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The Compactness Spectrum of Abstract Logics, Large Cardinals and Combinatorial Principles.

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Sunto. – *Si studiano le conseguenze, per estensioni elementari, del realizzare particolari tipi, che implicano l'esistenza di un elemento al di sopra di un cardinale. Si generalizzano così alcuni teoremi riguardanti gli ultrafiltri. Si danno poi applicazioni alla Teoria dei Modelli Astratta, dimostrando che ogni logica* (λ^+, λ^+)*-compatta è anche* (λ, λ)*-compatta, purchè* λ *sia un cardinale regolare.*

0. – Introduction.

We analyze the consequences of realizing a particular kind of types, saying that there exists an element above a cardinal, thus generalizing some theorems about regular ultrafilters. We give applications to Abstract Model Theory: if λ is regular, then every (λ^+, λ^+)-compact logic is (λ, λ)-compact.

Ultrapowers are one of the most important constructions in Model Theory: to any *ultrafilter* D and any structure $\mathfrak{A}$ one can associate $\prod_D \mathfrak{A}$, which is (isomorphic to a) *complete extension* of $\mathfrak{A}$. It turns out that some properties of D can be equivalently stated as properties of the ultrapower: the example which will play a major role in this paper, due to Keisler [CK, Exercise 4.3.34] is that, if λ is a regular cardinal, then D is (λ, λ)-regular iff in $\prod_D \langle \lambda, < \rangle$ there exists an element larger than all ordinals of λ (*regularity* is a quite natural property of ultrafilters introduced by Keisler; roughly, D is (μ, λ)-regular iff it can prove the (λ, μ)-compactness of first order logic; if λ is regular, then (λ, λ)-regularity can be equivalently defined as *λ-descending incompleteness or λ-decomposability*).

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Theorems about ultrafilters can be reformulated as theorems about complete extensions: as an example, for λ and μ regular, the statement « every (λ, λ)-regular ultrafilter is (μ, μ)-regular » is equivalent to « if $\mathfrak{A}$ is a complete extension of $\langle\lambda, <\rangle$ and in $\mathfrak{A}$ there is an element larger than all ordinals of λ, then in $\mathfrak{A}$ there is an element less than μ but larger than all ordinals of μ » (this property, known to be true if $\lambda = \mu^+$, in the general case depends on the axioms of Set Theory used: see e.g. [BF, XVIII, Theorem 1.5.6] or [Do]).

In this paper we mainly deal with the case when $\mathfrak{A}$ is no more assumed to be a complete extension, but just an elementary extension (of some finite expansion) of $\langle\lambda, <\rangle$; this modified notion shall be denoted by $\lambda \Rightarrow \mu$. It has applications to ***Abstract Model Theory*** (see [BF] for a conceivably complete introduction to the subject) by means of the following equation: « complete extensions are to $[\lambda, \mu]$-compactness what elementary extensions are to (λ, μ)-compactness », and by extending the techniques developed by Makowsky and Shelah connecting regularity of ultrafilters and $[\lambda, \mu]$-compactness of logics (see [BF, Chapter XVIII] for a review; $[\lambda, \mu]$-*compactness* is a natural and easy to handle strengthening of (λ, μ)-compactness: essentially, this amounts to assuming compactness for the class of models of every L-theory).

Indeed, the original problem we started from was: « for which cardinals λ and μ does (λ, λ)-compactness imply (μ, μ)-compactness? ». We prove that, for λ, μ regular, this is true provided that $\lambda \Rightarrow \mu$ holds (and sometimes the converse is also true); so that our relation $\lambda \Rightarrow \mu$ is helpful in the study of the *compactness spectra* of logics (that is classes having the form $\{\lambda \mid L \text{ is } (\lambda, \lambda)\text{-compact}\}$). Our main result is that, if λ is regular, then $\lambda^+ \Rightarrow \lambda$ holds, so that every (λ^+, λ^+)-compact logic is (λ, λ)-compact (with still possible but unlikely exceptions at singular λ). We also prove that if L is (λ, λ)-compact then L is $(\varkappa, \varkappa)$-compact for some cardinal $\varkappa$ such that $L_{\varkappa\omega}$ is $(\varkappa, \varkappa)$-compact (were $\varkappa$ inaccessible, it should be weakly compact). The analogous results for $[\lambda, \lambda]$-compactness (every λ, and a measurable $\varkappa$) have already been obtained in the 70's by Makowsky and Shelah, and are much easier; also the possible spectra for $[\lambda, \lambda]$-compactness can be easily characterized (with still some gaps at singular cardinals [Lp2]).

Of course, our results about Abstract Model Theory could have been proved (and in fact, they were originally proved) by directly incorporating in the proofs the needed facts about the relation $\lambda \Rightarrow \mu$, and without explicitly mentioning it; nevertheless, its isolation shows in a clearer way what makes things work; moreover, we believe that it is interesting for itself and from the point of view of «classical» Model Theory, and, anyway, there are some connections with the *combinatorial properties* E_k^λ and $\square_\varkappa$, and with *large cardinals* (see [KM] for an introduction and an exhaustive review of known results).

Because of this we have organized the paper in such a way that almost all the results about $\lambda \Rightarrow \mu$ are presented in § 1 before any application to Abstract Model Theory, though this may look rather unnatural. The reader interested in Abstract Model Theory only might begin reading § 2, going back to § 1 when needed. In addition to the main result Theorem 1.3, and after a remark about the cardinalities of some elementary extensions, in § 1 we prove the equivalence of $\lambda \Rightarrow \mu$ and its «relativized» version, and also of some regularity properties of filters. The proof of Proposition 1.6 is indeed a very first step towards a generalization for filters and (λ, μ)-compactness of what the family $UF(L)$ of [MS] is for ultrafilters and $[\lambda, \mu]$-compactness (the possibility of such a generalization is one of the most interesting problems left open in this paper).

Section 2 contains the immediate applications to the compactness spectrum. Under suitable cardinality hypotheses, in Theorem 2.5 we give several equivalents of the statement «every (λ, λ)-compact logic is (μ, μ)-compact»; there we also show that limiting oneself to cardinality logics is not too restrictive to this respect (part (viii) seems to be an application to cardinality logics of independent interest). Easier proofs of slightly less general results (but working also for singular cardinals) are given in Theorem 2.7 using ultrafilters only.

In § 3 we take up the study of the connections between compactness and *characterizability* (equivalently, existence of *maximal* models). We show that compactness of a logic implies compactness of infinitary logics; so that many large cardinals are characterized as first cardinals for which some logic is compact. Some of our methods are generalizations of methods already used for infinitary languages or cardinality logics (see e.g. [Dr] or [MR]); however many of our results in § 2 and § 3 seem to be new even in these very particular cases (see e.g. Corollary 3.11).

In § 4 we prove that the combinatorial principle $E_\varkappa^\lambda$ implies the property $\varkappa \Rightarrow \lambda$ and that this implication is strict. Hence, the set theoretical principle $\square_\varkappa$ affects the possible compactness spectrums; also, if λ is singular and there is a (λ^+, λ^+)-compact non (λ^+, ω)-compact logic, then there exists an inner model with a measurable cardinal (this can be indeed be improved to many measurable cardinals).

Section 5 is quite outside of the main theme of the paper. We exactly characterize the compactness properties of logics generated by monadic quantifiers (in terms of the compactness properties of cardinality logics), thus generalizing results of [Lp1, § 5]. We also show that for the compactness of such logics we do not need many large cardinals.

Finally, § 6 contains some general remarks about some aspects of the present state of Abstract Model Theory.

Most of the results proved in this paper were announced without proof in [Lp2], [Lp4] and [Lp6].

We use rather standard terminology and notations (see for example [BF], [CK], [KM], [Lp1]). $|A|$ is the cardinality of A; if A is a proper class, we put $|A| = \infty$; and we set $\lambda < \infty$, for every cardinal λ; the letters λ, μ, ν are reserved for *infinite cardinals*, while $\varkappa$ may be any cardinal or ∞. $\lambda^{<\mu}$ is $\sup\{\lambda^{\mu'} \mid \mu' < \mu\}$; $(<\lambda)^{\mu} = \sup\{\lambda'^{\mu} \mid \lambda' < \lambda\}$; $(<\lambda)^{<\mu}$ is defined similarly.

We call (*similarity*) *types* what [BF] calls *vocabularies*; in general, for sake of simplicity, we restrict ourselves to the *single-sorted* case, however most of our results have immediate generalizations to the *many-sorted case*; we use the word *type* also for *elementary types* (that is, a collection of first-order formulas with a free variable-usually x or y): when confusion may arise, we shall specify which kind of type we are dealing with.

We shall present here an alternative definition of what a logic is (cf. [KV]). We believe that this is a faster way to introduce the concept of a logic, moreover greatly simplifying notations; of course, everything has a translation for the more usual definition.

An (*abstract*) *sentence* φ is a class of models of the same similarity type $\tau\varphi$, closed under isomorphism (when $\varphi = \emptyset$ we have to consider a different-« false »-sentence for each type τ; alternatively, see [Mu]). If $\mathfrak{A}$ is a structure of type τ, and $\tau \supset \tau\varphi$, then we set $\mathfrak{A} \models \varphi$ iff $\mathfrak{A}_{\restriction \tau\varphi} \in \varphi$.

$\neg$, $\wedge$, $\vee$, $\exists c$ are operators on sentences defined in the natural way (as an example, $\varphi \vee \psi$ is the sentence $\{\mathfrak{A} \mid \tau\mathfrak{A} = \tau\varphi \cup \tau\psi, \mathfrak{A} \models \varphi$ or $\mathfrak{A} \models \psi\}$).

A *logic*, now, is just a collection of sentences, closed with respect to suitable operators. The closure properties we require a logic to satisfy are somewhat less than the ones of a *regular* logic [BF], in fact, the properties listed in [Lp1, § 1] are enough (see e.g. [Ca], [Lp1, Counterexamples 6.2 and 6.3] or [Lp5, Remark 3.5] for things that may happen without regularity). By *quantifier* we always mean a relativizing quantifier as in [BF, II.4.1.4].

We wish to express our gratitude to P. Giannini for making us understand the importance of the ultraproduct construction.

1. – On realizing types bounding a cardinal.

In this section we deal with the model theory of first order logic only. We analyze the consequences of realizing (elementary) types of the kind $\{\alpha < x < \lambda \mid \alpha < \lambda\}$, for λ a regular cardinal; a model realizing such a type is said to *bound* λ. In the case of complete extensions of a model, as follows from a variation on [Lp1, Lemma 3.2], this corresponds to taking a (limit) ultrapower modulo a (λ, λ)-regular (limit) ultrafilter.

First we show that bounding λ influences cardinalities.

1.1. PROPOSITION. – (i) *If λ is regular, then there exists an expansion $\mathfrak{A}$ of $\langle\lambda, <\rangle$ such that $|\tau\mathfrak{A}| = \lambda^+$ and, whenever $\mathfrak{A} < \mathfrak{B}$ and $\mathfrak{B}$ realizes $\{\alpha < x \mid \alpha < \lambda\}$, then $\lambda < |B|$.*

(ii) *If λ is regular, then there exists a finite expansion $\mathfrak{A}$ of $\langle\lambda^+, <\rangle$ such that whenever $\mathfrak{A} < \mathfrak{B}$ and $\mathfrak{B}$ realizes $\{\alpha < x < \lambda \mid \alpha < \lambda\}$ then $|\{x \in B \mid \mathfrak{B} \models x < \lambda\}| > \lambda$.*

PROOF. – Since λ is regular, there is a sequence $(f_\beta)_{\beta\in\lambda^+}$ of functions from λ to λ increasing modulo eventual dominance (that is, for every $\beta < \gamma < \lambda^+$, there exists an $\alpha_{\beta\gamma}$ such that, $f_\gamma(\alpha) > f_\beta(\alpha)$, for every $\alpha \geqslant \alpha_{\beta\gamma}$).

(i) Put $\mathfrak{A} = \langle\lambda, <, f_\beta\rangle_{\beta\in\lambda^+}$: if $\mathfrak{B} > \mathfrak{A}$ and $\mathfrak{B} \models \alpha < b$ $(\alpha \in \lambda)$, then $f_\beta(b)$ are λ^+ different elements of B $(\beta \in \lambda^+)$.

(ii) Put $\mathfrak{A} = \langle\lambda^+, <, f\rangle$, where f is binary and $f(\beta, \alpha) = f_\beta(\alpha)$ $(\beta \in \lambda^+,\ \alpha \in \lambda)$ (f is defined arbitrarily in the other cases).

Note that in 1.1 we could also conclude that $\mathfrak{B}$ ($\{x \in B \mid \mathfrak{B} \models x < \lambda\}$, respectively) contains a subset well ordered by $<$ of type λ^+.

Proposition 1.1 is a generalization of [CK, Exercise 4.3.13] for λ regular. It is not clear to what extent Proposition 1.1 can be generalized. For example, if there are $\varkappa$ functions $(f_\beta)_{\beta\in\varkappa}$ from λ to μ such that for every $\beta \neq \gamma$, $\beta, \gamma \in \varkappa$ there is $\alpha_{\beta\gamma} \in \lambda$ such that $f_\beta(\alpha) \neq f_\gamma(\alpha)$ for every $\alpha \geqslant \alpha_{\beta\gamma}$, then bounding λ lifts a set of cardinality μ to one of cardinality $\varkappa$; also [CK, Exercise 4.3.17] can be generalized.

1.2. DEFINITION. – If $\lambda \geqslant \mu$ are infinite regular cardinals, and $\varkappa$ is a cardinal, we write $\lambda \overset{\varkappa}{\Rightarrow} \mu$ iff there exists an expansion $\mathfrak{A}$ of $\langle\lambda, <\rangle$ such that $|\tau\mathfrak{A} \setminus \{<\}| \leqslant \varkappa$ and whenever $\mathfrak{B} > \mathfrak{A}$ and $\mathfrak{B}$ realizes $\{\alpha < x \mid \alpha \in \lambda\}$, then $\mathfrak{B}$ realizes $\{\beta < y < \mu \mid \beta \in \mu\}$.

If $\varkappa = \omega$, we just write $\lambda \Rightarrow \mu$.

Thus, $\lambda \overset{\varkappa}{\Rightarrow} \mu$ iff, for some theory T for a similarity type with at most $\varkappa$ symbols (other than $<$), every model bounding λ also bounds μ.

Trivial facts about the relation $\lambda \overset{\varkappa}{\Rightarrow} \mu$ are that it is transitive (for $\varkappa$ fixed) and is preserved by increasing $\varkappa$. Moreover, $\lambda \Rightarrow \mu$ iff $\lambda \overset{\varkappa}{\Rightarrow} \mu$, for every $\varkappa$ with $0 < \varkappa \leqslant \lambda$ (λ relations can be coded by the ordinals of λ, using a pairing function to bound arities); obviously, also $\varkappa \geqslant 2^\lambda$ gives nothing new (this is also true, but not trivial, for the variant introduced in Proposition 1.6); for $\varkappa = 0$, cf. [CK, p. 303].

Thus the above notion is interesting only for the case $\lambda \leqslant \varkappa \leqslant 2^\lambda$. Because of Theorem 2.5 (iv), Corollary 3.10 (*b*) and [Do, Theorem 4.5], if λ is weakly compact and $\neg L^\mu$ then for every regular $\mu < \lambda$, $\lambda \Rightarrow \mu$ is false, but $\lambda \overset{2^\lambda}{\Rightarrow} \mu$ holds; so that (if it is consistent to have a weakly compact cardinal) the two notions are different. We do not know what may happen for intermediate $\varkappa$.

The next theorem generalizes, for λ regular, a result of Chang (using *GCH*) and of Kunen, Prikry and Cudnovskii-Cudnovskii independently without set-theoretical hypotheses: every (λ^+, λ^+)-regular ultrafilter is (λ, λ)-regular (see [CN, Theorem 8.35]; we use notations as similar to this proof as possible in order to make the comparison clearer). Similar results holding also for λ-singular are stated without proof in [Lp4] and [Lp6].

1.3. Theorem. – *If α is an infinite regular cardinal, then $\alpha^+ \Rightarrow \alpha$.*

1.4. Lemma (Ulam, Prikry) [CN, Lemmata 8.33 and 8.34]. – *If $\alpha \geqslant \omega$, then there exists a family $(B_{\xi\eta})_{\xi<\alpha,\eta<\alpha^+}$ of subsets of α^+ such that $|\alpha^+ \setminus \bigcup_{\xi<\alpha} B_{\xi\eta}| \leqslant \alpha$, for $\eta < \alpha^+$, and if $A \subset \alpha^+$, $\xi < \alpha$ and $|A| > |\xi|$, then $\bigcap_{\eta\in A} B_{\xi\eta} = \emptyset$. Moreover, $\xi < \xi'$ implies that $B_{\xi\eta} \subset B_{\xi'\eta}$.*

Proof of Th. 1.3. – Let $B_{\xi\eta}$ ($\xi < \alpha$, $\eta < \alpha^+$) be as in Lemma 1.4. Expand $\langle \alpha^+, < \rangle$ to a model $\mathfrak{A}$ by adding:

(i) a ternary relation R such that $R(\eta, \xi, x)$ holds iff $\xi < \alpha$ and $x \in B_{\xi\eta}$;

(ii) a binary function f such that, for $\alpha \leqslant \eta < \alpha^+$, $f(\eta, -)$ is a bijection from η onto α;

($\eta'' < \alpha^+$)

(iii) a ternary function g such that, for $\eta' < \eta < \alpha^+$ and $\xi < \alpha$, if $R(\eta', \xi, \eta'')$, then $\alpha > g(\eta, \eta'', \xi) > f(\eta, \eta')$ (this is possible since, given η'', $|\{\eta' \mid \eta'' \in B_{\xi\eta'}\}| \leqslant |\xi| < \alpha$, and α is regular).

Let $\mathfrak{B} > \mathfrak{A}$, and suppose that $b \in B$ and $b > \eta$, for every $\eta < \alpha^+$. There are two cases:

(a) There exists a $\eta < \alpha^+$ such that for every $\xi < \alpha$, $\mathfrak{B} \not\models R(\eta, \xi, b)$.

But $|\alpha^+ \setminus \bigcup_{\xi<\alpha} B_{\xi\eta}| \leqslant \alpha$, so that there exists $\eta' < \alpha^+$ such that $\mathfrak{B} > \mathfrak{A} \models \forall x \geqslant \eta' \ \exists y < \alpha \ R(\eta, y, x)$, so that, as $b > \eta'$, for some $d \in B$, $\mathfrak{B} \models R(\eta, d, b)$.

But, since $\xi < \xi'$ implies $B_{\xi\eta} \subset B_{\xi'\eta}$, we have that $\xi < d < \alpha$, for every $\xi < \alpha$.

(b) For every $\eta < \alpha^+$ there is a $\xi_\eta < \alpha$ such that $\mathfrak{B} \models R(\eta, \xi_\eta, b)$.

Then there exists $\bar{\xi} < \alpha$ and a set $X \subset \alpha^+$, $|X| = \alpha$, such that $\xi_\eta = \bar{\xi}$, for $\eta \in X$. Put $\bar{\eta} = \sup X$: for every $\eta \in X$, $\mathfrak{B} \models R(\eta, \bar{\xi}, b)$, so that $\alpha > g(\bar{\eta}, b, \bar{\xi}) > f(\bar{\eta}, \eta)$; but $\{f(\bar{\eta}, \eta) \mid \eta \in X\}$ has cardinality α, hence is cofinal in α, so that $g(\bar{\eta}, b, \bar{\xi})$ realizes the type $\{\xi < y < \alpha \mid \xi < \alpha\}$.

Without the use of f and g we could only prove (in the terminology of [Lp4]): $(\alpha^+, \alpha^+) \Rightarrow$ almost (α, α); f and g give us the possibility of proving almost $(\alpha, \alpha) \Rightarrow (\alpha, \alpha)$. These two steps could also be performed separately.

Theorem 1.3 suggests that there exists some relationship between the relation $\lambda \Rightarrow \mu$ and regularity of ultrafilters (in fact, the exact connection is with some form of regularity of prime filters over fields of sets). Using this we can see that a seemingly weaker version of $\lambda \Rightarrow \mu$ (a «relativized» form) is indeed equivalent to it.

1.5. Definitions. – If F is a field of sets, we say that a collection $\mathfrak{X} = (X(\alpha))_{\alpha \in \lambda}$ of members of F is *(λ, λ)-regular* iff:

(i) $\alpha < \beta \in \lambda \Rightarrow X(\alpha) \supset X(\beta)$; and

(ii) $\bigcap_{\alpha \in \lambda} X(\alpha) = \emptyset$.

If D is a *prime* (i.e. maximal) filter of F, (i) and (ii) hold and, in addition:

(iii) $X(\alpha) \in D$, for every $\alpha \in \lambda$;

we say that $\mathfrak{X}$ *makes D (λ, λ)-regular.*

D is (λ, λ)-regular iff some collection makes it (λ, λ)-regular.

This definition clearly extends the usual notion of a (λ, λ)-regular ultrafilter.

If F is over λ, we say that F (or D) are *almost uniform* if, for every $\alpha \in \lambda$, then $\lambda \setminus \alpha = \{x \mid \alpha < x < \lambda\}$ belongs to F (or D).

1.6. PROPOSITION. – *If* $\lambda \geqslant \mu$ *are regular cardinals, and* $\varkappa \geqslant \lambda$, *then the following are equivalent*:

$(\mathrm{i})_\varkappa$ $\lambda \overset{\varkappa}{\Rightarrow} \mu$;

$(\mathrm{ii})_\varkappa$ ($\lambda \overset{\varkappa}{\Rightarrow} \mu$ *relativized*). *There exists a model* $\mathfrak{A} = \langle A, \lambda, <, \alpha, \ldots\rangle_{\alpha \in \lambda}$ *such that* $|\tau\mathfrak{A}| \leqslant \varkappa$, $<$ *is the order on* λ *and whenever* $\mathfrak{B} \equiv \mathfrak{A}$ *and* $\mathfrak{B}$ *realizes* $\{\alpha < x \mid \alpha \in \lambda\}$, *then* $\mathfrak{B}$ *realizes* $\{\beta < y < \mu \mid \beta \in \mu\}$.

$(\mathrm{iii})_\varkappa$ *There exists a field of sets* F *almost uniform over* λ *such that* $|F| \leqslant \varkappa$, *and there exist at most* $\varkappa$ *collections* $\mathfrak{X}_\delta$ $(\delta \in \varkappa)$ *of members of* F *such that for every almost uniform prime filter* D *of* F *there is* $\delta \in \varkappa$ *such that* $\mathfrak{X}_\delta$ *makes* D (μ, μ)*-regular.*

$(\mathrm{iv})_\varkappa$ *There exist a set* I, *a field* F *of subsets of* I *with* $|F| \leqslant \varkappa$, *a* (λ, λ)*-regular collection* $\mathfrak{Y}$ *and at most* $\varkappa$ *collections* $\mathfrak{X}_\delta (\delta \in \varkappa)$ *such that every prime filter* D *of* F *made* (λ, λ)*-regular by* $\mathfrak{Y}$ *is made* (μ, μ)*-regular by some* $\mathfrak{X}_\delta$.

If in addition either $\varkappa = \varkappa^\mu$ *or* $\varkappa \geqslant 2^\lambda$, *then the preceding are also equivalent to*:

$(\mathrm{v})_\varkappa$ *There exists a field of sets* F *almost uniform over* λ *such that* $|F| \leqslant \varkappa$ *and every almost uniform prime filter* D *of* F *is* (μ, μ)*-regular.*

Indeed, for every $\varkappa$ $(\mathrm{iv})_\varkappa \Rightarrow (\mathrm{v})_\varkappa$, $(\mathrm{v})_\varkappa \Rightarrow (\mathrm{iv})_{\varkappa^\mu}$; *and if* $\varkappa \geqslant 2^\lambda$, *then* $(\mathrm{v})_\varkappa \Rightarrow (\mathrm{iv})_\varkappa$.

PROOF. – $(\mathrm{i})_\varkappa \Rightarrow (\mathrm{ii})_\varkappa$, $(\mathrm{iii})_\varkappa \Rightarrow (\mathrm{iv})_\varkappa$ and $(\mathrm{iii})_\varkappa \Rightarrow (\mathrm{v})_\varkappa$ are trivial.

$(\mathrm{ii})_\varkappa \Rightarrow (\mathrm{iii})_\varkappa$: let $\mathfrak{A}$ satisfy (ii) and suppose w.l.o.g. that $\mathfrak{A}$ has Skolem functions. Take F to be the set of all subsets of λ definable in $\mathfrak{A}$ without parameters, and let $\{f_\delta \mid \delta \in \varkappa\}$ be all the definable functions from λ to μ; and, for every $\delta \in \varkappa$ and $\alpha \in \mu$, let $X_\delta(\alpha) = \{\beta \in \lambda \mid f_\delta(\beta) \geqslant \alpha\}$.

Let now D be a prime filter of F: D can be extended to an ultrafilter D' over λ; let $\mathfrak{B}$ be the substructure of $\prod_{D'} \mathfrak{A}$ generated by $\{d(\alpha) \mid \alpha \in \lambda\} \cup \{id_{D'}\}$, where d is the canonical embedding. Since $\mathfrak{A}$ has Skolem functions, $\mathfrak{B} \equiv \prod_{D'} \mathfrak{A} \equiv \mathfrak{A}$. Moreover, $\mathfrak{B} \models d(\alpha) < id_{D'}$, for every $\alpha \in \lambda$, as $\alpha \notin D$, so that, by $(\mathrm{ii})_\varkappa$, for some term t (depending on some constants from $\mathfrak{A}$), $d(\beta) < t(id_{D'}) < d(\mu)$, for every $\beta \in \mu$.

Let now f: $\lambda \to \mu$ be defined by $f(\beta) = t(\beta)$ if $t(\beta) < \mu$ and $f(\beta) = 0$ otherwise. f is clearly definable in $\mathfrak{A}$, so that $f = f_\delta$, for some $\delta \in \varkappa$; now, if $\alpha \in \mu$, then

$$X_\delta(\alpha) = \{\beta \in \lambda \mid f_\delta(\beta) \geqslant \alpha\} \supset \{\beta \in \lambda \mid t(\beta) \geqslant \alpha\} \cap \{\beta \in \lambda \mid t(\beta) < \mu\}$$

but, by Łoš theorem, the last two sets belong to D', hence to D (since they are already in F), so that $X_\delta(\alpha) \in D$ and $\mathfrak{X}_\delta$ makes D (μ, μ)-regular.

$(\mathrm{iv})_\varkappa \Rightarrow (\mathrm{ii})_\varkappa$: let $f\colon I \to \lambda$ be defined by:

$$f(i) = \beta \quad \text{iff } i \notin Y(\beta) \text{ and for every } \gamma < \beta \;\; i \in Y(\gamma)\,;$$

without loss of generality by rearranging the $Y(\beta)$'s we can assume that f is onto.

Let $\mathfrak{A}$ be the model obtained from $\langle I, \lambda, <, \alpha, f\rangle_{\alpha\in\lambda}$ by adding:

(a) a unary relation U_X for every $X \in F$;

(b) for every $\delta \in \varkappa$, a binary relation R_δ such that $R_\delta(i, \alpha)$ iff $i \in X_\delta(\alpha)$;

(c) a function $f_\delta\colon I \to \mu + 1$ such that $f_\delta(i) = \alpha$ iff not $R_\delta(i, \alpha)$, and $R_\delta(i, \alpha')$, for every $\alpha < \alpha'$.

Let $\mathfrak{B} \equiv \mathfrak{A}$, and suppose that $\beta < b \in \lambda^{\mathfrak{B}}$, for every $\beta \in \lambda$; then there exists a $c \in I^{\mathfrak{B}}$ such that $f(c) = b$. Let D be the prime filter over F defined by: $X \in D$ iff $\mathfrak{B} \not\models U_X(c)$; D is made (λ, λ)-regular by $\mathfrak{Y}$ since $Y(\beta) \in D$ iff $\mathfrak{B} \not\models U_{Y(\beta)}(c)$ iff $c \notin Y(\beta)^{\mathfrak{B}}$ iff $b = f(c) > \beta$.

Because of $(\mathrm{iv})_\varkappa$, for some $\gamma \in \varkappa$, $\mathfrak{X}_\gamma$ makes D (μ, μ)-regular; hence $\mathfrak{A} \models \forall x \in I \;\; f_\gamma(i) < \mu$ (by 1.5 (ii)), so that $\mathfrak{B} \models f_\gamma(c) < \mu$; moreover, for every $\alpha \in \mu$, $f_\gamma(c) \geqslant \alpha$ iff not $R_\gamma(c, \alpha)$ iff not $U_{X_\delta(\alpha)}(c)$ iff $X_\delta(\alpha) \in D$, and this is true, as $\mathfrak{X}$ makes D (μ, μ)-regular.

$(\mathrm{iii})_\varkappa \Rightarrow (\mathrm{i})_\varkappa$ is similar to $(\mathrm{iv})_\varkappa \Rightarrow (\mathrm{ii})_\varkappa$ and easier (indeed, it is the particular case when $I = \lambda$ and $Y(\beta) = \lambda \setminus \beta$).

$(\mathrm{v})_\varkappa \Rightarrow (\mathrm{iii})_{\varkappa^\mu}$; and if $\varkappa \geqslant 2^\lambda$ then $(\mathrm{v})_\varkappa \Rightarrow (\mathrm{iii})_\varkappa$ (this is because there can be at most $\varkappa^\mu$ or 2^λ, respectively, collections as in (iii)).

1.7. Corollary. – *If λ is regular, then there exists a field F of subsets of λ^+ such that $|F| = \lambda^+$ and every almost uniform prime filter over F is (λ, λ)-regular.*

Proof. – Theorem 1.3 and Proposition 1.6 (i) $\Rightarrow$ (v).

We do not know more direct proofs of (ii) $\Rightarrow$ (i) and of (iv) $\Rightarrow$ $\Rightarrow$ (iii) in Proposition 1.6. The possibility for (v) to be equivalent to the other conditions under weaker hypotheses is left open.

Other results about the relation $\lambda \overset{\varkappa}{\Rightarrow} \mu$ are given in § 4, and in Theorems 2.5 and 3.12.

PROBLEM. – Try to generalize other theorems about ultrafilters by expressing their properties in terms of elementary extensions. Saturation leaves some possibilities open: see, e.g., Proposition 2.8 in the particular case $L = L_{\omega\omega}$.

2. – Applications to the compactness spectrum.

First, we introduce $\varkappa$-(λ, μ)-compactness, a concept intermediate between $[\lambda, \mu]$-compactness and (λ, μ)-compactness; for $\varkappa \geqslant \lambda$ it represents what $[\lambda, \mu]$-compactness and $(\varkappa, \mu)$-compactness « have in common ». This new intermediate concept is not strictly necessary in this section, but gives our results a greater generality; it will play a major role in § 3, where it will be shown to be connected with maximal models and with characterizability of models; in particular it will be useful in Corollary 3.11, where it will measure what can be brought down from (θ, θ)-compactness to $(\varkappa, \varkappa)$-compactness for a weakly compact cardinal $\varkappa$. In most of the theorems, at first reading, one can take $\varkappa = 1$, so that $\varkappa$-(λ, μ)-compactness turns out to be equivalent to (λ, μ)-compactness (Proposition 2.2 (ii)); there is also the possibility of taking $\varkappa = \infty$, but in such a way some theorems become known results about $[\lambda, \mu]$-compactness.

A class K of models (of any type) is (λ, μ)-*compact* relative to L iff the following holds: whenever $\Gamma \subset L$, $|\Gamma| = \lambda$ and every subset of Γ of cardinality less than μ has a model in K, then Γ has a model in K.

PROPOSITION 2.1. – *If L is a logic, $\lambda \geqslant \mu$ are infinite cardinals, and $\varkappa$ is a non-zero cardinal or ∞, the following are equivalent:*

(i) *if $\Sigma, \Gamma \subset L$, $|\Sigma| \leqslant \varkappa$ and $|\Gamma| = \lambda$, then $\Sigma \cup \Gamma$ has a model, provided $\Sigma \cup \Gamma'$ has a model, for every $\Gamma' \subset \Gamma$ such that $|\Gamma'| < \mu$;*

(ii) *if $(\Sigma_\alpha)_{\alpha \in \lambda}$ is a collection of L-sentences and $|\Sigma_\alpha| \leqslant \varkappa$, for $\alpha \in \lambda$, then $\bigcup_{\alpha \in \lambda} \Sigma_\alpha$ has a model, provided that $\bigcup_{\alpha \in X} \Sigma_\alpha$ has a model, for every $X \subset \lambda$ with $|X| < \mu$;*

(iii) *if $|\Sigma| \leqslant \varkappa$, $\Sigma \subset L$, then* $\mathrm{Mod}(\Sigma)$ *is (λ, μ)-compact relative to L.*

Proposition 2.1 is proved as in [BF, XVIII, Proposition 1.1.1]. We say that L is $\varkappa$-(λ, μ)-*compact* iff any of the above conditions holds. The following proposition states some obvious facts about this concept; (ii) and (iii) may be taken as definitions of (λ, μ)-compactness and of $[\lambda, \mu]$-compactness.

PROPOSITION 2.2. – (i) *$\varkappa$-(λ, μ)-compactness is preserved by increasing μ and lowering $\varkappa$ and λ.*

(ii) *for every $\varkappa \leqslant \lambda$, $\varkappa$-(λ, μ)-compactness is equivalent to (λ, μ)-compactness.*

(iii) *$[\lambda, \mu]$-compactness is equivalent to ∞-(λ, μ)-compactness.*

(iv) *a logic is $\varkappa$-(λ, μ)-compact iff it is $\varkappa$-(ν, ν)-compact for every ν such that $\mu \leqslant \nu \leqslant \lambda$.*

(v) *if $\varkappa \geqslant \lambda$, then $\varkappa$-$(\mathrm{cf}(\lambda), \mathrm{cf}(\lambda))$-compactness implies $\varkappa$-(λ, λ)-compactness.*

(vi) *$\varkappa$-(λ, μ)-compactness is a consequence of either $[\lambda, \mu]$-compactness or $(\sup(\varkappa, \lambda), \mu)$-compactness.*

Note that, by (ii) above, without loss of generality we can always suppose $\varkappa \geqslant \lambda$. Moreover, by (iv), we can reduce the study of $\varkappa$-(λ, μ)-compactness to the study of $\varkappa$-(λ, λ)-compactness. Already λ^+-(λ, λ)-compactness considerably strengthens (λ, λ)-compactness:

PROPOSITION 2.3. – *If λ is regular and L is a λ^+-(λ, λ)-compact logic, then any L-theory T of cardinality $\leqslant \lambda^+$ having a model of cardinality λ has a model of cardinality $\geqslant \lambda^+$. In particular, $L_{\omega\omega}(Q_{\alpha+1})$ is not $\omega_{\alpha+1}$-$(\omega_\alpha, \omega_\alpha)$-compact, provided ω_α is regular.*

PROOF. – Let $\mathfrak{B}$ be a model of T of cardinality λ; without loss of generality suppose that $B = \lambda$. Let $\mathfrak{A}$ be as in Proposition 1.1 (i) and T' be the $L_{\omega\omega}$-theory of $\mathfrak{A}$, put $\Sigma = T \cup T'$, $\Gamma = \{\alpha \leqslant b \mid \alpha \in \lambda\}$ and use Proposition 2.1 (i).

Proposition 2.3 is true also for logics not having relativization (if we use the definition given in 2.1 (i)).

If L allows relativization, we clearly have a relativized form of Proposition 2.3 (that is, we consider the cardinality of $\{x \mid U(x)\}$ for some unary predicate U in a model, instead of the cardinality of the whole model).

Notice that the proof of Proposition 2.3 would be considerably easier assuming (λ^+, λ^+)-compactness (which is stronger than λ^+-(λ, λ)-compactness): let Γ say that $(c_\alpha)_{\alpha\in\lambda^+}$ are λ^+ different constants (and this works also for λ singular).

The conclusion of Proposition 2.3 in the particular case $L = L_{\omega\omega}(Q_1)$ is implicit in [MR]; indeed, they could prove that if $\omega_\alpha \leqslant 2^\omega$ then $L_{\omega\omega}(Q_\alpha)$ is not ω_α-(ω, ω)-compact. Also, if $\varkappa$, λ are as in Proposition 3.15, and $\omega_\alpha = \lambda < \lambda^\omega$, then $L_{\omega\omega}(Q_{\alpha+1})$ is not $\omega_{\alpha+1}$-(ω, ω) compact. Notice that, on the contrary, if $\omega_\alpha = (2^\omega)^+$ then $L_{\omega\omega}(Q_\alpha)$ is even $[\omega, \omega]$-compact.

The next theorem gives us the way for applying the concepts introduced in § 1 to the compactness spectrum of logics.

2.4. THEOREM. – *If $\lambda \geqslant \mu$ are regular cardinals, and $\lambda \overset{\varkappa}{\Rightarrow} \mu$, then every $\varkappa$-(λ, λ)-compact logic is $\varkappa$-(μ, μ)-compact.*

PROOF. – Suppose by contradiction that L is $\varkappa$-(λ, λ)-compact and Σ, $\{\sigma_\beta | \beta \in \mu\} = \Gamma$ is a counterexample to the $\varkappa$-(μ, μ)-compactness of L; and let $\mathfrak{A}$ be an expansion as in Definition 1.2.

Let Σ^* say that $\{x | U(x)\}$ is isomorphic to an $L_{\omega\omega}$-elementary extension of $\mathfrak{A}$ and that for every $\beta \in \mu$ $\mu \geqslant y \geqslant \beta$ implies that $\{x | f(x) = y\}$ is a model of $\Sigma \cup \{\sigma_\beta\}$; and put $\Gamma^* = \{c \geqslant \alpha | \alpha \in \lambda\}$.

Now, as λ and μ are regular, $\Sigma^* \cup \Gamma^*$ has a model $\mathfrak{B}$ by $\varkappa$-(λ, λ)-compactness, but then, because of Γ^*, and since $\lambda \overset{\varkappa}{\Rightarrow} \mu$, in $\mathfrak{B}$ there exists a b such that $\mu > b > \beta$, for every $\beta \in \mu$ and then $\Sigma^* \cup \Gamma^*$ implies that $\{x | f(x) = b\}$ is a model of $\Sigma \cup \Gamma$, a contradiction.

In some cases, a converse of Theorem 2.4 holds:

2.5. THEOREM. – *If $\lambda \geqslant \mu$ are regular cardinals, then for every $\varkappa$ and $\varkappa'$ such that $\infty \geqslant \varkappa' \geqslant \varkappa \geqslant 1$ and $\varkappa' \geqslant 2^\lambda$, the following are equivalent:*

(i) $\lambda \overset{2^\lambda}{\Rightarrow} \mu$;

(ii) $\lambda \overset{\varkappa'}{\Rightarrow} \mu$;

(iii) *every (λ, λ)-regular ultrafilter over λ is (μ, μ)-regular;*

(iv) *every (λ, λ)-regular ultrafilter over any set is (μ, μ)-regular;*

(v) *every $\varkappa'$-(λ, λ)-compact logic is $\varkappa$-(μ, μ)-compact;*

(vi) *every $[\lambda, \lambda]$-compact logic is (μ, μ)-compact;*

(vii) *every $[\lambda, \lambda]$-compact logic generated by at most μ cardinality quantifiers is (μ, μ)-compact;*

(viii) *if K is a set of ordinals, has order type μ and is such that $\nu < \omega_\alpha$ implies $\nu^\lambda < \omega_\alpha$, for every $\alpha \in K$, and such that either* $\mathrm{cf}(\omega_\alpha) > \lambda$ *or* $\mathrm{cf}(\omega_\alpha) = \mu$, *for every $\alpha \in K$, and if $\alpha' = \sup K$, and $K' = K \cup \{\alpha'\}$, then the logic $L = L_{\omega\omega}(Q_\alpha)_{\alpha \in K'}$ is not $[\lambda, \lambda]$-compact.*

PROOF. – The equivalence of conditions from (i) to (iv) is a consequence of Proposition 1.6; (ii) $\Rightarrow$ (v) is a consequence of Theorem 2.4 and Proposition 2.2 (i); (v) $\Rightarrow$ (vi) is a consequence of Proposition 2.2 (i) (iii); (vi) $\Rightarrow$ (vii) is trivial.

(vii) $\Rightarrow$ (viii) It is enough to show that L is not (μ, μ)-compact. Indeed, a counterexample is given by a set of sentences saying:

(a) there are less than $\omega_{\alpha'}$ elements;

$(b)_\alpha$ there are at least ω_α elements ($\alpha \in K$).

(viii) $\Rightarrow$ (iii) Suppose by contradiction that D is a (λ, λ)-regular not (μ, μ)-regular ultrafilter over λ, and let K, K', and L be as in the hypotheses of (viii); if $(\nu_\beta)_{\beta\in\lambda}$ are cardinals, and $\{\beta\in\lambda | |\nu_\beta < \omega_{\alpha'}\} \in D$, then, since $\mathrm{cf}(\omega_{\alpha'}) = \mu$ and D is not (μ, μ)-regular, there is $\nu < \omega_{\alpha'}$ such that $\{\beta | \nu_\beta < \nu\} \in D$, and hence

$$\left|\prod_D \nu_\beta\right| \leqslant \left|\prod_D \nu\right| \leqslant \nu^\lambda \leqslant \omega_\alpha^\lambda = \omega_\alpha < \omega_{\alpha'}, \quad \text{for some } \alpha \in K .$$

By [Lp1, Corollary 4.5 (iv)] this implies that $(D, V) \in F_\nu(L_{\omega\omega}(Q_{\alpha'}))$, for every cardinal ν, where $V = S(\lambda\times\lambda)$. In a similar fashion, the cardinality hypotheses on ω_α imply that $(D, V) \in F_\nu(L_{\omega\omega}(Q_\alpha))$, for every cardinal ν and for every $\alpha \in K$. Then, by [Lp1, Corollary 3.5], L would be $[\lambda, \lambda]$-compact, a contradiction.

Concerning Theorem 2.5, we do not know if in (vii) we can replace « at most μ » with « one » (if we ask for $[\mu, \mu]$-compactness instead of (μ, μ)-compactness, we can do this, as is shown by a variation-indeed, a simplification-of the proof). Since it is known that (iii) is a property of set-theoretical nature, it follows that so is $\lambda \overset{\varkappa}{\Rightarrow} \mu$: we shall see more detailed connections in § 4.

By the above remark, the $[\lambda, \lambda]$-compactness spectra of logics and the (λ, λ)-regularity spectra of ultrafilters are very similar. The situation changes if we consider $[\lambda, \mu]$-compactness and (μ, λ)-regularity: indeed, a (λ^+, λ^+)-regular ultrafilter is necessarily (λ, λ)-regular, but this does not imply (λ, λ^+)-regularity (see e.g. [Do]). Proposition 2.2 (iv) with $\varkappa = \infty$ shows that for logics the situation is radically different.

In view of Theorem 2.5, it is possible that the following holds: $\lambda \overset{\varkappa}{\Rightarrow} \mu$ iff every $\varkappa$-(λ, λ)-compact logic is $\varkappa$-(μ, μ)-compact; anyway, this is true if $V = L$ (see Corollary 4.4). It is possible also that the right notion for characterizing when (λ, λ)-compactness implies (μ, μ)-compactness involves filters (cf. Proposition 1.6).

Nevertheless, we can put together Theorem 2.4 and Theorem 1.3 (if $\lambda = \omega_\alpha$, let λ^{+n} be $\omega_{\alpha+n}$):

2.6. COROLLARY. – *If λ is regular, then every $\varkappa$-$(\lambda^{+n}, \lambda^{+n})$-compact logic is $\varkappa$-(λ, λ)-compact.*

In particular, if λ is regular then every $(\lambda^{+n}, \lambda^{+n})$-compact logic is (λ^{+n}, λ)-compact.

In some cases we have a similar result whose proof refers only to regularity of ultrafilter. Notice that in the following theorem λ and ν are not necessarily regular.

2.7. THEOREM. – *Suppose that $\varkappa \geqslant 2^{\lambda^{<\lambda}}$ and that every (λ, λ)-regular ultrafilter is (ν, ν)-regular. Then every $\varkappa$-$(\lambda^{+n}, \lambda^{+n})$-compact logic is $\varkappa$-(ν, ν)-compact.*

If $\varkappa \geqslant \sup\{\lambda^{+n}, 2^{\lambda^{<\lambda}}\}$, then every $(\varkappa, \lambda^{+n})$-compact logic is $(\varkappa, \lambda)$-compact.

PROOF. – Let us say that $H \subset S_\lambda(\lambda)$ is *cofinal* in $S_\lambda(\lambda)$ iff for every $x \in S_\lambda(\lambda)$ there exists $y \in H$ such that $x \subset y$ ($S_\lambda(\lambda)$ is the set of subsets of λ of cardinality less than λ).

Let $\mathrm{cf}\, S_\lambda(\lambda) = \inf\{|H| \,|\, H$ is cofinal in $S_\lambda(\lambda)\}$. It is not difficult to show that $\lambda \leqslant \mathrm{cf}\, S_\lambda(\lambda) \leqslant |S_\lambda(\lambda)| = \lambda^{<\lambda}$; and that if λ is regular, then $\mathrm{cf}\, S_\lambda(\lambda) = \lambda$. We shall actually prove the theorem with the weaker hypothesis $\varkappa \geqslant 2^{\sup(\mathrm{cf}\, S_\lambda(\lambda),\, \mathrm{cf}\, S_\nu(\nu))}$. Because of this remark, and [CN, Theorem 8.35], the theorem will follow just from the case $n = 0$.

By standard methods (see e.g. [Lp1, Lemma 3.2]) one can prove that if H is cofinal in $S_\lambda(\lambda)$ and $\{\alpha\} \in H$, for every $\alpha \in \lambda$, then an ultrafilter D is (λ, λ)-regular iff in $\prod_D \langle H, \subset, \{\alpha\}\rangle_{\alpha\in\lambda}$ there is x such that $d(\{\alpha\}) \subset x$, for every $\alpha \in \lambda$.

Suppose that $\Sigma, \Gamma = \{\sigma_\alpha \,|\, \alpha \in \nu\} \subset L$ are such that $|\Sigma| \leqslant \varkappa$ and if $\Gamma' \subset \Gamma$ and $|\Gamma'| < \nu$ then $\Sigma \cup \Gamma'$ has a model. We construct a model of $\Sigma \cup \Gamma$ as follows: let H, K be cofinal in $S_\lambda(\lambda)$, $S_\nu(\nu)$, respectively, of minimal cardinality; and let $\mathfrak{A}$ be the *completion* of the model $\langle H \cup K \cup \lambda, \subset, H, K, \{\alpha\}\rangle_{\alpha\in\lambda}$ (that is, the model obtained from it by adding a symbol for every constant, relation or function in $H \cup K \cup \lambda$). Let T be the $L_{\omega\omega}$-theory of $\mathfrak{A}$; and let Σ^* say that $\{x \,|\, U(x)\}$ is a model of T and, for all $\alpha \in \nu$, $\{\alpha\} \subset x$ and $K(x)$ imply that $\{y \,|\, f(y) = x\}$ is a model of $\Sigma \cup \{\sigma_\alpha\}$; and let $\Gamma^* = \{\{\alpha\} \subset c \,|\, \alpha \in \lambda\} \cup \{H(c)\}$.

By $\varkappa$-(λ, λ)-compactness, $\Sigma^* \cup \Gamma^*$ has a model $\mathfrak{B}$, and $\mathfrak{B}|_U$ is a complete extension of $\mathfrak{A}$; because of [CK, Theorem 6.4.4] there exists a model $\mathfrak{C}$ such that $\mathfrak{A} < \mathfrak{C}|_U < \mathfrak{B}|_U$, $c \in C$ and $\mathfrak{C}|_U \cong \prod_D \mathfrak{A}$, for some ultrafilter D. Because of c, D is (λ, λ)-regular, hence (ν, ν)-regular, so that in $\mathfrak{C}|_U$ there is a d such that $\{\alpha\} \subset d$, for every $\alpha \in \nu$; hence in $\mathfrak{B}$ $\{y \,|\, f(y) = d\}$ is a model of $\Sigma \cup \Gamma$.

It is possible that the hypothesis $\varkappa \geqslant 2^{\lambda^{<\lambda}}$ in Theorem 2.7 can be weakened (maybe to $\varkappa \geqslant 2^\lambda$; this is indeed true for the second statement [Lp4], as well for the case of both λ and ν regular. A slight variation on the proof shows that $\varkappa \geqslant \mathrm{cf}\, S_\nu(\nu)^{\mathrm{cf}\, S_\lambda(\lambda)}$ is enough.) Anyway, if λ is a weakly compact cardinal and there is no uncountable measurable cardinal smaller than λ, then every (λ, λ)-

regular ultrafilter is (ω, ω)-regular; but there is a (λ, λ)-compact logic not (ω, ω)-compact.

We remark that we can define also cf $S_\mu(\lambda)$ in a similar way: this notion seems interesting for the study of (λ, μ)-regularity.

A generalization of König lemma gives cf (cf $S_\varkappa(\lambda)) \geqslant$ cf $\varkappa$.

Finally, we mention without proof:

2.8. Proposition. – If $\mathfrak{A} = \langle A, \omega, <, \ldots\rangle$ and L is countably generated then there exists an expansion $\mathfrak{A}^+$ of $\mathfrak{A}$ such that $|\tau\mathfrak{A}^+| \leqslant |\tau\mathfrak{A}|^\omega$ and whenever $\mathfrak{B} \equiv_L \mathfrak{A}^+$ and $\mathfrak{B}$ realizes $\{n < x \mid n \in \omega\}$ then $\mathfrak{B}$ realizes every countable consistent L-type over any finite subset of B (that is, if $\{\varphi_i(x, \bar{c})\}_{i\in\omega}$ are L-formulas, $\bar{c} = (c_1, \ldots, c_n) \in B^n$, and for every finite $F \subset \omega$ there is $d \in B$ such that $\mathfrak{B} \models \varphi_i(d, \bar{c})$ $(i \in F)$ then there is $d \in B$ such that $\mathfrak{B} \models \varphi_i(d, \bar{c})$ $(i \in \omega)$).

Indeed, the only hypothesis needed about L is that if $|\tau| \leqslant |\tau\mathfrak{A}|^\omega$ then L has at most $|\tau\mathfrak{A}|^\omega$ sentences of type τ.

Notice that a $\mathfrak{B}$ realizing such a type exists in case L is $|\mathrm{Th}_L(\mathfrak{A}^+)|-(\omega, \omega)$-compact .

On the contrary, if $L = L_{\omega\omega}(\mathrm{Cf}^{=\omega})$, $\mathfrak{B} \equiv_L \langle\omega, <\rangle$ and $\{c_n\}_{n\in\omega}$ are infinitely many elements of B cofinal in $\mathfrak{B}$, then $\mathfrak{B}$ does not realize $\{c_n < d\}_{n\in\omega}$.

3. – Large cardinals and infinitary logics.

3.1. Definitions. – We say that a logic N *characterizes a cardinal* λ *(with* μ *sentences)* iff there is a (consistent) N-theory T with a unary predicate U (and $|T| \leqslant \mu$) such that in every model of T $\{x \mid U(x)\}$ is a model isomorphic to $\langle\lambda, <, \alpha\rangle_{\alpha\in\lambda}$. Clearly, if N characterizes λ, then N characterizes every cardinal $< \lambda$.

A model $\mathfrak{A}$ is *N-maximal* iff it has no proper N-elementary extension; maximal predicates are defined similarly (see [MS, section 1.6]). It is easy to show that if $\mathfrak{A}$ is N-maximal (or has an N-maximal predicate P) then N characterizes $|A|$ ($|P|$, respectively) with $|\mathrm{Th}_N(\mathfrak{A})|$ sentences; and, conversely, if N characterizes λ, then there is a model with an N-maximal predicate of cardinality λ.

We denote by $N_{\alpha\beta}$ the logic obtained from N allowing conjunctions and disjunctions of $< \alpha$ sentences, and universal or existential quantification over $< \beta$ constants. If N is also closed under applications of the (finitary) quantifier Q, we also allow $N_{\alpha\beta}$ to be closed under finitely many applications of Q.

3.2. THEOREM. – *If the logic N is μ-$(\nu, \varkappa)$-compact, and characterizes every $\lambda' < \lambda$ with μ sentences, then also $N_{\lambda\omega}$ is μ-$(\nu, \varkappa)$-compact.*

PROOF. – We first observe that, under the hypotheses, N characterizes every $\lambda' < \lambda$ with μ sentences, so that N cannot be μ-(λ', λ')-compact and hence $\varkappa \geqslant \lambda$. By Proposition 2.2 (ii) we may also suppose that $\mu \geqslant \nu$.

Suppose by contradiction that Σ, Γ is a counterexample to the μ-$(\nu, \varkappa)$-compactness of $N_{\lambda\omega}$ and, for every $\lambda' < \lambda$, let $U_{\lambda'}$ be a new unary relation symbol, and let $\Sigma_{\lambda'}$ be such that $|\Sigma_{\lambda'}| \leqslant \mu$ and in every model of $\Sigma_{\lambda'}$, $\{x \mid U_{\lambda'}(x)\}$ is a model isomorphic to $\{\lambda', <, \alpha\}_{\alpha\in\lambda'}$ (without loss of generality, we can suppose that the types of $\Sigma \cup \Gamma$ and of the $\Sigma_{\lambda'}$'s have no symbol in common).

For every subformula $\varphi(\bar{x}) = \bigwedge\limits_{\alpha\in\lambda'<\lambda} \varphi_\alpha(\bar{x})$ of some sentence σ of $\Sigma \cup \Gamma$, the $\varphi_\alpha(\bar{x})$'s being formulas of N, let R_φ be a new $(n+1)$-ary relation symbol, where n is the number of variables of $\varphi(\bar{x})$ (n is finite, since σ is a sentence, and we can quantify away only a finite number of variables). Now, for every such φ, substitute $\varphi^*(\bar{x}) = \forall y\big(U_{\lambda'}(y) \Rightarrow R_\varphi(y, \bar{x})\big)$ for $\varphi(\bar{x})$, for every occurrence of $\varphi(\bar{x})$ in $\Sigma \cup \Gamma$. Similarly, substitute $\exists y\big(U_{\lambda'}(y) \wedge R_\varphi(y, \bar{x})\big)$ for $\bigvee\limits_{\alpha\in\lambda'<\lambda} \varphi_\alpha(\bar{x})$.

Let Σ^* and Γ^* be obtained from Σ and Γ by iterating transfinitely this procedure of substitution: an easy induction on the complexity of sentences of $N_{\lambda\omega}$ shows that we need to introduce at most $\sup\big(|\Sigma|, \lambda\big) \leqslant \mu$ new relations R_φ's and then

$$\Sigma^* \cup \{\Sigma_{\lambda'} \mid \lambda' < \lambda\} \cup \big\{\forall\bar{x}(\varphi_\alpha(\bar{x}) \Leftrightarrow R_\varphi(\alpha, \bar{x})) \mid \varphi \text{ as above, } \alpha < \lambda'\big\}$$

and Γ^* give a counterexample to the μ-$(\nu, \varkappa)$-compactness of N.

We remark that in the proof of 3.2 we made a substantial use of the fact that in $N_{\lambda\omega}$ we do not allow quantification over infinite sets (even in the case that this is allowed in N). Indeed, Theorem 3.2 cannot be generalized:

3.3. EXAMPLE. – Let $\mathrm{Cf}^{=\mu}$ and WO be the quantifiers interpreted by:

$$\mathrm{Cf}^{=\mu}\, xy \;\; \varphi(x, y) \text{ iff } \varphi \text{ defines a linear order of cofinality } \mu$$

and

$$\mathrm{WO}\, xy \;\; \varphi(x, y) \text{ iff } \varphi(x, y) \text{ defines a well order}.$$

(Usually, $\mathrm{Cf}^{=\mu}$ is denoted by $Q^{\mathrm{Cf}=\mu}$; however, we believe that our notation is simpler and creates no confusion).

Then, if μ is a measurable cardinal, the logic $L_{\mu\omega}(\mathrm{Cf}^{=\mu})$ is $[\mu, \mu]$-compact and characterizes every $\mu' < \mu$ with μ' sentences; but $L_{\mu\omega_1}(\mathrm{Cf}^{=\mu})$ is not $[\mu, \mu]$-compact (indeed, already $L_{\omega\omega}(\mathrm{WO}, \mathrm{Cf}^{=\mu})$ is not (μ, μ)-compact).

Thus, with the order defined in [KV], $[\mu, \mu]$-compact logics do not form a lattice, but just a meet semilattice (this is true also for fully compact logics: it is possible to find an example of two compact logics whose union is not compact).

PROOF. – The $[\mu, \mu]$-compactness of $L_{\mu\omega}(\mathrm{Cf}^{=\mu})$ is similar to [Lp1, Example 6.1]: use a non principal μ-complete ultrafilter over μ, an anti-well-order of type μ with a well-order of type ω at the bottom (this is done in order to prevent the possibility of changing to μ cofinalities which are unboundedly $< \mu$).

On the contrary, $L_{\omega\omega}(\mathrm{WO}, \mathrm{Cf}^{=\mu})$ is not (μ, μ)-compact, since every model with a linear order satisfying the following sentences:

$$\mathrm{Cf}^{=\mu} xy \;\; x<y \; ; \qquad \forall x\, \mathrm{Cf}^{\neq\mu} yz \;\; y<z<x \; ; \qquad \mathrm{WO}\, xy \;\; x<y$$

must be well-ordered of type μ.

CONJECTURES. – The logic $L_{\mu\omega}(\mathrm{Cf}^{=\mu})$ has the same compactness properties of $L_{\mu\omega}$, for every cardinal μ.

If N is closed under applications of WO, then the conclusion of Theorem 3.2 can be extended to $N_{\lambda\lambda}$ (at least when λ is not a too large cardinal).

In some cases however, we have an analogue of Theorem 3.2 for infinitary quantifiers; the proof of the following proposition uses [Lp1, Proposition 6.5.1] (it is likely that also the version for $(\nu, \varkappa)$-compactness holds).

3.4. PROPOSITION. – *If N is $[\nu, \varkappa]$-compact, contains $L_{\beta\beta}$ and characterizes every $\lambda' < \lambda$, then also $N_{\lambda\beta}$ is $[\nu, \varkappa]$-compact.*

If N is a logic, let $N_{(\mu)\omega}$ be the least sublogic of $N_{\mu\omega}$ containing N as well as $\bigwedge\limits_{\alpha\in\mu} \varphi_\alpha$, whenever $\{\varphi_\alpha\}_{\alpha\in\mu}$ are N-sentences and $\models \varphi_\beta \Rightarrow \varphi_\alpha$, for every $\alpha \leqslant \beta \in \mu$. The proof of Theorem 3.1 can be adapted in order to give: *if $\lambda \geqslant \mu$ are regular cardinals and N is (λ, λ)-compact but not (μ, μ)-compact, then also $N_{(\mu)\omega}$ is (λ, λ)-compact.*

Indeed, the hypothesis that N is not (μ, μ)-compact could be replaced by the weaker « $\lambda \overset{N}{\Rightarrow} \mu$ fails », which is defined as in 1.6 (ii) but referring to N-elementary equivalence.

3.5. Lemma. – *If the logic N characterizes λ with μ sentences, then N characterizes λ^+ with* $\sup(\mu, \lambda^+)$ *sentences.*

Proof. – First, observe that N is not μ-(λ^+, λ^+)-compact, by the remark after Proposition 2.3.

Hence, a set of μ N-sentences cofinally characterizes λ^+ (cf. [BF, XVIII, Definition 1.2.1]); and, if for every α such that $\lambda \leqslant \alpha < \lambda^+$ we say that f_α is a bijection from α to λ, and we characterize λ, we have completely characterized λ^+.

3.6. Lemma. – *If λ is singular and N characterizes every $\lambda' < \lambda$ with μ sentences, then N characterizes λ with* $\sup(\lambda, \mu)$ *sentences.*

Proof. – Since $\mathrm{cf}(\lambda) < \lambda$, $\mathrm{cf}(\lambda)$ is characterizable, so that λ is characterized by characterizing every $\lambda' < \lambda$, as well as a cofinal sequence in λ.

3.7. Lemma. – *If λ is a regular limit cardinal, N characterizes every $\lambda' < \lambda$ with μ sentences and N is not μ'-(λ, λ)-compact, then N characterizes λ with* $\sup(\lambda, \mu', \mu)$ *sentences.*

Proof. – As in the preceding lemmas, we use the failure of μ'-(λ, λ)-compactness of L in order to cofinally characterize λ.

3.8. Lemma. – *If L characterizes λ with μ sentences and λ' with μ' sentences and $\varkappa$ is either $\lambda^{\lambda'}$, $(<\lambda)^{\lambda'}$, $\lambda^{<\lambda'}$ or $(<\lambda)^{<\lambda'}$, then L characterizes $\varkappa$ with* $\sup(\varkappa, \mu, \mu')$ *sentences.*

Proof. – For $\varkappa = \lambda^{\lambda'}$, characterize λ' and λ, and make correspond (using a ternary relation) a distinct ordinal of $\lambda^{\lambda'}$ to a distinct function from λ' to λ, saying that all the functions are distinct.

For the other cases use the first part and Lemma 3.6.

3.9. Theorem. – *If $\mu \leqslant \infty$, and N is a logic, then the following are equivalent:*

(i) *λ is the first cardinal such that N is μ-(λ, λ)-compact;*

(ii) *λ is the first cardinal which is not characterized by* $\sup(\lambda, \mu)$ *sentences of N.*

Moreover, if this is the case, then also $N_{\lambda\omega}$ is μ-(λ, λ)-compact; indeed, if $\mu' \geqslant \mu$, then $N_{\lambda\omega}$ is μ'-$(\nu, \varkappa)$-compact iff so is N. In particular, λ is a cardinal such that $L_{\lambda\omega}$ is μ-(λ, λ)-compact.

If, in addition, $2^{<\lambda} \leqslant \mu$, then λ is strongly inaccessible, hence weakly compact; and if $2^\lambda \leqslant \mu$, then λ is measurable.

PROOF. – From Theorem 3.2 and Lemmata 3.5, 3.6, 3.7 and 3.8. For the last conclusion, note that if N is 2^λ-(λ, λ)-compact, then $F_\lambda(N) \cap \mathrm{Reg}\,(\lambda, \lambda) \neq \emptyset$, by [Lp1, Proposition 3.3], so that there exists a λ-complete non principal ultrafilter over λ.

We can now characterize many large cardinals as first cardinals for which some logic satisfies some compactness property (other results in this direction are contained in [Ma]). Weakly and strongly compact cardinals were originally defined as cardinals for which infinitary languages are compact: Corollary 3.10 shows that the results would have been the same even if the starting point were different logics.

3.10. COROLLARY. – *(a) $L_{\varkappa\omega}$ is $(\varkappa, \varkappa)$-compact iff there is a logic L such that $\varkappa$ is the first cardinal for which L is $(\varkappa, \varkappa)$-compact.*

(b) $\varkappa$ is weakly compact iff there is a logic L such that $\varkappa$ is the first cardinal for which L is $2^{<\varkappa}$-$(\varkappa, \varkappa)$-compact iff $\varkappa$ is strong limit and there is a logic L such that $\varkappa$ is the first cardinal for which L is $(\varkappa, \varkappa)$-compact.

(c) [BF, XVIII, Theorem 1.5.2] *$\varkappa$ is measurable iff there is a logic L such that $\varkappa$ is the first cardinal for which L is $[\varkappa, \varkappa]$-compact iff there is a logic L such that $\varkappa$ is the first cardinal for which L is $2^\varkappa$-$(\varkappa, \varkappa)$-compact.*

(d) $\varkappa$ is λ-compact iff there is a logic L which is $[\lambda, \varkappa]$-compact but not $[\mu, \mu]$-compact for every $\mu < \varkappa$.

(e) $\varkappa$ is strongly compact iff there is a logic L which is $[\infty, \varkappa]$-compact but not $[\mu, \mu]$-compact for every $\mu < \varkappa$.

3.11. COROLLARY. – *If a logic N is (θ, θ)-compact, then there exists a $\varkappa \leq \theta$ such that $N_{\varkappa\omega}$ is θ-$(\varkappa, \varkappa)$-compact and either $\varkappa$ is measurable (or ω) or $\theta < 2^\varkappa$.*

PROOF. – Take $\varkappa$ to be the first cardinal such that N is θ-$(\varkappa, \varkappa)$-compact.

For the case $N = L_{\omega_1\omega}$ Corollary 3.11 strengthens a theorem of Bell [Di, p. 186]. Incidentally, this may be seen as another example of the success of Abstract Model Theory in solving concrete problems for very particular logics [MS, p. 292].

Weakly compact cardinals were originally defined by Hanf as cardinals for which $L_{\varkappa\varkappa}$ is $(\varkappa, \varkappa)$-compact: the eventuality that this definition does not imply (strong) inaccessibility of $\varkappa$ is seen in many texts as a defect, so that nowadays « weak compactness » incorporates inaccessibility right in the definition. However, in this paper we show that, to some extent, also Hanf's notion is very natural and interesting (unless the concept of $(\varkappa, \varkappa)$-compactness is obsolete—we believe this is not the case). We remark that we do not know of any example of a cardinal $\varkappa$ such that $L_{\varkappa\varkappa}$ is $(\varkappa, \varkappa)$-compact but $\varkappa$ is not inaccessible(*); we also do not know whether the $(\varkappa, \varkappa)$-compactness of $L_{\varkappa\omega}$, of its propositional part, and of $L_{\varkappa\varkappa}$ can be equivalent without assuming the inaccessibility(**) of $\varkappa$ (however, by an argument of [Si], it is possible that $\varkappa$ is weakly inaccessible, has the tree property but $L_{\varkappa\omega}$ is not $(\varkappa, \varkappa)$-compact).

Also the problem of characterizing relative minima in the compactness spectrum is completely open (and maybe very hard in the general case).

Problem. – Characterize those cardinals $\varkappa$ such that there exists a $\lambda < \varkappa$ and a logic L which is $(\varkappa, \varkappa)$-compact but not (ν, ν)-compact, for every regular ν, $\lambda \leqslant \nu < \varkappa$. From the statement of Corollary 3.10 we can obtain similar problems; e.g., for which cardinals $\varkappa$ there exists a logic L such that $\varkappa$ is the least cardinal for which L is $[\infty, \varkappa]$-compact?

We remark that, by Corollary 4.3, if there is no inner model with a measurable cardinal, $\lambda < \varkappa$ and there exists a logic $(\varkappa^+, \varkappa^+)$-compact not (ν, ν)-compact for every ν, $\lambda \leqslant \nu < \varkappa$, then $\varkappa$ must be weakly inaccessible.

On the other side, by a result of [BM], and [Lp2, Theorem 7] it is possible (assuming the consistency of a $\varkappa^+$-supercompact cardinal $\varkappa$) to have a logic $[\omega_{\omega+1}, \omega_\omega]$-compact not $[\omega_n, \omega_n]$-compact for every $n \geqslant 1$ (this can be improved to not (ω_n, ω_n)-compact, by a variation on Theorem 2.5 (viii)).

The relation $\lambda \overset{\varkappa}{\Rightarrow} \mu$ is connected with compactness properties of infinitary languages:

3.12. Theorem. – *If* $\varkappa \neq 0$, *then* $\lambda \overset{\varkappa}{\not\Rightarrow} \omega$ *iff* $L_{\omega_1\omega}$ *is* $\varkappa$-(λ, λ)-*compact.*

The proof of Theorem 3.12 is similar to the one of Theorem 3.2. Clearly, in Theorem 3.12 we can replace $L_{\omega_1\omega}$ with $L_{\mu\omega}$, if μ is the first cardinal such that $L_{\mu\omega}$ is $\varkappa$-(μ, μ)-compact.

Moreover, with suitable modifications, we can extend Theorem 3.12 to larger cardinals.

3.13. Definition. – If λ, μ_i $(i \in I)$ are infinite regular cardinals, and $\varkappa$ is a cardinal, we write $\lambda \overset{\varkappa}{\Rightarrow} \bigvee_{i \in I} \mu_i$ iff there exists an expansion $\mathfrak{A}$ of $\langle \lambda, < \rangle$ such that $|\tau\mathfrak{A} \setminus \{<\}| \leqslant \varkappa$ and whenever $\mathfrak{B} > \mathfrak{A}$

(*) AN EXAMPLE EXISTS: SEE [Bo].
(**) SEE AGAIN [Bo], IN PARTICULAR, THEOREM 2.4.

and $\mathfrak{B}$ realizes $\{\alpha < x \mid \alpha < \lambda\}$, then for some $i \in I$ $\mathfrak{B}$ realizes $\{\beta < y < \mu_i \mid \beta < \mu_i\}$.

3.14. THEOREM. – *If $\varkappa \neq 0$, then $\lambda \overset{\varkappa}{\Rightarrow} \bigvee_{\mu' < \mu} \mu'$ does not hold iff $L_{\mu\omega}$ is $\varkappa$-(λ, λ)-compact.*

We remark that similar methods give a (maybe new) generalization of a theorem by Rabin in a different direction than Keisler's version [CK, Theorem 6.4.5]:

3.15. PROPOSITION. – *If $\varkappa$ is the first cardinal for which $L_{\omega_1\omega}$ is $(\varkappa, \varkappa)$-compact, and $\lambda < \varkappa$, $\lambda \leqslant \nu \leqslant \lambda^\omega$, then there exists an expansion $\mathfrak{A}$ of $\langle \lambda, < \rangle$ such that $|\tau\mathfrak{A}| = \nu$ and every proper elementary extension of $\mathfrak{A}$ has cardinality $\geqslant \nu$.*

On the contrary, if μ is measurable and λ is the least cardinal of cofinality ω larger than 2^μ, then λ has a proper complete extension of cardinality λ, yet $\lambda^\omega > \lambda$ (cf. also [CK, Exercise 6.4.12]).

3.16. PROPOSITION. – *If $\mathfrak{A}$ is a model, $|A|$ is less than the first cardinal $\varkappa$ for which $L_{\omega_1\omega}$ is $(\varkappa, \varkappa)$-compact, and L is countably generated, then there exists an expansion $\mathfrak{A}^+$ of $\mathfrak{A}$ such that $|\tau\mathfrak{A}^+| \leqslant$* sup $(|A|^\omega, |\tau\mathfrak{A}|^\omega)$ *and every proper L-elementary extension $\mathfrak{B}$ of $\mathfrak{A}^+$ realizes every countable consistent L-type over any finite subset of B (cf. Proposition 2.8).*

4. – Consequences of $\mathbf{E}_\varkappa^\lambda$ and $\square_\varkappa$.

If $\lambda < \varkappa$ are regular cardinals, let $S_\varkappa^\lambda$ be $\{\alpha < \varkappa \mid \mathrm{cf}\, \alpha = \lambda\}$. The *combinatorial principle* $E_\varkappa^\lambda$ states that there exists a subset A of $S_\varkappa^\lambda$ stationary in $\varkappa$ such that, for all limit $\alpha < \varkappa$, $A \cap \alpha$ is non stationary in α.

Let $\gamma\colon S_\varkappa^\lambda \times \lambda \to \varkappa$ be such that for every $\delta \in S_\varkappa^\lambda$ $\{\gamma(\delta, \xi) \mid \xi < \lambda\}$ is an increasing closed cofinal sequence in δ. If $X \subset S_\varkappa^\lambda$, say that $f\colon X \to \lambda$ is a *disjointer* for X iff, whenever $\delta, \eta \in X$ and $\delta \neq \eta$, we have that $\{\gamma(\delta, \xi) \mid \xi > f(\delta)\} \cap \{\gamma(\eta, \xi) \mid \xi > f(\eta)\} = \emptyset$.

4.1. THEOREM. – *If $\lambda < \varkappa$ are regular cardinals and $E_\varkappa^\lambda$ holds, then $\varkappa \Rightarrow \lambda$.*

PROOF. – Suppose by contradiction that $\varkappa \Rightarrow \lambda$ fails and let γ as above be fixed. By [KM, p. 220], in order to prove that $E_\varkappa^\lambda$ fails it is enough to prove that if $A \subset S_\varkappa^\lambda$ is stationary and for every $\alpha < \varkappa$ there is a disjointer f_α for $A \cap \alpha$ then there is a disjointer for A.

Consider the model $\mathfrak{M} = \langle \varkappa, <, A, \gamma, f \rangle$, where $f\colon A \times \varkappa \to \lambda$ is defined by $f(\alpha, \delta) = f_\alpha(\delta)$ if $\delta \in A \cap \alpha$, and arbitrarily in the other cases.

Since $\varkappa \Rightarrow \lambda$ fails, there exists $\mathfrak{N} > \mathfrak{M}$ such that $x \in N$ and $x \geqslant \delta$, for every $\delta \in \varkappa$; but no element $y \in N$ is such that for every $\xi < \lambda$ $\xi < y < \lambda$ (without loss of generality we can assume that $x \in A^{\mathfrak{N}}$, as A is cofinal in $\varkappa$). Now, the fact that, for every $\alpha < \varkappa$, $f(\alpha, -)$ is a disjointer for $A \cap \alpha$ can be expressed as a first order statement, by saying that for every $\alpha \in A$ and $\delta \neq \eta \in A \cap \alpha$,

$$\gamma\big(\delta, (f(\alpha, \delta), \lambda)\big) \cap \gamma\big(\eta, (f(\alpha, \eta), \lambda)\big) = \emptyset .$$

So that in $\mathfrak{N}$, for every $\delta \neq \eta \in A^{\mathfrak{N}}$,

$$\gamma\big(\delta, (f(x, \delta), \lambda)\big) \cap \gamma\big(\eta, (f(x, \eta), \lambda)\big) = \emptyset ;$$

but the range of f is contained in λ, so that for every $\delta \in A$ there exists $\xi < \lambda$ such that $f(x, \delta) < \xi$. If $f^*(\delta)$ is the least such ξ, then clearly $f^*\colon A \to \lambda$ is a disjointer for A.

This proof is very similar to the one given in [KM] using a N.V-(λ, λ)-regular ultrafilter over $\varkappa$. Indeed, that proof can be performed by considering the above model $\mathfrak{M}$ and just taking its ultrapower.

4.2. Corollary. – *If $\square_\varkappa$ holds, then $\varkappa^+ \Rightarrow \lambda$ holds for every regular cardinal $\lambda < \varkappa$; and hence, every $(\varkappa^+, \varkappa^+)$-compact logic is (λ, λ)-compact; hence also $(\varkappa^+, \omega)$-compact.*

Proof. – If $\varkappa = \omega$ (or just $\varkappa < \omega_\omega$) Theorem 1.3 is enough and we do not need $\square_\varkappa$. Otherwise, $\square_\varkappa$ implies $E^\lambda_{\varkappa^+}$ [KM, p. 221], and the conclusion follows from Theorems 4.1 and 2.4, and Proposition 2.2 (iv) (v).

It is known that, for $\varkappa > \omega$, $\square_\varkappa$ is a consequence of either $V = L$ (or even $V = K$) or «$\varkappa^+$ is not Mahlo in L»; so that the conclusions of Corollary 4.2 also hold under such hypotheses. We also have:

4.3. Corollary. – *If $\varkappa$ is singular and there is no inner model with a measurable cardinal, then $\varkappa^+ \Rightarrow \lambda$ holds, for every regular $\lambda < \varkappa$. So that every $(\varkappa^+, \varkappa^+)$-compact logic is $(\varkappa^+, \omega)$-compact.*

Proof. – By Corollary 4.2 and some results by Dodd and Jensen ([KM, p. 222] or [Do, Theorem 3.12]).

4.4. COROLLARY. – ($V = L$). *For a regular cardinal $\varkappa$, the following are equivalent:*

(i) *$\varkappa$ is weakly compact.*

(ii) *For some (equivalently, every) regular $\lambda < \varkappa$ there is a $(\varkappa, \varkappa)$-compact logic not (λ, λ)-compact.*

(iii) *$\varkappa \Rightarrow \lambda$ fails for some (equivalently, every) regular $\lambda < \varkappa$.*

(iv) *$E^{\lambda}_{\varkappa}$ fails for some (equivalently, every) regular $\lambda < \varkappa$.*

In particular, if $V = L$, then for every regular λ and $\varkappa$, the properties $E^{\lambda}_{\varkappa}$, $\varkappa \Rightarrow \lambda$ and «every $(\varkappa, \varkappa)$-compact logic is (λ, λ)-compact» are all equivalent to each other.

PROOF. – (iv) $\Rightarrow$ (i) is due to Jensen [KM, p. 219]. The other implications follow from Corollary 3.10 (*b*) and Theorems 2.4 and 4.1.

It is conceivable that the assumption $V = L$ can be weakened in Corollary 4.4 (indeed, it is used only in (iv) $\Rightarrow$ (i)). Nevertheless, we cannot go too far: by a result of Baumgartner [KM, p. 222] we cannot prove the converse of Theorem 4.1.

4.5. COROLLARY. – *If ZFC + «There exists a weakly compact cardinal» is consistent, then it is consistent to assume that $E^{\omega}_{\omega_2}$ fails. On the contrary, $\omega_2 \Rightarrow \omega$ always holds, because of Theorem* 1.3.

It is very likely that the proof of [Do, Theorem 1.4] can be adapted in order to show that, for $\varkappa \geqslant \lambda$ regular cardinals, $\square^{-}_{\varkappa}$ implies $\varkappa \Rightarrow \lambda$.

5. – Logics generated by monadic quantifiers.

We now characterize the compactness properties of monadic quantifiers (more precisely, we reduce it to the problem—not yet completely solved—of the compactness properties of cardinality quantifiers). A *monadic quantifier* Q_H is completely determined by a class H of n-tuples of cardinals. Its interpretation is given by:

$$\mathfrak{A} \models Q_H x_1, \dots, x_n(\varphi_1, \dots, \varphi_n)$$
$$\text{iff } \big(|\{x \in A \mid \mathfrak{A} \models \varphi_1(x)\}|, \dots, |\{x \in A \mid \mathfrak{A} \models \varphi_n(x)\}|\big) \in H .$$

5.1. DEFINITION. – Let H be a fixed class of n-tuples of cardinals: we say that $\alpha \leqslant \infty$ is an *extreme* (*for the i-th coordinate*) iff for every $\beta < \alpha$ there exist β', $\gamma_1, \ldots, \gamma_{n-1}$ such that $\beta \leqslant \beta' < \alpha$ and $(\gamma_1, \ldots, \gamma_{i-1}, \alpha, \gamma_i, \ldots, \gamma_{n-1}) \in H$ iff $(\gamma_1, \ldots, \gamma_{i-1}, \beta', \gamma_i, \ldots, \gamma_{n-1}) \notin H$.

Thus, α and α' are consecutive extremes iff $[\alpha, \alpha')$ is a maximal interval with the property that for every $\gamma_1, \ldots, \gamma_{n-1}$ either $\{\gamma_1\} \times \times \ldots \times [\alpha, \alpha') \times \ldots \times \{\gamma_{n-1}\}$ is contained in H or is contained in the complement of H.

For the rest of this section, H will be an arbitrary but fixed class of n-tuples of cardinals. We first see that if $L_{\omega\omega}(Q_H)$ is (ω, ω)-compact, then H has a very particular form.

5.2. THEOREM. – *If the logic $L_{\omega\omega}(Q_H)$ is (ω, ω)-compact, then H is a finite union of products of intervals.*

PROOF. – It is enough to prove that, for every coordinate, there are only finitely many extremes: then H is a union of products of intervals of the kind $[\alpha, \alpha')$, where α and α' are extremes for the appropriate coordinate.

So, suppose by contradiction that there are infinitely many extremes for the i-th coordinate: from them we can extract an increasing sequence $(\alpha_k)_{k\in\omega}$ of consecutive extremes.

Let now $\mathfrak{A}$ be a model $\langle A, \omega, <, f_1, \ldots, f_n, g\rangle$ such that $<$ is the order on ω, each f_j is a function from A to ω and g is a binary function such that for every $k \in \omega$ $g(k, -)$ is injective from $f_i^{-1}(k)$ into $f_i^{-1}(k+1)$ and moreover, for every $k \in \omega$,

$$(f_1^{-1}(k), \ldots, f_i^{-1}(k), \ldots, f_n^{-1}(k)) \in H \quad \text{iff} \quad (f_1^{-1}(k), \ldots, f_i^{-1}(k+1), \ldots, f_n^{-1}(k)) \notin H$$

(this is possible since we can choose f_i in such a way that $|f_i^{-1}(k)| = \alpha_k$).

By (ω, ω)-compactness, there exists a model $\mathfrak{B} \equiv_L \mathfrak{A}$ with a non-standard element c in the order $<$: choose c in such a way that $|f_i^{-1}(c)|$ is minimum: by the L-theory of $\mathfrak{A}$ we have that

$$(f_1^{-1}(c-1), \ldots, f_i^{-1}(c-1), \ldots, f_n^{-1}(c-1)) \in H \quad \text{iff} \quad (f_1^{-1}(c-1), \ldots, f_i^{-1}(c), \ldots, f_n^{-1}(c-1)) \notin H\,,$$

but this is absurd, as $|f_i^{-1}(c)| = |f_i^{-1}(c-1)|$.

This method of proof is more general than the one given in [Lp1, Proposition 5.2], which uses ultraproducts; however the former is inspired by the latter. For the case $n = 1$ Theorem 5.2 was already stated with a hint of the proof in [Lp3, Theorem 2].

Now we prove that in most cases the Δ-closure of a logic generated by a monadic quantifier is equivalent to the Δ-closure of a logic generated by cardinality quantifiers (we conjecture that this is true for every monadic quantifier).

5.3. THEOREM. – *If H is a finite union of products of intervals, then $\Delta(L_{\omega\omega}(Q_H)) = \Delta(\vee\{L_{\omega\omega}(Q_\alpha) \mid \omega_\alpha$ is an extreme for H, for some coordinate$\})$. In particular, this is true if $L_{\omega\omega}(Q_H)$ is (ω, ω)-compact.*

PROOF. – $L_{\omega\omega}(Q_H)$ is indeed a sublogic of $\vee\{L_{\omega\omega}(Q_\alpha) \mid \omega_\alpha$ is an extreme for H, for some coordinate$\}$.

For the converse, for every i with $1 \leqslant i \leqslant n$, let $\alpha_{i,j}$ $(1 \leqslant j \leqslant m_{i+1})$ be the extremes for H for the i-th coordinate, arranged in increasing order: we say that the cardinals $\beta_{i,j}$ $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m_i)$ are a *grating* for H iff for every i and j with $1 \leqslant i \leqslant n$, and $1 \leqslant j \leqslant m_i$, $\beta_{i,j} \in [\alpha_{i,j}, \alpha_{i,j+1})$. Clearly, if $\varphi_{i,j}$ $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m_i)$ are formulas of $L_{\omega\omega}(Q_H)$, and $\varphi_{ij}(x) \Rightarrow \varphi_{ij'}(x)$ for $j < j'$, then the property that $|\{x \mid \varphi_{i,j}(x)\}|$ $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m_i)$ is a grating for H is expressible by a sentence of $L_{\omega\omega}(Q_H)$.

Now, the condition that $\gamma \in [\alpha_{h,k}, \alpha_{h,k+1})$ is equivalent to both:

(i) whenever $\beta_{i,j}$ $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m_i)$ is a grating for H, then it still remains a grating when we replace $\beta_{h,k}$ with γ, and leave the other β's unchanged; and

(ii) there is a grating $\beta_{i,j}$ $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m_i)$ such that we still have a grating when we replace $\beta_{h,k}$ with γ, and leave the other β's unchanged;

and this shows that, if $\alpha = \alpha_{h,k}$ and $\alpha' = \alpha_{h,k+1}$, then $L_{\omega\omega}(Q_{[\alpha,\alpha')})$ is a sublogic of $\Delta(L_{\omega\omega}(Q_H))$. Since this can be done for every h and k, we can express every sentences of $L_{\omega\omega}(Q_\alpha)$ by a sentence of $\Delta(L_{\omega\omega}(Q_H))$, if ω_α is an extreme for H, for some coordinate.

5.4. COROLLARY. – *If $L_{\omega\omega}(Q_H)$ is (ω, ω)-compact, then it enjoys exactly the same compactness properties of $\vee\{L_{\omega\omega}(Q_\alpha) \mid \omega_\alpha$ is an extreme for H, for some coordinate$\}$. Indeed, this is true for all properties preserved by Δ-closure and taking sublogics.*

PROOF. – This follows from Theorem 5.2 and the fact that Δ-closure preserves all compactness properties.

5.5. PROPOSITION. – *If H is a union of $< \varkappa$ products of intervals, then $L_{\varkappa\omega}(Q_H)$ enjoys exactly the same compactness properties of $\vee\{L_{\varkappa\omega}(Q_\alpha) \mid \omega_\alpha$ is an extreme for H, for some coordinate$\}$.*

The proof of Proposition 5.5 is a generalization of the above arguments, by using a form of Δ-closure allowing infinitely many constants, relations and functions. On the contrary, a generalization of Theorem 5.2 is not possible (at least from some point on): by results of [Ma], any logic is $[\mu, \mu]$-compact, for some cardinal μ, if we assume Vopenka's principle; so that, e.g., $L_{\omega\omega}(Rg)$ is $[\mu, \mu]$-compact, for some μ, where Rg is a quantifier such that $Rgx\varphi(x)$ means that $|\{x \mid \varphi(x)\}|$ is a regular cardinal; but both regular and singular cardinals are cofinal in the universe.

5.6. PROBLEM. – Find the first cardinal λ for which there exists a class H of cardinals such that both H and its complement are unbounded but $L_{\omega\omega}(Q_H)$ is (λ, λ)-compact. ($[\lambda, \lambda]$-compact, respectively.)

By theorems of Magidor and Stavi [BF, XVIII, 1.5.11 and 1.5.15], λ is less than the first measurable (extendible, respectively).

However, if we want cardinality logics or infinitary languages to be compact we need much weaker hypotheses than Vopenka's principle.

The following is immediate from Kunen's Lemma [KM, p. 156], [Lp1, Proposition 6.5.1], and [Lp2, Theorem 6]. The proof of this last theorem is obtained as in Theorem 2.5 (viii) $\Rightarrow$ (iii).

5.7. PROPOSITION. – *For every ordinal* α, $L_{\mu\mu}(Q_\alpha)$ *is* $[\mu, \mu]$*-compact for all but finitely many measurable cardinals* μ.

5.8. COROLLARY. – *If there are infinitely many measurable cardinals, then any logic generated by a finite number of cardinality quantifiers is* $[\mu, \mu]$*-compact for some* μ.

If the measurable cardinals form a proper class, then any infinitary logic $L_{\alpha\beta}$ *is* $[\mu, \mu]$*-compact for some* μ.

The second part of Corollary 5.8 follows from [Lp1, Theorem 6.5.2]. Problems: try to obtain results similar to Corollary 5.8 for other kinds of logics. Is $L_{\omega\omega}(Q_\alpha)$ $(\varkappa, \varkappa)$-compact for all but finitely many weakly compact cardinals?

Finally we notice an improvement of [Lp1, Theorem 3.9], whose proof is exactly the same (if $\mathrm{Oc}(L) = \omega$, then two models are L-elementarily equivalent iff so are their reducts to any finite type).

5.9. THEOREM. – *If* L *is* 2^ω-(ω, ω)*-compact*, $\mathrm{Oc}(L) = \omega$, $|A| = |B| = \omega$, *and, for every finite* $\tau \subset \tau\mathfrak{A}$, $\mathrm{Th}(\mathfrak{A}_{\restriction\tau})$ *is superstable and either* (i) *does not have the finite cover property or* (ii) *some countable* *model has a proper L-complete extension of cardinality* 2^{ω}, *then* $\mathfrak{A} \equiv_L \mathfrak{B}$ *iff* $\mathfrak{A} \equiv \mathfrak{B}$.

6. – Conclusions.

The results of [MS], as well as subsequent ones, might give the impression that $[\lambda, \mu]$-compactness is a much more natural notion than (λ, μ)-compactness. Indeed, the former is a very useful tool for the study of full compactness; moreover it may be very difficult to decide if a given logic L is (λ, μ)-compact, while the problem of its $[\lambda, \mu]$-compactness is generally easier: for cardinality logics the former problem is still open [BF, Chapter V], while the latter has a rather simple solution [Lp2]. Furthermore, of some theorems proved for $[\lambda, \mu]$-compactness no counterpart has been found yet for (λ, μ)-compactness; other theorems do have such counterparts, but their proofs are much more involved, and are deeply influenced by the earlier proofs for the $[\lambda, \mu]$-compact case; and—at least as we are concerned—could not even have been found without them.

Nevertheless, however desirable full compactness is, it is very rare and difficult to be found [Sh]: if we decide to renounce to it and limit ourselves to « fragments » of compactness in order to have at least a small amount of logics to deal with, we discover that changing (λ, μ)-compactness into $[\lambda, \mu]$-compactness gives a perhaps excessive strengthening and cuts out a lot of logics (for example, $L_{\omega\omega}(Q_1)$ is (ω, ω)-compact but not $[\omega, \omega]$-compact—the exact breaking point being explicitly given in Proposition 2.3), so that (λ, μ)-compactness still remains an interesting concept deserving further study, as it can be applied to a somewhat broader context.

More generally, our point of view is that restricting oneself to logics satisfying desirable properties seems a rather strong requirement (producing a collapse to a one-element set if countable compactness and the Löwenheim-Skolem theorem are between the desirable properties) and that looking for the notions and methods of first order Model Theory which can be generalized to the largest possible amount of logics is as interesting as searching for logics, maybe strange and unnatural, having the Model Theory as similar as possible to the one of $L_{\omega\omega}$.

Anyway, there is another reason for which (λ, μ)-compactness is worth of study, a reason in some way connected with large cardinal axioms in Set Theory (this is no surprise: weakly and strongly compact cardinals were indeed originally defined in terms of compactness properties of infinitary languages).

The results of Section 3 naturally lead to the following identity:

$$(\lambda, \lambda)\text{-compactness} : \text{weak compactness} = \\ = [\lambda, \lambda]\text{-compactness} : \text{measurability}\,.$$

So that it is conceivable that (λ, λ)-compactness may be useful for translating results involving measurable cardinals into properties of much smaller cardinals, as well as for finding generalizations of measurability suitable for these cardinals (for the interest of such a process we refer to [KM, e.g. p. 180 or the last lines of p. 105]). Concerning the present paper, the relation $\lambda \Rightarrow \mu$, as a strengthening of « every (λ, λ)-regular ultrafilter is (μ, μ)-regular » may be interesting also from a set-theoretical point of view.

Finally, let us observe that many non elementary properties have been already studied by classical Model Theory (cardinality, omitting types, ...); the new thing in Abstract Model Theory seems to be just in looking at what happens if one requires very few natural closure properties. This may explain why many « abstract » results are implicit in or forerun by older classical ones.

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