + 1. \tag{4.1}$$

As in Section 3.2, we can make the following:

Definition 4.1.2 *Let S be a smooth surface and $C \in Div(S)$. We denote by $\nu(C, H)$ the* Hodge number *of C and H, defined as*

$$\nu(C, H) := (CH)^2 - C^2H^2.$$

By the Algebraic index theorem (Theorem 1.1.24) this is a non-negative number, since H is a very ample divisor.

Remark 4.1.3 Let $S \subset \mathbb{P}^r$ be a smooth, non-degenerate linearly normal surface, and let H be the hyperplane section on S. If $C \in Div(S)$ is an effective divisor, suppose that C is smooth, non-degenerate and such that $C-H$ big and nef (see Definiton 1.1.28). Clearly $h^0(\mathcal{O}_S(H-C)) = 0$ (since C is non-degenerate), hence we have the following exact sequence

$$0 \to H^0(S, \mathcal{O}_S(H)) \to H^0(C, \mathcal{O}_C(H)) \to H^1(S, \mathcal{O}_S(H-C)) \to \cdots$$

By Serre duality, $h^1(S, \mathcal{O}_S(H-C)) = h^1(S, \mathcal{O}_S(K_S + C - H))$ and, by the Mumford vanishing (see Theorem 1.2.4), this equals 0. Hence, by (4.1), it follows that

$$h^0(C, \mathcal{O}_C(H)) = h^0(S, \mathcal{O}_S(H)) = r+1, \tag{4.2}$$

so C is linearly normal.

4.2 Geometric linear normality on projective, non-degenerate and linearly normal surfaces

In this section we discuss the problem of geometric linear normality for nodal curves on some smooth, projective and linearly normal surfaces. We characterize the geometric linear normality of a nodal curve C in a given linear system, only in terms of its set of nodes and of some cohomological properties of the surface.

Our results generalize what is proven in [27], where the case of smooth surfaces in $\mathbb{P}^3$ is treated. We shall briefly recall here their results.

Theorem 4.2.1 *(see Theorem 3.4. in [27]) Let S be a smooth surface of degree d in $\mathbb{P}^3$ and let $C \subset S$ be a complete intersection curve of type $(d,\ n)$ in $\mathbb{P}^3$, having only δ nodes as singularities. Then C is geometrically linearly normal if and only if the set of nodes N of C imposes independent conditions to the linear system $\mid (n+d-5)H \mid$, where H is the plane divisor of S. In particular, for $d=5$, C is geometrically linearly normal if and only if N imposes independent conditions to $\mid C \mid$, i.e if and only if the Severi variety $V_{|C|,\,\delta}$ is regular at $[C]$.*

Proof: The proof of this result is based on the fact that the canonical divisor of $\tilde{C}$ is $\omega_{\tilde{C}} \cong \varphi^*(\mathcal{O}_C((n+d-4)H))(-\tilde{N})$, where $\tilde{N}$ is the pull-back to $\tilde{C}$ of the set N of nodes of C. Therefore, $\varphi^*(\mathcal{O}_C(H))$ is residual to $\varphi^*(\mathcal{O}_C((n+d-5)H))(-\tilde{N})$. Since there is the following standard isomorphism (see Proposition 3.3. in [27]):

$$H^0(S, \mathcal{I}_{N/S}(mH))/H^0(S, \mathcal{I}_{C/S}(mH)) \cong H^0(\tilde{C}, \varphi^*(\mathcal{O}_C(mH))(-\tilde{N})),\ \forall m \in \mathbb{Z},$$

by Riemann-Roch theorem on $\tilde{C}$ one determines

$$h^0(\tilde{C}, \varphi^*(\mathcal{O}_C(H))) = 4 + dim(Coker(\psi)),$$

where

$$\psi : H^0(\mathcal{O}_S((n+d-5)H)) \rightarrow H^0(\mathcal{O}_N)$$

and

$$Coker(\psi) \subseteq H^1(\mathcal{I}_N((n+d-5)H)).$$

Therefore, C is geometrically linearly normal if and only if $dim(Coker(\psi)) = 0$. For details, the reader is referred to the original paper. □

By using their result on nodal curves numerically equivalent to pK_S (see Theorem 3.1.6), Chiantini and Sernesi prove the following:

Theorem 4.2.2 *Let S be a smooth surface of degree $d \geq 5$ in $\mathbb{P}^3$ and let H be its plane divisor; let $[C] \in \mid nH \mid$, $n \geq 2$, correspond to an irreducible curve having only δ nodes as singularities. If $\delta < \frac{1}{4}(nd(n-2))$ then C is geometrically linearly normal.*

Proof: For details, the reader is referred to Theorem 3.5. in [27]. □

They improve the previous result when S is a general smooth quintic surface.

Proposition 4.2.3 *(see Proposition 3.6. in [27]) Let S be a smooth quintic surface in $\mathbb{P}^3$, with Picard group isomorphic to $\mathbb{Z}$. Let $C \sim nH$ be a curve with only δ nodes as singularities. Assume n odd and $\delta < \frac{5}{4}(n-1)^2$. Then C is geometrically linearly normal.*

Our approach focuses more generally on the geometric linear normality of nodal curves in a complete linear system $\mid D \mid$ on a smooth surface $S \subset \mathbb{P}^r$, where S is assumed to be linearly normal and to satisfy the cohomological condition $h^1(S, \mathcal{O}_S(H)) = 0$. Let $[C] \in \mid D \mid$ correspond to a reduced, irreducible curve with only δ nodes as singular points and let N be the 0-dimensional scheme of its nodes.

Before going into details, we want to spend a few words on the cohomological conditions imposed.

Remark 4.2.4 The linear normality of S means that $h^1(\mathcal{I}_{S/\mathbb{P}^r}(H)) = 0$ and this is clearly necessary to consider our problem since, otherwise, we cannot hope to say too much on C. On the other hand, as it will be also clear from the proof of Theorem 4.2.5, the vanishing condition $h^1(S, \mathcal{O}_S(H)) = 0$ implies that the linear series $\mid \omega_{\tilde{C}}(\varphi^*(-H)) \mid$ is complete, where $\omega_{\tilde{C}}$ denotes the canonical sheaf on the smooth curve $\tilde{C}$ and $\varphi : \tilde{C} \rightarrow C \subset S$ is the normalization map of C. More precisely, if $\mu : \tilde{S} \rightarrow S$ denotes the blow-up of S along the set of nodes of C, such that $B = \sum_{i=1}^{\delta} E_i$ is the μ-exceptional divisor in $\tilde{S}$, the map μ induces the normalization map $\varphi : \tilde{C} \rightarrow C \subset S$. The exact sequence defining $\omega_{\tilde{C}}$ gives rise to

$$0 \rightarrow \mathcal{O}_{\tilde{S}}(\mu^*(K_S - H) + B) \rightarrow \mathcal{O}_{\tilde{S}}(\mu^*(K_S + C - H) - B) \rightarrow$$

$$\to \omega_{\tilde{C}}(\varphi^*(-H)) \to 0.$$

We observe that $h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(\mu^*(K_S - H) + B)) = 0$ implies that the map

$$H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\mu^*(K_S + C - H) - B)) \to H^0(\tilde{C}, \omega_{\tilde{C}}(\varphi^*(-H)))$$

is surjective. Indeed, observe that by Serre duality on $\tilde{S}$, $h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(\mu^*(K_S - H) + B)) = h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} - \mu^*(H))) = h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(\mu^*(H)))$, so the vanishing follows from Leray isomorphism and our assumption on $h^1(S, \mathcal{O}_S(H))$.

To prove our main theorem, we shall use the following preliminary result.

Theorem 4.2.5 *Let S be a smooth, non-degenerate and linearly normal surface in $\mathbb{P}^r$ such that $h^1(S,\ \mathcal{O}_S(H)) = 0$. Let $\mid D \mid$ be a linear system on S whose general element is supposed to be smooth, irreducible and linearly normal in $\mathbb{P}^r$. Assume that $[C] \in \mid D \mid$ corresponds to an irreducible curve with only δ nodes as singular points. Then C is geometrically linearly normal if and only if the set of nodes, N, imposes independent conditions to the linear system $\mid D + K_S - H \mid$.*

Proof: Let D be the general member of the linear system $\mid D \mid$. By the linear normality hypothesis and by Riemann-Roch on D, we have

$$h^1(D, \mathcal{O}_D(H)) = (r+1) - deg(D) + p_a(D) - 1,$$

hence, by Serre duality and by adjunction on S, we get

$$h^0(D, \mathcal{O}_D(D + K_S - H)) = (r+1) - deg(D) + p_a(D) - 1. \tag{4.3}$$

Now, let $C \sim D$ be a curve with only δ nodes as singularities. Denote by $\mu : \tilde{S} \to S$ the blow-up of S along the set of nodes of C, N, and let $B = \sum_{i=1}^{\delta} E_i$ be the exceptional divisor in $\tilde{S}$. The blow-up induces the normalization map $\varphi : \tilde{C} \to C \subset S$. By adjunction theory on $\tilde{S}$,

$$\omega_{\tilde{C}} = \mathcal{O}_{\tilde{C}}(K_{\tilde{S}} + \tilde{C}) = \mathcal{O}_{\tilde{C}}(\mu^*(K_S + C) - B) = \mathcal{O}_{\tilde{C}}(\varphi^*(K_S + C)(-\tilde{N})), \tag{4.4}$$

where $\mathcal{O}_{\tilde{C}}(\tilde{N}) = \mathcal{O}_{\tilde{C}}(B)$ is a divisor of degree 2δ on $\tilde{C}$, formed by the points which map to the nodes of C. From Riemann-Roch on $\tilde{C}$, it follows that

$$h^1(\tilde{C}, \mathcal{O}_{\tilde{C}}(\varphi^*(H))) = h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\varphi^*(H))) - deg(C) + p_a(C) - 1 - \delta.$$

By using (4.4) and the fact that $C \sim D$ on S, we get

$$h^0(\mathcal{O}_{\tilde{C}}(\varphi^*(K_S + D - H)(-\tilde{N}))) =$$

$$= h^0(\mathcal{O}_{\tilde{C}}(\varphi^*(H))) - deg(C) + p_a(C) - 1 - \delta. \tag{4.5}$$

Observe that $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\varphi^*(H))) = r + 1$ if and only if

$$h^0(\mathcal{O}_{\tilde{C}}(\varphi^*(K_S + D - H)(-\tilde{N}))) = (r+1) - deg(C) + p_a(C) - 1 - \delta.$$

By using (4.3) and the fact that the adjunction on S is independent from the chosen element in $\mid D \mid$, we obtain

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\varphi^*(H))) = r+1 \Leftrightarrow h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\varphi^*(K_S + D - H)(-\tilde{N}))) \quad (4.6)$$
$$= h^0(C, \mathcal{O}_C(D + K_S - H)) - \delta.$$

Now we use our assumption $h^1(S, \mathcal{O}_S(H)) = 0$. It implies, by duality on S, that

$$h^0(S, \mathcal{O}_S(D + K_S - H)) - h^0(S, \mathcal{O}_S(K_S - H)) = h^0(C, \mathcal{O}_C(D + K_S - H))$$

whereas, on $\tilde{S}$,

$$h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\mu^*(K_S + D - H) - B)) - h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\mu^*(K_S - H) + B) =$$
$$= h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\varphi^*(K_S + D - H)(-\tilde{N}))),$$

since, by Leray isomorphisms, $h^1(\mathcal{O}_{\tilde{S}}(\mu^*(K_S - H) + B)) = h^1(\mathcal{O}_S(H)) = 0$. Substituting in the second equality of (4.6), it gives

$$h^0(\mathcal{O}_{\tilde{C}}(\varphi^*(H))) = r+1 \Leftrightarrow h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\mu^*(K_S + D - H) - B)) =$$
$$h^0(S, \mathcal{O}_S(D + K_S - H)) - \delta.$$

The claim follows from the fact that $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\mu^*(K_S + D - H) - B)) = h^0(S, \mathcal{I}_{N/S}(K_S + D - H))$. $\square$

Remark. From (4.2) in Remark 4.1.3 and from the hypotheses on S, the same conclusion holds if we assume that the general element of $\mid D \mid$ is a smooth, irreducible and non-degenerate divisor such that $D - H$ is big and nef.

For what concerns the geometric linear normality problem, by considering Bogomolov-unstable vector bundles on S (see Definition 1.2.12) we can obtain an upper-bound δ_u on the number of nodes such that, if C has less than δ_u nodes, then it is geometrically linearly normal. Using a similar procedure of Theorem 3.2.3, we can prove the following result.

Theorem 4.2.6 *Let S be a smooth, non-degenerate and linearly normal surface in $\mathbb{P}^r$ such that $h^1(\mathcal{O}_S(H)) = 0$. Let C be a smooth, irreducible divisor on S. Suppose that:*

(i) $CH > H^2$;

(ii) $(C - 2H)^2 > 0$ *and* $C(C - 2H) > 0$;

(iii) $\nu(C, H) < 4(C(C - 2H) - 4)$, *where $\nu(C, H)$ is the Hodge number of C and H (see Definition 4.1.2);*

(iv) $\delta < \frac{C(C-2H)+\sqrt{C^2(C-2H)^2}}{8}$.

If $[C'] \in \mid C \mid$ parametrizes a reduced, irreducible curve with only δ nodes as singular points and if N denotes the 0-dimensional scheme of nodes of C', then N imposes independent conditions to $\mid C - H + K_S \mid$.

Proof: By contradiction, assume that N does not impose independent conditions to $\mid C - H + K_S \mid$. Let $N_0 \subset N$ be a minimal 0-dimensional subscheme of N for which this property holds and let $\delta_0 = \mid N_0 \mid$. This means that $h^1(S, \mathcal{I}_{N_0}(C - H + K_S)) \neq 0$ and that N_0 satisfies the Cayley-Bacharach condition; so, a non-zero element of $H^1(\mathcal{I}_{N_0}(C - H + K_S))$ gives rise to a non-trivial rank-two vector bundle $\mathcal{E} \in Ext^1(\mathcal{I}_{N_0}(C - H), \mathcal{O}_S)$ fitting in the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{I}_{N_0}(C - H) \to 0, \tag{4.7}$$

with $c_1(\mathcal{E}) = C - H$ and $c_2(\mathcal{E}) = \delta_0$ hence

$$c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) = (C - H)^2 - 4\delta_0. \tag{4.8}$$

By (iv)

$$(C - H)^2 - 4\delta_0 \geq (C - H)^2 - 4\delta = C^2 - 2CH + H^2 - 4\delta > H^2 > 0,$$

since $\delta_0 \leq \delta$ thus $\mathcal{E}$ is Bogomolov-unstable (see Theorem 1.2.13), hence $h^0(\mathcal{E}(-M)) \neq 0$. Tensoring (4.7) with $\mathcal{O}_S(-M)$, we obtain

$$0 \to \mathcal{O}_S(-M) \to \mathcal{E}(-M) \to \mathcal{I}_{N_0}(C - H - M) \to 0. \tag{4.9}$$

We claim that $h^0(\mathcal{O}_S(-M)) = 0$; otherwise, $-M$ would be an effective divisor, therefore $-MA > 0$, for each ample divisor A. From (1.9), in Definition 1.2.12, it follows that $c_1(\mathcal{E}) = M + B$, so by (4.7)

$$M - B = 2M - C + H \in N(S)^+, \tag{4.10}$$

where $N(S)^+$ denotes the ample divisor cone of S (see Section 1.1).

This implies

$$MH > \frac{(C - H)H}{2}; \tag{4.11}$$

thus, by (i), it follows that $H(C - H) > 0$, hence $-MH < 0$.

The cohomology sequence associated to (4.9) allows us to deduce that there exists a divisor Δ in $\mid C - H - M \mid$ such that $N_0 \subset \Delta$ and the irreducible nodal curve C' in $\mid C \mid$, whose set of nodes is N, is not a component of Δ. Otherwise, $-M - H$ would be an effective divisor whereas, by (4.11), we get

$$H(-M - H) = -H^2 - HM < -H^2 - \frac{(C - H)H}{2} = -\frac{(C + H)H}{2} < 0,$$

since $H(C + H) = (C - H)H + 2H^2 > 0$.

Next, by Bezout's theorem

$$C'\Delta = C'(C - H - M) \geq 2\delta_0. \tag{4.12}$$

On the other hand, taking M maximal, we may further assume that the general section of $\mathcal{E}(-M)$ vanishes in codimension 2. Denote by Z this vanishing-locus, thus, $c_2(\mathcal{E}(-M)) = deg(Z) \geq 0$; moreover,

$$c_2(\mathcal{E}(-M)) = c_2(\mathcal{E}) + M^2 + c_1(\mathcal{E})(-M) = \delta_0 + M^2 - M(C - H),$$

which implies

$$\delta_0 \geq M(C-H-M). \tag{4.13}$$

Applying the Index theorem to the divisor pair $(C,\ 2M-C+H)$, we get

$$C^2(2M-C+H)^2 \leq (C(C-H)-2C(C-H-M))^2. \tag{4.14}$$

Note now that, from hypothesis (i) and the second one of (ii) it follows that $C(C-H)>0$, since $C(C-2H)>0$ hence $C^2-HC>HC>0$. In the same way we find $C^2>0$. Since C is irreducible, this also implies that C is a nef divisor (see Lemma 3.2.2). From (4.12) and from the positivity of $C(C-H)$, it follows that

$$C(C-H)-2C(C-H-M) \leq C(C-H)-4\delta_0. \tag{4.15}$$

We observe that the left side member of (4.15) is non-negative, since $C(C-H)-2C(C-H-M)=C(2M-C+H)$, where C is effective and, by (4.10), $2M-C+H \in N(S)^+$. Squaring both sides of (4.15), together with (4.14), we find

$$C^2(2M-C+H)^2 \leq (C(C-H)-4\delta_0)^2. \tag{4.16}$$

On the other hand, by (4.13), we get

$$(2M-C+H)^2 = 4(M-\frac{(C-H)}{2})^2 =$$

$$(C-H)^2-4(C-H-M)M \geq (C-H)^2-4\delta_0,$$

i.e

$$(2M-C+H)^2 \geq (C-H)^2-4\delta_0. \tag{4.17}$$

Next, we define

$$F(\delta_0) := 16\delta_0^2-4C(C-2H)\delta_0+(CH)^2-C^2H^2. \tag{4.18}$$

Putting together (4.16) and (4.17), it follows that $F(\delta_0)\geq 0$. We will show that, with our numerical hypotheses, one has $F(\delta_0)<0$, proving the statement.

Indeed, the discriminant of the equation $F(\delta_0)=0$ is $16C^2(C-2H)^2$, and it is a positive number, since $(C-2H)^2>0$ by the first one of (ii) and $C^2>0$. We remark that $F(\delta_0)<0$ iff $\delta_0 \in (\alpha(C,\ H),\ \beta(C,\ H))$, where

$$\alpha(C,H) = \frac{C(C-2H)-\sqrt{C^2(C-2H)^2}}{8}$$

and

$$\beta(C,H) = \frac{C(C-2H)+\sqrt{C^2(C-2H)^2}}{8};$$

so we have to show that, $\delta_0 \in (\alpha(C,\ H),\ \beta(C,\ H))$.

From (iv), it follows that $\delta_0<\beta(C,H)$. Note that $\alpha(C,H)\geq 0$. Indeed, if $\alpha(C,H)<0$ then $C(C-2H)<\sqrt{C^2(C-2H)^2}$, which contradicts the Index Theorem, since $C(C-2H)>0$. In order to simplify the notation, we put $t := C(C-2H)$. Thus, $\alpha(C,H)<1$ if and only if $t-8<\sqrt{t^2-4\nu(C,H)}$.

If $t-8 < 0$, the previous inequality trivially holds, so $\delta_0 > \alpha(C,H)$. Note also that, by (iii), $4\nu(C,H) < 16t-64$, so that $\beta(C,H) > 1$, which ensures there exists at least a positive integral value for the number of nodes.

If $t-8 \geq 0$, $\alpha(C,H) < 1$ directly follows from (iii), whereas $\beta(C,H) > 1$ holds since it is equivalent to $t-8 > -\sqrt{t^2 - 4\nu(C,H)}$.

In conclusion, our numerical hypotheses contradict $F(\delta_0) \geq 0$, therefore the assumption $h^1(\mathcal{I}_N(D-H+K_S)) \neq 0$ leads to a contradiction. □

Corollary 4.2.7 *In the hypotheses of the previous theorem, if C is linearly normal in $\mathbb{P}^r$ then C' is geometrically linearly normal.*

Proof: See Theorem 4.2.5. □

Remark 4.2.8 Observe that, if $t-8 \geq 0$, then

$$\frac{C(C-2H)+C(C-2H)-8}{8} < \frac{C(C-2H)+\sqrt{C^2(C-2H)^2}}{8},$$

therefore we may change the bound $\delta < \beta(C,H)$ with the more "readable" one $\delta \leq \frac{C(C-2H)}{4} - 1$.

Indeed,

$$\frac{C(C-2H)+C(C-2H)-8}{8} < \frac{C(C-2H)+\sqrt{C^2(C-2H)^2}}{8} \leq \frac{C(C-2H)}{4}.$$

We remark that Theorem 4.2.6 gives purely numerical conditions on the divisors C and H in order to determine the geometric linear normality of nodal curves on S. As we shall see in the next section, these conditions can be directly checked in many cases, where other criteria fail.

4.3 Examples of obstructed curves on canonical complete intersection surfaces

This section will be devoted to the study of some examples, which also show the sharpness of our bound in Theorem 4.2.6 for smooth canonical complete intersection surfaces.

First of all, assume that S is a smooth, projective, non-degenerate and linearly normal surface, with Picard group $\mathbb{Z}$-generated by the hyperplane section H. Suppose also that $h^1(S, \mathcal{O}_S(H)) = 0$; then our results easily apply to the cases of nodal curves $C \sim nH$ on S, such that $n \geq 3$ and $deg(S) > \frac{4}{n(n-2)}$. Indeed, condition (ii) in Theorem 4.2.6 implies that $n > 2$, whereas condition (iii) gives that $\nu(nH,H) = 0$, so $C(C-2H) - 4 = n(n-2)H^2 - 4 > 0$ if and only if $H^2 > \frac{4}{n(n-2)}$; this means that the degree of S must be greater than or equal to 2, but with the further condition that $S \subset \mathbb{P}^r$ is non-degenerate.

If we go back to the case of a general surface $S \subset \mathbb{P}^3$, such that $deg(S) \geq 2$, the bound on the number of nodes is

$$\delta < \frac{n(n-2)}{4} deg(S);$$

this generalizes Chiantini and Sernesi's result (see Theorem 4.2.2), where the cases in which K_S is ample are considered.

Moreover, in [27], the authors prove the sharpness of their bound on δ for a general quintic surface in $\mathbb{P}^3$. In particular, since in this case the Neron-Severi group of S is such that $NS(S) \cong \mathbb{Z}[K_S]$ then, when $C \sim nH$ with n an odd integer, the bound on the number of nodes is $\delta < \frac{5(n-1)^2}{4}$ instead of $\frac{5n(n-2)}{4}$.

We will extend their results by showing the sharpness of bound (iv) in Theorem 4.2.6 by considering nodal curves $C \sim nH$ on general canonical complete intersection surfaces. To apply our result, observe that in these cases the necessary cohomological conditions trivially hold since a smooth complete intersection is arithmetically Cohen-Macaulay (see Definition 1.1.9); moreover, the Hodge number for divisors of type nH is always zero. Thus, as in the case of $S \subset \mathbb{P}^3$, the bound (iv) in Theorem 4.2.6 reduces to

$$\delta < \frac{n(n-2)}{4} deg(S); \tag{4.19}$$

moreover, when n is an odd integer, (4.19) can be replaced by

$$\delta < \frac{(n-1)^2}{4} deg(S),$$

as it follows from Theorem 2.2 in [27]. On the other hand, by applying the same procedure of [27], we will show that our bounds are almost-sharp for a sextic surface in $\mathbb{P}^3$.

To do this, we want to recall that the geometric linear normality property is equivalent to regularity of some Severi varieties on some smooth, projective surfaces.

Remark 4.3.1 Let S be a smooth, projective surface which is non-degenerate and linearly normal. Suppose that $K_S \sim H$ (i.e. S is canonical). In such a case, the fundamental condition $h^1(S, \mathcal{O}_S(H)) = 0$, used in the proof of Theorem 4.2.5, implies that S is a regular surface. Therefore, Theorem 4.2.6 determines purely numerical conditions on the nodal curve C ensuring that its set of nodes imposes independent conditions to $\mid C \mid$, i.e. that C corresponds to a regular point $[C]$ of the Severi variety $V_{|D|,\delta}$.

Examples of projective, regular, non-degenerate and linearly normal surfaces, such that $K_S \sim H$, are given by general complete intersections in $\mathbb{P}^r$ of type $(a_1, ..., a_{r-2})$, such that $(\sum_{i=1}^{r-2} a_i) = r + 2$; therefore, only finitely many cases may occur. More precisely, we have a general quintic surface in $\mathbb{P}^3$, surfaces of type $(2, 4)$ and $(3, 3)$ in $\mathbb{P}^4$, a surface of type (2,2,3) in $\mathbb{P}^5$, whereas in $\mathbb{P}^6$ we have the case $(2,2,2,2)$. In $\mathbb{P}^r$, for $r \geq 7$, no non-degenerate case can occur.

In the next example we consider the case of a general complete intersection of type $(2,4)$ in $\mathbb{P}^4$, since the first example of a smooth quintic surface in $\mathbb{P}^3$ is already treated in Section 4 of [27]. The following construction can be obviously generalized to the other cases in the list above.

Example. Let F_2, F_4 be two general hypersurfaces in $\mathbb{P}^4$ of degree 2 and 4, respectively; let S be the surface of degree 8, which is the complete intersection of F_2 and F_4. Denote by W_2 and W_4 the cones in $\mathbb{P}^5$, over F_2 and F_4 respectively, with the same vertex $p \in \mathbb{P}^5$. Let V_2 and V_m be two general 4-folds in $\mathbb{P}^5$ of degree 2 and m, respectively, where m is a positive integer greater than or equal to 3. Let T be the complete intersection 3-fold of V_2 and V_m and denote by $\pi_p : T \to T'$ the projection of T from the point p onto the variety T' of dimension 3. It is classically known that the degree of T' is $2m$ and that T' contains a double surface G.

In order to compute the degree of G, let us denote by E the curve obtained on T by taking two consecutive hyperplane sections; hence E is a complete intersection of type $(2,m,1,1)$ in $\mathbb{P}^5$ and so $p_g(E) = m(m-2)+1$. Using the same procedure for $T' \in \mathbb{P}^4$, we obtain a plane curve E' of degree $2m$; therefore, its arithmetic genus is $p_a(E') = 2m^3 - 3m + 1$. Hence, $deg(G) = m^2 - m$.

Let $\tilde{C}$ be the complete intersection curve in $\mathbb{P}^5$ determined by

$$\tilde{C} := V_2 \cap V_m \cap W_2 \cap W_4.$$

$\tilde{C}$ is a smooth curve of degree $16m$, which lies on the cone of dimension 3 $\tilde{S} := W_2 \cap W_4$. Denote by C the projection of $\tilde{C}$ from p; C has degree $16m$ and is complete intersection of S and T' in $\mathbb{P}^4$. Therefore, $C \in \mid 2mH \mid$ on S and its singularities coincide with the zero-dimensional scheme of $S \cap G$; thus C has a set N of $\delta = 8m^2 - 8m$ nodes and no other singularities. By construction, $\tilde{C}$ is the normalization of C which, therefore, cannot be geometrically linearly normal. Observe that, the bound in (4.19) becomes, in this case, $\delta < 8m^2 - 8m$, hence it is sharp.

Remark 4.3.2 The above construction shows that our result is sharp for general canonical complete intersection surfaces. Furthermore, from Theorem 4.2.5 it follows that, in this example, N cannot impose independent conditions to $\mid C \mid$, so that the Severi variety $V_{|2mH|,8m^2-8m}$ is not smooth of the expected dimension (i.e. $dim(\mid 2mH \mid) - 8m^2 - 8m$) at $[C]$.

Proposition 4.3.3 *The curve C constructed above is a singular point of $V_{|2mH|,8m^2-8m}$, which is generically smooth, of the expected dimension.*

Proof: The previous construction, together with Theorem 4.2.5, shows that the tangent space of $V_{|2mH|,8m^2-8m}$ at $[C]$ has codimension $8m^2 - 8m - 1$ in the tangent space of $\mid 2mH \mid$ at $[C]$. Hence, $h^1(S, \mathcal{I}_N(2mH)) = 1$, since C is the projection of a smooth, complete intersection in $\mathbb{P}^5$.

Let $[C']$ be a point in a neighbourhood of $[C]$ in $V_{|2mH|,8m^2-8m}$, for which the set of nodes N' of the correspondent curve C' does not impose independent conditions to $\mid 2mH \mid$. Then, by semicontinuity, $h^1(S, \mathcal{I}_{N'}(2mH)) = 1$; therefore, also C' is the projection of a curve $\tilde{C}'$ in $\mathbb{P}^5$ which "lives" in a neighbourhood of $\tilde{C}$ in the Hilbert scheme of $\mathbb{P}^5$. It follows that also $\tilde{C}'$ must be a smooth, complete intersection of the cone $\tilde{S}$ with some complete intersection 3-fold of type $(2, m)$. If we denote by $\mathcal{G}$ the subvariety of $V_{|2mH|,8m^2-8m}$, formed by these projected curves, we can find an upperbound for $dim(\mathcal{G})$. By keeping the cones W_2 and W_4 fixed, the normalizations of the elements of $\mathcal{G}$ fill a variety of dimension at most

$$h^0(\tilde{C}, \mathcal{N}_{\tilde{C}/\tilde{S}}) = h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(2) \oplus \mathcal{O}_{\tilde{C}}(m)) = 8m^2 - 16m + 38.$$

If we let also the vertex p vary in $\mathbb{P}^5$, we get a variety of dimension at most $8m^2 - 16m + 43$. On the other hand, $V_{|2mH|,8m^2-8m}$ has dimension at least

$$h^0(S, \mathcal{O}_S(2m)) - 1 - 8m^2 + 8m = 8m^2 + 5.$$

Since $m \geq 3$, then $8m^2 + 5 > 8m^2 - 16m + 43$, which means that the general element of the Severi variety does not arise from this construction and is a smooth point of $V_{|2mH|,8m^2-8m}$. □

Remark 4.3.4 We remark that there exist non-canonical surfaces for which the bound is almost-sharp. Indeed, let us consider a nonsingular sextic surface S in $\mathbb{P}^3$. Let C be a curve on S equivalent to nH, with n an even integer greater than 4. Arguing with cones as in the previous example, we can prove that C has $\frac{3}{2}(n^2 - 2n)$ nodes, whereas the bound in this case is given by the number $\frac{3}{2}n(n-4)$, and C is the projection of a curve in $\mathbb{P}^4$. It remains to understand what happens in the range $[\frac{3}{2}n(n-4), \frac{3}{2}n(n-2)-1]$.

We end this section by considering some examples of surfaces to which our numerical criterion easily applies, whereas other criteria fail. We shall focus on surfaces in $\mathbb{P}^3$ which contain a line L. The computations are similar to the ones in the examples given in Chapter 3.

Example. Let S be a smooth quintic surface in $\mathbb{P}^3$ which contains a line L. Denote by $\Gamma \subset S$ a plane quartic which is coplanar to L, so that $\Gamma \sim H - L$. Thus,

$$H^2 = 5,\ HL = 1,\ L^2 = -3,\ H\Gamma = 4,\ \Gamma^2 = 0 \textit{ and } \Gamma L = 4.$$

Choose $C \sim 3H + L$, so that $\mid C \mid$ contains curves which are residue to Γ in the complete intersection of S with the smooth quartic surfaces of $\mathbb{P}^3$ containing Γ. $\mid 3H + L \mid$ is base-point-free and not composed with a pencil, since $(3H + L)L = 0$ and $3H$ is an ample divisor. By Bertini's theorems, its general member is smooth and irreducible; but C and $C - K_S$ can not be both either ample or, even, nef divisors. In fact, $CL = 0$ and $(C - K_S)L = (2H + L)L = -1$. Moreover, C is not numerically equivalent to a rational multiple of K_S. Therefore, the results in [27] and in [53] cannot be applied.

Neverthless, observe that for such a S one trivially has

$$h^1(\mathcal{I}_{S/\mathbb{P}^3}(H)) = h^1(\mathcal{O}_S(H) = 0;$$

furthermore, $CH = C(C-2H) = \nu(C,H) = 16$, $(C-2H)^2 = 4$, $H^2 = 5$, $4(C(C-2H)-4) = 48$; we then obtain $\delta < \frac{16}{4} = 4$. Thus, if $\mid 3H + L \mid$ contains some nodal, irreducible curves, then, if $\delta \leq 3$, such a singular curve is geometrically linearly normal; since $K_S \sim H$, this is equivalent to saying that such a curve corresponds to a regular point of $V_{|3H+L|,\delta}$, which will be everywhere smooth of the expected dimension.

Chapter 5

Moduli of Nodal Curves on Smooth Projective Surfaces of General Type

In this chapter we shall study families of curves, on a smooth, projective and regular surface S of general type, from the point of view of their moduli behaviour. More precisely, consider an effective divisor D on S and suppose that the general element of the complete linear system $\mid D \mid$ is a smooth, irreducible curve of geometric genus $p_a(D) = \frac{D(D+K_S)}{2} + 1$, where K_S is a canonical divisor on S. If $g = p_a(D) - \delta$ denotes the geometric genus of an irreducible, δ-nodal curve in $\mid D \mid$, assume that $g \geq 2$[1], for each $\delta \geq 0$. In this case, one can consider the morphisms

$$\pi_{|D|,\,\delta} : V_{|D|,\delta} \longrightarrow \mathcal{M}_g, \tag{5.1}$$

for each $\delta \geq 0$, where $V_{|D|,\delta}$ is the Severi variety of irreducible, δ-nodal curves in $\mid D \mid$, as in Definition 2.2.28, and $\mathcal{M}_g$ is the moduli space of smooth curves of geometric genus g.

As we shall more precisely explain in Section 5.1, the morphisms $\pi_{|D|,\,\delta}$ are functorially defined since one can consider a simultaneous desingularization of all the δ-nodal curves parametrized by $V_{|D|,\delta}$.

The problem is to study, for each morphism, the dimension of its image, which is called the *number of moduli* of the family (see Definition 5.1.1). Apart from some particular cases, if we assume that $V_{|D|,\,\delta}$ is generically regular (in the sense of Definition 2.2.30), what we expect is that the number of moduli equals $dim(V_{|D|,\,\delta})$; in other words, we expect that a regular point $[X] \in V_{|D|,\,\delta}$ is birationally isomorphic to finitely many curves in $V_{|D|,\,\delta}$.

In some cases we will prove that this actually happens (the reader will find precise statements in the sequel). On the other hand, we shall also discuss some examples which show that the problem has not an immediate

[1] We shall observe in the sequel that this assumption is not so restrictive for the problems we are interested in.

answer, but it depends on the kind of considered divisors.

Our approach is analogous to that of Sernesi, [119], where he applied infinitesimal deformation theory to families of plane nodal curves to study their number of moduli (see also Section 2.3 for a brief overview of his results).

In Section 5.1 we make the precise definition of what is meant by *moduli problem* and we discuss clarifying examples, some of which give affirmative answers to the moduli problem whereas others are counterexamples. Section 5.2 in the core of the chapter, where we discuss our new results. Since we have used many different approaches to arrive at a final statement, for clarity sake we shall separately prove all the partial results which give affirmative answers to the moduli problem. In Section 5.3, we will summarize all these results in a unique precise statement.

5.1 The infinitesimal approach to the moduli problem

As mentioned in the introduction of this chapter, here we fix some notation and we precisely state the problem we are interested in. We also discuss some examples which give both some positive and negative answers to our moduli problem.

In this section S will denote a smooth, projective surface which is regular and of general type, unless otherwise specified. Let D be an effective divisor on S and let $|D|$ be the complete linear system, whose general member is supposed to be a smooth, irreducible curve. Denote by X an irreducible curve in $|D|$ which has only δ nodes as singularities, where $\delta \geq 0$. The map

$$\varphi : C \to X \subset S$$

denotes its normalization, where C is a smooth curve of geometric genus $g = p_a(X) - \delta$.

We recall that one can associate to φ the following exact sequence on C (see (2.5), in Section 2.2),

$$0 \to \mathcal{T}_C \to \varphi^*(\mathcal{T}_S) \to \mathcal{N}_\varphi \to 0, \tag{5.2}$$

where $\mathcal{T}_C$ is the tangent bundle of C, $\mathcal{T}_S$ the tangent bundle of S and $\mathcal{N}_\varphi$ is the normal bundle of the map φ, in the sense of Horikawa's theory.

If we deal with $V_{|D|,0}$ and if we denote always by X the general (smooth) element of $|D|$, the exact sequence (5.2) reduces to the standard normal sequence of X in S,

$$0 \to \mathcal{T}_X \to \mathcal{T}_S|_X \to \mathcal{N}_{X/S} \to 0.$$

From Section 2.2., we know that the study of first-order deformations of the pair (C, φ) is equivalent to the study of first-order embedded deformations of X in S which preserve the singularities of X, i.e. the equisingular first-order embedded deformations. Moreover, the tangent space to the functor of such deformations is isomorphic to $H^0(C,\ \mathcal{N}_\varphi)$, whereas its obstruction space is a subspace of $H^1(C,\ \mathcal{N}_\varphi)$. On the other hand, $H^0(\mathcal{N}_{X/S})$ and $H^1(\mathcal{N}_{X/S})$ play the same role when we consider first-order embedded deformations of the smooth curve X in S.

We have already proven in Section 2.2. (see (2.20) and (2.24)) that

$$H^0(\mathcal{O}_S(D))/ < X >$$

is the Zariski tangent space to $\mid D \mid$ at $[X]$ whereas

$$H^0(\mathcal{I}_{N/S}(D))/ < X >$$

is the Zariski tangent space to $V_{|D|,\,\delta}$ at $[X]$, where N denotes the set of nodes of X. Moreover, since S is assumed to be regular, these spaces coincide with the spaces of all first-order embedded deformations (i.e. $H^0(\mathcal{N}_{X/S})$) and all first-order equisingular deformations (i.e. $H^0(C,\ \mathcal{N}_\varphi)$) of X in S, respectively (see Remark 2.2.29).

When $N = \emptyset$, the Zariski tangent space to $V_{|D|,0}$ at $[X]$ coincides with

$$H^0(\mathcal{O}_S(D))/ < X >,$$

reflecting the fact that $V_{|D|,0}$ is an open subscheme of $\mid D \mid$.

In this situation, suppose $g \geq 2$. In most of the cases we are interested in, this is not a restriction, since we shall mainly consider very-ample divisors of type mH, where $m \in \mathbb{N}$ and H the hyperplane section of $S \subset \mathbb{P}^r$. Thus, the arithmetic genus of a general (smooth) section is at least 2. On the other hand, if for example we consider nodal, very-ample sections which are elements of a regular Severi variety, we know that Theorem 3.2.3 gives a sufficient condition for the regularity of such a Severi variety, so that this condition determines an upper-bound on the admissible number of nodes of curves parametrized by it and so a lower-bound on the values of the admissible geometric genera; this lower-bound is actually greater than 1.

However, in the sequel, we shall not restrict ourselves to consider only Severi varieties $V_{|D|,\delta}$ for which D is a positive integral multiple of the hyperplane divisor H or which are everywhere regular (see Theorems 5.2.2 and 5.2.7).

If we consider a smooth element X in $\mid D \mid$, we have

$$0 \to H^0(\mathcal{T}_S\mid_X) \to H^0(\mathcal{N}_{X/S}) \xrightarrow{\partial} H^1(\mathcal{T}_X) \to \cdots, \tag{5.3}$$

where $h^0(\mathcal{N}_{X/S}) = dim(\mid D \mid)$ and $h^1(\mathcal{T}_X) = 3p_a(X) - 3$.

On the other hand, if we consider $[X] \in V_{|D|,\delta}$, $\delta \geq 1$, a regular point and $\varphi : C \to X \subset S$ its normalization, we get

$$0 \to H^0(\varphi^*(\mathcal{T}_S)) \to H^0(\mathcal{N}_\varphi) \xrightarrow{\partial} H^1(\mathcal{T}_C) \to \cdots, \tag{5.4}$$

where $h^0(\mathcal{N}_\varphi) = dim_{[X]}(V_{|D|,\delta})$ and $h^1(\mathcal{T}_C) = 3g - 3 = 3(p_a(X) - \delta - 1)$.

From Remark 2.2.6, the coboundary maps in the above sequences apply Horikawa's classes to Kodaira-Spencer's classes of the corresponding families

$p_\delta : \mathcal{C}_\delta \to \Delta_\epsilon$[2], of smooth curves, parametrized by $V_{|D|,\delta}$, $\delta \geq 0$; moreover, these maps can be identified with the differentials of the morphisms $\pi_{|D|,\delta}$, $\delta \geq 0$, at the points $[X]$ and $[C \to X \subset S]$, respectively.

When $\delta \geq 1$, these families of smooth curves are determined by the normalizations of the nodal curves on S parametrized by the same base scheme. Indeed, as in Section 2.2, denote by $\mathcal{X}_\delta$ the universal family of δ-nodal curves in S parametrized by $V_{|D|,\delta}$, i.e.

$$\begin{array}{cc} \mathcal{X}_\delta & \subset S \times V_{|D|,\delta} \\ \downarrow & \\ V_{|D|,\delta} & . \end{array}$$

The elements in $\mathcal{X}_\delta$ can be simultaneously desingularized, so there exists a diagram of proper morphisms

$$\begin{array}{ccccc} \mathcal{C}_\delta & \xrightarrow{\Phi} & \mathcal{X}_\delta & \subset S \times V_{|D|,\delta} \\ & \searrow^{f} & \downarrow & \\ & & V_{|D|,\delta} & \end{array}$$

where Φ is fibrewise the normalization map. The map Φ is the blow up of $\mathcal{X}_\delta$ along its codimension-one singular locus and, for each $\delta \geq 1$, the morphism

$$\pi_{|D|,\delta} : V_{|D|,\delta} \to \mathcal{M}_g$$

is functorially defined by f.

We can make the following definition.

Definition 5.1.1 *Let S be a smooth, projective surface of general type (not necessarily regular) and let D be a smooth curve on S. Let $\delta \geq 0$ be such that $V_{|D|,\delta} \neq \emptyset$. The* number of moduli *of the family $V_{|D|,\delta}$ is*

$$dim(\pi_{|D|,\delta}(V_{|D|,\delta})).$$

The expected number of moduli *of $V_{|D|,\delta}$ is*

$$expmod(V_{|D|,\delta}) := dim(V_{|D|,\delta}),$$

for each $\delta \geq 0$.

Remark 5.1.2 Let X be a curve corresponding to a regular point $[X]$ of $V_{|D|,\delta}$ (when $\delta = 0$, $[X]$ is automatically a smooth point of $V_{|D|,0}$) and let $\varphi : C \to X \subset S$ be its normalization map. Assume that S is as in the definition above and suppose that it is also regular. If we further assume that $[X]$ is a general point of $V_{|D|,\delta}$, $\delta \geq 0$, from (5.3), (5.4) and from the regularity of S we have that

$$dim(\pi_{|D|,0}(V_{|D|,0})) = h^0(\mathcal{N}_{X/S}) - h^0(\mathcal{T}_S \mid_X)$$

and

$$dim(\pi_{|D|,\delta}(V_{|D|,\delta})) = h^0(\mathcal{N}_\varphi) - h^0(\varphi^*(\mathcal{T}_S)),$$

respectively.

[2]Recall that, as in (1.24), Section 1.5, Δ_ϵ denotes the affine scheme $Spec(\mathbb{C}[\epsilon]/(\epsilon^2))$.

In the sequel, when $\delta = 0$ (so $N = \emptyset$) we shall use in the same way the symbols X and D to denote a smooth curve in $\mid D \mid$.

Definition 5.1.3 *Given S a smooth, projective surface of general type (not necessarily regular), the* moduli problem *consists in determining for which kind of S's and which kind of divisor classes D in $Div(S)$ the number of moduli of $V_{|D|,\delta}$ coincides with the expected number of moduli of the family, i.e. when*

$$expmod(V_{|D|,\delta}) = dim(\pi_{|D|,\delta}(V_{|D|,\delta}))$$

holds.

Remark 5.1.4 From Remark 5.1.2, when S is assumed to be also a regular surface, the moduli problem reduces to find for which S and for which divisor classes D on S $h^0(D, \mathcal{T}_S \mid_D) = 0$ and $h^0(C, \varphi^*(\mathcal{T}_S)) = 0$ hold, respectively.

As mentioned in the introduction of this chapter, there are some examples which show that the expected dimension for the moduli problem posed, for example, by the map

$$\pi_{|D|,0} : V_{|D|,0} \longrightarrow \mathcal{M}_{p_a(D)},$$

is not always achieved.

Negative answers:

1) Even if we are mainly interested in regular surfaces, the first trivial example is given by particular irregular surfaces. Indeed, consider C and D two smooth curves of geometric genera $g(C)$ and $g(D)$, respectively, each greater than 1. The surface $S = C \times D$ is a smooth surface of general type, with irregularity $q(S) = g(C) + g(D) \geq 4$. Such a surface contains two isotrivial (irrational) pencils of smooth curves.

2) In [122], Serrano studies, more generally, isotrivial fibrations. By *fibration*, there it is meant a morphism $\Psi : S \rightarrow C$, where S is a smooth, projective surface and C a smooth, projective curve, where the fibres of ψ are connected. When all smooth fibres are isomorphic to each other and non-rational, the fibration ψ is said to be *isotrivial*. This means that, if ψ is a fibration with general fibre isomorphic to a curve A, then there exists a smooth curve B and a finite group G, acting algebraically on A and B, such that S is birational to $(A \times B)/G$, $C \cong B/G$ and ψ commutes with the map $(A \times B)/G \rightarrow B/G$. The case of *quasi-bundle* is given when each singular fibre is a "multiple" in $Div(S)$ of a smooth curve. Such a surface determines an example of a *surface strongly isogenous to a product* and it is the case when the group G freely acts on the product. We shall discuss, later on, a simple case of a surface strongly isogenous to a product, which will give us an example of a smooth, regular, projective surface of general type which contains isotrivial families.

Turning back to the work of Serrano, he gives numerical characterization of surfaces which contain an isotrivial fibration. A key tool for his analysis is a result of Miyaoka, [93], who proves that if S is a surface with *positive index* then Ω^1_S is *almost-everywhere ample* (in short, *a.e. ample*), which means that it is a rank-two vector bundle that is ample outside a proper Zariski closed subset of S. We recall that the *index of a variety* is a linear combination of Chern or Pontrjagin numbers of X. In the case of a surface S, the index is defined by

$$\tau(S) := \frac{1}{3}(c_1^2(S) - 2c_2(S)) = 2 + 2p_g(S) - h^{1,1}(S). \tag{5.5}$$

What Serrano proves is the following numerical result.

Proposition 5.1.5 *(see [122], Proposition 5.3) If S admits an isotrivial fibration, then $K_S^2 \leq 2c_2(S)$.*

Proof: If $K_S^2 > 2c_2(S)$, then $\tau(S) > 0$; from Miyaoka's result Ω^1_S is a.e. ample. Let F be the generic fibre of the isotrivial map

$$\psi : S \longrightarrow C;$$

then $\Omega^1_S \mid_F$ is a.e. ample on F. Therefore $H^0(F, (\Omega^1_S \mid_F)^\vee) = 0$, which implies that $\psi_*(T_S) = 0$, which concludes the proof. □

The last assertion follows from Serrano's Lemma 3.2., in which he shows that, given a fibration $\psi : S \longrightarrow C$ with fibres of genus $g \geq 2$, then ψ is isotrivial if and only if $\psi_*(\mathcal{T}_S)$ is not the zero sheaf.

3) As announced before, we discuss a particular example of a surface strongly isogenous to a product. It belongs to a class of surfaces that Catanese has recently studied (see [19]) called *Beauville's surfaces* or *fake quadrics* (see [126], page 195). This consists in a regular surface of general type containing isotrivial fibrations and having the following numerical properties:

$$p_g(S) = q(S) = 0,\ b_2(S) = 2,\ K_S^2 = 8.$$

Observe that, since $b_1(S) = 2q(S) = 0$ and $p_g(S) = 0$, S is projective, as it follows from the Kodaira embedding theorem (see [75]).

To construct a concrete example of such a surface, we shall briefly discuss some useful details before.

Consider C_i a smooth curve of genus $g_i \geq 2$, $1 \leq i \leq 2$. Take S to be the surface

$$S := (C_1 \times C_2)/G,$$

where G is a finite group acting on each C_i and also freely acting on the product $C_1 \times C_2$. Therefore, the quotient is a smooth surface and the projection

$$p : C_1 \times C_2 \longrightarrow S$$

is a topological covering. From standard results on topological coverings of (algebraic) varieties, we get that $\kappa(S) = \kappa(C_1 \times C_2)$ where $\kappa(-)$ denotes the Kodaira dimension (see Section 1.3). This means that S is of general type.

It will be useful to recall the following well-known result.

Lemma 5.1.6 *Let X and Y be non-singular varieties and denote by $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ the canonical projections.*

(i) If $\mathcal{F}$ (resp. $\mathcal{E}$) is a vector bundle on X (resp. Y), then

$$H^0(X, \mathcal{F}) \otimes H^0(Y, \mathcal{E}) \cong H^0(X \times Y, p_1^*\mathcal{F} \otimes p_2^*\mathcal{E}).$$

(ii) If X and Y are smooth curves,

$$\Omega^1_{X \times Y} \cong p_1^*\Omega^1_X \oplus p_2^*\Omega^1_Y;$$

$$\omega_{X \times Y} \cong p_1^*\omega_X \otimes p_2^*\omega_Y.$$

Proof: See Fact III.22 in [10], page 36. □

By the previous lemma, in our case we obtain $K_{C_1 \times C_2} = p_1^*(K_{C_1}) + p_2^*(K_{C_2})$, so $K^2_{C_1 \times C_2} = 8(g_1 - 1)(g_2 - 1)$; on the other hand, since $p : C_1 \times C_2 \to S$ is a topological covering, then $K_{C_1 \times C_2} = p^*(K_S)$ therefore $K^2_{C_1 \times C_2} = deg(p)K^2_S$. If we take a suitable finite group, acting as stated above and of cardinality

$$| G |= (g_1 - 1)(g_2 - 1),$$

we find $K^2_S = 8$. If the action of G on C_i is such that $C_i/G \cong \mathbb{P}^1$, we immediately obtain $q(S) = p_g(S) = 0$. To prove this, it is useful to recall the following result.

Lemma 5.1.7 *Let X be a non-singular variety and let G be a finite subgroup of $Aut(X)$. Let $\pi : X \to Y = X/G$ be the natural projection. If π is ètale, then Y is non-singular; moreover*

$$H^0(Y, (\Omega^p_Y)^{\otimes k}) \cong H^0(X, (\Omega^p_X)^{\otimes k})^G, \;\; p \geq 0, \; k \geq 0.$$

If X is a curve and π is a ramification covering, the same is true only for $k = 1$.

Proof: See Lemma VI.11 and Exercise VI.12 in [10], page 78. □

In our case, from Lemma 5.1.7, we get

$$H^0(S, \Omega^1_S) \cong H^0(T, \Omega^1_T)^G;$$

by Lemma 5.1.6 and by the Leray isomorphism,

$$H^0(T, \Omega^1_T) \cong H^0(C_1, \omega_{C_1}) \oplus H^0(C_2, \omega_{C_2}).$$

Therefore,

$$H^0(S, \Omega^1_S) \cong H^0(C_1, \omega_{C_1})^G \oplus H^0(C_2, \omega_{C_2})^G.$$

Always by Lemma 5.1.6,

$$H^0(C_i, \omega_{C_i})^G \cong H^0(C_i/G, \omega_{C_i/G}), \; 1 \leq i \leq 2.$$

So, if $C_i/G \cong \mathbb{P}^1$, then $h^0(C_i, \omega_{C_i})^G = h^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}) = 0$, therefore $0 = h^0(S, \Omega^1_S) = h^1(S, \mathcal{O}_S) = q(S)$, by Dolbeault isomorphisms. Now, since $p : T \to S$ is a topological covering, then

$$\chi(\mathcal{O}_T) = (g_1 - 1)(g_2 - 1)\chi(\mathcal{O}_S);$$

from the fact that $T = C_1 \times C_2$, it follows that $\chi(\mathcal{O}_T) = (g_1 - 1)(g_2 - 1)$; therefore $\chi(\mathcal{O}_S) = 1$. Since we already know that $q(S) = 0$, we get $1 = \chi(\mathcal{O}_S) = 1 + p_g(S)$, so $p_g(S) = 0$.

For the last numerical invariant, we can consider Noether's formula

$$12\chi(\mathcal{O}_S) = e(S) + K_S^2,$$

where $e(S)$ denotes the Euler topological characteristic of S. In our case, we obtain

$$K_S^2 = 10 - h^{1,1}(S), \; e(S) = 2 + h^{1,1}(S).$$

Since $K_S^2 = 8$, then $h^{1,1}(S) = 2$. Thus, for the given hypotheses, the second Betti number $b_2(S) := 2p_g(S) + h^{1,1}(S)$ equals 2.

To sum up, by having supposed to have a finite group G, acting on the single curves and freely acting on the given curve product, such that

- $T := C_1 \times C_2$,
- $\mid G \mid = (g_1 - 1)(g_2 - 1)$,
- $S = T/G$ and $C_i/G \cong \mathbb{P}^1$

we have found that $p_g(S) = q(S) = 0$, $K_S^2 = 8$, $b_2(S) = 2$ and $\kappa(S) = 2$, which are actually the numerical invariants of a fake quadric.

Now we want to exhibit a concrete example of a surface $T = C_1 \times C_2$ and a finite subgroup $G < Aut(T)$, with the necessary properties of the actions on each C_i and on T, in such a way that the quotient $S = T/G$ satisfies all the above numerical conditions.

Consider $C_1 = C_2 = C$ a *Fermat plane quintic* of equation

$$x_0^5 + x_1^5 + x_2^5 = 0.$$

Then consider $T = C \times C$, where $g(C) = 6$.

Notation: For clarity sake, we shall write $C_{(i)}$ to denote the i^{th}-factor of the product, $1 \leq i \leq 2$, even if each factor is equal to the same curve C.

Let $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5$. Write $g = (a, b) \in \mathbb{Z}_5 \oplus \mathbb{Z}_5$ to denote an arbitrary element of the group.

G acts on:

1) $C_{(1)}$ by the following rule:

$$(a, b)[x^0, x^1, x^2] := [x^0, \zeta^a x^1, \zeta^b x^2],$$

where $[x^0, x^1, x^2]$ are the homogeneous coordinates of an arbitrary point on the curve $C_{(1)} \subset \mathbb{P}^2$ and ζ is a 5^{th}-primitive root of unity in $\mathbb{C}$;

2) $C_{(2)}$ by a similar rule of the one on $C_{(1)}$; indeed, the action of the element (a, b) on a point $[x_0, x_1, x_2]$ of $C_{(2)}$ is defined as in 1) but via the element of

the group $\varphi((a,b)) \in G$, where $\varphi \in Aut(\mathbb{Z}_5 \oplus \mathbb{Z}_5) \cong GL(2, \mathbb{Z}_5)$, represented by the following matrix:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

This φ is an automorphism of G which determines a free action of G on $T = C_{(1)} \times C_{(2)}$. Indeed, the action on T is defined by

$$(a,b)(P,Q) := ((a,b)P, \varphi((a,b))Q),$$

where $P \in C_{(1)}$ and $Q \in C_{(2)}$. Thus, $S = T/G$ is a smooth surface and $p : T \to S$ is an ètale covering of degree 25.

We now study the action of G on $C_{(1)}$ and then on the product T.

Let $\overline{P} \in C_{(1)}/G$ and $\epsilon_{(1)} : C_{(1)} \to C_{(1)}/G$ the canonical projection; then $P,\ Q \in \epsilon_{(1)}^{-1}(\overline{P})$ if and only if there exists $g \in G$ such $gP = Q$. Observe that:

(i) the points on $C_{(1)}$, of the form $Q_k = [0,\ 1,\ e_k]$, where $0 \leq k \leq 4$ and e_k a 5^{th}-root of -1, are stabilized by the following subgroup of G,

$$K := \{(a,a) \ \mid \ a \in \mathbb{Z}_5\} \cong \mathbb{Z}_5;$$

each point Q_k has ramification index equal to 5.

(ii) the points on $C_{(1)}$, of the form $T_h = [1,\ 0,\ e_h]$, where $0 \leq h \leq 4$ and e_h a 5^{th}-root of -1, are stabilized by the following subgroup of G,

$$H := \{(a,0) \ \mid \ a \in \mathbb{Z}_5\} \cong \mathbb{Z}_5;$$

each point T_h has ramification index equal to 5.

(iii) the points on $C_{(1)}$, of the form $P_j = [1,\ e_j,\ 0]$, where $0 \leq j \leq 4$ and e_j a 5^{th}-root of -1, are stabilized by the following subgroup of G,

$$J := \{(0,b) \ \mid \ b \in \mathbb{Z}_5\} \cong \mathbb{Z}_5;$$

each point P_j has ramification index equal to 5.

By Riemann-Hurwitz formula,

$$2g(C_{(1)}) - 2 = deg(\epsilon_{(1)})(2g(C_{(1)}/G) - 2) + deg(R),$$

where $deg(\epsilon_{(1)}) = 25$, $g(C_{(1)}) = 6$ whereas R denotes the ramification divisor; its degree is $deg(R) = 3 \cdot 4 \cdot 5 = 60$. Therefore, $g(C_{(1)}/G) = 0$ so $C_{(1)}/G \cong \mathbb{P}^1$.

Analogously, since the action of G on $C_{(2)}$ coincides, up to the automorphism φ of G, with the previous action, we have $C_{(2)}/G \cong \mathbb{P}^1$, which determines the ramification covering $\epsilon_{(2)} : C_{(2)} \to C_{(2)}/G$ (with the "translated" action of G via φ).

Now consider the action of G on the product $T = C_{(1)} \times C_{(2)}$. If $P_{(i)} \in C_{(i)}$, denote by $[x^0_{(i)}, x^1_{(i)}, x^2_{(i)}]$ its homogeneous coordinates. Thus,

$$(a,b)(P_{(1)}, P_{(2)}) = ((a,b)P_{(1)}, \varphi((a,b))P_{(2)}) =$$

$$= ([x^0_{(1)}, \zeta^a x^1_{(1)}, \zeta^b x^2_{(1)}], [x^0_{(2)}, \zeta^{\tilde{a}} x^1_{(2)}, \zeta^{\tilde{b}} x^2_{(2)}]),$$

where

$$\begin{cases} \tilde{a} \equiv a + 2b \mod 5, & \tilde{a} \in \mathbb{Z}_5 \\ \tilde{b} \equiv 3a + 4b \mod 5, & \tilde{b} \in \mathbb{Z}_5 \end{cases} \quad (*)$$

Therefore, one has that

$$(a,b)(P_{(1)}, P_{(2)}) = (P_{(1)}, P_{(2)}) \quad (**)$$

if and only if $P_{(1)}$ and $P_{(2)}$ are ramification points of $\epsilon_{(1)} : C_{(1)} \to C_{(1)}/G$ and $\epsilon_{(2)} : C_{(2)} \to C_{(2)}/G$, respectively. If we go back to the ramification points of $\epsilon_{(1)} : C_{(1)} \to C_{(1)}/G$, in cases (i), (ii) and (iii) discussed above, one easily deduces that $(**)$ can never occur. To see this, we consider, for example, case (i); the others are completely the same.

We know that the points on $C_{(1)}$ of the form $Q_k = [0,\ 1,\ e_k]$, $0 \leq k \leq 4$, $(e_k)^5 = -1$, are ramification points for $\epsilon_{(1)} : C_{(1)} \to C_{(1)}/G$, being stabilized by the subgroup $K = \{(a,a) \mid a \in \mathbb{Z}_5\}$. If we consider the transforms of the pairs in K, via the automorphism φ, from the congruence system $(*)$ we get all pairs of the form $\{(3a, 2a) \mid a \in \mathbb{Z}_5\}$, which determine the subgroup

$$\{(0,0),\ (3,2),\ (1,4),\ (4,1),\ (2,3)\} < \mathbb{Z}_5 \oplus \mathbb{Z}_5.$$

Therefore, if there existed ramification points also for $\epsilon_{(2)} : C_{(2)} \to C_{(2)}/G$, such points should be stabilized by this subgroup; but this subgroup does not have ramification points on $C_{(2)}$.

By considering also the other cases, we determine that the action of G on T, defined above, is free. Since $g(C_{(i)}) = 6$, the quotients $C_{(i)}/G \cong \mathbb{P}^1$, $S = T/G$ is a smooth surface of general type and $\mid G \mid = 25$, so S is a Beauville surface (equiv. a fake quadric).

Since there is an action of G on each $C_{(i)}$ and on the product T, we also have an action of G on the fibres of the canonical projections. Indeed, denote by $\pi_{(1)}$ the first projection

$$\begin{array}{c} T = C_{(1)} \times C_{(2)} \\ \downarrow \\ C_{(1)}. \end{array}$$

We get a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{G} & T \\ \downarrow & & \downarrow \\ C_{(1)} & \xrightarrow{G} & C_{(1)}, \end{array}$$

where the vertical arrows are given by $\pi_{(1)}$, such that

$$\pi_{(1)}^{-1}(g(P_{(1)})) = g(\pi_{(1)}^{-1}(P_{(1)})),$$

for all $g \in G$, $P_{(1)} \in C_{(1)}$, where

$$\pi_{(1)}^{-1}(P_{(1)}) = \{P_{(1)}\} \times C_{(2)} = \{(P_{(1)}, P_{(2)}) \mid P_{(2)} \in C_{(2)}\} \cong C_{(2)}.$$

If $P_{(1)} \in C_{(1)}$ is not a ramification point for $\epsilon_{(1)} : C_{(1)} \to C_{(1)}/G$, then its isotropy subgroup is trivial; therefore, the orbit of $P_{(1)}$ consists of 25 distinct points. This means that the set

$$\{\pi_{(1)}^{-1}(g(P_{(1)}))\}_{g \in G}$$

consists of 25 distinct copies of $C_{(2)}$. If, as before, we denote by R the ramification divisor of $\epsilon_{(1)} : C_{(1)} \to C_{(1)}/G$, then the family

$$\{\pi_{(1)}^{-1}((P_{(1)}))\}_{P_{(1)} \in C_{(1)} \setminus R}$$

is an irrational pencil of curves isomorphic to $C_{(2)}$, parametrized by an open dense subset of $C_{(1)}$. In the topological covering

$$p : T \to S$$

the 25 copies of $C_{(2)}$, in $\{\pi_{(1)}^{-1}(g(P_{(1)}))\}_{g \in G}$, identify to a smooth curve $\overline{C}_{(2)} \subset S$ which is isomorphic to $C_{(2)}$.

If $P_{(1)} \in R \subset C_{(1)}$ is a ramification point for $\epsilon_{(1)} : C_{(1)} \to C_{(1)}/G$, such a point is stabilized by a subgroup of index 5 in G. Denote by $G_{P_{(1)}}$ this isotropy subgroup; then

$$\{\pi_{(1)}^{-1}(g(P_{(1)}))\}_{g \in G_{P_{(1)}}}$$

is a non-reduced curve consisting of 5 identified copies of $C_{(2)}$. Therefore, the set

$$\{\pi_{(1)}^{-1}(g(P_{(1)}))\}_{g \in G}$$

is an orbit consisting of five elements, each of these being a non-reduced curve of the form $D_{(2)} = 5C_{(2)}$. In the topological covering $p : T \to S$, these five elements in the orbit will be identified to a non reduced curve of the form $\overline{D}_{(2)} = 5\overline{C}_{(2)}$ in S.

In conclusion, we have determined in S an isotrivial rational pencil of smooth curves of genus 6, parametrized by an open dense subset of $C_{(1)}/G \cong \mathbb{P}^1$. The singular fibres are non-reduced curves, each isomorphic to five copies of a general, smooth fibre. Observe that $\overline{C}_{(i)}^2 = 0$, for each i, since they are fibres. Moreover, for each $1 \leq i \leq 2$, we have the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(\overline{C}_{(i)}) \to \mathcal{O}_{\overline{C}_{(i)}}(\overline{C}_{(i)}) \to 0.$$

From the regularity of S and the fact that $deg(\mathcal{O}_{\overline{C}_{(i)}}(\overline{C}_{(i)})) = 0$, we get that $dim(\mid \mathcal{O}_S(\overline{C}_{(i)}) \mid) = 1$, so the complete linear system coincides with the constructed isotrivial family.

Remark. The previous example shows that we cannot expect to have, "tout court", an affirmative answer to the moduli problem, even in the case of projective, minimal and regular surfaces of general type. Indeed, note that the surface constructed above is also minimal; otherwise, if there existed a rational smooth curve E in S, $p^{-1}(E)$ should be a disjoint union of rational curves in T which, being not contained in the fibres, should project isomorphically to the bases $C_{(i)}$, $1 \leq i \leq 2$, that is a contradiction. Observe that the divisors of the form $\overline{C}_{(i)}$ are not ample on S, since they are fibres in S. Moreover, these divisors are not even (integral or rational) multiples of the canonical divisor, since they have zero self-intersection, whereas $K_S^2 = 8$. This is an important remark, as it will be clear after having considered the next examples and our new results in Section 5.2.

We now discuss some examples of families of curves on a smooth and regular surface S of general type which give, on the contrary, positive answers to the moduli problem posed in Definition 5.1.3.

Some cases of positive answers:

A direct approach which can be used for the study of the moduli problem consists in using the notion of stability of vector bundles on smooth surfaces and curves. Let's restrict ourselves to the smooth case, i.e. let's consider smooth curves in a given linear system $\mid D \mid$, where D is, for example, an ample divisor on S. We want to find conditions, on the chosen divisor class, in such a way that the dimension of the general fibre of the map

$$\pi_{|D|,0} : V_{|D|,0} \longrightarrow \mathcal{M}_{p_a(D)}$$

is zero. From assumptions on S and from Remark 5.1.4, computing the dimension of the general fibre ${\pi_{|D|,0}}^{-1}([D])$ reduces to determining the dimension of $H^0(D, \mathcal{T}_S \mid_D)$. So we are interested in cases in which this space vanishes.

To this aim, one partially successful method is to use the semi-stability of vector bundles on S restricted to D. We have recalled, in Definition 1.2.11, the notion of (Mumford-Takemoto) stability.

Denote by D the general element in $\mid D \mid$, which is supposed to be a smooth, irreducible curve on S. Assume that K_S is an effective divisor. If $\mathcal{T}_S \mid_D$ were stable on D, it couldn't have global sections. Indeed, from the normal sequence

$$0 \to \mathcal{T}_D \to \mathcal{T}_S \mid_D \to \mathcal{N}_{D/S} \to 0$$

and from adjunction formula, we get that $deg(\mathcal{T}_S \mid_D) = 2 - 2g + D^2 = -DK_S < 0$, since D is very ample and K_S is assumed to be effective. So, if we compute the slope of this rank-two vector bundle on D, we get

$$\mu(\mathcal{T}_S \mid_D) = \frac{-DK_S}{2} < 0.$$

Thus, if $\mathcal{T}_S \mid_D$ is stable, there cannot exist a non-zero global section, i.e. an injective sheaf morphism

$$0 \to \mathcal{O}_D \to \mathcal{T}_S \mid_D,$$

otherwise we would have $\mu(\mathcal{O}_D) = 0 > \mu(\mathcal{T}_S \mid_D)$ which contradicts the stability property.

We recall that the tangent bundle of a surface of general type is Bogomolov-stable (see Section 1.3). So it would be sufficient to know some criteria which establish that, given a smooth curve D on S, the restriction of a Bogomolov-stable rank-two vector bundle on S to D remains stable (in the sense of Mumford-Takemoto). There are some results answering to this question (see [43] for a detailed overview and analysis): Mehta and Ramanathan, [90], proved that for a given semistable vector bundle $\mathcal{E}$ on S, there always exists a number d_0 such that the restriction of $\mathcal{E}$ to a general curve of degree $d \geq d_0$ is semistable. Unfortunately, the bound d_0 dependes on $\mathcal{E}$ itself and not only on numerical invariants of $\mathcal{E}$ such as the rank or the Hilbert polynomial. In a later work, [42], Flenner found numerical conditions in order that a given semistable vector bundle on a n-dimensional, normal, projective variety X restricts to a general hypersurface $H \in | \mathcal{O}_X(d) |$ in such a way that it remains semistable. These conditions determine a lower-bound d_0 such that, for each $d > d_0$ the restriction property holds. However this bound d_0, depending on n, is very large. Other papers, which deal with this subject are, for example, the work of Miyaoka, [94], and a recent article of Hein, [65]. Both these authors use arguments in prime characteristic to deduce some results of semistability in characteristic zero. However, the bounds are always very large.

Another approach that could be used consists in applying discriminants and elementary modifications of vector bundles on S along D (see Definitions 1.2.15, 1.2.16 and Proposition 1.2.17). Since the dual of a stable vector bundle is stable, we can prove that, for some regular surfaces S of general type, the restriction to a canonical divisor of Ω^1_S is stable.

Proposition 5.1.8 *Let S be a smooth, projective surface of general type (not necessarily regular), with positive index, i.e. $\tau(S) = \frac{1}{3}(c_1^2 - 2c_2) > 0$, and effective canonical divisor. Let $D \sim K_S$ on S. Then, $\Omega^1_S |_D$ is stable.*

Proof: By contradiction, assume that $\Omega^1_S |_D$ is unstable. Therefore, there must exist a destabilizing projection

$$\Omega^1_S |_D \xrightarrow{h} \mathcal{F}.$$

If we compose this projection with the restriction map to D, we obtain

$$\Omega^1_S \xrightarrow{r_D} \Omega^1_S |_D \xrightarrow{h} \mathcal{F}.$$

This gives the following exact sequence on S:

$$0 \to T_{D,\mathcal{F}}(\Omega^1_S) \to \Omega^1_S \to i_*(\mathcal{F}) \to 0,$$

where $i : D \hookrightarrow S$ is the inclusion of D in S and $T_{D,\mathcal{F}}(\Omega^1_S)$ is the elementary modification of Ω^1_S along D via $\mathcal{F}$. If $\delta(\mathcal{E}) = \frac{c_1(\mathcal{E})^2}{4} - c_2(\mathcal{E})$ denotes the discriminant of a vector bundle $\mathcal{E}$, then by Bogomolov's criterion (Theorem 1.2.13) $\mathcal{E}$ is unstable if and only if $\delta(\mathcal{E}) > 0$. From Proposition 1.2.17,

$$c_1(T_{D,\mathcal{F}}(\Omega^1_S)) = c_1(\Omega^1_S) - D = K_S - D \sim 0,$$

since $D \sim K_S$. Moreover,

$$\delta(T_{D,\mathcal{F}}(\Omega^1_S)) = \delta(\Omega^1_S) + \frac{1}{2}c_1(\Omega^1_S)D - c_1(\mathcal{F}) + \frac{1}{4}D^2$$

$$= \frac{1}{4}K_S^2 - c_2(S) + \frac{1}{4}K_S^2 + (\frac{1}{2}K_S^2 - deg(\mathcal{F})) = \frac{1}{2}K_S^2 - c_2(S) + \alpha,$$

with $\alpha \geq 0$ since, by hypothesis, $\mathcal{F}$ destabilizes $\Omega^1_S |_D$ so $deg(\mathcal{F}) \geq \frac{1}{2}K_S^2$. Therefore, by assumptions on S, $\delta(T_{D,\mathcal{F}}(\Omega^1_S)) = \tau(S) + \alpha > 0$. This implies that $T_{D,\mathcal{F}}(\Omega^1_S)$ is Bogomolov-unstable, i.e. there exists a line bundle $L \subset T_{D,\mathcal{F}}(\Omega^1_S)$ such that $L^2 > 0$ and $L \in N^+(S)$ the ample divisor cone. Since $L \subset T_{D,\mathcal{F}}(\Omega^1_S) \subset \Omega^1_S$, this determines the instability of Ω^1_S, which is a contradiction, being S of general type. □

Corollary 5.1.9 *Let S be as in Proposition 5.1.8 and suppose that it is also regular with $K_S^2 > 0$. Then the family $V_{|K_S|,0}$ has the expected number of moduli.*

Proof: From the previous proposition and the fact that the stability is invariant under duality, we conclude that the restriction of the tangent bundle $\mathcal{T}_S = (\Omega_S^1)^\vee$ to a smooth canonical divisor is a stable vector bundle. For what we have observed before Proposition 5.1.8, $h^0(\mathcal{T}_S \mid_D) = 0$. From Remark 5.1.4, we get the statement. □

Remark 5.1.10 Observe that, if S is as in Corollary 5.1.9, it can be neither a surface of degree $d \geq 5$ in $\mathbb{P}^3$, since for such surfaces $c_1^2 < c_2$, nor a complete intersection in $\mathbb{P}^{r+2}$, $r \geq 2$, since in these cases we have $c_1^2 \leq \frac{2r}{r+1} c_2$, which contradicts the positivity of the index. Moishezon and Teicher proved, in [96], that there exists an infinite class of examples of surfaces of general type with positive index which are simply connected. The triviality of the fundamental group implies that, for such a S, the first Betti number $b_1(S)$ equals 0, so the regularity of S.

The numerical techniques of Proposition 5.1.8 can be easily extended to surfaces of general type, which are not assumed to have necessarily positive index, and to $\mathbb{Q}$-divisors which are numerically equivalent to rational multiples of the canonical divisor.

Proposition 5.1.11 *Let S be a smooth, projective and regular surface of general type. Assume that K_S is effective with $K_S^2 > 0$. Let D be a divisor which is numerically equivalent to $\frac{p}{q} K_S$, where p and q are positive and relatively prime integers, such that*

$$(*) \quad \frac{p^2 + q^2}{4q^2} K_S^2 - e(S) > 0,$$

where $e(S)$ denotes the Euler-Poincare' characteristic of S. Then, the family $V_{|D|,0}$ has the expected number of moduli.

Proof: By applying the same procedure of Proposition 5.1.8, assume by contradiction that there exists a destibilizing sheaf $\mathcal{F}$ for $\Omega_S^1 \mid_D$; thus, $deg(\mathcal{F}) \leq \frac{p}{2q} K_S^2$. Therefore,

$$\delta(T_{D,\mathcal{F}}(\Omega_S^1)) = \frac{1}{4} K_S^2 - c_2(S) + \frac{1}{2} K_S (\frac{p}{q} K_S) + (\frac{p^2}{2q^2} K_S^2 - deg(\mathcal{F})).$$

By $(*)$ and by hypothesis on $\mathcal{F}$, $\delta(T_{D,\mathcal{F}}(\Omega_S^1)) > 0$ which implies the Bogomolov-instability of $\delta(T_{D,\mathcal{F}}(\Omega_S^1))$ and so of Ω_S^1, which is absurd. Then one concludes as in Corollary 5.1.9. □

Remark 5.1.12 If, for example, $S \subset \mathbb{P}^3$ is a general quintic and $q = 1$, then $e(S) = 55$ and $D \sim pH$, H the plane divisor of S, the above result holds as soon as $p \geq 7$. If $S \subset \mathbb{P}^3$ is general and of degree 6, then $e(S) = 108$, so $(*)$ reduces to $p^2 > 17q^2$. Thus, the previous result holds if $D \sim aH$, with $a \geq 9$. Similar computations can be done for $S_d \subset \mathbb{P}^3$, $d \geq 7$ and/or $q \geq 2$.

5.2 Moduli of curves in a given linear system on a surface of general type

In this section we shall study the number of moduli of families of irreducible, nodal curves on some surfaces of general type.

For what concerns other classes of projective surfaces, we only want to mention that, apart from the paper of Sernesi for the plane case ([119], see also Section 2.3), there are also some results related to this subject in the rational surface case. Indeed, Beauville ([9]) studied the problem of classifying smooth, projective hypersurfaces with isomorphic smooth hyperplane sections and proved that (in characterictic zero) the only hypersurfaces with such a property are hyperquadrics and hyperplanes. Some properties of projective varieties with isomorphic or birational hyperplane sections have been shown by McKernan, [89]. Pardini ([105]), on the other hand, has studied the projective varieties with the property that there exists a projective isomorphism between two of their generic hyperplane sections. In the case of surfaces, she has given the correct proof of classical statements of Fubini and Fano, by using the approach of Ciliberto, [28], based on the fact that the "special" hyperplane sections, in particular all the tangent sections, of a variety with projectively isomorphic hyperplane sections, admit infinitely many projective automorphisms. This classical result establishes that if S is a surface of degree d in $\mathbb{P}^r$ (not necessarily smooth), with projectively equivalent hyperplane sections, then S must be: a cone, a projectively ruled rational surface or a projection of a Veronese surface.

We now turn back to our problem. From now on, S will be a smooth, projective and regular surface of general type, unless otherwise specified. $\mid D \mid$ will denote a complete linear system on S, whose general element, D, is assumed to be smooth and irreducible. Our approach consists in the infinitesimal study of the morphism

$$\pi_{|D|,\delta} : V_{|D|,\,\delta} \to \mathcal{M}_g,$$

where $\delta \geq 0$, and $g = p_a(D) - \delta$ is the geometric genus of the smooth curve C obtained via the normalization map $\varphi : C \to X \subset S$ of the nodal curve X corresponding to $[X] \in V_{|D|,\,\delta}$.

For what observed in Remark 5.1.4, our aim is to find conditions on S and D to get the vanishing of $H^0(\mathcal{T}_S\mid_D)$ and of $H^0(\varphi^*(\mathcal{T}_S))$, respectively.

We start by proving the following crucial and more general result.

Theorem 5.2.1 *Let S be a smooth, projective surface of general type (not necessarily regular). Let $X \sim D$ be an irreducible, δ-nodal curve, $\delta \geq 0$, whose set of nodes is denoted by N. Then,*

$$h^1(S,\, \mathcal{I}_N \otimes \Omega^1_S(D + K_S)) = 0 \Rightarrow h^0(C,\, \varphi^*(\mathcal{T}_S)) = 0. \qquad (5.6)$$

In particular, when $\delta = 0$,

$$h^1(S,\ \Omega^1_S(D+K_S))=0 \Rightarrow h^0(D,\ \mathcal{T}_S\mid_D)=0 \tag{5.7}$$

Proof: If $N \neq \emptyset$, denote by $\mu : \tilde{S} \to S$ the blow-up of S in N, so that one can consider the following diagram of morphisms

$$\begin{array}{ccc} C & \subset & \tilde{S} \\ \downarrow^{\varphi} & & \downarrow^{\mu} \\ X & \subset & S \end{array} .$$

Thus, we obtain

$$H^0(\varphi^*(\mathcal{T}_S)) = H^0(\mu^*(\mathcal{T}_S)\mid_C).$$

If we tensor the exact sequence defining C in $\tilde{S}$ with $\mu^*(\mathcal{T}_S)$, we get the following exact sequence on $\tilde{S}$:

$$0 \to \mu^*(\mathcal{T}_S)(-C) \to \mu^*(\mathcal{T}_S) \to \mu^*(\mathcal{T}_S)\mid_C \to 0. \tag{5.8}$$

By the Leray isomorphism, we have

$$H^0(\mu^*(\mathcal{T}_S)) = H^0(\mathcal{T}_S) = 0,$$

since $H^0(\mathcal{T}_S)$ is isomorphic to the Lie algebra of the Lie group $Aut(S)$, which is finite by assumption on S (see Section 1.3). Thus, the cohomological exact sequence associated to (5.8) reduces to

$$0 \to H^0(\mu^*(\mathcal{T}_S)\mid_C) \to H^1(\mu^*(\mathcal{T}_S)(-C)) \to \cdots.$$

A sufficient condition for $h^0(\varphi^*(\mathcal{T}_S))=0$ is therefore $h^1(\mu^*(\mathcal{T}_S)(-C))=0$. By Serre duality on $\tilde{S}$, we have the following equality

$$h^1(\mu^*(\mathcal{T}_S)(-C)) = h^1((\mu^*(\mathcal{T}_S))^\vee \otimes \mathcal{O}_{\tilde{S}}(K_{\tilde{S}}+C)). \tag{5.9}$$

Since $\mathcal{T}_S$ is locally free on S, then

$$\mu^*(\mathcal{T}_S)^\vee = \mu^*(\mathcal{T}_S^\vee) = \mu^*(\Omega^1_S);$$

so (5.9) becomes

$$h^1(\mu^*(\mathcal{T}_S)(-C)) = h^1(\mu^*(\Omega^1_S)(K_{\tilde{S}}+C)). \tag{5.10}$$

Denote by B the μ-exceptional divisor in $\tilde{S}$ such that $B=\Sigma_{i=1}^{\delta}E_i$. From standard computations with blow-ups, we get

$$\mu^*(X) = C + 2B$$

and

$$\mu^*(K_S) = K_{\tilde{S}} - B,$$

so

$$K_{\tilde{S}} + C = \mu^*(K_S+X) - B.$$

Therefore, the right-hand side of the equality (5.10) becomes

$$h^1(\mu^*(\Omega^1_S)(K_{\tilde{S}}+C)) = h^1(\mu^*(\Omega^1_S(K_S+X)) \otimes \mathcal{O}_{\tilde{S}}(-B)).$$

By using Lemma 1.2.6, we have

$$H^1(\mu^*(\Omega^1_S(K_S + X)) \otimes \mathcal{O}_{\tilde{S}}(-B)) \cong H^1(\mathcal{I}_N(X + K_S) \otimes \Omega^1_S).$$

Since $X \sim D$ on S, we get statement (5.6).

For (5.7), i.e. $\delta = 0$, one can directly use the exact sequence

$$0 \to \mathcal{T}_S(-D) \to \mathcal{T}_S \to \mathcal{T}_S\mid_D \to 0.$$

□

In the sequel, we will be concerned about finding conditions on divisors D on $S \subset \mathbb{P}^r$ in order that

$$H^1(S,\ \Omega^1_S(K_S + D)) = 0 \tag{5.11}$$

and

$$H^1(S,\ \mathcal{I}_N \otimes \Omega^1_S(K_S + D)) = 0 \tag{5.12}$$

hold, where N is the set of nodes of an irreducible, nodal curve $X \sim D$.

Our first result in this direction is the following.

Theorem 5.2.2 *Let $S \subset \mathbb{P}^r$ be a smooth and regular surface of general type with hyperplane divisor H. Suppose that the linear system $\mid D \mid$ on S has general element which is a smooth, irreducible curve. Let $X \sim D$ be an irreducible, δ-nodal curve, $\delta \geq 0$, of geometric genus $g = p_a(D) - \delta$. Assume that:*

(i) $\Omega^1_S(K_S)$ is globally generated;

(ii) $D \sim K_S + 6H + L$, where L is an ample divisor;

(iii) the Severi variety $V_{|D|,\delta}$ is regular at $[X]$ (in the sense of Definition 2.2.30).

Then, the morphism

$$\pi_{|D|,\delta} : V_{|D|,\delta} \to \mathcal{M}_g$$

has injective differential at $[X]$. In particular, $\pi_{|D|,\delta}$ has finite fibres on each regular component of $V_{|D|,\delta}$, so each such component parametrizes a family having the expected number of moduli.

Proof: First of all, we want to show that hypothesis (ii) implies the vanishing in (5.11). To prove this, we will use Griffiths vanishing results (see Theorems 1.2.7, 1.2.8 and Corollary 1.2.10). Therefore, the first step of our analysis is to apply such vanishing results to the vector bundle

$$\mathcal{E} = \Omega^1_S(mH),$$

where m is a positive integer. We have $det(\mathcal{E}) = K_S + 2mH$.

If we want to apply Theorem 1.2.7, with $l = q = 1$ and $\mathcal{E} = \Omega^1_S(mH)$, we get that

$$K_S \otimes \mathcal{E} \otimes det(\mathcal{E}) \otimes \mathcal{O}_S(L) = \Omega^1_S \otimes \mathcal{O}_S(K_S + K_S + 3mH + L),$$

where L is an ample divisor. In this case, if we find an m_0 such that, for $m \geq m_0$, $\Omega^1_S(mH)$ is globally generated on S and if we take L an ample divisor on S, then the hypotheses of Theorem 1.2.7 are satisfied and for each curve $D \sim K_S + 3mH + L$ the vanishing in (5.11) holds, as soon as $m \geq m_0$.
If we want to apply Corollary 1.2.10, with $k = l = q = 1$ and $\mathcal{E} = \Omega^1_S(mH)$, we get that

$$K_S \otimes \mathcal{E} \otimes det(\mathcal{E}) = \Omega^1_S \otimes \mathcal{O}_S(K_S + K_S + 3mH).$$

In this case, if we find an m_1 such that, for $m \geq m_1$, $\Omega^1_S(m)$ is an ample vector bundle on S, then the hypotheses of Corollary 1.2.10 are satisfied; so it is sufficient to consider divisors $D \sim K_S + 3mH$, to get the vanishing in (5.11), as soon as $m \geq m_1$.

The problem reduces to finding which "twists" of Ω^1_S are globally generated or ample on a smooth, projective surface.

In the sequel we shall shortly write $\Omega^1_S(m)$ instead of $\Omega^1_S(mH)$. Now, since $S \subset \mathbb{P}^r$ is a smooth surface, we have the conormal sequence

$$0 \to Con_{S/\mathbf{P}^r}(m) \to \Omega^1_{\mathbf{P}^r}(m) \mid_S \to \Omega^1_S(m) \to 0.$$

From Proposition 1.2.1, we have to compute for which positive integers m

$$\Omega^1_{\mathbf{P}^r}(m)$$

is ample or globally generated.
If we dualize the Euler sequence of $\mathbb{P}^r$ (see (1.2) in Section 1.1),

$$0 \to \mathcal{O}_{\mathbf{P}^r} \to \mathcal{O}_{\mathbf{P}^r}^{\oplus(r+1)}(1) \to \mathcal{T}_{\mathbf{P}^r} \to 0,$$

we get

$$0 \to \Omega^1_{\mathbf{P}^r} \to \mathcal{O}_{\mathbf{P}^r}^{\oplus(r+1)}(-1) \to \mathcal{O}_{\mathbf{P}^r} \to 0. \tag{5.13}$$

Since an exact sequence of vector bundles of the form

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{O}_{\mathbf{P}^r} \to 0$$

yields exact sequences

$$0 \to \bigwedge^p \mathcal{E} \to \bigwedge^p \mathcal{F} \to \bigwedge^{p-1} \mathcal{E} \to 0,$$

for $p \geq 1$ (see [104], page 6, and [128], page 73), from (5.13) and from the choice $p = 2$, we obtain

$$0 \to \Omega^2_{\mathbf{P}^r} \to \mathcal{O}_{\mathbf{P}^r}^{\oplus\frac{r(r+1)}{2}}(-2) \to \Omega^1_{\mathbf{P}^r} \to 0.$$

Therefore, one trivially has

$$0 \to \Omega^2_{\mathbf{P}^r}(2) \to \mathcal{O}_{\mathbf{P}^r}^{\oplus\frac{r(r+1)}{2}} \to \Omega^1_{\mathbf{P}^r}(2) \to 0$$

and also

$$0 \to \Omega^2_{\mathbb{P}^r}(3) \to \mathcal{O}_{\mathbb{P}^r}^{\oplus \frac{r(r+1)}{2}}(1) \to \Omega^1_{\mathbb{P}^r}(3) \to 0.$$

From these exact sequences we get that

(a) $\Omega^1_{\mathbb{P}^r}(2)$ is globally generated;

(b) $\Omega^1_{\mathbb{P}^r}(m)$ is globally generated and ample, for $m \geq 3$.

By turning back to the vanishing in (5.11), we get the following possibilities.

(1) If $D \sim K_S + 6H + L$ (i.e. when m=2), where L an ample divisor on S, then the vanishing holds by Theorem 1.2.7;

(2) If $D \sim K_S + 3mH$, with $m \geq 3$, then the vanishing holds by Corollary 1.2.10. However, since $m \geq 3$, $K_S + 3mH = K_S + 6H + (3m-6)H$, where $3m - 6 \geq 3$. So, once again, we are in case (1).

To sum up, if $D \sim K_S + 6H + L$, L an ample divisor, $H^1(S,\ \Omega^1_S(K_S+D)) = 0$. This vanishing is a fundamental tool for the next second part of the proof.

On S we can consider the exact sequence

$$0 \to \mathcal{I}_N(D) \to \mathcal{O}_S(D) \to \mathcal{O}_N(D) \to 0 \tag{5.14}$$

which determines the map ρ_D,

$$0 \to H^0(\mathcal{I}_N(D)) \to H^0(\mathcal{O}_S(D)) \stackrel{\rho_D}{\to} H^0(\mathcal{O}_N(D)) \to H^1(\mathcal{I}_N(D)) \to \cdots.$$

By hypothesis (iii), ρ_D is surjective.

Next, if we tensor the exact sequence (5.14) with $\Omega^1_S(K_S)$ and if we take the associated cohomology sequence, we get

$$0 \to H^0(\mathcal{I}_N(D) \otimes \Omega^1_S(K_S)) \to H^0(\Omega^1_S(K_S + D)) \stackrel{\rho_{\Omega^1_S(K_S+D)}}{\longrightarrow}$$

$$\stackrel{\rho_{\Omega^1_S(K_S+D)}}{\longrightarrow} H^0(\mathcal{O}_N(\Omega^1_S(K_S + D))) \cong \mathbb{C}^{2\delta} \to H^1(\mathcal{I}_N(D) \otimes \Omega^1_S(K_S)) \to$$

$$\to H^1(\Omega^1_S(K_S + D)) \to 0.$$

Thus, the map $\rho_{\Omega^1_S(K_S+D)}$ is surjective if and only if $H^1(\mathcal{I}_N(D) \otimes \Omega^1_S(K_S)) \cong H^1(\Omega^1_S(K_S + D))$. For what we have shown in the first part of this proof, hypothesis (ii) implies that $H^1(\Omega^1_S(K_S + D)) = 0$. So, if D is as in (ii), we have

$$h^1(\mathcal{I}_N(D) \otimes \Omega^1_S(K_S)) = 0 \Leftrightarrow \ \rho_{\Omega^1_S(K_S+D)} \textit{ surjective.}$$

By (5.6) of Theorem 5.2.1, the surjectivity of $\rho_{\Omega^1_S(K_S+D)}$ implies therefore that $h^0(\varphi^*(\mathcal{T}_S)) = 0$ and so the statement.

The last step is to determine if, with the given hypotheses, the map $\rho_{\Omega^1_S(K_S+D)}$ is surjective. Recall that hypotesis (iii) implies the surjectivity of ρ_D. Consider the map

$$H^0(\Omega^1_S(K_S + D)) \stackrel{\rho_{\Omega^1_S(K_S+D)}}{\longrightarrow} H^0(\mathcal{O}_N(\Omega^1_S(K_S + D)) \cong \mathbb{C}^{2\delta} \cong \bigoplus_{i=1}^{\delta} \mathbb{C}^2_{(i)}. \tag{5.15}$$

By hypothesis (i), for each $p \in S$, the map

$$H^0(\Omega^1_S(K_S)) \otimes \mathcal{O}_{S,p} \to \Omega^1_S(K_S) \mid_p \cong \mathcal{O}_{S,p}^{\oplus 2}$$

is surjective; thus, for each $p \in S$ there exist two global sections

$$s_1^p,\ s_2^p \in H^0(\Omega^1_S(K_S))$$

which generate the stalk $\Omega^1_S(K_S)\mid_p$ as an $\mathcal{O}_S$-module, i.e.

$$s_1^p(p) = (1,0) \in \mathcal{O}_{S,p}^{\oplus 2}\ ,$$

$$s_2^p(p) = (0,1) \in \mathcal{O}_{S,p}^{\oplus 2}\ .$$

If $N = \{p_1,\ p_2,\ \ldots,\ p_\delta\}$ is the set of nodes of X, then

$$H^0(\mathcal{O}_N(D)) = \mathbb{C}^\delta \cong \mathbb{C}_{(1)} \oplus \mathbb{C}_{(2)} \oplus \cdots \oplus \mathbb{C}_{(\delta)}.$$

The surjectivity of ρ_D implies there exist global sections $\sigma_i \in H^0(\mathcal{O}_S(D))$ such that

$$\sigma_i(p_j) = (0,\ 0,\ \ldots,\ 0),\ if\ 1 \leq i \neq j \leq \delta,$$

$$\sigma_i(p_i) = (0,\ \ldots,\ 0,\ 1,\ 0,\ \ldots,\ 0),\ 1 \in \mathbb{C}_{(i)},\ 1 \leq i \leq \delta.$$

Therefore, $s_1^{p_i} \otimes \sigma_i,\ s_2^{p_i} \otimes \sigma_i \in H^0(\Omega^1_S(D+K_S))$ and

$$s_1^{p_i} \otimes \sigma_i(p_j) = s_2^{p_i} \otimes \sigma_i(p_j) = (0,\ \ldots,\ 0) \in \mathbb{C}^2_{(1)} \oplus \cdots \oplus \mathbb{C}^2_{(\delta)} \cong \mathbb{C}^{2\delta},\ 1 \leq i \neq j \leq \delta,$$

$$s_1^{p_i} \otimes \sigma_i(p_i) = ((0,0),\ \ldots,\ (1,0),\ \ldots,\ (0,0)) = (0,\ \ldots,\ 1,\ 0,\ \ldots,\ 0) \in \mathbb{C}^{2\delta},$$

where $(1,0) \in \mathbb{C}^2_{(i)}$ and

$$s_2^{p_i} \otimes \sigma_i(p_i) = ((0,0),\ \ldots,\ (0,1),\ \ldots,\ (0,0)) = (0,\ \ldots,\ 0,\ 1,\ \ldots,\ 0) \in \mathbb{C}^{2\delta},$$

where $(0,\ 1) \in \mathbb{C}^2_{(i)}$, for $1 \leq i \leq \delta$. This means that the map (5.15) is surjective. Moreover, since the condition for a point $[X] \in V_{|D|,\delta}$ to be regular is an open condition in the family, it follows that the component of $V_{|D|,\delta}$ containing $[X]$ has the expected number of moduli. □

Corollary 5.2.3 *If $S_d \subset \mathbb{P}^3$ is a smooth surface of degree $d \geq 6$, the regular components of $V_{|nH|,\delta}$, $n \geq d+3$, have the expected number of moduli*

Observe that, in the case $\delta = 0$ and S_d general, once again we find what computed in Remark 5.1.12 (see the case S_6).

Remark 5.2.4 The use of Griffiths vanishing results has been a fundamental tool for our previous analysis; so the condition $\Omega^1_S(m)$ globally generated for $m \geq 2$ (resp. ample, for $m \geq 3$) plays a crucial role to apply such vanishings. Thus, with this approach, the results in Theorem 5.2.2 are, in a certain sense, sharp. Indeed, since we are dealing with regular surfaces, the regularity of S implies that Ω^1_S cannot be globally generated. In fact, with notation as in (1.8), Section 1.1, $\overline{H^{1,0}}(S) = H^{0,1}(S)$ and $H^{1,0}(S) \cong H^0(S, \Omega^1_S)$ whereas $H^{0,1}(S) \cong H^1(S, \mathcal{O}_S) = 0$; therefore Ω^1_S has no global sections.

Moreover, we have also the following results of Schneider ([117]).

Theorem. *Let $X \subset \mathbb{P}^n$ be a d-dimensional variety whose cotangent bundle is nef. Then $d \leq n/2$.*

Corollary. *Let $X \subset \mathbb{P}^n$ be a d-dimensional variety.*

(i) Assume that its cotangent bundle is ample. Then X cannot be embedded in $\mathbb{P}^{2d-1}$;

(ii) If $n = 2d-1$, $\Omega^1_X(1)$ cannot be ample.

Therefore, even in the most natural case of smooth surfaces S in $\mathbb{P}^3$ of degree $d \geq 5$, we have that Ω^1_S and $\Omega^1_S(1)$ are not ample.

From the first part of the proof of Theorem 5.2.2, observe that in the case of smooth curves one can eliminate hypotheses (i) and (iii). Indeed, we have the following more general result.

Theorem 5.2.5 *Let $S \subset \mathbb{P}^r$ be a smooth surface (not necessarily regular and of general type) and let D be an effective divisor on S. Denote by H the hyperplane section of S. Assume that*

$$D \sim K_S + 6H + L,$$

where L an ample divisor on S. In this case, the vanishing in (5.11) actually occurs.

If moreover S is regular and of general type and $\mid D \mid$ contains smooth, irreducible elements, the family of smooth curves $V_{|D|,0}$ has the expected number of moduli.

Proof: For the first part of the statement, one can repeat the procedure at the beginning of the proof of Theorem 5.2.2. Then, when S is assumed to be also regular and of general type, from (5.7) we get the second part of the statement. □

Remark 5.2.6 Observe that Theorem 5.2.5 applies, for example, to linear system of the form $\mid mH \mid$, $m \geq d+3$, on surfaces $S_d \subset \mathbb{P}^3$ of degree $d \geq 5$. Thus we have positive answers to the moduli problem for smooth curves, in such linear systems, improving Corollary 5.2.3 in the cases $V_{|mH|,0}$.

By making the further assumption that S is also a non-degenerate complete intersection surface, we can give further improvements of Theorem 5.2.2.

Theorem 5.2.7 *Let $S \subset \mathbb{P}^r$ be a smooth, non-degenerate complete intersection surface of type $(a_1, a_2, \ldots, a_{r-2})$ (where $a_1 \leq a_2 \leq \ldots \leq a_{r-2}$). Let $D \sim mH$ on S. If*

$$m > a_{r-2}$$

then

$$H^1(\Omega^1_S(K_S + mH)) = 0. \tag{5.16}$$

Suppose moreover that S is of general type and that $\delta \geq 0$ is such that $V_{|mH|,\delta} \neq \emptyset$. With notation as in Theorem 5.2.2, if we also assume that *(i)* $\Omega^1_S(K_S)$ *is globally generated,*

(ii) the Severi variety $V_{|D|,\delta}$ *is regular at* $[X]$*,*

then, the morphism

$$\pi_{|D|,\delta} : V_{|D|,\delta} \to \mathcal{M}_g$$

has injective differential at $[X]$*. In particular,* $\pi_{|D|,\delta}$ *has finite fibres on each regular component of* $V_{|D|,\delta}$*, so each such component parametrizes a family having the expected number of moduli.*

Proof: We only have to show that the hypothesis on m implies the vanishing in (5.16). After that, one can repeat the procedure of the second part of the proof of Theorem 5.2.2 and conclude since S, being a complete intersection, is a.C.M. (see Definition 1.1.9) and so automatically regular.

Consider the conormal sequence

$$0 \to Con_{S/\mathbf{P}^r} \to \Omega^1_{\mathbf{P}^r}|_S \to \Omega^1_S \to 0.$$

By tensoring this exact sequence with $\mathcal{O}_S(K_S + mH)$, we get

$$0 \to Con_{S/\mathbf{P}^r}(K_S+mH) \to \Omega^1_{\mathbf{P}^r}|_S \otimes \mathcal{O}_S(K_S+mH) \to \Omega^1_S(K_S+mH) \to 0;$$

we want to find conditions on m in order that
(a) $H^1(S,\ \Omega^1_{\mathbf{P}^r}|_S \otimes \mathcal{O}_S(K_S + mH)) = 0$ and
(b) $H^2(S,\ Con_{S/\mathbf{P}^r}(K_S + mH)) = 0$
hold. The vanishing in (b) is equivalent, by Serre duality, to
(b') $H^0(S,\ \mathcal{N}_{S/\mathbf{P}^r}(-mH)) = 0$.
We now use the hypothesis on S, i.e. we assume that S is a smooth, non-degenerate complete intersection.
For what concerns the vanishing in (a), consider the Euler exact sequence restricted to S,

$$0 \to \mathcal{O}_S \to \mathcal{O}_S^{\oplus(r+1)}(1) \to \mathcal{T}_{\mathbf{P}^r}|_S \to 0.$$

By tensoring its dual sequence with $\mathcal{O}_S(K_S + mH)$, we get

$$0 \to \Omega^1_{\mathbf{P}^r}|_S \otimes \mathcal{O}_S(K_S+mH) \to \mathcal{O}_S^{\oplus(r+1)}(K_S+(m-1)H) \to \mathcal{O}_S(K_S+mH) \to 0.$$

Since S is a smooth, complete intersection in $\mathbb{P}^r$, there exists an integer α, depending on the degrees of the $(r-2)$ hypersurfaces determining S, such that $K_S \sim \alpha H$. Therefore, we get

$$0 \to H^0(\Omega^1_{\mathbf{P}^r}|_S \otimes \mathcal{O}_S((\alpha+m)H) \to H^0(\mathcal{O}_S((\alpha+m-1)H) \otimes \mathbb{C}^{(r+1)} \xrightarrow{\mu_m}$$

$$\xrightarrow{\mu_m} H^0(\mathcal{O}_S((\alpha+m)H)) \to H^1(\Omega^1_{\mathbf{P}^r}|_S \otimes \mathcal{O}_S((\alpha+m)H) \to 0,$$

since $H^1(\mathcal{O}_S(t)) = 0$, for all $t \in \mathbb{Z}$, being S a complete intersection in $\mathbb{P}^r$. It is well known that, for each m, the map

$$H^0(\mathcal{O}_S((\alpha+m-1)H)) \otimes H^0(\mathcal{O}_S(H)) \xrightarrow{\mu_m} H^0(\mathcal{O}_S(\alpha+m)H)$$

is surjective if and only if S is projectively nomal. Since a smooth complete intersection is in particular projectively normal, the map μ_m is surjective, for $m \in \mathbb{Z}$. Therefore,

$$H^1(\Omega^1_{\mathbf{P}^r} |_S \otimes \mathcal{O}_S((\alpha + m)H) = 0,$$

for each $m \geq 1$. So the vanishing in (a) holds.

For what concerns the vanishing in (b'), once again we use the fact that S is a complete intersection. Its normal bundle in $\mathbb{P}^r$ splits in

$$\mathcal{N}_{S/\mathbf{P}^r} = \bigoplus_{i=1}^{r-2} \mathcal{O}_S(a_i),$$

where, by hypothesis, $a_1 \leq a_2 \leq \ldots \leq a_{r-2}$, $a_i = deg(V_i)$, where V_i hypersurface in $\mathbb{P}^r$ such that $S = \bigcap_{i=1}^{r-2} V_i$, and each $a_i > 1$, since S is assumed to be non-degenerate. Therefore,

$$H^0(S,\ \mathcal{N}_{S/\mathbf{P}^r}(-mH)) = \bigoplus_{i=1}^{r-2} H^0(\mathcal{O}_S((a_i - m)H)).$$

We have to distinguish two different cases.

(1) Suppose $m > a_{r-2} = max_{1 \leq i \leq r-2}\{a_i\}$. Therefore, one immediately finds that $H^0(S,\ \mathcal{N}_{S/\mathbf{P}^r}(-mH)) = 0$. This implies that $H^1(\Omega^1_S((\alpha + m)H)) = H^1(\Omega^1_S(K_S + mH)) = 0$.

(2) Suppose that there exists at least an $i_0 \in \{1, \ldots,\ r-2\}$ such that $a_{i_0} - m \geq 0$. From this hypotesis, we get that $H^0(S,\ \mathcal{N}_{S/\mathbf{P}^r}(-mH)) \neq 0$. Thus, $H^2(S,\ Con_{S/\mathbf{P}^r}(K_S + mH)) \neq 0$. If we reconsider the exact sequence

$$0 \to Con_{S/\mathbf{P}^r} \otimes \mathcal{O}_S(K_S + mH) \to \Omega^1_{\mathbf{P}^r} |_S\ \mathcal{O}_S(K_S + mH) \to \Omega^1_S(K_S + mH) \to 0$$

we get that

$$(*) \qquad 0 \to H^1(\Omega^1_S(K_S + mH)) \to H^2(Con_{S/\mathbf{P}^r} \otimes \mathcal{O}_S(K_S + mH)) \to$$

$$\to H^2(\Omega^1_{\mathbf{P}^r} |_S\ \mathcal{O}_S(K_S + mH)) \to H^2(\Omega^1_S(K_S + mH)) \to 0.$$

If we use Serre duality, we get

$$H^2(\Omega^1_S(K_S + mH)) \cong H^0(\mathcal{T}_S(-mH))^\vee$$

and

$$H^2(\Omega^1_{\mathbf{P}^r} |_S\ \mathcal{O}_S(K_S + mH)) \cong H^0(\mathcal{T}_{\mathbf{P}^r} |_S\ (-mH))^\vee.$$

Now, if we consider the twist, by $\mathcal{O}_S(-mH)$, of the Euler exact sequence restricted to S, we obtain

$$H^0(\mathcal{T}_{\mathbf{P}^r} |_S\ (-mH)) \cong \mathbb{C}^{(r+1)} \otimes H^0(\mathcal{O}_S(1 - mH)).$$

Therefore, two subcases can occurr.

- $m > 1$: in this case $h^0(\mathcal{O}_S(1-mH)) = h^0(\mathcal{T}_{\mathbb{P}^r} \mid_S (-mH)) = 0$. By Serre duality and by the cohomological exact sequence $(*)$, we get $h^0(T_S(-mH)) = 0$, which implies that $h^2(\Omega^1_S(K_S + mH)) = 0$. This means that

$$H^1(\Omega^1_S(K_S + mH)) \cong H^0(S,\, \mathcal{N}_{S/\mathbb{P}^r}(-mH)).$$

  So, since we are in the hypothesis that there exists i_0, $1 \leq i_o \leq r-2$, such that $a_{i_0} - m \geq 0$, this isomorphism implies that $h^1(\Omega^1_S(K_S + mH)) \neq 0$.

- $m = 1$: in this case, $H^0(\mathcal{T}_{\mathbb{P}^r} \mid_S (-H)) \cong \mathbb{C}^{(r+1)}$. By using the cohomological exact sequence $(*)$ and the equality $h^1(\Omega^1_S(K_S + H)) = h^1(\mathcal{T}_S(-H))$ determined from Serre duality, we have

$$h^1(\Omega^1_S(K_S + H)) = h^0(\mathcal{N}_{S/\mathbb{P}^r}(-H)) - (r+1) + h^0(\mathcal{T}_S(-H)).$$

  Therefore, $h^1(\Omega^1_S(K_S + H)) = 0$ if and only if

$$h^0(\mathcal{N}_{S/\mathbb{P}^r}(-H)) = (r+1) - h^0(\mathcal{T}_S(-H));$$

  the last equality implies that $h^0(\mathcal{N}_{S/\mathbb{P}^r}(-H)) \leq (r+1)$. This contradicts our hypotheses on S; indeed, since S is a non-degenerate complete intersection in $\mathbb{P}^r$, $\mathcal{N}_{S/\mathbb{P}^r}(-H) = \bigoplus_{i=1}^{r-2} \mathcal{O}_S((a_i - 1)H)$ and $a_i - 1 \geq 1$, for each $1 \leq i \leq r-2$. So, $\sum_{i=1}^{r-2} h^0(\mathcal{O}_S((a_i - 1)H)) > (r+1)$. Therefore, even in this case we conclude that $h^1(\Omega^1_S(K_S + H)) \neq 0$.

□

The previous result improves Theorem 5.2.2 in the case of linear systems of the form $\mid mH \mid$. Indeed, we have the following:

Corollary 5.2.8 *If $S_d \subset \mathbb{P}^3$ is a smooth surface of degree $d \geq 6$, then the regular components of $V_{|mH|,\delta}$ on S_d, with $m \geq d+1$, have the expected number of moduli.*

Thus, this result improves Corollary 5.2.3, where the condition on m was $m \geq d+3$.

As in Theorem 5.2.5, observe that in the case of smooth curves in $\mid mH \mid$ one can eliminate hypotheses (i) and (ii) in Theorem 5.2.7.

Theorem 5.2.9 *Let $S \subset \mathbb{P}^r$ be a smooth, non-degenerate complete intersection surface of type $(a_1,\ a_2,\ \ldots,\ a_{r-2})$ (where $a_1 \leq a_2 \leq \ldots \leq a_{r-2}$). Then,*

$$H^1(\Omega^1_S(K_S + mH)) = 0,\ for\ m > a_{r-2}.$$

If S is also assumed to be of general type, the family of smooth curves in $\mid mH \mid$, with $m > a_{r-2}$, has the expected number of moduli.

Proof: This is a consequence of the computations in Theorem 5.2.7 and of (5.7) in Theorem 5.2.1. □

Remark 5.2.10 Observe that, in the case of divisors of the form mH, H a hyperplane section of S, we have improved the results of Theorem 5.2.5; indeed, if, for example, S is a smooth quintic in $\mathbb{P}^3$, from Theorem 5.2.9, we get that $h^1(S, \Omega^1_S(K_S + mH)) = 0$ for $m \geq 6$, instead of $m \geq 8$ (see Remark 5.2.6). Moreover, this vanishing result is sharp, as it can be directly checked from the following explicit example. Let $D \sim H$ (i.e. $m = 1$) be a smooth curve on S. Thus, $r = 3$, $a_1 = 5 > 1 = m$. Since $\mathcal{N}_{S/\mathbf{P}^3} \cong \mathcal{O}_S(5H)$, then $h^0(\mathcal{N}_{S/\mathbf{P}^3}(-H)) = 35 \neq 4$ $(= r+1)$. Therefore, in this case, we must have $h^1(S, \Omega^1_S(K_S + H)) \neq 0$. If we want to explicitely compute this dimension, it is sufficient to use the fact that S is of general type; thus, $h^0(\mathcal{T}_S(-H)) = h^0(\mathcal{T}_S) = 0$, and, by previous computations, we get $h^1(\mathcal{T}_S(-H)) = h^1(S, \Omega^1_S(K_S + H)) = 35 - 4 = 31$, whereas $h^1(\mathcal{T}_S) = 40$.

We remark that, for $S \subset \mathbb{P}^r$, the results in this section give affirmative answers to the moduli problem, posed in Definition 5.1.3), for example for divisors mH on S of sufficiently high-order class m (with respect to the degree of S). If we want to consider the moduli problem for classes of divisors of low degrees, we can restrict ourselves to the case of a general smooth, complete intersection surface $S \subset \mathbb{P}^r$ and immediately conclude by using a recent result of Schoen, [127]. In his paper, he studies (algebraic) varieties which are dominated by products of varieties of smaller dimension (abbreviated *DPV*); in the case of products of curves, one writes *DPC*. Since we are concerned with surfaces, we shall recall only *DPC*-property.

Definition 5.2.11 *A variety W, of dimension n, is said to be* dominated by products of curves *(*DPC*) if there exist curves $C_1, \ldots, C_s$ and a dominant rational map*

$$F : C_1 \times \ldots \times C_s \to W.$$

Some known examples of *DPC*-varieties are: unirational varieties, abelian varieties, ruled and hyperelliptic surfaces and Fermat hypersurfaces.

The main goal of Schoen's paper is to discuss some obstructions to *DPC* and *DPV* properties; more precisely, by using Hodge theory, he constructs an invariant, τ, which gives an obstruction to a variety being *DPV* (this is based on the Deligne analysis in [34], Sect. 7).

Theorem 5.2.12 *(see [127], Theorem 1.1) There is a function τ which assigns to each birational isomorphism class of varieties a non-negative integer and which has the following properties:*

(1) If $dim(W) = n$, then $\tau(W) \leq n$;

(2) If $W_1 \to W_2$ is a dominant rational map, then $\tau(W_1) \geq \tau(W_2)$;

(3) $\tau(W_1 \times \ldots \times W_s) = max_{1 \leq i \leq s}\{\tau(W_i)\}$

Corollary 5.2.13 *If $\tau(W) > 1$, then W is not* DPC.

By using Hodge structure and real algebraic group theory, he shows that some varieties cannot be *DPC*; for example, if $W \subset \mathbb{P}^N$ is a sufficiently general complete intersection of degree $d > N + 1$, then $\tau(W) = dim(W) = n$; so, by corollary above, if W is such a general complete intersection and if $n > 2$, then W cannot satisfy *DPC*-property.

To relate this result to our moduli problem, observe that it implies that a general complete intersection surface $S \subset \mathbb{P}^r$, of degree $d \geq r + 2$, cannot be dominated by a product of curves $C_1 \times C_2$. Therefore, if H denotes the hyperplane section of S, in the complete linear system $\mid mH \mid$ on S there cannot exist isotrivial (rational or irrational) pencils of smooth or δ-nodal curves, otherwise, after a suitable base change, such a surface would be *DPC*. In particular, the fibres of the morphism

$$\pi_{|mH|,\delta} : V_{|mH|,\delta} \to \mathcal{M}_g$$

are finite, for $\delta \geq 0$ and for each $m \geq 1$.

Remark 5.2.14 Observe that, via Schoen's results, we can answer the moduli problem, for smooth and nodal curves in the linear system $\mid mH \mid$, on a general complete intersection surface $S \subset \mathbb{P}^r$ of degree $d \geq r + 2$ and for each $m \geq 1$.

Our results are more generally valid for divisors D on S, where S can have a wildly complicated $Div(S)$. For the particular cases of smooth and nodal curves in the complete linear system $\mid mH \mid$ on $S \subset \mathbb{P}^r$, they give positive answers to the moduli problem for $m \geq m(S)$, $m(S)$ depending on the structure of S.

On the other hand, our techniques involve only vanishing results and vector bundle theory on smooth, projective surfaces, so they are of a more elementary nature and give simpler proofs.

Our work in progress is to give positive answers to the moduli problem, at least for surfaces $S \subset \mathbb{P}^3$, for divisors of type mH, $m \geq 1$, without using Hodge theory and Schoen's result but only using degeneration techniques. Some preliminary results have been already obtained by the author.

5.3 The final statement

This section contains a final statement of what we have proven and recalled in this chapter concerning the moduli problem of Definition 5.1.3 and Remark 5.1.4. It can be viewed as the collection of all partial results of Section 5.2.

Theorem 5.3.1 *Let $S \subset \mathbb{P}^r$ be a smooth, regular surface of general type and denote by H the hyperplane section on S. Let D be an effective divisor* *and assume that the general member of* $| D |$ *is a smooth, irreducible curve of geometric genus* $p_a = p_a(D)$. *Let* $X \sim D$ *be a* δ*-nodal curve,* $\delta \geq 0$, *determining a regular point of the Severi variety* $V_{|D|,\delta}$. *Denote by* $\varphi : C \to X \subset S$ *the normalization map of* X, *where* C *is a smooth curve of geometric genus* $g = p_a - \delta$. *In this case:*

(1) if $\Omega^1_S(K_S)$ *is globally generated and* $D \sim K_S + 6H + L$, *where* L *an ample divisor on* S, *then the morphism*

$$\pi_{|D|,\delta} : V_{|D|,\delta} \to \mathcal{M}_g$$

has injective differential at $[X]$. *In particular,* $\pi_{|D|,\delta}$ *has finite fibres over each regular component of* $V_{|D|,\delta}$, *so each such component parametrizes a family having the expected number of moduli.*

When $\delta = 0$, *the same conclusion holds for the open subset* $V_{|D|,0}$ *of* $| D |$ *without the hypothesis on* $\Omega^1_S(K_S)$*;*

(2) If S *is moreover a non-degenerate complete intersection of type*

$$(a_1,\ a_2,\ \ldots,\ a_{r-2})$$

(where $a_1 \leq a_2 \leq \ldots \leq a_{r-2}$, $a_i > 1$, *for each* i*) and* $D \sim mH$ *such that* $m > a_{r-2}$ *and if* $\Omega^1_S(K_S)$ *is globally generated, then the morphism*

$$\pi_{|mH|,\delta} : V_{|mH|,\delta} \to \mathcal{M}_g$$

has injective differential at $[X]$. *In particular,* $\pi_{|mH|,\delta}$ *has finite fibres over each regular component of* $V_{|mH|,\delta}$, *so each such component parametrizes a family having the expected number of moduli.*

Furthermore, if we restrict to the open subset $V_{|mH|,0}$ *of* $| mH |$, *the same conclusion holds without the hypothesis on* $\Omega^1_S(K_S)$*;*

(3) If, moreover, S *is assumed to be a general complete intersection of degree* $d \geq r + 2$ *and* $D \sim mH$, *where* $m \geq 1$, *then the the fibres of the morphism*

$$\pi_{|mH|,\delta} : V_{|mH|,\delta} \to \mathcal{M}_g$$

are finite, for $\delta \geq 0$. *In particular, the regular components of* $V_{|mH|,\delta}$ *have the expected number of moduli.*

Proof: See Theorems 5.2.2, 5.2.5, 5.2.7, 5.2.9 and Corollary 5.2.13 .

□

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