Some trends and problems in quantum probability

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0.) From the experimental point of view probability enters quantum theory just like classical statistical physics, i.e. as an expected relative frequency. However it is well known that the statistical formalism of quantum theory is quite different from the usual Kolmogorovian one involving, for example, complex numbers, amplitudes, Hilbet spaces... The quantum statistical formalism has been described, developped, applied, generalized with the contributions of many authors; however its theoretical status remained, until recently, quite obscure, as shown by the widely contrasting statements that one can find in the vast literature concerning the following questions.

Question I.) Is it possible to justify the choice of the classical or the quantum statistical formalism, for the description of a given set of statistical data, on rigorous mathematical criteria rather than on empirical ones?

In particular, is the quantum statistical formalism in some sense necessary, or (as some authors seem to believe) is it an historical accident and the whole quantum theory can be developed within the framework of the classical Kolmogorovian model?

Question II.) If (as it will be shown to be the case) it is possible to devise a rigorous mathematical criterion which allows us to discriminate between the two statistical models, then the Kolmogorovian model must include a hidden postulate which limits its applicability to the statistical description of the natural phenomena. Which of the Kolmogorov axioms plays, for probability, the role played by the parallel axiom for geometry?

Question III.) Which new physical requirements should substitute the hidden axiom of the Kolmogorovian model mentioned in Question II.)? Will such requirements be sufficient to account for all the specific features of the quantum statistical model?

Question IV.) Are the Kolmogorovian, the usual quantum model, and its known generalizations, the only statistical models which can arise in the description of nature?

In the present paper we describe how the above mentioned problems can be formulated and solved in a rigorous mathematical way. In particular the negative answer to Question IV.), strongly supports the point of view that, in analogy with geometry, one should look at probability theory not as the study of the laws of chance but of the possible, mutually inequivalent models for the laws of chance, the choice among which for the description of natural phenomena, being a purely experimental question. The main goal of quantum probability should then be recognized in the inner development of these models, their classification, the analysis of their mutual relations, and in no case reduced to a translation of classical probabilistic results into a quantum (or "non–commutative") language. On the contrary, the deepest problems of quantum probability are just those in which this translation becomes impossible, as a consequence of the different axioms which lie at the foundation of the two theories. For lack of space our discussion will be limited to Question I.),..., IV.) and to the problems which arise in connection with them. In particular we will not discuss the relevance of the answers to these questions for the old *interpretational* problems of quantum theory (cf. [7]), nor the important steps that have been made in the last years towards the inner development of the usual quantum probabilistic model (for these we refer to the papers in these proceedings). We believe that once understood the origins and the meaning of quantum probability, it will be easier to go further along the lines of the analysis of more general classes of nonkolomogorovian probabilistic models the deepening of the usual quantum quantum one, as well as the study of the connections between the kolmogorovian and the quantum model.

In Sections (1.) and (2.) we review some known results concerning the answer to Question I.). Question II.) has been dealt with extensively in [7] and will not be discussed here; in Sections (3.) and (4.) we answer Question III.) giving the proofs of some results announced in previous papers [5], [6]; finally in Sections (5.) we outline a geometrical generalization of the quantum probabilistic formalism which, in view of some recent progresses in theoretical physics and in the theory of operator algebras, seems to be the most promising line along which to investigate the answer to Question IV.).

1 The statistical invariants

The answer to Question I.) of § (0.) is based on the following idea: the existence of a Kolmogorovian (resp. a quantum) model which describes a given set of statistical sata imposes some constraints on them which can be explicitely computed. It happens that in nature one can find some sets of statistical data satisfying the constraints which characterize the existence of a quantum statistical model but not those characterising the existence of a Kolmogorovian model (or conversely!). This proves that, as long as we want to describe these statistical data within a single mathematical model, giving up the classical Kolmogorovian model is a mathematical and experimental necessity. No new experiment is needed to prove this statement: it is sufficient to apply the results formulated in the following to the oldest and well established data of quantum theory. Locally (i.e. when restricted to statistical data concerning sets of compatible observables) the quantum probability arise in the description of statistical data concerning mutually incompatible sets observables.

The following notations will be used throughout the paper: let T be a set (index set); let, for each $x \in T$, be given an observable quantity $A(x)$ whose values will be denoted $a_1(x), \ldots, a_n(x)$. Unless explicitely state n will be a finite positive integer independent on $x \in T$. Heuristically $A(x)$ should be thought as a complete set of compatible observables. For each $x \in T$ and α, $\beta = 1, \ldots, n$, let us denote

$$P(A(x) = a_\alpha(x)|A(y) = a_\beta(y) = p_{\alpha\beta}(x, y) \tag{1}$$

the conditional probability that the observable $A(x)$ takes the value $a_\alpha(x)$ given that the observable $A(y)$ takes the value $a_\beta(y)$. For fixed $x, y \in T$, then $n \times n$ matrix defined by (1) will be denoted $P(x, y)$ and called the transition probability matrix relative to the observables $A(x)$, $A(y)$.

Definition 1 *In the notations above, the family $\{P(x, y) : x, y \in T\}$ of transition probability matrices is said to admit a Kolmogorovian model if there exist:*

– a probability space (Ω, θ, μ)

– for each $x \in T$, a measurable partition $A_1(x), \ldots, A_n(x)$ of Ω

$$p_{\alpha\beta}(x, y) = \frac{\mu(A_\alpha(x) \cap A_\beta(y))}{\mu(A_\beta(y))} \tag{2}$$

Definition 2 *In the notations above, the family $\{P(x, y) : x, y \in T\}$ of transition probability matrices is said to admit a complex Hilbert space model if there exists:*

– a complex Hilbert space $\mathcal{H}$

– for each $x \in T$, an orthonormal basis $\phi_1(x), \ldots, \phi_n(x)$ of $\mathcal{H}$

$$p_{\alpha\beta}(x, y) = |\langle \phi_\alpha(x), \phi_\beta(y) \rangle|^2 \tag{3}$$

Remark (1.). Renyi models or real or quaternion Hilbert space models are defined in obvious analogy with Definitions (1.1), (1.2).

Remark (2.). The symmetry conditions:

$$p_{\alpha\beta}(x,y) = p_{\beta\alpha}(y,x) \ ; \quad x,y \in T \ , \quad \alpha,\beta = 1,\ldots,n \tag{4}$$

is a necessary condition for the existence of a (real or) complex Hilbert space model.

For a given set T we will denote $P(t)$ the family of all sets of the form $\{P(x,y) : x,y \in T\}$ where each $P(x,y)$ is a $n \times n$ stochastic matrix. Thus a point of the space $P(T)$ is a set of $n \times n$ stochastic matrices indexed by $T \times T$.

Definition 3 *A Kolmogorovian (resp. complex o real Hilbert,...) statistical invariant for the family $P(T)$ is defined by:*
– a family of functions $F_j : P(T) \to \mathbb{R}(j \in I; I$–a given set)
– a family $\{B_j : j \in I\}$ of sub–sets of $\mathbb{R}$
such that a set of transition probability matrices $\{P(x,y) : x,y \in T\}$ admits a Kolmogorovian model (resp. $\mathbb{C}$– or $\mathbb{R}$–Hilbert space models,...) if and only if for each $j \in I$, $F_j(P(x,y) : x,y \in T) \in B_j$.

Any probability preserving transformation of a given model for $\{P(x,y) : x,y \in T\}$ will preserve the values of the functions F, which are model independent. In this sense we speak the values of the functions F, which are model independent. In this sense we speak of statistical invariants. Once known the probabilities of the events $[A(x) = a_\alpha(x)]$ $(x \in T, \alpha = 1,\ldots,n)$, the problem of determing the Kolmogorovian statistical invariants is reduced to a linear one: one just writes the usual compatibility conditions for the (unknown) joint probabilities of the random variables $A(x)$ $(x \in T)$ and looks for conditions under which the resulting linear system (whose coefficients depend only on the probabilities $p_{\alpha\beta}(x,y)$, $P(a(x) = a_\alpha(x))$) has a positive normalized solution. Thus, if T is a finite set, there is always a finite algorithm which allows to determine the Kolmogorovian invariant for the family $P(T)$ (a precise formulation for $T = \{1,2,3\}$ can be found in [4], Proposition (1.1)). There is no mystery in the non existence of a Kolmogorovian model for a given set of statistical data: the fact is that the joint probabilities mentioned above are in principle unobservable (due to Heisenberg's principle) and, moreover, in all concrete examples in which one knows that the Kolmogorovian model doesn't exist; the transition probabilities $P(A(x) = a_\alpha(x)|A(y) = a_\beta(y))$ refer to physically different and mutually incompatible physical situations. Thus it is not obvious physically, (besides being mathematically wrong) that these conditioned probabilities could be derived in the usual way by a set of (unobservable) joint probabilities. Examples of physically meaningful statistical data not admitting a Kolmogorovian model were known since the early days of quantum mechanics (cf. The discussion of the two–slit experiment in [7]); in [9] Bell pointed out another simple example (based on correlations rather than conditinal probabilities)of statistical data not admitting a Kolmogorovian model. This examples was at the origin of a vast literature (cr. for example [10], [13], [14], [19], [20], [21]) whose results can be framed in the general scheme described above. In fact one can show (cf. [7]) that all the so–called paradoxes of quantum theory arise from the application of the usual rules of the Kolmogorovian model to sets of statistical data which do nota dmit such a model.

Concerning the statistical invariants for the Kolmogorovian model at the moment the following results are known:

Case $T = \{1, 2\}$ *(two observables).* Without the symmetry condition (4), the Kolmogorovian model might not exist and the statistical invariants are explicitely known for $n < +\infty$. With the symmetry condition (4), the Kolmogorovian model always exists for $n < +\infty$ – never for $n = +\infty$; the Renyi model always exists. In all cases the Kolmogorovian model is unique up to stochastic equivalence [6].

Case $T = \{1, 2, 3\}$*;* $n = 2$ *(Three, two–valued observables).* The statistical invariants are explicitely known [4], [6], [14].

Case T *– arbitrary finite set;* $n < +\infty$ *(a finite number of finitely valued observables).* Various algorithms have been proposed for the numerical solution of the problem [13], [14].

Moreover various kinds of necessary conditions are known for the existence of the Kolmogorovian model in the case of three of four observables with values ± 1. In particular the resutls of Suppes and Zanotti [20] based on Bell's inequality can be considered as the expression of the statistical invariants for the Kolmogorovian model in terms of pair correaltions rather than transition probabilities.

The problem of determining the statistical invariants for the complex (or real,...) Hilbert space model is more difficult, being intrinsecally nonlinear: here one looks for unitary matrices $U(x,y) = (u_{\alpha\beta}(x,y))$ satisfying

$$u_{\alpha\beta}(x,y) = \sqrt{p_{\alpha\beta}(x,y)}e^{i\theta_{\alpha\beta}(x,y)} \tag{5}$$

(where $\theta_{\alpha\beta}(x,y)$ is an unknown phase), and the compatibility conditions, coming from the orthogonality relations, lead to nonlinear equations in the unknown phases. Concerning these models the known results are the following:

Case $T = \{1,2\}$ (Two observables). The complex or real Hilbet space model always exists fro $n = 2$, but in general it doesn't exist for $n \geq 3$. For $n = 3$ the statistical invariants for the complex Hilbert space model are explicitely known [6]. For $n > 3$ the problem is open and can be formulated as follows:

Problem (I.a). Given a $n \times n$ bi–stochastic matrix $P = (p_{\alpha\beta})$:

$$p_{\alpha\beta} \geq 0 \ ; \quad \sum_\alpha p_{\alpha\beta} = \sum_\beta p_{\alpha\beta} = 1$$

find necessary and sufficient conditions on the $p_{\alpha\beta}$'s for the existence of a complex (resp. real) $n \times n$ unitary matrix $U = (u_{\alpha\beta})$ satisfying: $p_{\alpha\beta} = |u_{\alpha\beta}|^2$; $\alpha, \beta = 1, \dots, n$.

Problem (I.a) above has been first studied by M. Roos [22], [23], in connection with the problem of the decay of the K–meson. In the language of quantum probability, Roos' investigations concerned the existence of a $\mathbb{C}$–Hilbert space model for the statistical data relative to the decay of the K–meson. Finally, if $n = +\infty$, one can show that the random walk bi–stochastic matrices:

$$p = \{p_{\alpha\beta} : \alpha, \beta \in \mathbb{Z}\} \ ; \quad p_{\alpha,\beta} = 0 \ , \quad \text{if} \quad \alpha \neq \beta \pm 1$$

do not admit a complex Hilbert space model.

Case $T = \{1,2,3\}$*;* $n = 2$. (Three, two–valued observables). The complex, real and quaterion Hilbert space statistical invariants are explicitely known [4]. The complex and real invariants are different; the complex and the quaterion invariants coincide; the complex and the Kolmogorovian invariants are different. In particular, comparing these invariants with the existing and well established experimental data on spin $-1/2$ particles, and considering the transition probabilities among values of the spin along three non parallel directions, one obtains an experimental proof, not only of the insufficiency of the Kolmogorovian model, but also of the necessity of using complex rather than real Hilbert spaces.

Finally one can construct mathematical counterexamples showing the existence of triples of 2×2 bi–stochastic matrices which do not admit neither a Kolmogorovian nor a complex Hilbert spae model. Even if at the moment it is unclear if these mathematical models might have a physical meaning, it would be interesting to answer the following question:

Problem (I.b). Do there exist triples of 2×2 bi–stochastic matrices which do not admit neither $\mathbb{C}$–Hilbert space nor a Kolmogorovian model, but which admit an Heisenberg algebra model in the sense specified by Section (3.) below?

For an arbitrary set T even the following problem is still open:

Problem (I.c). For finite T and n, can the statistical invariants for $P_n(T)$ be determined by a finite algorithm?

We conjecture that also the use of vector valued wave functions instead of scalar valued ones has a purely statistical origin which can be read out from appropriate statistical invariants. Finally, in connection with the generalized quantum formalism proposed by Ludwig and his school, let us mention the following problem:

Problem (I.d). Can one distinguish (at least in some simple models) the generalized Ludwig formalism by the usual $\mathbb{C}$–Hilbert space formalism by means of statistical invariants?

2 Complementary pairs

In the preceeding section we have shown that many features of the quantum mechanical formalism can be read out directly from the (experimentally measurable) transition probabilities.

Let us now consider pairs of n–valued observables A, B whose transition probability matrix corresponds to the maximum indeterminacy, i.e.:

$$P(A = a_\alpha | B = b_\beta) = \frac{1}{n} \ ; \quad \alpha, \beta = 1, \ldots, n \tag{1}$$

in such a case we will say that the two observables A, B are *complementary* or that thay form a complementary pair.

The bi–stochastic matrix (1) always admit a $\mathbb{C}$–Hilbert space model. Moreover, denoting still A, B the operators corresponding to the observables A, B, it can be shown that, if A, B satisfy the discrete version of the CCR, namely [18]:

$$e^{ihA}e^{ikB} = e^{i\frac{hK}{n}}e^{ikB}e^{ihA} \ ; \quad h,k = 1,\dots,n \tag{2}$$

then they are complementary. Unfortunately the converse is not true: there are complementary pairs A, B which do not satisfy the CCR, i.e. (2). Thus means that the algebraic relations between observables satisfying the CCR cannot be read out from the corresponding statistics. The following problem thus arises:

Problem (I.f). Classify the complementary pairs in $\mathcal{B}(\mathcal{H})$ ($\dim\mathcal{H} < +\infty$) up to the natural unitary equivalence.

There is a continuous version of this problem. In fact one easily sees that condition (1) is equivalent to:

$$\tau(E_A(I)\cdot E_B(J)) = |I|\cdot|J| \tag{3}$$

where I, J denote sub–sets of $\{1,\dots,n\}$; $E_A(\cdot)$, $E_B(\cdot)$ are the spectral measures of A, B respectively; $|\cdot|$ denotes the uniform probability measure on $\{1,2,\dots,n\}$ (i.e. $(1/n, 1/n,\dots,1:n)$; and $\tau-$ the normalized trace on $\mathcal{B}(\mathcal{H})$. Thus, if $\dim\mathcal{H} = +\infty$, it is natural to call *complementary* two self–adjoint operators A, B satisfying (3) with I, $J\subseteq\mathbb{R}$; $\tau-$ the trace on $\mathcal{B}(\mathcal{H})$; and $|\cdot|$ – the Lebesgue measure on $\mathbb{R}$. By Fourier analysis manipulations show that if A, B denote respectively the position and momentum operators on $L^2(\mathbb{R}, dx)$, then they are complementary. Hence by the Stone–von Neumann theorem any pair A, B of self–adjoint operators satisfying the CCR:

$$e^{ihA}e^{ikB} = e^{ihK}e^{ikB}e^{ihA} \ ; \quad h,k\in\mathbb{R} \tag{4}$$

is a complementary pair. It is reasonable to expect that also in the infinite–dimensional case complemtary pairs are not unique up to unitary isomorphism, thus the problem of classifying complementary pairs on $\mathcal{B}(\mathcal{H})$ (or more generally, on a semi–finite von Neumann algebra) is open.

3 Deduction of the quantum formalism from Heisenberg principle

In order to answer to Question (III.) in § (0.) let us recall that, as shown in [5], [6], [7], the axiomatic formalization of the notion of first kind measurement (or filter), and of the operations which can be perfomed on them, leads naturally to the notion of *Schwinger algebra of measurements* which, in the notations introduced in § (1.), is defined as follows:

Definition 4 *A Schwinger algebra of measurements associated to the family of observables $\{A(x) : x \in T\}$ is a real, associative algebra $\mathcal{A}$ with an identity 1 and an involution *, satisfying the following conditions:*

i) *to each $x \in T$ and to each value $a_\alpha(x)$ of $A(x)$ one can associate an element $A_\alpha(x)$ of $\mathcal{A}$.*

ii) *For each $x \in T$:*

$$A_{\mathcal{A}}(x) = A_\alpha(x)^* \ ; \quad \alpha = 1, \ldots, n \tag{1}$$

$$\sum_{\alpha=1} A_\alpha(x) = 1 \tag{2}$$

$$A_\alpha(x) \cdot A_{\alpha'}(x) = \delta_{\alpha\ \alpha} A_\alpha(x) \ ; \quad \alpha, \alpha' = 1, \ldots, n \tag{3}$$

iii) *The algebra $\mathcal{A}$ is generated by the set $\{A_\alpha(x) : x \in T;\ \alpha = 1, \ldots, n\}$.*

The main ideas on Schwinger algebras were introduced by J. schwinger [17]. The $A_\alpha(x)$ are called elementary (or atomic, or maximal,...) filters. Heuristically $A_\alpha(x)$ corresponds to the (first kind) measurement of $A(x)$ which, from an ensemble of system selects those for which the observable $A(x)$ takes the value $a_\alpha(x)$; multiplication of elementary filters corresponds to the performance in series of the corresponding operations and yields a filter, or apparatus; the involution corresponds to the application in series of the same sequence of elementary filters, but and is extended by a purely mathematical procedure (cf. [5] for more details). Two elements of $\mathcal{A}$ are called *compatible* if they commute. The centre of $\mathcal{A}$, i.e. the set of those elements of $\mathcal{A}$ which are compatible with all elements of $\mathcal{A}$, will be denoted κ. The observable $A(x)$ will be called maximal if the algebra $\mathcal{A}(x)$ generated by $A_\alpha(x)$ ($\alpha = 1, \ldots, n$) over κ (i.e. the set of all linear combinations of the form $\sum_{\alpha=1}^n \rho_\alpha A_\alpha(x)$, with $\rho_\alpha(\in,\ \alpha = 1, \ldots, n)$ is maximal abelian, that is: an element of $\mathcal{A}$ commuting with all elements of $\mathcal{A}(x)$ must belong to $\mathcal{A}(x)$. In the classical case all the observables are mutually compatible and $\mathcal{A}$ itself is abelian. In the quantum case, since any set of mutually compatible observables can be completed to a maximal set of compatible observables, it is not a physical restriction to consider only maximal sets of compatible obervables.

Proposition 1 *In the notations above, if each observable $A(x)$ $(x \in T)$ is maximal then for any $x, y \in T$ and $\alpha, \beta = 1, \ldots, n$, there exists an element $p_{\alpha\beta}(x, y) \in \kappa$ satisfying:*

$$A_\alpha(x) \cdot A_\beta(y) \cdot A_\alpha(x) = p_{\alpha\beta} A_\alpha(x) \tag{4}$$

$$p_{\alpha\beta}(x, y) A_\alpha(x) \geq 0 \ ; \quad \sum_{\beta=1}^{n} p_{\alpha\beta}(x, y) A_\alpha \in x) = A_\alpha(x) \tag{5}$$

$$p_{\alpha\beta}(x, y) \cdot A_\alpha(x) \cdot A_\beta(y) = p_{\beta\alpha}(y, x) A_{\alpha_1}(x) A_\beta(y) \tag{6}$$

Proof. For each $x, y \in T$, $\alpha, \beta = 1, \ldots, n$, $A_\alpha(x) A_\beta(y) A_\alpha(x)$ belongs to the ocmmutant of $\mathcal{A}(x)$, hence by maximality to $\mathcal{A}(x)$ itself, therefore it must have the form (4). (5) is a trivial consequence of (4). Moreover:

$$p_{\alpha\beta}(x, y) A_\alpha(x) A_\beta(y) = A_\alpha(x) [A_\beta(y) A_\alpha(x) A_\beta(y)] = p_{\beta\alpha}(y, x) A_\alpha(x) A_\beta(y)$$

and this proves (5).

Remark. Since $x \in \mathcal{A} \to \sum_\alpha A_\alpha(x) X A_\alpha(x)$ is a projection onto the commutant of $\mathcal{A}(x)$, if the products $\{A_\alpha(x) A_\beta(y) : \alpha, \beta = 1, \ldots, n$ linearly span $\mathcal{A}$ over κ, then condition (4) is also sufficient for the maximal abelianity of $\mathcal{A}(x)$.

The identity (4) has a natural physical interpretation, described in [5] – § (4.). It implies in particular that for any pair $x, y \in T$ the sub–algebra of $\mathcal{A}$ generated over κ by $A_\alpha(x)$, $A_\beta(y)$ $(\alpha, \beta = 1, \ldots, n)$ is finite dimensional: in fact, due to (4), all the products $A_{j_I}(x) A_{j_l}(y) \cdot A_{i_2}(x) A_{i_2}(y) \cdot \ldots \cdot A_{i_k}(x) A_{i_k}(y)$ can be expressed as linear combinations of products of this type with $k = n$ and all the indices $i_1, \ldots, i_n$, (resp. $j_1, \ldots, j_n$) mutually different.

A Schwinger algebra satifying conditions (I1.), (I.2.) below will be called *generic*:

(I1.) For each $\gamma \in \kappa$ and $x \in \mathcal{A}$

$$\gamma \cdot X = 0 \Leftrightarrow \gamma = 0 \text{ or } \quad X = 0$$

(I.2.) For each $x \in T$, $\alpha = 1, \ldots, n$, and $\gamma \in \kappa$:

$$\gamma \cdot A_\alpha(x) \geq 0 \Rightarrow \gamma \geq 0$$

For a generic Schwinger algebra, the conditions (5) and (6)) in Proposition (3.2) become:

$$p_{\alpha\beta}(x, y) \geq 0 \; ; \qquad \sum_{\beta=1}^{n} p_{\alpha\beta}(x, y) = 1 \tag{7}$$

$$p_{\alpha\beta}(x, y) = p_{\beta\alpha}(x, y) \tag{8}$$

A $n \times n$ matrix $(p_{\alpha\beta})$ with coefficients in an abelian $*$–algebra κ (real, associative and with identity) and satisfying:

$$p_{\alpha\beta} \geq 0 \; ; \qquad \sum_{\beta=1}^{n} p_{\alpha\beta} = 1 \tag{9}$$

will be called a *κ–valued stochastic matrix* (*bi–stochastic* if also $\sum_{\beta=1}^{n} p_{\alpha\beta} = 1$). Occasionally we will also use the term "κ–valued transition probability matrix". If κ is finite–dimensional then it must be the algebra of diagonal matrices over the real or complex numbers. In both cases condition (9) implies that the j–th coefficients $(p^j_{\alpha\beta})$ $(\alpha, \beta = 1, \ldots, n)$ of the diagonal matrix $(p_{\alpha\beta})$ is a stochastic (resp. bi–stochastic) matrix in the usual sense.

Thus a generic Schwinger algebra associated to a given set of maximal observables $\{A(x) : x \in T\}$ has an intrinsic stochasticity built into its algebraic structure and represented by κ–valued bi–stochastic matrices $(p_{\alpha\beta}(x, y))$. This elements that is, whenever there exist at least two different maximal observables. But the existence of pairs of maximal observables is a direct consequence of Heisenberg indeterminacy principle. We conclude that the notion of first kind measurement and the Heisenberg principle leads naturally to an interplay between algebraic and statistical structures. Our goal is to study this interplay and in particular the following questions:

– to what extent does the stochastic structure given by the transition probability matrices $(p_{\alpha\beta}(x, y))$ determine the algebraic structure of the Schwinger algebra and conversely:

– which restrictions are imposed on the transition probability matrices $(p_{\alpha\beta}(x,y))$ from the property of being canonically associated to a Schwinger algebra in the sense of Proposition (3.2)?

We will give a complete answer to these questions for a particular class of Schwinger algebras: the Heisenberg algebras.

Definition 5 *A Schwinger algebra, associated to a set $\{A(x) : x \in T\}$ of maximal observables will be called an Heisenberg algebra if it satisfies the genericity conditions (I.1), (I.2) and has minimal dimension over its centre κ.*

Remarking that the genericity conditions (I.1) and (I.2) imply that, for each $x, y \in T$, the products $\{A_\alpha(x) \cdot A_\beta(y) : \alpha, \beta = 1, \dots, n\}$ are always linearly independent over κ, one can conclude that an Heisenberg algebra (associated to n–valued observables) has dimension n^2 over its centre κ and that for each $x, y \in T$ the products $\{A_\alpha(x) \cdot A_\beta(y) : \alpha, \beta = 1, \dots, n\}$ are a κ–basis.

The following problem is open:

Problem (III.a). Does there exist a Schwinger algebra associated to a family $\{A(x) : x \in T\}$ of maximal observables (T–some set), which is not an Heisenberg algebra?

As we will see Heisenberg algebras include the usual algebras which appear in quantum theory (of finite–valued observables). Hence an affermative answer to Problem (III.a) should be based on some algebraic structure unsual for quantum theory. Of course Problem (III.a) is a particular case of

Problem (III.b). Classify the Schwinger algebras which can be associated to a given set $\{A(x) : x \in T\}$ of maximal observables.

4 The classification theorem

Let us first consider the case of a two–element index set: $T = \{1, 2\}$. In this case one has two maximal observables, denoted A and B and, by Proposition (3.2), one κ–valued transition probability matrix:

$$p_{\alpha\beta} = p_{\alpha\beta}(x,y) = p_{\beta\alpha}(y,x) \tag{1}$$

The algebra $\mathcal{A}$ is generated over its centre κ by the products $\{A_\alpha B_\beta : \alpha, \beta = 1, \ldots, n\}$–thus, in particular

$$B_\beta A_\alpha = \sum_{\alpha', \beta'=1}^{n} \gamma_{\alpha\beta}^{\alpha'\beta'} A_\alpha B_\beta \tag{2}$$

with $\gamma_{\alpha\beta}^{\alpha'\beta'} \in \kappa(\alpha, \beta, \alpha', \beta' = 1, \ldots, n)$. The elements $\gamma_{\alpha\beta}^{\alpha'\beta'}$ of κ will be called the structure constants of the Heisenberg algebra $\mathcal{A}$ in the $(A_\alpha B_\beta)$–basis. They uniquely determine the algebraic structure of $\mathcal{A}$ in view of the following proposition whose proof, which is done by routine arguments, is omitted.

Proposition 2 *Let $\mathcal{A}$ be an associative $\mathbb{R}$–algebra with identity and with centre κ. Let (A_α), (B_β) $(\alpha, \beta = 1, \ldots, n)$ be elements of $\mathcal{A}$ such that:*

$$\{A_\alpha \cdot B_\beta : \alpha, \beta = 1, \ldots, n\} \quad \textit{is a } \kappa\textit{–basis of } \mathcal{A} \tag{3}$$

$$A_\alpha A_{\alpha'} = \delta_{\alpha\alpha'} A_\alpha \ ; \quad B_\beta B_{\beta'} = \delta_{\beta\beta'} B_\beta \ ; \quad \sum_\alpha A_\alpha = \sum_\beta B_\beta = 1 \tag{4}$$

and assume that the genericity condition (I.1) of § (3.) is satisfied. Then, if $(\gamma_{\alpha\beta}^{\alpha'\beta'})$ are the structure constants of $\mathcal{A}$ in the $(A_\alpha B_\beta)$–basis:

$$\sum_{\alpha=1}^{n} \gamma_{\alpha\beta}^{\alpha'\beta'} = \delta_{\beta\beta'} \tag{5}$$

$$\sum_{\beta=1}^{n} \gamma_{\alpha\beta}^{\alpha'\beta'} = \delta_{\alpha\alpha'} \tag{6}$$

$$\gamma_{\alpha'\beta}^{\alpha\beta'} \cdot \gamma_{\alpha''\beta'}^{\alpha\beta''} = \gamma_{\alpha'\beta}^{\alpha\beta''} \cdot \gamma_{\alpha''\beta'}^{\alpha'\beta''} \tag{7}$$

Conversely if κ is a commutative, associative $\mathbb{R}$–algebra with identity and $(\gamma_{\alpha\beta}^{\alpha'\beta'})$ $(\alpha, \beta, \alpha', \beta' = 1, \ldots, n)$ are n^4 elements of κ satisfying (5), (6), (7), then there exist an associative κ–algebra with identity $\mathcal{A}$ and elements (A_α), (B_β) of $\mathcal{A}$ satisfying conditions (3) and (4).

Example. In the notations above, $\mathcal{A}$ is abelian if and only if:

$$\gamma_{\alpha\beta}^{\alpha'\beta'} = \delta_{\alpha\alpha'} \delta_{\beta\beta'}$$

Theorem 1 *Let $\mathcal{A}$ be an Heisenberg algebra associated to the two observables $A = (A_\alpha)$, $B = (B_\beta)$, and let $\gamma_{\alpha\beta}^{\alpha'\beta'}$ be the structure constants of $\mathcal{A}$ in the $(A_\alpha B_\beta)$–basis. If each $\gamma_{\alpha\beta}^{\alpha'\beta'}$ is invertible, then there exists a κ–valued matrix $U = u(A|B) \equiv (u_{\alpha\beta})$ such that:*

$$u \quad \text{is invertible for } \alpha, \beta = 1, \ldots, n \tag{8}$$

$$\sum_{\alpha=1}^{n} \left(\frac{p_{\alpha\beta}}{u_{\alpha\beta}} \right) u_{\alpha\beta'} = \delta_{\beta\beta'} \tag{9}$$

$$\sum_{\beta=1}^{n} u_{\alpha'\beta} \left(\frac{p_{\alpha\beta}}{u_{\alpha\beta}} \right) = \delta_{\alpha\alpha'} \tag{10}$$

$$\gamma_{\alpha\beta}^{\alpha'\beta'} = \frac{u(\alpha,\beta') \cdot u(\alpha',\beta)}{u(\alpha,\beta) \cdot u(\alpha',\beta')} \cdot p_{\alpha\beta} \tag{11}$$

where $(p_{\alpha\beta})$ is the transition probability matrix associated to the pair (A_α), (B_β) according to Proposition (3.2). Conversely, given a κ–valued bi–stochastic matrix $P = (p_{\alpha\beta})$ such that $p_{\alpha\beta}$ is invertible for each $\alpha, \beta = 1, \ldots, n$, any κ–valued matrix $U = (u_{\alpha\beta})$ satisfying (8), (9), (10) defines an Heisenberg algebra $\mathcal{A}$ with centre κ, associated to two maximal observables (A_α), (B_β) and such that $P = (p_{\alpha\beta})$ is the transition probability matrix canonically associated to the pair (A_α), (B_β) and the structure constants of $\mathcal{A}$ in the $(A_\alpha B_\beta)$–basis are given by (11).

Proof. Sufficiency. Let $P = (p_{\alpha\beta})$ be a κ–alued bi–stochastic matrix, and let $U = (u_{\alpha\beta})$ be a κ–valued matrix satisfying (8), (9), (10). Then, defining $\gamma_{\alpha\beta}^{\alpha'\beta'}$ by to Proposition (4.1), there exists an associative κ–algebra $\mathcal{A}$ with identity and elements (A_α), (B_β) $(\alpha, \beta = 1, \ldots, n)$ of $\mathcal{A}$ satisfying conditions (3), (4). Any involution $*$ on κ can be extended to $\mathcal{A}$ by: $(\lambda A_\alpha B_\beta)* = \lambda * B_\beta A_\alpha$ $(\lambda \in \kappa)$, and under this extension $A_\alpha = A_\alpha^*$; $B_\beta = B_\beta^*$ $(\alpha, \beta = 1, \ldots, n)$. Finally:

$$A_\alpha B_\beta A_\alpha = A_\alpha \left(\sum_{\alpha'\beta'} \gamma_{\alpha\beta}^{\alpha'\beta'} A_{\alpha'} B_{\beta'} \right) = p_{\alpha\beta} A_\alpha \left(\sum_{\beta'} B_{\beta'} \right) = p_{\alpha\beta} A_\alpha$$

and similarly for $B_\beta A_\alpha B_\beta$. Thus, in view of the remark following Proposition (3.2) both (A_α) and (B_β) generate maximal abelian sub–algebras of $\mathcal{A}$, and the associated bi–stochastic matrix is P.

Necessity. Under our assumptions, one easily verifies that, for any α', $\beta' = 1, \dots, n$:

$$\gamma^{\alpha'\beta}_{\alpha\beta} = p_{\alpha\beta} \ ; \qquad \gamma^{\alpha\beta'}_{\alpha\beta} = p_{\alpha\beta} \tag{12}$$

Hence, under our assumptions, $p_{\alpha\beta}$ is invertible and we can define the normalized structure constants:

$$\Gamma^{\alpha'\beta'}_{\alpha\beta} = (p_{\alpha\beta})^{-1} \cdot \gamma^{\alpha'\beta'}_{\alpha\beta} \tag{13}$$

Now, using twice the identity (7) one obtains:

$$\gamma^{\alpha''\beta}_{\alpha'\beta'} \cdot \gamma^{\alpha'\beta}_{\alpha\beta'} \cdot \gamma^{\alpha''\beta''}_{\alpha\beta} = \gamma^{\alpha''\beta'}_{\alpha'\beta'} : \gamma^{\alpha''\beta}_{\alpha\beta'} \cdot \gamma^{\alpha''\beta''}_{\alpha\beta} = \gamma^{\alpha''\beta'}_{\alpha'\beta'} \cdot \gamma^{\alpha''\beta''}_{\alpha\beta'} \cdot \gamma^{\alpha\beta''}_{\alpha\beta}$$

hence, dividing both sides by $p_{\alpha\beta}$ and using (12) and (13):

$$\Gamma^{\alpha'\beta}_{\alpha\beta'} \cdot \Gamma^{\alpha''\beta}_{\alpha'\beta'} \cdot \Gamma^{\alpha''\beta''}_{\alpha\beta} = \Gamma^{\alpha''\beta''}_{\alpha\beta'} \tag{14}$$

Choosing $\beta'' = \beta$ in (14), one finds:

$$\Gamma^{\alpha'\beta}_{\alpha\beta'} \cdot \Gamma^{\alpha''\beta}_{\alpha'\beta'} = \Gamma^{\alpha''\beta}_{\alpha\beta'} \tag{15}$$

hence in particular:

$$\Gamma^{\alpha\beta}_{\alpha'\beta'} = (\Gamma^{\alpha'\beta}_{\alpha\beta'})^{-1} \tag{16}$$

Now fix an index α_0 arbitrarily and denote

$$\gamma\left(\alpha; \begin{matrix}\beta \\ \beta'\end{matrix}\right) = \Gamma^{\alpha_0\beta}_{\alpha\beta'} \tag{17}$$

with these notations, using (15) and (16), (14) becomes

$$\frac{\gamma\left(\alpha; \begin{matrix}\beta'' \\ \beta'\end{matrix}\right)}{\gamma\left(\alpha''; \begin{matrix}\beta'' \\ \beta'\end{matrix}\right)} = \frac{\gamma\left(\alpha; \begin{matrix}\beta \\ \beta'\end{matrix}\right)}{\gamma\left(\alpha'; \begin{matrix}\beta \\ \beta'\end{matrix}\right)} \cdot \frac{\gamma\left(\alpha'; \begin{matrix}\beta \\ \beta'\end{matrix}\right)}{\gamma\left(\alpha''; \begin{matrix}\beta \\ \beta'\end{matrix}\right)} \cdot \frac{\gamma\left(\alpha; \begin{matrix}\beta'' \\ \beta\end{matrix}\right)}{\gamma\left(\alpha''; \begin{matrix}\beta'' \\ \beta\end{matrix}\right)}$$

This implies that the expression:

$$\gamma\left(\alpha; \begin{matrix}\beta \\ \beta'\end{matrix}\right) \cdot \gamma\left(\alpha; \begin{matrix}\beta'' \\ \beta\end{matrix}\right) \cdot \gamma\left(\alpha; \begin{matrix}\beta'' \\ \beta'\end{matrix}\right)^{-1}$$

is independent on α an therefore equal to 1 (since we can choose $\alpha = \alpha_0$). Thus, fixing an index β_0 arbitrarily, and denoting:

$$u(\alpha, \beta) = \gamma \left(\alpha; \begin{matrix} \beta \\ \beta_0 \end{matrix} \right) \tag{18}$$

we obtain:

$$\gamma_{\alpha\beta}^{\alpha'\beta'} = p_{\alpha\beta}\Gamma_{\alpha\beta}^{\alpha'\beta'} = p_{\alpha\beta} \cdot \gamma \left(\alpha; \begin{matrix} \beta' \\ \beta \end{matrix} \right) \cdot \left(\gamma \left(\alpha'; \begin{matrix} \beta' \\ \beta \end{matrix} \right) \right)^{-1} = p_{\alpha\beta} \frac{u(\alpha, \beta') \cdot u(\alpha', \beta)}{u(\alpha, \beta) \cdot u(\alpha', \beta')}$$

and this proves (11). Once proved (11), (9) and (10) are immediate consequences of (5), (6) respectively, while (8) follows from (18) and the invertibility of the $\gamma_{\alpha\beta}^{\alpha'\beta'}$. And this ends the proof.

Remark 1.). Introducing the notations

$$u_{\alpha\beta}(A|B) = u(\alpha, \beta) \; ; \; u_{\beta\alpha}(B|A) = p_{\alpha\beta}/u(\alpha, \beta); \; u(A/B) = \{u_{\alpha\beta}(A/B)\}; \; u(B/A) = \{u_{\beta\alpha}(B/A)\} \tag{19}$$

conditions (9) and (10) become respectively:

$$U(B|A)U(A|B) = 1 \; ; \qquad U(A|B)U(B|A) = 1 \tag{20}$$

Remark 2.). One easily recognizes in (9), (10) a generalization of the usual orthogonality relations of quantum theory. In the usual quantum case the structure constants have the form:

$$\gamma_{\alpha\beta}^{\alpha'\beta'} = \frac{\langle \phi_{\alpha'}, \psi_\beta \rangle \langle \psi_\beta, \phi_\alpha \rangle \langle \phi_\alpha, \psi_{\beta'} \rangle}{\langle \phi_{\alpha'}, \psi_{\beta'} \rangle} \tag{21}$$

where (ϕ_α) and (ψ_β) are two ortho–normal bases of a complex Hilbert space $\mathcal{H}$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathcal{H}$. This shows in what sense condition (8) is a genericity condition.

If $P = (p_{\alpha\beta})$ is a κ–valued bi–stochastic matrix, a κ–valued matrix $U = (u_{\alpha\beta})$ satisfying conditions (8), (9), (10) will be called a κ–valued transition amplitude matrix for p.

A family (C_α) of elements of $\mathcal{A}$ satisfying:

$$C_\alpha \cdot C_{\alpha'} = \delta_{\alpha\alpha'} C_\alpha \; ; \quad \sum_\alpha C_\alpha = 1 \tag{22}$$

will be called a partition of the identity in $\mathcal{A}$. If the κ–algebra generated by (C_α) is maximal abelian, the partition of the identity will be called maximal abelian. If (C_α), (D_β) $(\alpha, \beta = 1, \ldots, n)$ are partitions of the identity in $\mathcal{A}$, such hat $C_\alpha D_\beta \neq 0$ for each $\alpha, \beta = 1, \ldots, n$, then clearly the set $(C_\alpha D_\beta)$ is a κ–basis of $\mathcal{A}$.

Lemma 1 *Let (C_α), (D_β) $(\alpha, \beta = 1, \ldots, n)$ be two maximal abelian partitions of the identity in $\mathcal{A}$ such that $C_\alpha D_\beta \neq 0$ for each $\alpha, \beta = 1, \ldots, n$. Denote $\gamma_{\alpha\beta}^{\alpha'\beta'}$ the structure constants of $\mathcal{A}$ in the $(C_\alpha D_\beta)$–basis, and $P(C|D) = (p_{\alpha\beta})$, the κ–valued bi–stochastic matrix associated to the pair (C_α), (D_β). If for any α, β, $p_{\alpha\beta}$ is invertible, then for any set $x(\alpha, \beta)$ of invertible elements of κ, the following assertions are equivalent:*

i) $\delta_{\alpha\beta}^{\alpha'\beta'} = \frac{x(\alpha',\beta)x(\alpha,\beta')}{x(\alpha',\beta')\cdot x(\alpha,\beta)}\, p_{\alpha\beta}$

ii) $E_{\alpha\beta}E_{\alpha'\beta'} = \frac{p(\alpha',\beta)}{x(\alpha',\beta)}\, E_{\alpha\beta'}$

where, by definition $E_{\alpha\beta} = C_\alpha D_\beta / x(\alpha, \beta)$.

Proof. By definition of structure constants:

$$C_{\alpha'} D_\beta C_\alpha D_{\beta'} = \delta_{\alpha\beta}^{\alpha'\beta'} C_{\alpha'} D_{\beta'}$$

or equivalently:

$$E_{\alpha'} \cdot E_{\alpha\beta'} = \delta_{\alpha\beta}^{\alpha'\beta'} \frac{x(\alpha', \beta')}{x(\alpha', \beta) \cdot x(\alpha, \beta')} E_{\alpha'\beta'}$$

thus (ii) is equivalent to:

$$\delta_{\alpha\beta}^{\alpha'\beta'} \cdot \frac{x(\alpha', \beta')x(\alpha, \beta)}{x(\alpha', \beta) \cdot x(\alpha, \beta')} \cdot \frac{1}{p_{\alpha\beta}} = 1$$

which is (i).

In the following we will determine the structure of the maximal abelian partitions of the identity (C_γ) which are generic in the sense that, if

$$C_\gamma = \sum_{\alpha\beta} C_{\alpha\beta}^{\gamma} A_\alpha B_\beta \ ; \quad C_{\alpha\beta}^{\gamma} \in \kappa$$

then all the $C_{\alpha\beta}^{\gamma}$'s are invertible.

Theorem 2 *In the notations and the assumptions of Theorem (4.1) let* (C_γ) *be a generic maximal abelian partition of the identity in* $\mathcal{A}$*. Then there exists two* κ*–valued matrices* $U(A|C) = (\rho(\alpha,\beta))$ *and* $U(B|C) = (\tau(\beta,\gamma))$ $(\alpha,\beta,\gamma = 1,\dots,n)$ *satisfying*

$$\sum_{\gamma=1}^{n} \rho(\alpha,\gamma)\tau(\gamma,\beta) = u(\alpha,\beta) \tag{23}$$

$$\sum_{\beta,\alpha} \tau(\gamma,\beta)u(\beta,\alpha)\rho(\alpha,\gamma') = \delta_{\gamma\gamma'} \tag{24}$$

$$C_\gamma = \sum_{\alpha,\beta} \frac{\rho(\alpha,\gamma)\cdot\tau(\gamma,\beta)}{u(\alpha,\beta)} A_\alpha B_\beta \tag{25}$$

Conversely, any pair of κ*–valued matrices* $U(A|C) = (\rho_{\alpha\gamma})$*,* $U(B|C) = (\tau_{\beta\gamma})$ *satisfying (23), (24) defines, through (25) a generic maximal abelian partition of the identiyt in* $\mathcal{A}$*.*

Proof. Sufficiency. Let $(\rho_{\alpha\gamma})$, $(\tau_{\beta\gamma})$ be κ–valued matrices satisfying (23), (24). Defining (C_γ) by (25), the conditions (22) are easily verified by direct calculation. Denote $\mathcal{A}(C)$ the κ–algebra generated by the C_γ's. To prove that $\mathcal{A}(C)$ is maximal abelian, it will be sufficient to show that th image of $\mathcal{A}$ for the conditional expectation $x \in \mathcal{A} \to \sum_\gamma C_\gamma X C_\gamma$ is $\mathcal{A}(C)$ or, equivalently, that for each α,β, $C_\gamma A_\alpha B_\beta C_\gamma = \lambda C_\gamma$ for some $\lambda \in \kappa$. And this identity is verified by direct computation too.
Necessity. Let (C_γ) be a generic partition of the identity in $\mathcal{A}$, and let

$$C_\gamma = \sum_{\alpha\beta} C^\gamma_{\alpha\beta} E_{\alpha\beta} \ ; \quad C^\gamma_{\alpha\beta} \in \kappa$$

where we put for convenience: $E_{\alpha\beta} = A_\alpha B_\beta / u_{\alpha\beta}$. Then:

$$(A_\alpha C_\gamma B_\beta)\cdot(A_\alpha, C_\gamma B_\beta) = C^\gamma_{\alpha\beta} C^\gamma_{\alpha'\beta} u_{\beta\alpha'} E_{\alpha\beta} \tag{26}$$

and, by the maximal abelianity of (C_γ):

$$C_\gamma B_\beta A_{\alpha'} C_\gamma = \Gamma_{\gamma,\alpha'\beta} C_\gamma \tag{27}$$

Comparing (26) and (27), one deduces:

$$C_\gamma B_\beta A_{\alpha'} C_\gamma = C^\gamma_{\alpha'\beta} u(\beta,\alpha') C_\gamma \tag{28}$$

But one has also:

$$C_\gamma B_\beta A_{\alpha'} C_\gamma = \sum_{\alpha\beta''} C^\gamma_{\alpha\beta} u(\beta,\alpha') C^\gamma_{\alpha'\beta''} E_{\alpha\beta''} \tag{29}$$

Thus, comparing (28) and (29) one obtains:

$$C^\gamma_{\alpha\beta} \cdot (C^\gamma_{\alpha'\beta})^{-1} = C^\gamma_{\alpha\beta'} \cdot (C^\gamma_{\alpha'\beta'})^{-1}$$

independently on α, β, α', β'. Hence, by the same arguments as in Theorem (4.2) there exist two κ–valued matrices $(\rho(\alpha,\gamma))$, $(\tau(\gamma,\beta))$, such that:

$$C^\gamma_{\alpha\beta} \cdot (C^\gamma_{\alpha'\beta})^{-1} = \rho(\alpha,\gamma)/\rho(\alpha',\gamma) \tag{30}$$

$$C^\gamma_{\alpha\beta} \cdot (C^\gamma_{\alpha\beta'})^{-1} = \tau'(\gamma,\beta)/\tau'(\gamma,\beta') \tag{31}$$

In particular the quantity:

$$\varepsilon_\gamma = C^\gamma_{\alpha\beta}/[\rho(\alpha,\gamma)\cdot\tau'(\gamma,\beta)]$$

is independent on α, $\beta = 1,\dots,n$, and denoting $\tau(\gamma,\beta) = \varepsilon_\gamma \tau'(\gamma,\beta)$, one deduces:

$$C^\gamma_{\alpha\beta} = \rho(\alpha,\gamma)\cdot\tau(\gamma,\beta)$$

and this proves (25). Moreover:

$$1 = \sum_{\gamma=1}^n C_\gamma = \sum_{\alpha,\beta}\left[\sum_\gamma \frac{\gamma(\alpha,\gamma)\tau(\alpha,\beta)}{u(\alpha,\beta)}\right] A_\alpha B_\beta$$

and this implies (23). Finally, the orthogonality relation $C_\gamma C_{\gamma'} = \delta_{\gamma\gamma'} C_\gamma$ is equivalent to:

$$\sum_{\alpha'\beta} C^\gamma_{\alpha,\beta}\cdot u(\beta,\alpha')\cdot C^{\gamma'}_{\alpha',\beta'} = \delta_{\gamma\gamma'} C^\gamma_{\alpha,\beta'}$$

and this, due to (31) and the genericity assumption is equivalent to (24). the theorem is proved.

Theorem 3 *Let $\mathcal{A}$ be an Heisenberg algebra of dimensions n^2 over its centre κ. For any triple (A_α), (B_β), (C_γ) of maximal abelian generic partitions of the identity in $\mathcal{A}$ there exist κ–valued matrices $\{U(X|Y) = (u_{\alpha\beta}(X|Y) : X,Y = A,B,C\}$ satisfying:*

$$U(X|X) = 1 \tag{32}$$

$$U(X|Y) \cdot U(Y|Z) = U(X|Z) \tag{33}$$

and such that, if $X \neq Y$; then the matrix $P(X|Y) = (p_{\alpha\beta}(X|Y))$ defined by:

$$p_{\alpha\beta}(X|Y) = u_{\alpha\beta}(X|Y) u_{\beta\alpha}(Y|X) \tag{34}$$

is the transition probability matrix canonically associated to the $(X_\alpha Y_\beta)$–basis according to Proposition (3.2). Conversely, any set $\{U(X|Y) : X, Y = A, B, C\}$ of κ–valued matrices satisfying (32), (33), (34) canbe obtained in this way.

Proof. Necessity. Because of Proposition (3.2) the matrices $P(A|C)$, $P(B|C), \ldots$ are characterized by the properties:

$$C_\gamma A_\alpha C_\gamma = p_{\gamma\alpha}(A|C) C_\gamma ; \;\; A_\alpha C_\gamma A_\alpha = p_{\alpha\gamma}(C|A) \tag{35}$$

$$C_\gamma B_\beta C_\gamma = p_{\gamma\beta}(B|C) C_\gamma \; ; \quad B_\beta C_\gamma B_\beta = p_{\beta\gamma}(C|B) \tag{36}$$

Using (35) and (25) one obtains:

$$p_{\alpha\gamma}(C|A) A_\alpha = A_\alpha C_\gamma A_\alpha = \sum_\beta \rho(\alpha,\gamma) \tau(\gamma,\beta) \frac{p_{\alpha\beta}(A|B)}{u_{\alpha\beta}(A|B)}$$

and this, due to (19) is equivalent to:

$$\sum_\beta \tau(\gamma,\beta) u_{\beta\alpha}(B|A) = p_{\alpha\gamma}(A|C)/\rho(\alpha,\gamma) \tag{37}$$

In a similar way one shows that:

$$\sum_\alpha u_{\beta\alpha}(B|A) \rho(\alpha,\gamma) = p_{\gamma\beta}(C|B)/\tau(\gamma,B) \tag{38}$$

Thus, defining the κ–valued matrices:

$$u_{\alpha\gamma}(A|C) = \rho(\alpha,\gamma) \; ; \quad u_{\gamma\alpha}(C|A) = p_{\alpha\gamma}(A|C)/\rho(\alpha,\gamma) \tag{39}$$

$$u_{\gamma\beta}(C|B) = \tau(\gamma,\beta) \; ; \quad u_{\beta\gamma}(B|C) = p_{\gamma B}(C|B)/\tau(\gamma,\beta) \tag{40}$$

(37) and (38) become respectively:

$$U(C|B) \cdot U(B|A) = U(C|A) \; ; \quad U(B|A) \cdot U(A|C) = U(B|C) \tag{41}$$

which, in view of (20) imply:

$$U(C|B) = U(C|A) \cdot U(A|B) \ ; \quad U(A|C) = U(A|B)U(B|C) \tag{42}$$

and this proves (33) for $X \neq Z$. Now for $X, Y = A, B, C$, denote:

$$E^{X,Y}_{\gamma\beta} = X_\alpha X_\beta / u_{\gamma\beta}(X|Y) \tag{43}$$

Using (25) and (41) one finds:

$$C_\gamma B_\beta \cdot C_{\gamma'} B_{\beta'} = \sum_{\alpha_2, \gamma'} u_{\alpha'\gamma}(A|C) \cdot u_{\gamma\beta}(C|B) \cdot u_{\alpha_2\gamma'}(A|C) \cdot u_{\gamma'\beta'}(C|B) \cdot u_{\beta,\alpha_2}(B|A) \cdot E^{AB}_{\alpha'\beta'} =$$

$$= \sum_{\alpha'} u_{\alpha'\gamma}(A|C) \cdot u_{\gamma\beta}(C|B) u_{\beta\gamma'}(B|C) u_{\gamma'\beta'}(C|B) E^{AB}_{\alpha'\beta'}$$

or, equivalently, using again (25) in appropriate notations:

$$E^{CB}_{\gamma\beta} \cdot E^{CB}_{\gamma'\beta'} = \left\{ \sum_{\alpha'\beta} u_{\alpha'\gamma}(A|C) \cdot u_{\gamma\beta}(C|B) \cdot E^{AB}_{\alpha'\beta} \right\} B_{\beta'} = u_{\beta\gamma'}(B|C) \cdot E^{CB}_{\gamma\beta'}$$

Therefore, from Lemma (4.3) we conclude that

$$\frac{u_{\gamma'\beta}(C|B) \cdot u_{\gamma\beta'}(C|B)}{u_{\gamma'\beta'}(C|B) \cdot u_{\gamma\beta}(C|B)} \cdot p_{\gamma\beta}(C|B) = \delta^{\gamma'\beta'}_{\gamma\beta}$$

are the structure constants of $\mathcal{A}$ in the $(C_\gamma B_\beta)$–basis and, in view of Theorem (4.1) this implies, in particular:

$$U(C|B) \cdot U(B|C) = U(B|C) \cdot U(C|B) = 1$$

Similarly one shows that:

$$U(A|C) \cdot U(C|A) = U(C|A) \cdot U(A|C) = 1$$

Sufficiency. Let $U(X|Y)$, $P(X|Y)$ $(X, Y = A, B, C)$ be as in the formulation of the theorem. Denote $\mathcal{A}$ the Heisenberg algebra generated over its centre κ by the partitions of the identity (A_α), (B_β) $(\alpha, \beta = 1, \ldots, n)$, whose structure constants in the $(A_\alpha B_\beta)$–basis are:

$$\gamma^{\alpha'\beta'}_{\alpha\beta} = \frac{u_{\alpha\beta'}(A|B) \cdot u_{\alpha'\beta}(A|B)}{u_{\alpha\beta}(A|B) \cdot u_{\alpha'\beta'}(A|B)} p_{\alpha\beta}(A|B)$$

Because of (33) (with $X = Z$) and (34) and of Theorem (4.2), these are the structure constants of an Heisenberg algebra with centre κ. Define, for $\gamma = 1, \dots, n$:

$$C_\gamma = \sum_{\alpha,\beta} \frac{u_{\alpha\gamma}(A|C) \cdot u_{\gamma\beta}(C|B)}{u_{\alpha\beta}(A|B)} A_\alpha B_\beta$$

Then, according to Theorem (4.4), (C_γ) is a maximal abelian partition of the identity in $\mathcal{A}$ and it is easy to convince oneself that the transition amplitude and probability matrices, associated to the triple (A_α), (B_β), (C_γ) according to the first part of the theorem are the given ones.

The results above allow to solve the following problem: given a set T and a family $\{P(x,y) : x, y \in T\}$ of real valued (or more generally, κ_0–valued, where κ_0 is a commutative, associative $*$–algebra with identity) $n \times n$ transition probability matrices, under which conditions do there exists:

i) An Heisenberg algebra $\mathcal{A}$ of dimension n^2 over its centre $\kappa (\supset \kappa_0)$.

ii) For each $x \in T$ a maximal abelian generic partition of the identity $\{A_\alpha(x) : \alpha = 1, \dots, n\}$ in $\mathcal{A}$ such that for each $x, y \in T$ the transition probability matrix associated to the pair $(A_\alpha(x))$, $(A_\beta(y))$ is $P(x,y)$?

Remark that, since the symmetry condition

$$p_{\alpha\beta}(x, y) = p_{\beta\alpha}(y; x) \tag{44}$$

has been shown in Proposition (3.2) to be a necessary condition for the solution of the problem, we can assume that it is satisfied. Moreover we will assume that:

$$P(x,x) = 1 \; ; \qquad \forall\, x \in T \tag{45}$$

$$P_{\alpha,\beta}(x,y) > 0 \; ; \quad \forall\, \alpha, \beta = 1, \dots, n \; ; \quad \forall\, x, y \in T \; ; \quad x \neq y \tag{46}$$

and look only for generic solutions (i.e. such that the structure constants of $\mathcal{A}$ in all the $(A_\alpha(x) B_\beta(y))$–bases are invertible. Under these assumptions we have:

Theorem 4 *The following assertion are equivalent:*

i1.) There exists an Heisenberg algebra with centre κ satisfying conditions (i), (ii) above.

i2.) For each $x, y \in T$ there exists a κ–valued transition amplitude matrix $U(x,y)$ for $P(x,y)$ such that:

$$U(x,x) = 1 \; ; \qquad \forall\, x \in T \tag{47}$$

$$U(x,y)\cdot U(y,z)=U(x,z)\ ;\quad \forall\, x,y,z\in T \tag{48}$$

i3.) There exists a κ–module $\mathcal{H}$ and for each $x\in T$ – a κ–basis $(a_\alpha(x))$ $(\alpha=1,\dots,n)$ of $\mathcal{H}$ such that the operators $A_\alpha(x)$ defined by

$$A_\alpha(x)\cdot a_{\alpha'}(x)=\delta_{\alpha'\alpha}a_\alpha(x) \tag{49}$$

satisfy

$$A_\alpha(x)\cdot A_\beta(y)\cdot A_\alpha(x)=p_{\alpha\beta}(x,y)A_\alpha(x) \tag{50}$$

Proof. The implication (i1.) $\Rightarrow$ (i2.) follows from Theorem (4.5). To prove that implication (i2.) $\Rightarrow$ (i3.), fix $x_0\in T$ arbitrarily and denote $\mathcal{H}$ the free κ–module generated by the symbols $a_1(x_0),\dots,a_n(x_0)$. Define, for every $x\in T$ and $\alpha=1,\dots,n$, the vector:

$$a_\alpha(x)=\sum_{\beta=1}^{n}u_{\beta\alpha}(x_0,x)a_\beta(x_0)\in\mathcal{H}$$

and the operator $A_\alpha(x):\mathcal{H}\to\mathcal{H}$

$$A_\alpha(x)a_\beta(x_0)=u_{\alpha\beta}(x,x_0)a_\alpha(x)$$

Then it is easy to verify hat for each $x\in T$ $(a_\alpha(x))$ is a κ–basis of $\mathcal{H}$ and (49) and (50) hold. Finally, if (i3.) holds, then, because of (46) and (50), for each $x,y\in T$ and $\alpha,\beta=1,\dots,n$, $A_\alpha(x)A_\beta(y)\neq 0$, hence by Lemma (4.3) the set $\{A_\alpha(x)A_\beta(\ )\}$ $(\alpha,\beta=1,\dots,n)$ is a κ–basis of the (Heisenberg) algebra of all κ–linear operators on $\mathcal{H}$. But then, due to the Remark after Proposition (3.2) condition (50) implies the maximal abelianity of the κ–algebra generated by $\{A_\alpha(x):\alpha=1,\dots,n\}$. Thus (i3.) $\Rightarrow$ (i1.) and the theorem is proved.

In the notations of Theorem (4.6) above, we say that the family of transitions probability matrices $\{P(x,y):x,y\in T\}$ admits a κ–Hilbert space model if in the κ–moduele $\mathcal{H}$, defined in point (i3.) of Theorem (4.6) one can define a κ–valued scalar product $\langle\cdot,\cdot\rangle$ for which all the κ–bases $(a_\alpha(x))$ are orthonormal bases. That is, if there is map $u,\,v\in\mathcal{H}\times\mathcal{H}\to\langle u,v\rangle\in\kappa$, such that $\forall\,u,v'\in\mathcal{H}$

$$\langle u,v+v'\rangle=\langle u,v\rangle+\langle u,v'\rangle \tag{51}$$

$$\langle u,\lambda v\rangle=\lambda\langle u,v\rangle\ ;\qquad \lambda\in\kappa \tag{52}$$

$$\langle u, v\rangle = \langle v, u\rangle^* \tag{53}$$

$$\langle u, u\rangle \geq 0 \tag{54}$$

$$\langle a_\alpha(x), a_{\alpha'}(x)\rangle = \delta_{\alpha\alpha'} \ ; \quad \forall\, x \in T \ ; \quad \forall\, \alpha = 1, \ldots, n \tag{55}$$

If this is the case, the transition amplitude matrices $U(x, y)$ are defined by:

$$\langle a(x), a(y)\rangle = u_{\alpha\beta}(x, y) \tag{56}$$

and satisfy:

$$u_{\alpha\beta}(x, y)^* = u_{\beta\alpha}(y, x) = \frac{p_{\alpha\beta}(x, y)}{u_{\alpha\beta}(x, y)} \tag{57}$$

$$\sum_{\substack{\alpha\beta=1,\ldots,n \\ jk=1,\ldots,m}} \lambda^*_{\alpha j}\lambda_{\beta k} u_{\alpha\beta}(x_j, x_k) \geq 0 \tag{58}$$

for any $m \in N$, $\lambda_{\alpha j} \in \kappa$ $(\alpha = 1, \ldots, n; j = 1, \ldots, m)$. Conversely, if the transition amplitude matrices $\mathrm{U}(x, y)$ defined in point (i2.) of Theorem (4.6) satisfy conditions (57) and (58), then the family of transition probability matrices $P(x, y)$ $(x, y \in T)$ admit a κ–Hilbert space model.

Remark 1.). Equation (48) is clearly a generalization of Schrödinger's evolution. In our theory it appears as a compatibility condition for a set of transition probability matrices $\{P(x, y)\}$ to admit an Heisenberg algebra model. If the index set T is acted upon by a group G so that probabilities are preserved $(P(x, y) = P(qx, qy))$, one might study the corresponding generalized unitary representaion of G on $\mathcal{H}$. Thus equation (48) is also a generalization of the notion of "unitary representation". Examples are easily constructed.

Remark 2.). The reversibility of the "generalized evolution" $\{U(x, y)\}$, implicit in equation (48), has a purely statistical origin, stemming from the symmetric role that two maximal observables $A(x)$ and $A(y)$ play in their mutual conditioning.

5 Geometric extensions of the quantum probabilistic formalism

We keep the notations of Sections (3.) and (4.). To fix the ideas our considerations will be restricted to κ–Hilbert space models, and we assume that the set of transition probability matrices $\{P(x, y) : x, y\vec{\in}T\}$ admits a κ–Hilbert space model, i.e. – cf. Theorem (4.6) – that there exists a κ–Hilbert space $\mathcal{H}$, $\forall\, x \in T$ – an orthonormal basis $\{a_\alpha(x) : \alpha = 1, \ldots, n\}$ of $\mathcal{H}$, and for any $x, y \in T$ – a unitary operator $U(x, y)$ (with coefficients in κ) satisfying:

$$U(x,y) : \mathcal{H} \to \mathcal{H} \tag{1}$$

$$U(x,x) = 1 \tag{2}$$

$$U(x,y) \cdot U(y,z) = U(x,z) \ ; \quad \forall\, x, y, z \in T \tag{3}$$

$$|\langle a_\beta(y), Y(x,y) a_\alpha(x)\rangle|^2 = p_{\alpha\beta}(x,y) \ ; \quad \alpha, \beta = 1, \ldots, n \tag{4}$$

If T is a manifold, it is a natural to introduce a path dependent generalization of the "evolution equation" (3) along the following lines: one considers a κ–Hilbert bundle, i.e. a fibre bundle $\mathcal{H}(T)$ with base T and fiber $\mathcal{H}(x)$ $(x \in T)$ isomorphic to a κ–Hilbet space $\mathcal{H}$. Introducing the space $\Omega(T)$ of all piecewise smooth paths $[0, 1] \to T$, we denote γ_{xy} a generic path $\gamma \in \Omega(T)$ such that $\gamma(0) = x$, $\gamma(1) = y$ $(x, y \in T)$. With these notations the notion of Heisenberg algebra model for a set of transition probability matrices $\{P(x, y)\}$ can be generalized as follows:

Definition 6 *Let T be a manifold. A family of $n \times n$ transition probability matrices $\{P(\gamma_{xy}) : \gamma_{x,y} \in \Omega(T),\ x, y \in T\}$ is said to admit a κ–Hilbert bundle model if there exist:*

i) A Hilbert bundle $\mathcal{H}(T)$ with base T.

ii) A unitary connection on $\mathcal{H}(T)$, i.e. a map $U : \gamma_{xy} \in \Omega(T) \to U(\gamma_{x,y}) \in$ Unitaries $\mathcal{H}(x) \to \mathcal{H}(y)$ such that

$$U(\gamma_{yz}) \cdot U(\gamma_{xy}) = U(\gamma_{xy} \circ \gamma_{yz}) \ ; \quad x, y, z \in T \tag{5}$$

$$U(\gamma) = U(\gamma)^{-1} \tag{6}$$

$$U(\gamma) = U(\gamma') \ \textit{if}\ \gamma' \ \textit{is a riparametrization of}\ \gamma \tag{7}$$

Here: $\gamma^{-1}(t) = \gamma(1 - t)$ and $\gamma^8\gamma'(t) = \gamma(2t)$ if $0 \leq t < 1/2$; $= \gamma'(2t - 1)$, if $1/2 \leq t \leq 1$.

iii) For any $x \in T$ an orthonormal basis $\{a_\alpha(x) : \alpha = 1, \ldots, n\}$ of $\mathcal{H}(x)$ such that, for any $\gamma_{xy} \in \Omega(T)$ $(x, y \in T)$

$$|\langle a_\beta(y), U(\gamma_{xy}) a_\alpha(x)\rangle|^2 = p_{\alpha\beta}(\gamma) \tag{8}$$

Two Hilbert bundle models $\{\mathcal{H}(T), U(\cdot)\}$, $\{\mathcal{H}'(T), U'(\cdot)\}$ are called isomorphic if there exists a vector bundle isomorphism V; $\mathcal{H}(T) \to \mathcal{H}'(T)$ which *intertwines the connections. It is not clear at the moment to what extent the transition probabilities $p_{\alpha\beta}(\gamma_{xy})$ fix the isomorphic type of the bundle. How ever, for a trivial bundle, the triviality of the holonomy group of the connection $U(\cdot)$ is easily seen to be a necessary and sufficient condition for an Hilbert bundle model to be isomorphic to a usual quantum model. This suggests the conjecture that also in the general case the statistical invariants in terms of the topological and geometrical invariants of the pair $\{\mathcal{H}(T), U(\cdot)\}$. Finally, let us remark that point (iii) in Definition (5.1) means that we are fixing a cross section into the frame bundle $\mathcal{F}(\mathcal{H}(T))$, i.e. the bundle of orthonormal frames of $\mathcal{H}(T)$, and consequently an identification of $\mathcal{F}(\mathcal{H}(T))$ with the principal bundle $P(T, U(n;\kappa))$ where $U(n;\kappa)$ denotes the unitary group with cofficients in κ. Once the cross section $a : T \to \mathcal{F}(\mathcal{H}(T))$ is fixed, the connection $V(\cdot)$ on $\mathcal{F}(\mathcal{H}(T))$ or, through the formula*

$$V(\gamma) = \vec{\exp}\left\{\int_\gamma A_a\right\} \quad ; \quad \gamma \in \Omega(T)$$

(where $\vec{\exp}$ means time–ordered exponential), to the assignment of a matrix valued 1–form A_a–the connection matrix of the connection $U(\cdot)$ in the frame field a (or simply the "potential"). The curvature form associated to A (i.e. $F = dA + \frac{1}{2}A\Lambda A$) is usually called a gauge field. Thus as the Heisenberg models deduced in Theorem (4.6) extend the usual quantum model, the corresponding bundle models, obtained with an obvious modification of Definition (5.1), generalize, in the same direction, the gauge field theories.

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