Uniqueness of braidings of quasitriangular Lie bialgebras and lifts of classical r-matrices

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Abstract

It is known that any quantization of a quasitriangular Lie bialgebra $\mathfrak{g}$ gives rise to a braiding on the dual Poisson-Lie formal group G^*. We show that this braiding always coincides with the Weinstein-Xu braiding. We show that this braiding is the "time one automorphism" of a Hamiltonian vector field, corresponding to a certain formal function on $G^* \times G^*$, the "lift of r", which can be expressed in terms of r by universal formulas. The lift of r coincides with the classical limit of the rescaled logarithm of any R-matrix quantizing it.

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§ 0 Outline of results

- a - Quasitriangular Lie algebras

We fix a base field $\mathbb{K}$ of characteristic zero. Let $(\mathfrak{g}, r)$ be a finite dimensional quasitriangular Lie bialgebra. Recall that this means that

- $(\mathfrak{g}, [-,-], \delta)$ is a Lie bialgebra;
- $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a solution of the classical Yang-Baxter equation (CYBE), i.e.,
$$[r^{1,2}, r^{1,3}] + [r^{1,2}, r^{2,3}] + [r^{1,3}, r^{2,3}] = 0;$$
- we have $\delta(x) = [r, x \otimes 1 + 1 \otimes x]$ for any $x \in \mathfrak{g}$, so in particular, $r + r^{2,1}$ is $\mathfrak{g}$-invariant.

- b - Quant($\mathfrak{g}$)

A *quantization* of $(\mathfrak{g}, r)$ is a quantized universal enveloping (QUE) algebra $(U_\hbar(\mathfrak{g}), m, \Delta)$ quantizing $(\mathfrak{g}, [-,-], \delta)$, together with an element $R \in U_\hbar(\mathfrak{g})^{\widehat{\otimes} 2}$, such that if $x \mapsto (x \bmod \hbar)$ is the canonical projection $U_\hbar(\mathfrak{g})^{\widehat{\otimes} 2} \to U(\mathfrak{g})^{\widehat{\otimes} 2}$ then

- $\Delta^{\mathrm{op}} = R \Delta R^{-1}$,
- $(\Delta \otimes \mathrm{id})(R) = R^{1,3} R^{2,3}$, $(\mathrm{id} \otimes \Delta)(R) = R^{1,3} R^{1,2}$,
- $(\epsilon \otimes \mathrm{id})(R) = (\mathrm{id} \otimes \epsilon)(R) = 1$ where $\epsilon : U_\hbar(\mathfrak{g}) \to \mathbb{K}[[\hbar]]$ is the counit of $U_\hbar(\mathfrak{g})$,
- $(R \bmod \hbar) = 1$, $\left(\frac{R-1}{\hbar} \bmod \hbar\right) = r \in \mathfrak{g} \otimes \mathfrak{g} \subset U(\mathfrak{g}) \otimes U(\mathfrak{g})$.

We denote by Quant($\mathfrak{g}$) the set of all quantizations of $(\mathfrak{g}, r)$. According to [EK], we have a map Assoc($\mathbb{K}$) $\to$ Quant($\mathfrak{g}$) (where Assoc($\mathbb{K}$) is the set of all Lie associators defined over $\mathbb{K}$), so Quant($\mathfrak{g}$) is nonempty.

- c - Braid($\mathfrak{g}$)

Let G^* be the formal group corresponding to the dual Lie bialgebra $\mathfrak{g}^*$, and let $\mathcal{O}_{G^*}$ be its function ring; so $\mathcal{O}_{G^*} = (U(\mathfrak{g}^*))^*$; this is a formal series Hopf algebra, equiped with coproduct $\Delta_{\mathcal{O}} : \mathcal{O}_{G^*} \to \mathcal{O}_{G^*} \bar{\otimes} \mathcal{O}_{G^*}$ ($\bar{\otimes}$ is the tensor product of the formal series algebras, $\mathcal{O}_{G^*} \bar{\otimes} \mathcal{O}_{G^*} = \mathcal{O}_{G^* \times G^*}$ is the function ring of $G^* \times G^*$) and counit $\epsilon_{\mathcal{O}} : \mathcal{O}_{G^*} \to \mathbb{K}$.

Definition 0.1. *A braiding of G^* is a Poisson algebra automorphism $\mathcal{R}$ of $\mathcal{O}_{G^*} \bar{\otimes} \mathcal{O}_{G^*}$ satisfying the conditions:*

(α) $(\epsilon_{\mathcal{O}} \otimes \mathrm{id}) \circ \mathcal{R} = \epsilon_{\mathcal{O}} \otimes \mathrm{id}$, $(\mathrm{id} \otimes \epsilon_{\mathcal{O}}) \circ \mathcal{R} = \mathrm{id} \otimes \epsilon_{\mathcal{O}}$,

(β) $\Delta_{\mathcal{O}}^{\mathrm{op}} = \mathcal{R} \circ \Delta_{\mathcal{O}}$,

(γ) $\mathcal{R}^{1,3} \circ \mathcal{R}^{2,3} \circ (\Delta_{\mathcal{O}} \otimes \mathrm{id}) = (\Delta_{\mathcal{O}} \otimes \mathrm{id}) \circ \mathcal{R}$,
$\mathcal{R}^{1,3} \circ \mathcal{R}^{1,2} \circ (\mathrm{id} \otimes \Delta_{\mathcal{O}}) = (\mathrm{id} \otimes \Delta_{\mathcal{O}}) \circ \mathcal{R}$,

(δ) *if* $\mathfrak{m}_{G^* \times G^*}$ *is the maximal ideal of* $\mathcal{O}_{G^*} \bar{\otimes} \mathcal{O}_{G^*}$*, then*

- *the automorphism* $\mathfrak{m}_{G^* \times G^*}/\mathfrak{m}^2_{G^* \times G^*} \to \mathfrak{m}_{G^* \times G^*}/\mathfrak{m}^2_{G^* \times G^*}$ *induced by* $\mathcal{R}$ *is the identity,*
- *therefore* $\mathcal{R} - \mathrm{id}$ *induces a linear map* $[\mathcal{R} - \mathrm{id}] : \mathfrak{m}_{G^* \times G^*}/\mathfrak{m}^2_{G^* \times G^*} \to \mathfrak{m}^2_{G^* \times G^*}/\mathfrak{m}^3_{G^* \times G^*}$*, and if we use the natural identifications*

$$\mathfrak{m}_{G^* \times G^*}/\mathfrak{m}^2_{G^* \times G^*} \xrightarrow{\sim} \mathfrak{g} \oplus \mathfrak{g}$$
$$\mathfrak{m}^2_{G^* \times G^*}/\mathfrak{m}^3_{G^* \times G^*} \xrightarrow{\sim} \left(S^2(\mathfrak{g})\right) \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \left(S^2(\mathfrak{g})\right),$$

then $[\mathcal{R} - \mathrm{id}]$ *coincides with the map*

$$(x, y) \mapsto (0, [r, x \otimes 1 + 1 \otimes y], 0).$$

We denote by $\mathrm{Braid}(\mathfrak{g})$ the set of all braidings of G^*.

- d - The Weinstein-Xu braiding

Define $\widetilde{\mathcal{R}}_{\mathrm{WX}} : G^* \times G^* \to G^* \times G^*$ by

$$\widetilde{\mathcal{R}}_{\mathrm{WX}}(u, v) = (\lambda_{R_-(v)}(u), \rho_{R_+(u)}(v)), \tag{0.1}$$

where $R_\pm : G^* \to G$ are the formal group morphisms exponentiating the Lie algebra morphisms $r_\pm : \mathfrak{g}^* \to \mathfrak{g}$, where $r_+(\xi) = \langle r, \xi \otimes \mathrm{id}\rangle$ and $r_-(\xi) = -\langle r, \mathrm{id} \otimes \xi\rangle$, and λ, ρ are the left and right dressing actions of G on G^* (regular action on $G^* = D/G$ and on $G^* = D\backslash G$, where D is the double group of G).

Let $\mathcal{R}_{\mathrm{WX}} \in \mathrm{Aut}(\mathcal{O}_{G^*} \bar{\otimes} \mathcal{O}_{G^*})$ be the algebra automorphism induced be $\widetilde{\mathcal{R}}_{\mathrm{WX}}$. Then

$$\mathcal{R}_{\mathrm{WX}} \in \mathrm{Braid}(\mathfrak{g}) \text{ (see [WX] and [GH2]).}$$

- e - The Gavarini-Halbout map

If $(U_\hbar(\mathfrak{g})(\mathfrak{g}), m, \Delta, R)$ is a quantization of $(\mathfrak{g}, r)$, define $\mathcal{O}_\hbar$ as a quantized function algebra associated to $U_\hbar(\mathfrak{g})$. So

$$\mathcal{O}_\hbar = \{f \in U_\hbar(\mathfrak{g}) | \ \forall n \geq 0, \ \delta^{(n)}(f) \in \hbar^n U_\hbar(\mathfrak{g})^{\widehat{\otimes} n}\};$$

where $\delta^{(n)} : U_\hbar(\mathfrak{g}) \to U_\hbar(\mathfrak{g})^{\widehat{\otimes} n}$ is defined by $\delta^{(n)} = (\mathrm{id} - \eta \circ \epsilon)^{\otimes n} \circ \Delta^{(n)}$. Then $\mathcal{O}_\hbar$ is a topological Hopf subalgebra of $U_\hbar(\mathfrak{g})$, and it is a quantization of the Hopf-Poisson algebra $\mathcal{O}_{G^*}$ (see [Dr,Ga]). In particular, $\mathcal{O}_\hbar/\hbar\mathcal{O}_\hbar \simeq \mathcal{O}_{G^*}$.

Theorem 0.2. *(see [GH] and also [EH]) The inner automorphism* $\mathrm{Ad}(R): x \mapsto RxR^{-1}$ *of* $U_\hbar(\mathfrak{g})^{\widehat{\otimes}2}$ *restricts to an automorphism* $\mathcal{R}_\hbar$ *of* $\mathcal{O}_\hbar^{\bar{\otimes}2}$. *The reduction* $\mathcal{R}$ *of* $\mathcal{R}_\hbar$ *modulo* $\hbar$ *is an outer automorphism of* $\mathcal{O}_{G^*}\bar{\otimes}\mathcal{O}_{G^*}$*, and* $\mathcal{R} \in \mathrm{Braid}(\mathfrak{g})$.

The main part of this result was proved in [GH] (see also [EH]). The remaining part is a consequence of Proposition 0.7. Therefore we have a map:

$$\mathrm{GH}: \ \mathrm{Quant}(\mathfrak{g}) \to \mathrm{Braid}(\mathfrak{g}).$$

- f - Unicity of braidings

Theorem 0.3. $\mathrm{Braid}(\mathfrak{g})$ *contains only one element, so*

$$\mathrm{Braid}(\mathfrak{g}) = \{\mathcal{R}_{\mathrm{WX}}\}.$$

In particular, the braiding $\mathcal{R}$ constructed in Theorem 0.2 coincides with $\mathcal{R}_{\mathrm{WX}}$.

- g - Formal Poisson manifolds

Let A be an arbitrary Poisson formal series algebra; let us denote by $\mathfrak{m}_A$ the maximal ideal of A, and let us assume that $\{A, A\} \subset \mathfrak{m}_A$. Then we have $\{\mathfrak{m}_A^k, \mathfrak{m}_A^l\} \subset \mathfrak{m}_A^{k+l-1}$, for any $k, l \geq 0$. For $f, g \in \mathfrak{m}_A^2$, the Campbell-Baker-Hausdorff (CBH) series

$$f \star g = f + g + \frac{1}{2}\{f, g\} + \cdots + B_k(f, g) + \cdots$$

converges in A with respect to the $\mathfrak{m}_A$-adic topology.

There is a unique Lie algebra morphism

$$\begin{aligned} V \ : \ & A \to \mathrm{Der}(A) \\ & f \mapsto (V_f \ : \ g \mapsto \{f, g\}). \end{aligned}$$

Define $\mathrm{Der}^+(A)$ as the Lie subalgebra of $\mathrm{Der}(A)$ of all derivations taking each $\mathfrak{m}_A^k$ to $\mathfrak{m}_A^{k+1}$. Then V restricts to a Lie algebra morphism $\mathfrak{m}_A^2 \to \mathrm{Der}^+(A)$. Moreover, for any derivation $D \in \mathrm{Der}^+(A)$, the series $\exp(D)$ is a well defined automorphism of A; this defines an exponential map

$$\begin{aligned} \exp \ : \ & \mathrm{Der}^+(A) \to \mathrm{Aut}(A) \\ & D \mapsto \exp(D). \end{aligned}$$

The series $\exp(D)$ is a well-defined automorphism of A. Let us denote by $\mathrm{Aut}^+(A)$ the subgroup of $\mathrm{Aut}(A)$ of all Poisson automorphisms θ such that the map $[\theta] : \mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2$ induced by θ is the identity (i.e., $\mathrm{Aut}^+(A)$ is the subgroup of $\mathrm{Aut}(A)$ of all Poisson automorphisms which are tangent to the identity). Then $\exp(D)$ belongs to $\mathrm{Aut}^+(A)$, and the map $\exp : \mathrm{Der}^+(A) \to \mathrm{Aut}^+(A)$ is a bijection.

- h - Lifts of the classical r-matrix

Using the previous section for the formal Poisson manifold $\mathcal{O}_{G^*}$, we can define lifts of the classical r-matrix r of the quasitriangular Lie bialgebra $\mathfrak{g}$:

Definition 0.4. *A lift of r is an element $\rho \in \mathcal{O}_{G^*}\bar{\otimes}\mathcal{O}_{G^*}$, such that:*

(α) $(\epsilon \otimes \mathrm{id})(\rho) = (\mathrm{id} \otimes \epsilon)(\rho) = 0$,

(β) $\Delta^{\mathrm{op}} = \mathrm{Ad}(\exp(V_\rho)) \circ \Delta$ *(equality of automorphisms of $\mathcal{O}_{G^*\times G^*}$),*

(γ) $(\Delta \otimes \mathrm{id})(\rho) = \rho^{1,3} \star \rho^{2,3}$, $(\mathrm{id} \otimes \Delta)(\rho) = \rho^{1,3} \star \rho^{1,2}$, *where $\rho^{i,j}$ is the image of ρ by the map $(\mathcal{O}_{G^*})^{\bar{\otimes}2} \to (\mathcal{O}_{G^*})^{\bar{\otimes}3}$ associated with (i,j),*

(δ) *the class $[\rho]$ of ρ in $\left(\mathfrak{m}_{G^*}/\mathfrak{m}_{G^*}^2\right)^{\otimes 2} = \mathfrak{g} \otimes \mathfrak{g}$ satisfies*

$$[\rho] = r.$$

<u>Remark:</u> Condition (β) may be rewritten as follows:

$$\forall f \in \mathcal{O}_{G^*},\ \Delta^{\mathrm{op}}(f) = \rho \star \Delta(f) \star (-\rho).$$

It will follow from the proof of Theorem 0.8 that this condition may be dropped from the definition of $\mathrm{Lift}(\mathfrak{g})$ (see Lemma 3.2). We denote by $\mathrm{Lift}(\mathfrak{g})$ the set of all lifts of r.

- i - Sequence of maps $\mathrm{Quant}(\mathfrak{g}) \to \mathrm{Lift}(\mathfrak{g}) \to \mathrm{Braid}(\mathfrak{g})$

Let us recall an $\hbar$-adic valuation result for R-matrices:

Theorem 0.5. *([EH]) If $(U_\hbar(\mathfrak{g}), m, \Delta, R)$ is a quantization of $(\mathfrak{g}, r)$, and if we set $\rho_\hbar = \hbar \log(R)$, then $\rho_\hbar \in \mathcal{O}_\hbar^{\bar{\otimes}2}$. If $\mathfrak{m}_\hbar$ is the kernel of the counit map $\mathcal{O}_\hbar \to \mathbb{K}[[\hbar]]$, we even have $\rho_\hbar \in \mathfrak{m}_\hbar^{\bar{\otimes}2}$.*

Corollary 0.6. *The reduction ρ of $\rho_\hbar$ modulo $\hbar$ belongs to $\mathrm{Lift}(\mathfrak{g})$. So the assignment $(U_\hbar(\mathfrak{g}), m, \Delta, R) \mapsto (\rho_\hbar \bmod \hbar)$ defines a map $\mathrm{Quant}(\mathfrak{g}) \to \mathrm{Lift}(\mathfrak{g})$.*

Proposition 0.7. *There is a unique map $\mathrm{Lift}(\mathfrak{g}) \to \mathrm{Braid}(\mathfrak{g})$, taking ρ to $\exp(V_\rho)$. Then the composed map $\mathrm{Quant}(\mathfrak{g}) \to \mathrm{Lift}(\mathfrak{g}) \to \mathrm{Braid}(\mathfrak{g})$ coincides with $\mathrm{GH} : \mathrm{Quant}(\mathfrak{g}) \to \mathrm{Braid}(\mathfrak{g})$.*

- j - Unicity of lifts

Theorem 0.8. Lift($\mathfrak{g}$) *consists of only one element.*

The unicity part of this theorem uses an elementary argument. The existence part uses the nonemptiness of Quant($\mathfrak{g}$), so it relies on the theory of associators and transcendental arguments. In the last part of the paper, we outline an algebraic proof of the existence part of Theorem 0.8, relying on co-Hochschild cohomology arguments.

- k - Universal versions

If $\mathfrak{a}$ is a finite dimensional Lie bialgebra and $\mathfrak{g}$ is the double of $\mathfrak{a}$ (so $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$, $\mathfrak{b} = \mathfrak{a}^*$), then we have the algebra isomorphisms

$$\mathcal{O}_{G^*} \simeq \widehat{S}^{\cdot}(\mathfrak{g}) \simeq \widehat{S}^{\cdot}(\mathfrak{a}) \bar{\otimes} \widehat{S}^{\cdot}(\mathfrak{b})$$

where $\widehat{S}^{\cdot}$ is the graded completion of the symmetric algebra. The last isomorphism is dual to the composed map

$$S^{\cdot}(\mathfrak{b}) \otimes S^{\cdot}(\mathfrak{a}) \stackrel{\mathrm{Sym} \otimes \mathrm{Sym}}{\longrightarrow} U(\mathfrak{a}) \otimes U(\mathfrak{b}) \stackrel{m}{\longrightarrow} U(\mathfrak{g})$$

where Sym is the symmetrization map and m is the multiplication map.

Therefore $\mathcal{O}_{G^*}^{\bar{\otimes} n} \simeq \widehat{S}^{\cdot}(\mathfrak{a})^{\bar{\otimes} n} \bar{\otimes} \widehat{S}^{\cdot}(\mathfrak{b})^{\bar{\otimes} n}$. Now if F and G are any Schur functors, one can define a universal version of the space $F(\mathfrak{a}) \otimes G(\mathfrak{b})$, namely $(F(\mathfrak{a}) \otimes G(\mathfrak{b}))_{\mathrm{univ}} = \underline{\mathrm{LBA}}(G, F)$, where $\underline{\mathrm{LBA}}$ is the prop of Lie bialgebras (see, e.g., [EE]). We then define Hopf algebras $\left(\mathcal{O}_{G^*}^{\bar{\otimes} n}\right)_{\mathrm{univ}} = \left(\widehat{S}^{\cdot}(\mathfrak{a})^{\bar{\otimes} n} \bar{\otimes} \widehat{S}^{\cdot}(\mathfrak{b})^{\bar{\otimes} n}\right)_{\mathrm{univ}}$, together with insertion-coproduct morphisms relating them.

Definition 0.9. *A universal lift is an element* $\rho_{\mathrm{univ}} \in \left(\mathcal{O}_{G^*}^{\bar{\otimes} 2}\right)_{\mathrm{univ}}$, *satisfying the universal versions of the conditions of Definition 0.4 (with* r *being the canonical element of* $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^*$*).*

We denote by $\mathrm{Lift}_{\mathrm{univ}}$ the set of all universal lifts.

When $\mathfrak{g}$ is any finite-dimensional quasitriangular Lie bialgebra, we have algebra morphisms $\left(\mathcal{O}_{G^*}^{\bar{\otimes} n}\right)_{\mathrm{univ}} \to \mathcal{O}_{G^*}^{\bar{\otimes} n}$. It follows that for any $\mathfrak{g}$, we have a map $\mathrm{Lift}_{\mathrm{univ}} \to \mathrm{Lift}(\mathfrak{g})$.

Theorem 0.10. $\mathrm{Lift}_{\mathrm{univ}}$ *consists of only one element* ρ_{univ}.

So the unique lift $\rho_{\mathfrak{g}}$ of a quasitriangular Lie bialgebra $\mathfrak{g}$ is obtained from the element

$$r \in \mathfrak{g} \otimes \mathfrak{g} \subset \widehat{S}^{\cdot}(\mathfrak{g}) \bar{\otimes} \widehat{S}^{\cdot}(\mathfrak{g}) = \mathcal{O}_{G^*}^{\bar{\otimes} 2}$$

by universal formulas. In [Re], Reshetikin computed $\rho_{\mathfrak{g}}$ when $\mathfrak{g} = \mathfrak{sl}_2$. His formulas involve the dilogarithm function. We do not know an explicit formula for ρ_{univ}. It might be simpler to express the pairing $\langle -, - \rangle : U(\mathfrak{g}^*)^{\otimes 2} \to \mathbb{K}$, defined by $\langle x, y \rangle = \langle \rho_{\mathfrak{g}}, x \otimes y \rangle$; this way one avoids the unnatural use of symmetrization maps.

- l - Plan of the paper

In Section 1, we construct a map $\text{Quant}(\mathfrak{g}) \to \text{Lift}(\mathfrak{g})$ (Corollary 0.6) and prove the unicity of lifts (Theorem 0.8).

In Section 2, we construct the map $\text{Quant}(\mathfrak{g}) \to \text{Braid}(\mathfrak{g})$ (Proposition 0.7), and then prove the unicity of braidings (Theorem 0.3). The proof of this theorem uses only a part of the arguments of Section 1 (essentially only the existence of a sequence of maps $\text{Quant}(\mathfrak{g}) \to \text{Lift}(\mathfrak{g}) \to \text{Braid}(\mathfrak{g})$).

In Section 3, we outline a proof of Theorem 0.8 not depending on the theory of associators.

In Section 4, we sketch a proof of Theorem 0.10.

In Section 5 (appendix), we construct a commutative diagram related to the duality theory of quantized universal enveloping algebras, which we use in the Sections 1 and 2.

§ 1 Lifts of classical r-matrices

Proposition 1.1. *There exists a map* $\text{Quant}(\mathfrak{g}) \to \text{Lift}(\mathfrak{g})$.

Proof. Let $(U_\hbar(\mathfrak{g}), m, \Delta_\hbar)$ be an element of $\text{Quant}(\mathfrak{g})$. Let $\mathcal{O}_\hbar \subset U_\hbar(\mathfrak{g})$ be the quantized formal series Hopf (QFSH) subalgebra sitting in $U_\hbar(\mathfrak{g})$ (see §0.e). Let $\mathfrak{m}_\hbar$ be the augmentation ideal of $\mathcal{O}_\hbar$; then $\mathfrak{m}_\hbar \subset \hbar U_\hbar(\mathfrak{g})$. In [EH], we showed that there exists a unique $\rho_\hbar \in \mathfrak{m}_\hbar^{\bar{\otimes} 2}$ such that $R = \exp\left(\frac{\rho_\hbar}{\hbar}\right)$ (this exponential is well-defined because $\left(\frac{\rho_\hbar}{\hbar}\right) \in \hbar U_\hbar(\mathfrak{g})^{\widehat{\otimes} 2}$). Then the quasitriangular identities of R can be translated as follows: for $a, b \in \mathcal{O}_\hbar^{\bar{\otimes} 3}$, we set $\{a, b\}_\hbar = \frac{1}{\hbar}[a, b]$. Let $\mathfrak{m}_\hbar^{(3)}$ be the augmentation ideal of $\mathcal{O}_\hbar^{\bar{\otimes} 3}$: then

$$\{\mathfrak{m}_\hbar^{(3)}, \mathfrak{m}_\hbar^{(3)}\}_\hbar \subset \mathfrak{m}_\hbar^{(3)},$$

therefore $\left\{\left(\mathfrak{m}_\hbar^{(3)}\right)^k, \left(\mathfrak{m}_\hbar^{(3)}\right)^l\right\}_\hbar \subset \left(\mathfrak{m}_\hbar^{(3)}\right)^{k+l-1}$. Now if $a, b \in \left(\mathfrak{m}_\hbar^{(3)}\right)^2$, the series

$$a \star_\hbar b = a + b + \frac{1}{2}\{a, b\}_\hbar + \cdots$$

(CBH series, where the Lie bracket is $\{-,-\}_\hbar$) is convergent in $\mathcal{O}_\hbar^{\bar{\otimes}3}$. Then we have :

$$(\Delta_\hbar \otimes \mathrm{id})(\rho_\hbar) = \rho_\hbar^{1,3} \star_\hbar \rho_\hbar^{2,3}, \qquad (\mathrm{id} \otimes \Delta_\hbar)(\rho_\hbar) = \rho_\hbar^{1,3} \star_\hbar \rho_\hbar^{1,2}. \tag{1.2}$$

$\Delta_\hbar$ restricts to a map $\mathcal{O}_\hbar \to \mathcal{O}_\hbar^{\bar{\otimes}2}$, the reduction of which modulo $\hbar$ is the coproduct map Δ of $\mathcal{O}_{G^*}$. Define ρ as the reduction modulo $\hbar$ of $\rho_\hbar$, so $\rho \in \mathfrak{m}_{G^*}^{\bar{\otimes}2}$. Taking the reduction of (1.2) modulo $\hbar$, we get (γ) of Definition 0.4.

On the other hand, we have $\Delta_\hbar^{\mathrm{op}} = \mathrm{Ad}\left(\exp(\frac{\rho_\hbar}{\hbar})\right) \circ \Delta$. Set $\mathrm{ad}_\hbar(a)(b) = \{a,b\}_\hbar$. The automorphisms $\mathrm{Ad}\left(\exp(\frac{\rho_\hbar}{\hbar})\right)$ and $\exp(\mathrm{ad}_\hbar(\rho_\hbar))$ coincide. So we get the identity :

$$\Delta_\hbar^{\mathrm{op}} = \exp(\mathrm{ad}_\hbar(\rho_\hbar)) \circ \Delta_\hbar \tag{1.3}$$

(equality of two morphisms $\mathcal{O}_\hbar \to \mathcal{O}_\hbar^{\bar{\otimes}2}$). Taking the reduction of (1.3) modulo $\hbar$, we get (β) of Definition 0.4.

To show that ρ satisfies (δ) of Definition 0.4, we use the following result (which will be proved in Section 4) :

Lemma 1.2. *Let σ be an arbitrary element of $\mathfrak{m}_\hbar \bar{\otimes} \mathfrak{m}_\hbar$ and $[\sigma]$ be its class in $\left(\mathfrak{m}_\hbar \big/ (\hbar \mathfrak{m}_\hbar + \mathfrak{m}_\hbar^2)\right)^{\bar{\otimes}2}$. Since $\mathfrak{m}_\hbar \big/ (\hbar \mathfrak{m}_\hbar + \mathfrak{m}_\hbar^2)$ identifies with $\mathfrak{g}$, $[\sigma] \in \mathfrak{g}^{\otimes 2}$. Since $\mathfrak{m}_\hbar \subset \hbar U_\hbar(\mathfrak{g})$, σ is an element of $\hbar^2 U_\hbar(\mathfrak{g})^{\widehat{\otimes}2}$. Then $\left(\frac{\sigma}{\hbar^2} \bmod \hbar\right)$ is an element of $U(\mathfrak{g})^{\otimes 2}$. We have the following identity in $U(\mathfrak{g})^{\otimes 2}$:*

$$\left(\frac{\sigma}{\hbar^2} \bmod \hbar\right) = [\sigma].$$

Since $\mathfrak{m}_\hbar \subset \hbar U_\hbar(\mathfrak{g})$, we have $\frac{\rho_\hbar}{\hbar} \in \hbar U_\hbar(\mathfrak{g})^{\widehat{\otimes}2}$, so in the right-hand-side of the identity

$$\frac{R-1}{\hbar} = \frac{\rho}{\hbar^2} + \frac{1}{2\hbar}\left(\frac{\rho}{\hbar}\right)^2 + \frac{1}{6\hbar}\left(\frac{\rho}{\hbar}\right)^3 + \cdots$$

the terms $\frac{1}{2\hbar}\left(\frac{\rho}{\hbar}\right)^2$, $\frac{1}{6\hbar}\left(\frac{\rho}{\hbar}\right)^3$, ..., all belong to $\hbar U_\hbar(\mathfrak{g})^{\widehat{\otimes}2}$, hence

$$\left(\frac{R-1}{\hbar} \bmod \hbar\right) = \left(\frac{\rho}{\hbar^2} \bmod \hbar\right). \tag{1.4}$$

Now

$$\begin{aligned}
[\rho] &= \left(\frac{\rho}{\hbar^2} \bmod \hbar\right) && \text{(by Lemma 1.2)} \\
&= \left(\frac{R-1}{\hbar} \bmod \hbar\right) && \text{(by identity (1.4))} \\
&= r && \text{(by hypothesis on } R).
\end{aligned}$$

Therefore ρ satisfies property (δ) of Definition 0.4.

Now we have proved that the reduction ρ of $\rho_\hbar$ modulo $\hbar$ satisfies all the conditions of Definition 0.4. $\square$

Proposition 1.3. $\mathrm{Lift}(\mathfrak{g})$ *contains at most one element.*

PROOF. Let us denote by $\mathfrak{m}_{G^*\times G^*}$ the maximal ideal of $\mathcal{O}_{G^*\times G^*}$, so $\mathfrak{m}_{G^*\times G^*} = \mathfrak{m}_{G^*}\bar{\otimes}\mathcal{O}_{G^*} + \mathcal{O}_{G^*}\bar{\otimes}\mathfrak{m}_{G^*}$. Then we have for any $N \geq 0$,

$$\mathfrak{m}_{G^*}^{\bar{\otimes}2} \cap \mathfrak{m}_{G^*\times G^*}^{N} = \sum_{\substack{a,b\geq 1\\ a+b=N}} \mathfrak{m}_{G^*}^{a}\bar{\otimes}\mathfrak{m}_{G^*}^{b}.$$

Let ρ and ρ' be two lifts or r. The classes of ρ and ρ' are the same in $\mathfrak{m}_{G^*}^{\bar{\otimes}2}/\left(\mathfrak{m}_{G^*}^{\bar{\otimes}2} \cap \mathfrak{m}_{G^*\times G^*}^{2}\right)$ and equal to r, by assumption.

Let N be an integer ≥ 2; assume that we have proved that ρ and ρ' are equal modulo $\mathfrak{m}_{G^*}^{\bar{\otimes}2} \cap \mathfrak{m}_{G^*\times G^*}^{N}$. Let us show that they are equal modulo $\mathfrak{m}_{G^*}^{\bar{\otimes}2} \cap \mathfrak{m}_{G^*\times G^*}^{N+1}$. Write $\rho' = \rho + \sigma$; then $\sigma \in \mathfrak{m}_{G^*}^{\bar{\otimes}2} \cap \mathfrak{m}_{G^*\times G^*}^{N}$. We get

$$\begin{aligned}(\Delta \otimes \mathrm{id})(\sigma) =& (\rho+\sigma)^{1,3} \star (\rho+\sigma)^{2,3} - \rho^{1,3} \star \rho^{2,3}\\ =& \sigma^{1,3} + \sigma^{2,3}\\ &+ \sum_{k>1} \left(B_k\left(\rho^{1,3}+\sigma^{1,3}, \rho^{2,3}+\sigma^{2,3}\right) - B_k\left(\rho^{1,3}, \rho^{2,3}\right)\right),\end{aligned} \tag{1.5}$$

where B_k is the total degree k homogeneous Lie polynomial of the CBH series.

Lemma 1.4. *If* $k > 1$*,* $B_k\left(\rho^{1,3}+\sigma^{1,3}, \rho^{2,3}+\sigma^{2,3}\right) - B_k\left(\rho^{1,3}, \rho^{2,3}\right)$ *is an element of* $\mathfrak{m}_{G^*\times G^*}^{N+1}$*.*

PROOF. This difference may be expressed as a sum of terms of the form

$$P_k\left(\sigma^{i_1,3}, \ldots, \sigma^{i_l,3}, \rho^{i_{l+1},3}, \ldots, \rho^{i_k,3}\right),$$

where P_k is a Lie polynomial, homogeneous of degree 1 in each variable $i_1, \ldots, i_k \in \{1,2\}$, and $l \geq 1$. This expression belongs to $\mathfrak{m}_{G^*\times G^*}^{l(N-2)+k+1}$. So it belongs to $\mathfrak{m}_{G^*\times G^*}^{N+k-1} \subset \mathfrak{m}_{G^*\times G^*}^{N+1}$. $\square$

Now $\mathcal{O}_{G^*}$ is equipped with a decreasing Hopf filtration $\mathcal{O}_{G^*} \supset \mathfrak{m}_{G^*} \supset \mathfrak{m}_{G^*}^2 \supset \cdots$: we have

$$\Delta\left(\mathfrak{m}_{G^*}^{k}\right) \subset \sum_{\alpha,\beta|\alpha+\beta=k} \mathfrak{m}_{G^*}^{\alpha}\bar{\otimes}\mathfrak{m}_{G^*}^{\beta}.$$

Its associated graded is therefore also a Hopf algebra; it is isomorphic to the formal completion $\widehat{S}^{\cdot}(\mathfrak{g})$ of the commutative and cocommutative symmetric algebra $S^{\cdot}(\mathfrak{g})$, the coproduct of which is defined by the condition that the elements of degree 1 are primitive. The tensor square $\mathcal{O}_{G^*}^{\bar{\otimes}2}$ is also filtered: the i-th term of the decreasing filtration is

$$\mathrm{Fil}^i\left(\mathcal{O}_{G^*}^{\bar{\otimes}2}\right) = \sum_{\alpha,\beta|\alpha+\beta=i} \mathfrak{m}_{G^*}^{\alpha}\bar{\otimes}\mathfrak{m}_{G^*}^{\beta};$$

and we have

$$\mathrm{gr}\left(\mathcal{O}_{G^*}^{\bar{\otimes}2}\right) = \widehat{S}^{\cdot}(\mathfrak{g})\bar{\otimes}\widehat{S}^{\cdot}(\mathfrak{g}).$$

Moreover, let $[\sigma]$ be the class of σ in $\mathrm{gr}^N\left(\mathcal{O}_{G^*}^{\bar{\otimes}2}\right)$; according to identity (1.5) and Lemma 1.4, we have

$$(\Delta\otimes\mathrm{id})([\sigma]) = [\sigma]^{1,3}+[\sigma]^{2,3}, \quad (\mathrm{id}\otimes\Delta)([\sigma]) = [\sigma]^{1,3}+[\sigma]^{1,2}.$$

The first identity implies that $[\sigma]\in\mathfrak{g}\otimes S^{N-1}(\mathfrak{g})$, the second identity implies that $[\sigma]\in S^{N-1}(\mathfrak{g})\otimes\mathfrak{g}$; since $\left(\mathfrak{g}\otimes S^{N-1}(\mathfrak{g})\right)\cap\left(S^{N-1}(\mathfrak{g})\otimes\mathfrak{g}\right)=\{0\}$, we get $[\sigma]=0$, therefore $\sigma\in\mathfrak{m}_{G^*\times G^*}^{N+1}$. So σ belongs to the intersection of all $\mathfrak{m}_{G^*\times G^*}^{N}$, $N\geq 0$, thus $\sigma=0$. This proves that $\rho=\rho'$. □

Corollary 1.5. *If $(\mathfrak{g},r)$ is a quasitriangular Lie bialgebra, there exists a unique element $\rho\in\mathrm{Lift}(\mathfrak{g})$.*

PROOF. The unicity follows from Proposition 1.3, and the existence follows from Proposition 1.1, and from the fact that $\mathrm{Quant}(\mathfrak{g})$ is nonempty: in [EK], Etingof and Kazhdan constructed a map $\mathrm{Assoc}(\mathbb{K})\to\mathrm{Quant}(\mathfrak{g})$, where $\mathrm{Assoc}(\mathbb{K})$ is the set of associators over the ground field $\mathbb{K}$; this set is introduced by Drinfeld in [Dr], where it is also shown that $\mathrm{Assoc}(\mathbb{K})$ is nonempty. □

Remark 1.6. Corollary 1.5 relies on the existence of associators, so it actually relies on transcendental arguments. Another proof of this Corollarary will be given in Section 3; this proof is algebraic and is based on the further use of co-Hochschild cohomology groups.

§ 2 Quasitriangular braidings

In this section, we construct the map $\mathrm{Quant}(\mathfrak{g})\to\mathrm{Braid}(\mathfrak{g})$ (Subsection 2.a). We then prove that $\mathcal{R}_{\mathrm{WX}}\in\mathrm{Braid}(\mathfrak{g})$ (Subsection 2.b). In Subsection 2.c, we prove that $\mathrm{Braid}(\mathfrak{g})$ contains at most one element. So the image of any element of $\mathrm{Quant}(\mathfrak{g})$ in $\mathrm{Braid}(\mathfrak{g})$ coincides with $\mathcal{R}_{\mathrm{WX}}$ (Theorem 0.3).

- a - The map $\mathrm{Quant}(\mathfrak{g}) \to \mathrm{Braid}(\mathfrak{g})$ (proof of Proposition 0.7)

Let us prove the map $\rho \mapsto \exp(V_\rho)$ actually maps $\mathrm{Lift}(\mathfrak{g}) \to \mathrm{Braid}(\mathfrak{g})$. If $\rho \in \mathrm{Lift}(\mathfrak{g})$, the fact that ρ satisfies axioms (α), (β) and (γ) of Definition 0.4 respectively implies that $\exp(V_\rho)$ satisies axioms (α), (β) and (γ) of Definition 0.1. Let us now prove that the fact that ρ satisfies axiom (δ) of Definition 0.4 implies that $\exp(V_\rho)$ satisfies axiom (δ) of Definition 0.1.

By definition, ρ is an element of $\mathfrak{m}_{G^*}\bar{\otimes}\mathfrak{m}_{G^*}$. We have $\mathfrak{m}_{G^*}\bar{\otimes}\mathfrak{m}_{G^*} \subset \mathcal{O}_{G^*}\bar{\otimes}\mathcal{O}_{G^*} = \mathcal{O}_{G^*\times G^*}$; actually, we have $\mathfrak{m}_{G^*}\bar{\otimes}\mathfrak{m}_{G^*} \subset \mathfrak{m}^2_{G^*\times G^*}$, so $\rho \in \mathfrak{m}^2_{G^*\times G^*}$. Since we have $\{\mathfrak{m}^2_{G^*\times G^*}, \mathfrak{m}_{G^*\times G^*}\} \subset \mathfrak{m}^2_{G^*\times G^*}$, the map

$$V_\rho \ : \ \mathfrak{m}_{G^*\times G^*}/\mathfrak{m}^2_{G^*\times G^*} \to \mathfrak{m}_{G^*\times G^*}/\mathfrak{m}^2_{G^*\times G^*}$$

induces the zero map. Therefore, so do all the $(V_\rho)^k$, $k \geq 1$. So $\exp(V_\rho)$ induces the identity map of $\mathfrak{m}_{G^*\times G^*}/\mathfrak{m}^2_{G^*\times G^*}$. Let us now compute the map

$$[\exp(V_\rho) - \mathrm{id}] \ : \ \mathfrak{m}_{G^*\times G^*}/\mathfrak{m}^2_{G^*\times G^*} \to \mathfrak{m}^2_{G^*\times G^*}/\mathfrak{m}^3_{G^*\times G^*}$$

using the identifications $\mathfrak{m}^2_{G^*\times G^*}/\mathfrak{m}^3_{G^*\times G^*} = S^2(\mathfrak{g}) \oplus (\mathfrak{g}\otimes\mathfrak{g}) \oplus S^2(\mathfrak{g})$ and $\mathfrak{m}_{G^*\times G^*}/\mathfrak{m}^2_{G^*\times G^*} = \mathfrak{g}\oplus\mathfrak{g}$. We have

$$(\epsilon\otimes\mathrm{id})\circ(\exp(V_\rho) - \mathrm{id}) = (\mathrm{id}\otimes\epsilon)\circ(\exp(V_\rho) - \mathrm{id}) = 0$$

(identity of maps $\mathcal{O}_{G^*\times G^*} \to \mathcal{O}_{G^*}$), because $\{\mathfrak{m}_{G^*}, \mathcal{O}_{G^*}\} \subset \mathfrak{m}_{G^*}$. The class of $\mathrm{Ker}(\epsilon\otimes\mathrm{id}) \cap \mathrm{Ker}(\mathrm{id}\otimes\epsilon) \cap \mathfrak{m}^2_{G^*\times G^*}$ in $\mathfrak{m}^2_{G^*\times G^*}/\mathfrak{m}^3_{G^*\times G^*}$ is the subspace $(\mathfrak{g}\otimes\mathfrak{g}) \subset S^2(\mathfrak{g}) \oplus (\mathfrak{g}\otimes\mathfrak{g}) \oplus S^2(\mathfrak{g})$.

On the other hand, if $f \in \mathfrak{m}_{G^*\times G^*}$, and $k \geq 2$, then $(V_\rho)^k(f) \in \mathfrak{m}^3_{G^*\times G^*}$. So the class of $(\exp(V_\rho) - \mathrm{id})(f)$ in $\mathfrak{m}^2_{G^*\times G^*}/\mathfrak{m}^3_{G^*\times G^*}$ coincides with that of $V_\rho(f)$. So we now compute the map

$$[V_\rho] \ : \ \mathfrak{g}\oplus\mathfrak{g} \to (\mathfrak{g}\otimes\mathfrak{g}).$$

Let $x_1, x_2 \in \mathfrak{g}$ and let $f_1, f_2 \in \mathfrak{m}_{G^*}$ be such that their classes in $\mathfrak{m}_{G^*}/\mathfrak{m}^2_{G^*} = \mathfrak{g}$ are x_1, x_2. Let us set $f = f_1\otimes 1 + 1\otimes f_2$, and let us compute $V_\rho(f)$. Set $\rho = \sum_\alpha \rho'_\alpha \otimes \rho''_\alpha$, with $\rho'_\alpha, \rho''_\alpha \in \mathfrak{m}_{G^*}$. Then

$$V_\rho(f) = \sum_\alpha \{\rho'_\alpha, f_1\} \otimes \rho''_\alpha + \rho'_\alpha \otimes \{\rho''_\alpha, f_2\}.$$

Now we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{m}_{G^*}\otimes\mathfrak{m}_{G^*} & \xrightarrow{\text{Poisson bracket}} & \mathfrak{m}_{G^*} \\ \downarrow & & \downarrow \\ \mathfrak{g}\otimes\mathfrak{g} & \xrightarrow{\text{Lie bracket}} & \mathfrak{g} \end{array}$$

where the vertical arrows correspond to the projection $\mathfrak{m}_{G^*} \to \mathfrak{m}_{G^*}/\mathfrak{m}_{G^*}^2 = \mathfrak{g}$. So the class of $V_\rho(f)$ in $\mathfrak{g}\otimes\mathfrak{g}$ is $[r, x_1\otimes 1 + 1\otimes x_2]$. Therefore $[\exp(V_\rho) - \mathrm{id}] : \mathfrak{g}\oplus\mathfrak{g} \to (\mathfrak{g}\otimes\mathfrak{g})$ is the map

$$(x_1, x_2) \mapsto (0, [r, x_1\otimes 1 + 1\otimes x_2], 0)\,,$$

which proves that $\exp(V_\rho)$ satisfies condition (δ) of Definition 0.1 and so belongs to $\mathrm{Braid}(\mathfrak{g})$. □

- b - Proof of $\mathcal{R}_{\mathrm{WX}} \in \mathrm{Braid}(\mathfrak{g})$

In [WX], it is proved that $\mathcal{R}_{\mathrm{WX}}$ satisfies conditions (α), (β) and (γ) of Definition 0.1. In [GH2], it is proved that it satisfies the first part of (δ) of this definition, namely $\mathcal{R}_{\mathrm{WX}}$ induces the identity endomorphism of $\mathfrak{m}_{G^*\times G^*}/\mathfrak{m}^2_{G^*\times G^*}$. Then $\mathcal{R}_{\mathrm{WX}} - \mathrm{id}$ induces a map $\mathfrak{m}_{G^*\times G^*}/\mathfrak{m}^2_{G^*\times G^*} \to \mathfrak{m}^2_{G^*\times G^*}/\mathfrak{m}^3_{G^*\times G^*}$, which we now compute.

Identify G^* with $\mathfrak{g}^*$ using the exponential map. We get, from (0.1), the expansion at second order of the map $\widetilde{\mathcal{R}}_{\mathrm{WX}}$:

$$\begin{aligned}
&\mathfrak{g}^*\oplus\mathfrak{g}^* \to \mathfrak{g}^*\oplus\mathfrak{g}^* \\
&\quad (\xi,\eta) \mapsto (\xi,\eta) + (\mathrm{ad}^*(r_+(\eta))(\xi), \mathrm{ad}^*(r_-(\xi))(\eta))\,.
\end{aligned}$$

View $(x,y)\in\mathfrak{g}\oplus\mathfrak{g}$ as a function of $\mathfrak{g}^*\oplus\mathfrak{g}^*$, taking (ξ,η) to $\langle\xi,x\rangle + \langle\eta,y\rangle$. Then $\mathcal{R}_{\mathrm{WX}}(x,y)$ takes (ξ,η) to

$$\begin{aligned}
&\langle \xi + \mathrm{ad}^*(r_+(\eta))(\xi), x\rangle + \langle \eta + \mathrm{ad}^*(r_-(\xi))(\eta), y\rangle \\
&\quad = \langle\xi,x\rangle + \langle\eta,y\rangle + \langle\xi,[r_+(\eta),x]\rangle + \langle\eta,[r_-(\xi),y]\rangle \\
&\quad = \langle\xi,x\rangle + \langle\eta,y\rangle + \sum_i \langle a_i,\xi\rangle\langle\eta,[b_i,y]\rangle + \sum_i \langle b_i,\eta\rangle\langle\xi,[a_i,x]\rangle \\
&\quad = \langle\xi,x\rangle + \langle\eta,y\rangle + \sum_i \langle \xi\otimes\eta, [a_i,x]\otimes b_i + a_i\otimes[b_i,y]\rangle + \big(\text{order 3 in } (\xi,\eta)\big)
\end{aligned}$$

where we set $r = \sum_i a_i\otimes b_i$, so that $r_+(\xi) = \sum_i \langle b_i,\xi\rangle a_i$, and $r_-(\xi) = \sum_i\langle a_i,\xi\rangle b_i$. Therefore

$$\begin{aligned}
[\mathcal{R}_{\mathrm{WX}} - \mathrm{id}] \ :\ &\mathfrak{g}\oplus\mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g} \\
&(x,y) \mapsto [r, x\otimes 1 + 1\otimes y].
\end{aligned}$$

Then $\mathcal{R}_{\mathrm{WX}}$ satisfies all the conditions of Definition 0.1. □

- c - Unicity of braidings

Let $\mathcal{R}$ and $\mathcal{R}'$ be two elements of Braid($\mathfrak{g}$). We know that the maps $[\mathcal{R}-\mathrm{id}]$ and $[\mathcal{R}'-\mathrm{id}] : \mathfrak{m}_{G^*\times G^*}/\mathfrak{m}^2_{G^*\times G^*} \to \mathfrak{m}^2_{G^*\times G^*}/\mathfrak{m}^3_{G^*\times G^*}$ coincide, so $(\mathcal{R}-\mathcal{R}')(\mathfrak{m}_{G^*\times G^*}) \subset \mathfrak{m}^3_{G^*\times G^*}$. Let us prove by induction over $k \geq 3$ that

$$(\mathcal{R}-\mathcal{R}')(\mathfrak{m}_{G^*\times G^*}) \subset \mathfrak{m}^k_{G^*\times G^*}. \tag{2.6}$$

As we have seen, (2.6) holds for $k=3$. Assume that it holds for some k and let us prove it for $k+1$. Let us set $\mathcal{S} = \mathcal{R}-\mathcal{R}'$. Then $\mathcal{S}$ is a linear map $\mathcal{O}_{G^*\times G^*} \to \mathfrak{m}^k_{G^*\times G^*}$. Moreover, we have for $f, g \in \mathcal{O}_{G^*\times G^*}$

$$\mathcal{S}(fg) = \mathcal{R}(f)\mathcal{S}(g) + \mathcal{S}(f)\mathcal{R}'(g). \tag{2.7}$$

Identity (2.7) allows to show by induction:

Lemma 2.1. *For any $a \geq 1$, we have $\mathcal{S}(\mathfrak{m}^a_{G^*\times G^*}) \subset \mathfrak{m}^{a+k-1}_{G^*\times G^*}$.*

Proof. This holds when $a=1$ due to (2.6).
Assume that we proved $\mathcal{S}(\mathfrak{m}^a_{G^*\times G^*}) \subset \mathfrak{m}^{a+k-1}_{G^*\times G^*}$; then for $f \in \mathfrak{m}^a_{G^*\times G^*}$ and $g \in \mathfrak{m}_{G^*\times G^*}$,

$$\mathcal{S}(fg) = \mathcal{R}(f)\mathcal{S}(g) + \mathcal{S}(f)\mathcal{R}'(g) \in \\ \in \mathfrak{m}^a_{G^*\times G^*} \cdot \mathfrak{m}^k_{G^*\times G^*} + \mathfrak{m}^{a+k-1}_{G^*\times G^*} \cdot \mathfrak{m}_{G^*\times G^*} \subset \mathfrak{m}^{a+k}_{G^*\times G^*}.$$

Therefore $\mathcal{S}(\mathfrak{m}^{a+1}_{G^*\times G^*}) \subset \mathfrak{m}^{a+k}_{G^*\times G^*}$. □

Let us now use the fact that $\mathcal{O}_{G^*}$ is a topological Hopf algebra, equipped with a decreasing Hopf filtration $\mathcal{O}_{G^*} \supset \mathfrak{m}_{G^*} \supset \mathfrak{m}^2_{G^*} \supset \cdots$. The completion of the associated graded of $\mathcal{O}_{G^*}$ is a commutative and cocommutative Hopf algebra

$$\widehat{\mathrm{gr}}(\mathcal{O}_{G^*}) = \widehat{\oplus}_i \, \mathrm{gr}^i(\mathcal{O}_{G^*}) = \widehat{S}(\mathfrak{g}).$$

$\mathcal{O}_{G^*\times G^*}$ is also filtered and $\widehat{\mathrm{gr}}(\mathcal{O}_{G^*\times G^*}) = \widehat{S}(\mathfrak{g})^{\bar{\otimes} 2}$. Then Lemma 2.1, together with identity (2.7), implies:

Lemma 2.2. *Define $\mathrm{gr}(\mathcal{S}) : \widehat{\mathrm{gr}}\left(\mathcal{O}^{\bar{\otimes}2}_{G^*}\right) \to \widehat{\mathrm{gr}}\left(\mathcal{O}^{\bar{\otimes}2}_{G^*}\right)$ as the degree k map such that $\mathrm{gr}(\mathcal{S}) : \mathfrak{m}^a_{G^*\times G^*}/\mathfrak{m}^{a+1}_{G^*\times G^*} \to \mathfrak{m}^{a+k-1}_{G^*\times G^*}/\mathfrak{m}^{a+k}_{G^*\times G^*}$ is induced by $\mathcal{S}$ for $a \geq 0$. Then $\mathrm{gr}(\mathcal{S})$ is a derivation of degree $k-1$ of $\mathrm{gr}(\mathcal{O}_{G^*\times G^*})$.*

Comparing the analogues of the identities (γ) for $\mathcal{R}$ and $\mathcal{R}'$, we get :

$$\left(\mathcal{S}^{1,3} \circ \mathcal{R}^{2,3} + \mathcal{R}'^{1,3} \circ \mathcal{S}^{2,3}\right) \circ (\Delta_{\mathcal{O}} \otimes \mathrm{id}) = (\Delta_{\mathcal{O}} \otimes \mathrm{id}) \circ \mathcal{S},$$

and

$$\left(\mathcal{S}^{1,3} \circ \mathcal{R}^{1,2} + \mathcal{R}'^{1,3} \circ \mathcal{S}^{1,2}\right) \circ (\mathrm{id} \otimes \Delta_{\mathcal{O}}) = (\mathrm{id} \otimes \Delta_{\mathcal{O}}) \circ \mathcal{S}$$

Both sides of each identity are algebra morphisms $\mathcal{O}_{G^*\times G^*} \to \mathcal{O}_{G^*\times G^*\times G^*}$ taking $\mathfrak{m}^a_{G^*\times G^*}$ to $\mathfrak{m}^{a+k-1}_{G^*\times G^*\times G^*}$. The associated graded morphisms are degree $k-1$ algebra morphisms $\widehat{\mathrm{gr}}(\mathcal{O}_{G^*\times G^*}) \to \widehat{\mathrm{gr}}(\mathcal{O}_{G^*\times G^*\times G^*})$. The corresponding identities between these morphisms are

$$\left(\mathrm{gr}\,(\mathcal{S})^{1,3} + \mathrm{gr}\,(\mathcal{S})^{2,3}\right) \circ (\Delta_0 \otimes \mathrm{id}) = (\Delta_0 \otimes \mathrm{id}) \circ \mathrm{gr}(\mathcal{S}),$$

and

$$\left(\mathrm{gr}\,(\mathcal{S})^{1,3} + \mathrm{gr}\,(\mathcal{S})^{1,2}\right) \circ (\mathrm{id} \otimes \Delta_0) = (\mathrm{id} \otimes \Delta_0) \circ \mathrm{gr}(\mathcal{S}), \tag{2.8}$$

where $\Delta_0 : \widehat{S}(\mathfrak{g}) \to \widehat{S}(\mathfrak{g})\bar{\otimes}\widehat{S}(\mathfrak{g})$ is the coproduct map of $\mathrm{gr}(\mathcal{S}) = \widehat{\mathrm{gr}}(\mathcal{O}_{G^*})$. These identities imply that the image of $\mathrm{gr}(\mathcal{S})$ is contained in $\mathrm{Prim}(\widehat{S}(\mathfrak{g})) \otimes \mathrm{Prim}(\widehat{S}(\mathfrak{g}))$. Since $\mathrm{Prim}(\widehat{S}(\mathfrak{g})) = S^1(\mathfrak{g}) = \mathrm{gr}^1(\mathcal{O}_{G^*})$, the image of $\mathrm{gr}(\mathcal{S})$ is therefore contained in $\mathrm{gr}^1(\mathcal{O}_{G^*})^{\otimes 2} \subset \mathrm{gr}^2(\mathcal{O}_{G^*\times G^*})$. Since the image of $\mathrm{gr}(\mathcal{S})$ is also contained in $\widehat{\otimes}_{i\geq 3}\,\mathrm{gr}^i(\mathcal{O}_{G^*\times G^*})$, we get $\mathrm{gr}(\mathcal{S}) = 0$. It follows that $\mathcal{S}(\mathfrak{m}_{G^*\times G^*}) \subset \mathfrak{m}^{k+1}_{G^*\times G^*}$. This proves the induction step of (2.6). Therefore $\mathcal{S}(\mathfrak{m}_{G^*\times G^*}) \subset \cap_{k\geq 0}\mathfrak{m}^k_{G^*\times G^*} = 0$. Since $\mathcal{S}$ is a derivation, we get $\mathcal{S} = 0$. Therefore $\mathcal{R} = \mathcal{R}'$. This proves that $\mathrm{Braid}(\mathfrak{g})$ contains at most one element. □

§ 3 Cohomological construction of ρ

Let $(\mathfrak{g}, r)$ be a finite-dimensional quasitriangular Lie bialgebra. The purpose of this section is to construct the unique element ρ of $\mathrm{Lift}(\mathfrak{g})$ by cohomological arguments, thus avoiding the use of associators. Our main result is:

Theorem 3.1. $\mathrm{Lift}(\mathfrak{g})$ *contains an element* ρ.

This result will be proved in Subsection 3.c. In Subsection 3.a, we introduce variants and truncations of the sets $\mathrm{Lift}(\mathfrak{g})$ and $\mathrm{Braid}(\mathfrak{g})$. Subsection 3.b contains the cohomological results allowing to construct ρ by successive approximations.

- a - Variants of the sets $\mathrm{Braid}(\mathfrak{g})$ and $\mathrm{Lift}(\mathfrak{g})$

We denote by $\mathrm{Braid}'(\mathfrak{g})$ the set of all Poisson automorphisms of $\mathcal{O}_{G^*\times G^*}$, satisfying conditions (α), (γ) and (δ) of Definition 0.1. We denote by $\mathrm{Lift}'(\mathfrak{g})$ the set of all elements ρ of $\mathcal{O}_{G^*\times G^*}$, satisfying conditions (α), (γ) and (δ) of Definition 0.4. The map $\rho \mapsto \exp(V_\rho)$ then restrict to a map $\mathrm{Lift}'(\mathfrak{g}) \to \mathrm{Braid}'(\mathfrak{g})$.

If n is an integer, we define $\mathrm{Braid}'_{\leq n}(\mathfrak{g})$ (resp., $\mathrm{Braid}_{\leq n}(\mathfrak{g})$) as the set of all Poisson automorphisms of $\mathcal{O}_{G^*\times G^*}/\mathfrak{m}^n_{G^*\times G^*}$, satisfying conditions (α), (γ) and (δ) (resp. (α), (β), (γ) and (δ)) of Definition 0.1, where $\mathcal{O}_{(G^*)^k}$ is replaced by $\mathcal{O}_{(G^*)^k}/\mathfrak{m}^n_{(G^*)^k}$, $k = 1, 2, 3$.

Similarly, we define $\mathrm{Lift}'_{\leq n}(\mathfrak{g})$ (resp., $\mathrm{Lift}_{\leq n}(\mathfrak{g})$) as the set of all lifts $\rho \in \mathcal{O}_{G^*\times G^*}/\mathfrak{m}^n_{G^*\times G^*}$, satisfying conditions (α), (γ) and (δ) (resp. (α), (β), (γ) and (δ)) of Definition 0.4, where $\mathcal{O}_{(G^*)^k}$ is replaced by $\mathcal{O}_{(G^*)^k}/\mathfrak{m}^n_{(G^*)^k}$, $k = 1, 2, 3$. Then $\rho \mapsto \exp(V_\rho)$ defines a map $\mathrm{Lift}'_{\leq n}(\mathfrak{g}) \to \mathrm{Braid}'_{\leq n}(\mathfrak{g})$.

Lemma 3.2. *We have:*

1. *The natural inclusions* $\mathrm{Lift}(\mathfrak{g}) \subset \mathrm{Lift}'(\mathfrak{g})$, $\mathrm{Braid}(\mathfrak{g}) \subset \mathrm{Braid}'(\mathfrak{g})$, $\mathrm{Lift}_{\leq n}(\mathfrak{g}) \subset \mathrm{Lift}'_{\leq n}(\mathfrak{g})$ *and* $\mathrm{Braid}_{\leq n}(\mathfrak{g}) \subset \mathrm{Braid}'_{\leq n}(\mathfrak{g})$ *are all equalities.*
2. *The set* $\mathrm{Braid}_{\leq n}(\mathfrak{g})$ *consists of only one element,* $\bar{\mathcal{R}}^{(n)}_{\mathrm{WX}}$*, which is the automorphism of* $\mathcal{O}_{G^*\times G^*}/\mathfrak{m}^n_{G^*\times G^*}$ *induced by the Weinstein-Xu automorphism.*

Proof. One can repeat the proof of the unicity part of Theorem 0.3 to show that the sets $\mathrm{Braid}'_{\leq n}(\mathfrak{g})$, $\mathrm{Braid}_{\leq n}(\mathfrak{g})$ and $\mathrm{Braid}'(\mathfrak{g})$ all contain at most one element. Since $\mathcal{R}_{\mathrm{WX}}$ is an element of $\mathrm{Braid}(\mathfrak{g})$, we get $\mathrm{Braid}(\mathfrak{g}) = \mathrm{Braid}'(\mathfrak{g}) = \{\mathcal{R}_{\mathrm{WX}}\}$. In the same way, the automorphism $\bar{\mathcal{R}}^{(n)}_{\mathrm{WX}}$ of $\mathcal{O}_{G^*\times G^*}/\mathfrak{m}^n_{G^*\times G^*}$ induced by $\mathcal{R}_{\mathrm{WX}}$ is an element of $\mathrm{Braid}'_{\leq n}(\mathfrak{g})$ and of $\mathrm{Braid}_{\leq n}(\mathfrak{g})$, so $\mathrm{Braid}'_{\leq n}(\mathfrak{g}) = \mathrm{Braid}_{\leq n}(\mathfrak{g}) = \{\bar{\mathcal{R}}^{(n)}_{\mathrm{WX}}\}$. This proves part 2 and the equalities beween the sets of braidings of part 1.
Now $\mathrm{Lift}(\mathfrak{g})$ is nothing but the preimage of $\mathrm{Braid}(\mathfrak{g})$ by the map

$$\begin{aligned} \exp\ &: \mathrm{Lift}'(\mathfrak{g}) \to \mathrm{Braid}'(\mathfrak{g}) \\ &\qquad\qquad \rho \mapsto \exp(V_\rho); \end{aligned}$$

similarly, $\mathrm{Lift}_{\leq n}(\mathfrak{g})$ is the preimage of $\mathrm{Braid}_{\leq n}(\mathfrak{g})$ by the map $\exp : \mathrm{Lift}'_{\leq n}(\mathfrak{g}) \to \mathrm{Braid}'_{\leq n}(\mathfrak{g})$. So we get $\mathrm{Lift}(\mathfrak{g}) = \mathrm{Lift}'(\mathfrak{g})$ and $\mathrm{Lift}_{\leq n}(\mathfrak{g}) = \mathrm{Lift}'_{\leq n}(\mathfrak{g})$. □

- b - A map $\mathrm{Lift}'_{\leq n}(\mathfrak{g}) \to \mathrm{Lift}'_{\leq n+1}(\mathfrak{g})$

We have canonical projection maps $\mathrm{Lift}'_{\leq n}(\mathfrak{g}) \xrightarrow{\pi_{n-1}} \mathrm{Lift}'_{\leq n-1}(\mathfrak{g}) \to \cdots$. Then

$$\mathrm{Lift}'(\mathfrak{g}) = \lim_{\substack{\longleftarrow \\ n}} \big(\mathrm{Lift}'_{\leq n}(\mathfrak{g})\big).$$

To construct an element of $\mathrm{Lift}'(\mathfrak{g})$, we will therefore construct a sequence of maps

$$\lambda_n \ : \ \mathrm{Lift}'_{\leq n}(\mathfrak{g}) \to \mathrm{Lift}'_{\leq n+1}(\mathfrak{g}), \ n \geq 3,$$

such that $\pi_n \circ \lambda_n = \mathrm{id}$.

Let $\rho_n \in \mathcal{O}_{G^*\times G^*}/\mathfrak{m}^n_{G^*\times G^*}$ be an element of $\mathrm{Lift}'_{\leq n}(\mathfrak{g})$. We have then

$$(\epsilon\otimes\mathrm{id})(\rho_n) = (\mathrm{id}\otimes\epsilon)(\rho_n) = 0,$$

$$(\Delta\otimes\mathrm{id})(\rho_n) = \rho_n^{1,3}\star\rho_n^{2,3},\ (\mathrm{id}\otimes\Delta)(\rho_n) = \rho_n^{1,3}\star\rho_n^{1,2},$$

$$[\rho_n] = r.$$

Let us take a lift $\widetilde{\rho}_n \in \mathcal{O}_{G^*\times G^*}/\mathfrak{m}^{n+1}_{G^*\times G^*}$ of ρ_n such that $(\epsilon\otimes\mathrm{id})(\widetilde{\rho}_n) = (\mathrm{id}\otimes\epsilon)(\widetilde{\rho}_n) = 0$. Set

$$\begin{aligned}\alpha &= (\Delta\otimes\mathrm{id})(\widetilde{\rho}_n) - \widetilde{\rho}_n^{1,3}\star\widetilde{\rho}_n^{2,3},\\ \beta &= (\mathrm{id}\otimes\Delta)(\widetilde{\rho}_n) = \widetilde{\rho}_n^{1,3}\star\widetilde{\rho}_n^{1,2}.\end{aligned} \tag{3.9}$$

Then $\alpha,\beta\in\mathfrak{m}^n_{(\mathcal{O}^*)^3}/\mathfrak{m}^{n+1}_{(\mathcal{O}^*)^3}$. Moreover

$$\begin{aligned}(\epsilon\otimes\mathrm{id}\otimes\mathrm{id})(\alpha) = (\mathrm{id}\otimes\epsilon\otimes\mathrm{id})(\alpha) = (\mathrm{id}\otimes\mathrm{id}\otimes\epsilon)(\alpha) = 0,\\ (\epsilon\otimes\mathrm{id}\otimes\mathrm{id})(\beta) = (\mathrm{id}\otimes\epsilon\otimes\mathrm{id})(\beta) = (\mathrm{id}\otimes\mathrm{id}\otimes\epsilon)(\beta) = 0.\end{aligned}$$

Let σ be an element of $\mathfrak{m}^n_{G^*\times G^*}/\mathfrak{m}^{n+1}_{G^*\times G^*}$. Set $\rho_{n+1} = \widetilde{\rho}_n + \sigma$. Then ρ_{n+1} belongs to $\mathrm{Lift}'_{\leq n+1}(\mathfrak{g})$ if and only if:

$$(\epsilon\otimes\mathrm{id})(\sigma) = (\mathrm{id}\otimes\epsilon)(\sigma) = 0, \tag{3.10}$$

$$(\mathrm{d}\otimes\mathrm{id})(\sigma) = -\alpha,\ (\mathrm{id}\otimes\mathrm{d})(\sigma) = -\beta. \tag{3.11}$$

Here, we identify $\mathfrak{m}^n_{G^*\times G^*}/\mathfrak{m}^{n+1}_{G^*\times G^*}$ with $S^n(\mathfrak{g}\oplus\mathfrak{g})$ and $\mathfrak{m}^n_{(\mathcal{O}^*)^3}/\mathfrak{m}^{n+1}_{(\mathcal{O}^*)^3}$ with $S^n(\mathfrak{g}\oplus\mathfrak{g}\oplus\mathfrak{g})$. Then the map $\mathrm{d} : S^{\cdot}(\mathfrak{g}) \to S^{\cdot}(\mathfrak{g}\oplus\mathfrak{g})$ is the Hochschild coboundary map, taking f to $\Delta_0(f) - f\otimes 1 - 1\otimes f$ (Δ_0 is the cocommutative coproduct of $S^{\cdot}(\mathfrak{g})$). Identities (3.10) and (3.11) follow from the identities

$$f\star(h+g) = f\star h + g, \qquad (f+g)\star h = f\star h + g, \tag{3.12}$$

when $f,h\in\mathfrak{m}^2_{(G^*)^k}/\mathfrak{m}^{n+1}_{(G^*)^k}$ and $g\in\mathfrak{m}^n_{(G^*)^k}/\mathfrak{m}^{n+1}_{(G^*)^k}$.

Let us now recall some results of co-Hochschild cohomology. Let $\mathrm{d}^{(2)} : S^{\cdot}(\mathfrak{g}\oplus\mathfrak{g}) \to S^{\cdot}(\mathfrak{g}\oplus\mathfrak{g}\oplus\mathfrak{g})$ be defined by $\mathrm{d}^{(2)}(f) = (\mathrm{d}\otimes\mathrm{id})(f) - (\mathrm{id}\otimes\mathrm{d})(f)$ (we identify $S^{\cdot}(\mathfrak{g}^{\oplus k})$ with $S^{\cdot}(\mathfrak{g})^{\otimes k}$). Then $\mathrm{d}^{(2)}\circ\mathrm{d} = 0$. The cohomology group $H^1_{\text{co-Hoch}} = \mathrm{Ker}(\mathrm{d}^{(2)})/\mathrm{Im}(\mathrm{d})$ identifies with $\wedge^2(\mathfrak{g})$. The canonical map $\mathrm{Ker}(\mathrm{d}^{(2)}) \to \wedge^2(\mathfrak{g})$ is given by the antisymmetrization $f \mapsto f - f^{2,1}$. The 0-th cohomology group $H^0_{\text{co-Hoch}} = \mathrm{Ker}(\mathrm{d})$ is equal to $\mathfrak{g}$. We then prove:

Lemma 3.3. *There exists a solution* $\sigma \in \mathfrak{m}^n_{G^*\times G^*}/\mathfrak{m}^{n+1}_{G^*\times G^*}$ *of equations (3.10) and (3.11) if and only if* α, β *satisfy the equations:*

$$(\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{d})(\alpha) = (\mathrm{d}\otimes\mathrm{id}\otimes\mathrm{id})(\beta), \tag{3.13}$$

$$(\mathrm{d}^{(2)}\otimes\mathrm{id})(\alpha) = (\mathrm{id}\otimes\mathrm{d}^{(2)}(\beta) = 0, \tag{3.14}$$

$$\alpha^{1,2,3} = \alpha^{2,1,3},\ \beta^{1,2,3} = \beta^{1,3,2}. \tag{3.15}$$

If these conditions are satisfied, then the solution is unique.

Proof. Assume that σ exists. Then both sides of (3.13) are equal to $-(\mathrm{d}\otimes\mathrm{d})(\sigma)$, so we have (3.13). (3.14) follows from $\mathrm{d}^{(2)}\circ\mathrm{d} = 0$ and (3.15) follows from the fact that the image of $\mathrm{d} : S^{\cdot}(\mathfrak{g}) \to S^{\cdot}(\mathfrak{g}\oplus\mathfrak{g})$ is contained in the subspace of invariants under the permutation of both summands of $\mathfrak{g}\oplus\mathfrak{g}$. So (3.13), (3.14) and (3.15) are satisfied.

Assume now that these identities are satisfied. The equalities $(\mathrm{d}^{(2)}\otimes\mathrm{id})(\alpha) = 0$ and $\alpha^{1,2,3} = \alpha^{2,1,3}$ imply that α corresponds to zero in $H^1_{\text{co-Hoch}}\otimes S^{n-2}(\mathfrak{g}) = \wedge^2(\mathfrak{g})\otimes S^{n-2}(\mathfrak{g})$, hence there exists $\sigma' \in S^n(\mathfrak{g}\oplus\mathfrak{g})$, such that $(\mathrm{d}\otimes\mathrm{id})(\sigma') = -\alpha$. In the same way, there exists $\sigma'' \in S^n(\mathfrak{g}\oplus\mathfrak{g})$, such that $(\mathrm{id}\otimes\mathrm{d})(\sigma'') = -\beta$. σ' is well-defined only up to addition of an element of $\mathfrak{g}\otimes S^{n-1}(\mathfrak{g})$, and σ'' is well-defined up to addition of an element of $S^{n-1}(\mathfrak{g})\otimes\mathfrak{g}$. Now (3.13) implies that $(\mathrm{d}\otimes\mathrm{d})(\sigma'-\sigma'') = 0$. Since $\mathrm{Ker}(\mathrm{d}) = \mathfrak{g}$, we get $\sigma' - \sigma'' \in \mathfrak{g}\otimes S^{n-1}(\mathfrak{g}) + S^{n-1}(\mathfrak{g})\otimes\mathfrak{g}$. Let $\sigma' - \sigma'' = \sigma'_0 + \sigma''_0$, $\sigma'_0 \in \mathfrak{g}\otimes S^{n-1}(\mathfrak{g})$, $\sigma''_0 \in S^{n-1}(\mathfrak{g})\otimes\mathfrak{g}$. Set $\sigma = \sigma' - \sigma'_0 = \sigma'' + \sigma''_0$. Then

$$(\mathrm{d}\otimes\mathrm{id})(\sigma) = (\mathrm{d}\otimes\mathrm{id})(\sigma') = -\alpha$$
$$\text{and}\ \ (\mathrm{id}\otimes\mathrm{d})(\sigma) = (\mathrm{id}\otimes\mathrm{d})(\sigma'') = -\beta.$$

So equations (3.13), (3.14) and (3.15) imply the existence of a solution σ of (3.11). Then $(\epsilon\otimes\epsilon)\circ\mathrm{d} = -\epsilon$, so

$$(\epsilon\otimes\mathrm{id})(\sigma) = -(\epsilon\otimes\epsilon)\circ(\mathrm{d}\otimes\mathrm{id})(\sigma) = (\epsilon\otimes\epsilon\otimes\mathrm{id})(\alpha) = 0,$$

and in the same way $(\mathrm{id}\otimes\epsilon)(\sigma) = 0$. So σ satisfies also equation (3.10).

The unicity of σ then follows from the fact that $\mathrm{Ker}(\mathrm{d}\otimes\mathrm{id})\cap\mathrm{Ker}(\mathrm{id}\otimes\mathrm{d}) = \mathfrak{g}\otimes\mathfrak{g} \subset S^2(\mathfrak{g}\oplus\mathfrak{g})$, so the intersection of $\mathrm{Ker}(\mathrm{d}\otimes\mathrm{id})\cap\mathrm{Ker}(\mathrm{id}\otimes\mathrm{d})$ with $S^n(\mathfrak{g}\oplus\mathfrak{g})$ is zero as $n \geq 3$. □

Proposition 3.4. *The elements* α *and* β *defined by (3.9) satisfy the identities (3.13), (3.14) and (3.15).*

Proof. Apply $\Delta\otimes\mathrm{id}\otimes\mathrm{id} - \mathrm{id}\otimes\Delta\otimes\mathrm{id}$ to the identity

$$(\Delta\otimes\mathrm{id})(\widetilde{\rho}_n) = \widetilde{\rho}_n^{1,3}\star\widetilde{\rho}_n^{2,3} + \alpha. \tag{3.16}$$

This yields an identity in $\mathcal{O}_{(G^*)^4}/\mathfrak{m}^{n+1}_{(G^*)^4}$. Its left side vanishes since $(\Delta\otimes\mathrm{id}-\mathrm{id}\otimes\Delta)\circ\Delta=0$. Using again (3.16), we get

$$0=(\widetilde{\rho}_n^{1,4}\star\widetilde{\rho}_n^{2,4}+\alpha^{1,2,4})\star\widetilde{\rho}_n^{3,4}+\alpha^{12,3,4}-\widetilde{\rho}_n^{1,4}\star(\widetilde{\rho}_n^{2,4}\star\widetilde{\rho}_n^{3,4}+\alpha^{2,3,4})-\alpha^{1,23,4}$$

(we use the notation $\gamma^{12,3,4}=(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\gamma)$, $\gamma^{1,23,4}=(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\gamma)$, etc.). Now identities (3.12) yield:

$$0=(\widetilde{\rho}_n^{1,4}\star\widetilde{\rho}_n^{2,4}\star\widetilde{\rho}_n^{3,4}+\alpha^{1,2,4})+\alpha^{12,3,4}-(\widetilde{\rho}_n^{1,4}\star\widetilde{\rho}_n^{2,4}\star\widetilde{\rho}_n^{3,4}+\alpha^{2,3,4})-\alpha^{1,23,4}$$

that is

$$\big(\mathrm{d}^{(2)}\otimes\mathrm{id}\big)(\alpha)=0.$$

Applying $\mathrm{id}\otimes\Delta\otimes\mathrm{id}-\mathrm{id}\otimes\mathrm{id}\otimes\Delta$ to the identity

$$(\mathrm{id}\otimes\Delta)(\widetilde{\rho}_n)=\widetilde{\rho}_n^{1,3}\star\widetilde{\rho}_n^{1,2}+\beta, \tag{3.17}$$

we get in the same way

$$\big(\mathrm{id}\otimes\mathrm{d}^{(2)}\big)(\beta)=0.$$

So α and β satisfy (3.14).

Apply now $\mathrm{id}\otimes\mathrm{id}\otimes\Delta$ to (3.16), $\Delta\otimes\mathrm{id}\otimes\mathrm{id}$ to (3.17), and substract the resulting equalities. Using again (3.16) and (3.17), we get

$$\begin{aligned}0=&(\widetilde{\rho}_n^{1,4}\star\widetilde{\rho}_n^{1,3}+\beta^{1,3,4})\star(\widetilde{\rho}_n^{2,4}\star\widetilde{\rho}_n^{2,3}+\beta^{2,3,4})+\alpha^{12,3,4}\\&-(\widetilde{\rho}_n^{1,4}\star\widetilde{\rho}_n^{2,4}+\alpha^{1,2,4})\star(\widetilde{\rho}_n^{1,3}\star\widetilde{\rho}_n^{2,3}+\alpha^{1,2,3})-\beta^{1,2,34}.\end{aligned}$$

Using again identities (3.12), and the fact that $\widetilde{\rho}_n^{1,3}\star\widetilde{\rho}_n^{2,4}=\widetilde{\rho}_n^{2,4}\star\widetilde{\rho}_n^{1,3}$, we get

$$\alpha^{12,3,4}-\alpha^{1,2,3}-\alpha^{1,2,4}=\beta^{1,2,34}-\beta^{1,2,3}-\beta^{1,2,4},$$

that is $(\mathrm{d}\otimes\mathrm{id}\otimes\mathrm{id})(\alpha)=(\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{d})(\beta)$. So α and β satisfy (3.13). To prove that they also satisfy (3.15), let us set

$$\psi=\widetilde{\rho}_n^{1,2}\star\widetilde{\rho}_n^{1,3}\star\widetilde{\rho}_n^{2,3}-\widetilde{\rho}_n^{2,3}\star\widetilde{\rho}_n^{1,3}\star\widetilde{\rho}_n^{1,2}$$

and let us prove:

Lemma 3.5. *We have $\psi=0$.*

Proof of Lemma. Since $\rho_n^{1,2}\star\rho_n^{1,3}\star\rho_n^{2,3}=\rho_n^{1,2}\star\rho_n^{12,3}=\rho_n^{21,3}\star\rho_n^{1,2}=\rho_n^{2,3}\star\rho_n^{1,3}\star\rho_n^{1,2}$, the class of ψ in $\mathcal{O}_{(G^*)^3}/\mathfrak{m}^n_{(G^*)^3}$ is zero, so $\psi\in\mathfrak{m}^n_{(G^*)^3}/\mathfrak{m}^{n+1}_{(G^*)^3}$. We identify ψ with an element of $S^n(\mathfrak{g}\oplus\mathfrak{g}\oplus\mathfrak{g})$. Then

$$\begin{aligned}
\psi^{12,3,4} =& \widetilde{\rho}_n^{12,3} \star \widetilde{\rho}_n^{12,4} \star \widetilde{\rho}_n^{3,4} - \widetilde{\rho}_n^{3,4} \star \widetilde{\rho}_n^{12,4} \star \widetilde{\rho}_n^{12,3} \\
=& (\widetilde{\rho}_n^{1,3} \star \widetilde{\rho}_n^{2,3} + \alpha^{1,2,3}) \star (\widetilde{\rho}_n^{1,4} \star \widetilde{\rho}_n^{2,4} + \alpha^{1,2,4}) \star \widetilde{\rho}_n^{3,4} \\
& - \widetilde{\rho}_n^{3,4} \star (\widetilde{\rho}_n^{1,4} \star \widetilde{\rho}_n^{2,4} + \alpha^{1,2,4}) \star (\widetilde{\rho}_n^{1,3} \star \widetilde{\rho}_n^{2,3} + \alpha^{1,2,3}) \\
=& \widetilde{\rho}_n^{1,3} \star \widetilde{\rho}_n^{2,3} \star \widetilde{\rho}_n^{1,4} \star \widetilde{\rho}_n^{2,4} \star \widetilde{\rho}_n^{3,4} - \widetilde{\rho}_n^{3,4} \star \widetilde{\rho}_n^{1,4} \star \widetilde{\rho}_n^{2,4} \star \widetilde{\rho}_n^{1,3} \star \widetilde{\rho}_n^{2,3} \\
& \qquad \text{by virtue of (3.12)} \\
=& \widetilde{\rho}_n^{1,3} \star \widetilde{\rho}_n^{1,4} \star (\widetilde{\rho}_n^{3,4} \star \widetilde{\rho}_n^{2,4} \star \widetilde{\rho}_n^{2,3} + \psi^{2,3,4}) \\
& + (-\widetilde{\rho}_n^{1,3} \star \widetilde{\rho}_n^{1,4} \star \widetilde{\rho}_n^{3,4} + \psi^{1,3,4}) \star \widetilde{\rho}_n^{2,4} \star \widetilde{\rho}_n^{2,3} \\
& \qquad \text{since } \widetilde{\rho}^{\,1,3} \star \widetilde{\rho}^{\,2,4} = \widetilde{\rho}^{\,2,4} \star \widetilde{\rho}^{\,1,3} \text{ and } \widetilde{\rho}^{\,1,4} \star \widetilde{\rho}^{\,2,3} = \widetilde{\rho}^{\,2,3} \star \widetilde{\rho}^{\,1,4} \\
=& \psi^{1,3,4} + \psi^{2,3,4} \qquad \text{by virtue of (3.12).}
\end{aligned}$$

We have also

$$\begin{aligned}
\psi^{1,23,4} =& (\widetilde{\rho}_n^{1,3} \star \widetilde{\rho}_n^{1,2} + \beta^{1,2,3}) \star \widetilde{\rho}_n^{1,4} \star (\widetilde{\rho}_n^{2,4} \star \widetilde{\rho}_n^{3,4} + \alpha^{2,3,4}) \\
& - (\widetilde{\rho}_n^{2,4} \star \widetilde{\rho}_n^{3,4} + \alpha^{2,3,4}) \star \widetilde{\rho}_n^{1,4} \star (\widetilde{\rho}_n^{1,3} \star \widetilde{\rho}_n^{1,2} + \beta^{1,2,3}) \\
=& \widetilde{\rho}_n^{1,3} \star \widetilde{\rho}_n^{1,2} \star \widetilde{\rho}_n^{1,4} \star \widetilde{\rho}_n^{2,4} \star \widetilde{\rho}_n^{3,4} - \widetilde{\rho}_n^{2,4} \star \widetilde{\rho}_n^{3,4} \star \widetilde{\rho}_n^{1,4} \star \widetilde{\rho}_n^{1,3} \star \widetilde{\rho}_n^{1,2} \\
=& \widetilde{\rho}_n^{1,3} \star (\widetilde{\rho}_n^{2,4} \star \widetilde{\rho}_n^{1,4} \star \widetilde{\rho}_n^{1,2} + \psi^{1,2,4}) \star \widetilde{\rho}_n^{3,4} \\
& + \widetilde{\rho}_n^{2,4} \star (-\widetilde{\rho}_n^{1,3} \star \widetilde{\rho}_n^{1,4} \star \widetilde{\rho}_n^{3,4} + \psi^{1,3,4}) \star \widetilde{\rho}_n^{1,2} \\
=& \psi^{1,2,4} + \psi^{2,3,4}.
\end{aligned}$$

In the same way, one proves that $\psi^{1,2,34} = \psi^{1,2,3} + \psi^{1,2,4}$. Therefore, we get

$$(\mathrm{d} \otimes \mathrm{id} \otimes \mathrm{id})(\psi) = (\mathrm{id} \otimes \mathrm{d} \otimes \mathrm{id})(\psi) = (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{d})(\psi) = 0,$$

so $\psi \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$. When $n > 3$, this implies $\psi = 0$. When $n = 3$, ψ is equal to $[r^{1,2}, r^{1,3}] + [r^{1,2}, r^{2,3}] + [r^{1,3}, r^{2,3}]$ and is also zero. □

End of proof of Proposition 3.4 Let us now prove that α and β satisfy equation (3.15). For this, we first prove:

Lemma 3.6. *We have*

$$\widetilde{\rho}_n^{1,2} \star \widetilde{\rho}_n^{12,3} = \widetilde{\rho}_n^{21,3} \star \widetilde{\rho}_n^{1,2} \tag{3.18}$$

(equality in $\mathcal{O}_{(G^*)^3} / \mathfrak{m}^{n+1}_{(G^*)^3}$*).*

Proof of Lemma. We will prove that if

$$f \in (\mathfrak{m}_{G^*} \bar{\otimes} \mathfrak{m}_{G^*}) \Big/ \big(\mathfrak{m}^{n+1}_{G^* \times G^*} \cap (\mathfrak{m}_{G^*} \bar{\otimes} \mathfrak{m}_{G^*})\big),$$

the equality

$$\widetilde{\rho}_n^{1,2} \star f^{12,3} = f^{21,3} \star \widetilde{\rho}_n^{1,2} \tag{3.19}$$

holds in $\mathcal{O}_{(G^*)^3}/\mathfrak{m}^{n+1}_{(G^*)^3}$. There exist element f_i', f_i'' of $\mathfrak{m}_{G^*}$, such that f is equal to the class of $\sum_i f_i' \otimes f_i''$. Then we have $\widetilde{\rho}_n^{1,2} \star (f_i')^{12} = (f_i')^{21} \star \widetilde{\rho}_n^{1,2}$ (equality in $\mathcal{O}_{(G^*)^3}/\mathfrak{m}^{n}_{(G^*)^3}$), because $\rho_n \in \mathrm{Lift}'_{\leq n}(\mathfrak{g}) = \mathrm{Lift}'_{\leq n}(\mathfrak{g})$ by virtue of Lemma 3.2. Tensoring this identity with $f_i'' \in \mathfrak{m}_{G^*}$, we get

$$\big(\widetilde{\rho}_n^{1,2} \star (f_i')^{12}\big)(f_i'')^3 = \big((f_i')^{21} \star \widetilde{\rho}_n^{1,2}\big)(f_i'')^3,$$

an equality in

$$\big(\mathcal{O}_{G^* \times G^*} \bar{\otimes} \mathfrak{m}_{G^*}\big) \Big/ \big(\mathfrak{m}^n_{G^* \times G^*} \bar{\otimes} \mathfrak{m}_{G^*}\big)$$

and therefore also in $\mathcal{O}_{(G^*)^3}/\mathfrak{m}^{n+1}_{(G^*)^3}$. Summing over all indices i, we get (3.19), which implies (3.18) by taking $f = \widetilde{\rho}_n$. □Plugging (3.16) into (3.18), we get

$$\widetilde{\rho}_n^{1,2} \star (\widetilde{\rho}_n^{1,3} \star \widetilde{\rho}_n^{2,3} + \alpha^{1,2,3}) = (\widetilde{\rho}_n^{2,3} \star \widetilde{\rho}_n^{1,3} + \alpha^{2,1,3}) \star \widetilde{\rho}_n^{1,2}.$$

Then (3.12) yields:

$$\alpha^{2,1,3} - \alpha^{1,2,3} = \widetilde{\rho}_n^{1,2} \star \widetilde{\rho}_n^{1,3} \star \widetilde{\rho}_n^{2,3} - \widetilde{\rho}_n^{2,3} \star \widetilde{\rho}_n^{1,3} \star \widetilde{\rho}_n^{1,2},$$

so Lemma 3.5 gives:

$$\alpha^{2,1,3} - \alpha^{1,2,3} = 0.$$

In the same way, we have

$$\widetilde{\rho}_n^{2,3} \star \widetilde{\rho}_n^{1,23} = \widetilde{\rho}_n^{1,32} \star \widetilde{\rho}_n^{2,3},$$

so by (3.17), we get

$$\widetilde{\rho}_n^{2,3} \star (\widetilde{\rho}_n^{1,3} \star \widetilde{\rho}_n^{1,2} + \beta^{1,2,3}) = (\widetilde{\rho}_n^{1,2} \star \widetilde{\rho}_n^{1,3} + \beta^{1,3,2}) \star \widetilde{\rho}_n^{2,3},$$

so $\beta^{1,2,3} - \beta^{1,3,2} = \widetilde{\rho}_n^{1,2} \star \widetilde{\rho}_n^{1,3} \star \widetilde{\rho}_n^{2,3} - \widetilde{\rho}_n^{2,3} \star \widetilde{\rho}_n^{1,3} \star \widetilde{\rho}_n^{1,2}$, so by Lemma 3.5, $\beta^{1,2,3} - \beta^{1,3,2} = 0$. So α and β satisfy equation (3.15). □

Let us now construct the map $\lambda_n : \mathrm{Lift}'_{\leq n}(\mathfrak{g}) \to \mathrm{Lift}'_{\leq n+1}(\mathfrak{g})$. Let $\rho \mapsto \widetilde{\rho}$ be any map

$$\Big\{ \rho \in \mathcal{O}_{G^* \times G^*}/\mathfrak{m}^n_{G^* \times G^*} \,\Big|\, (\epsilon \otimes \mathrm{id})(\rho) = (\mathrm{id} \otimes \epsilon)(\rho) = 0 \Big\}$$
$$\to \Big\{ \widetilde{\rho} \in \mathcal{O}_{G^* \times G^*}/\mathfrak{m}^{n+1}_{G^* \times G^*} \,\Big|\, (\epsilon \otimes \mathrm{id})(\widetilde{\rho}) = (\mathrm{id} \otimes \epsilon)(\widetilde{\rho}) = 0 \Big\},$$

which is a section of the canonical projection map (we may take $\rho \mapsto \widetilde{\rho}$ linear). If $\rho_n \in \mathrm{Lift}'_{\leq n}(\mathfrak{g})$, we define α and β by (3.9). Then Proposition 3.4 and Lemma 3.3 allow us to construct a unique element $\sigma \in \mathfrak{m}^n_{G^* \times G^*} / \mathfrak{m}^{n+1}_{G^* \times G^*}$, such that $\widetilde{\rho}_n + \sigma \in \mathrm{Lift}'_{\leq n+1}(\mathfrak{g})$. We then set

$$\lambda_n(\rho_n) = \widetilde{\rho}_n + \sigma.$$

This defines the desired map $\lambda_n : \mathrm{Lift}'_{\leq n}(\mathfrak{g}) \to \mathrm{Lift}'_{\leq n+1}(\mathfrak{g})$.

- c - Proof of Theorem 3.1

$r \in \mathfrak{g} \otimes \mathfrak{g}$ defines an element ρ_3 of $\mathrm{Lift}'_{\leq 3}(\mathfrak{g})$. Applying to it $\lambda_3, \lambda_4, \ldots,$ we define a sequence ρ_n of elements of $\mathrm{Lift}'_{\leq n}(\mathfrak{g})$, and therefore an element $\rho \in \mathrm{Lift}'(\mathfrak{g})$. According to Lemma 3.2, ρ is then an element of $\mathrm{Lift}(\mathfrak{g})$. □

§ 4 Construction of universal lifts (proof of Theorem 0.10)

Theorem 0.10 can be proved in the same way as its "non-universal" counterpart Theorem 0.8:

1. the unicity part is proved using the same argument;
2. the existence part can be proved either using the map $\mathrm{Quant}(\mathbb{K}) \to \mathrm{Lift}_{\mathrm{univ}}$, and the nonemptiness of $\mathrm{Quant}(\mathbb{K})$ (see [EK]); or it can be proved following the arguments of Section 4.

§ 5 Appendix: a commutative diagram related to QFSH algebras

The aim of this section is to prove the following lemma:

Lemma 5.1. *Let σ be an arbitrary element of $\mathfrak{m}_\hbar$ and $[\sigma]$ be its class in $\mathfrak{m}_\hbar/(\hbar\mathfrak{m}_\hbar + \mathfrak{m}_\hbar^2)$. Since $\mathfrak{m}_\hbar/(\hbar\mathfrak{m}_\hbar + \mathfrak{m}_\hbar^2)$ identifies with $\mathfrak{g}$, we have $[\sigma] \in \mathfrak{g}$. Since $\mathfrak{m}_\hbar \subset \hbar U_\hbar(\mathfrak{g})$, σ is an element of $\hbar U_\hbar(\mathfrak{g})$. Then $\left(\frac{\sigma}{\hbar} \bmod \hbar\right)$ is an element of $U(\mathfrak{g})$. We have the following identity in $U(\mathfrak{g})$:*

$$\left(\frac{\sigma}{\hbar} \bmod \hbar\right) = [\sigma].$$

This lemma clearly implies Lemma 1.2: if $\sigma = \sum_i \sigma_i^1 \otimes \sigma_i^2 \in \mathfrak{m}_\hbar \bar{\otimes} \mathfrak{m}_\hbar$ satisfies the hypothesis of Lemma 1.2, then $[\sigma] = \sum_i [\sigma_i^1] \otimes [\sigma_i^2] \in \mathfrak{g}^{\otimes 2}$ and

$$
\begin{aligned}
\frac{\sigma}{\hbar^2}\Big|_{U_\hbar^{\widehat{\otimes} 2}\to U^{\otimes 2}} &= \sum_i \left(\frac{\sigma_i^1}{\hbar}\otimes\frac{\sigma_i^2}{\hbar}\right)\Big|_{U_\hbar^{\widehat{\otimes} 2}\to U^{\otimes 2}} \\
&= \sum_i \frac{\sigma_i^1}{\hbar^2}\Big|_{U_\hbar\to U} \otimes \frac{\sigma_i^2}{\hbar^2}\Big|_{U_\hbar\to U} \\
&= \sum_i [\sigma_i^1]\otimes[\sigma_i^2] \qquad \text{(by Lemma 5.1)} \\
&= [\sigma].
\end{aligned}
$$

We use the notation $x|_{U_\hbar^{\widehat{\otimes} k}\to U^{\otimes k}}$ for $(x \bmod \hbar)$, when $x \in U_\hbar(\mathfrak{g})^{\widehat{\otimes} k}$.

More generally, in this section, we will consider a Lie bialgebra $(\mathfrak{g}, \delta)$ over a field $\mathbb{K}$, $(U_\hbar(\mathfrak{g}), \Delta)$ a quantization of $U(\mathfrak{g})$ and the subalgebra $U_\hbar(\mathfrak{g})' \subset U_\hbar(\mathfrak{g})$, where

$$U_\hbar(\mathfrak{g})' = \left\{ x \in U_\hbar(\mathfrak{g}) \,\middle|\, \forall n \in \mathbb{N},\ \delta^{(n)}(x) \in \hbar^n U_\hbar(\mathfrak{g})^{\widehat{\otimes} n} \right\}$$

(where $\delta^{(n)} = (\mathrm{id} - \eta\circ\varepsilon)^{\otimes n} \circ \Delta^{(n)}$). The definition of $U_\hbar(\mathfrak{g})'$ yields, when $n = 1$ or 2:

Lemma 5.2. *Let f be an element of $U_\hbar(\mathfrak{g})'$. We have:*

- $f - \varepsilon(f) \in \hbar U_\hbar(\mathfrak{g})$
- $\frac{f-\varepsilon(f)}{\hbar}\Big|_{U_\hbar\to U} \in \mathfrak{g} \subset U(\mathfrak{g})$

According to a theorem of Drinfeld (see [Dr] and also [Ga]) the quantized formal series Hopf algebra $U_\hbar(\mathfrak{g})'$ is a quantization of the function algebra $\mathcal{O}_{G^*}$. The projection $U_\hbar(\mathfrak{g})'/\hbar U_\hbar(\mathfrak{g})' \to \mathcal{O}_{G^*} \simeq U(\mathfrak{g}^*)^*$ may be described as follows:

Theorem 5.3. *For $f \in U_\hbar(\mathfrak{g})'$, let $f|_{\mathcal{O}_\hbar\to\mathcal{O}}$ be its class in $U_\hbar(\mathfrak{g})'/\hbar U_\hbar(\mathfrak{g})'$. There exists a unique Hopf pairing $U(\mathfrak{g}^*) \otimes \big(U_\hbar(\mathfrak{g})'/\hbar U_\hbar(\mathfrak{g})'\big) \to \mathbb{K}$, such that*

$$\forall \xi \in \mathfrak{g}^*, \forall f \in U_\hbar(\mathfrak{g})' \ \left\langle \xi, f|_{\mathcal{O}_\hbar\to\mathcal{O}} \right\rangle = \left\langle \xi, \frac{f-\varepsilon(f)}{\hbar}\Big|_{U_\hbar\to U} \right\rangle_{\mathfrak{g}\times\mathfrak{g}^*}.$$

This pairing induces an isomorphism $U_\hbar(\mathfrak{g})'/\hbar U_\hbar(\mathfrak{g})' \xrightarrow[\lambda]{\sim} U(\mathfrak{g}^*)^*$.

We can now reformulate Lemma 5.1 (in this section, we will use the notation $(H)_0$ for the maximal ideal $\mathrm{Ker}(\varepsilon)$ of a Hopf algebra H):

Proposition 5.4. *The following diagram commutes*

$$\begin{array}{ccc} \big(U_\hbar(\mathfrak{g})'\big)_0\big/\big(\hbar\big(U_\hbar(\mathfrak{g})'\big)_0+\big(U_\hbar(\mathfrak{g})'\big)_0^2\big) & \overset{(a)}{\overset{\sim}{\longrightarrow}} & \big(U(\mathfrak{g}^*)^*\big)_0\big/\big(U(\mathfrak{g}^*)^*\big)_0^2 \\ \text{(b)}\downarrow & & \downarrow\text{(d)} \\ \hbar U_\hbar(\mathfrak{g})/\hbar^2U_\hbar(\mathfrak{g}) & \overset{(c)}{\overset{\sim}{\longrightarrow}} & U(\mathfrak{g}) \underset{\mathrm{can}_\mathfrak{g}}{\hookleftarrow} \mathfrak{g} \end{array}$$

where $\mathrm{can}_\mathfrak{g}: \mathfrak{g} \hookrightarrow U(\mathfrak{g})$ *is the canonical injection and the other maps are given by:*

(a) *is the composed map*

$$\begin{aligned} &\big(U_\hbar(\mathfrak{g})'\big)_0\Big/\big(\hbar\big(U_\hbar(\mathfrak{g})'\big)_0+\big(U_\hbar(\mathfrak{g})'\big)_0^2\big) \\ &\quad\overset{\sim}{\longrightarrow}\Big(\big(U_\hbar(\mathfrak{g})'\big)_0/\hbar\big(U_\hbar(\mathfrak{g})'\big)_0\Big)\Big/\Big(\big(U_\hbar(\mathfrak{g})'\big)_0/\hbar\big(U_\hbar(\mathfrak{g})'\big)_0\Big)^2 \\ &\qquad\qquad\to \big(U(\mathfrak{g}^*)^*\big)_0\Big/\big(U(\mathfrak{g}^*)^*\big)_0^2 \end{aligned}$$

where the last map is induced by $\lambda: U_\hbar(\mathfrak{g})'/\hbar U_\hbar(\mathfrak{g})' \underset{\lambda}{\overset{\sim}{\longrightarrow}} U(\mathfrak{g}^*)^*$ *(see Theorem 5.3),*

(b) *is the quotient map of the injection :* $\big(U_\hbar(\mathfrak{g})'\big)_0 \hookrightarrow \hbar U_\hbar(\mathfrak{g})$ *with respect to the ideals* $\hbar\big(U_\hbar(\mathfrak{g})'\big)_0+\big(U_\hbar(\mathfrak{g})'\big)_0^2 \subset \big(U_\hbar(\mathfrak{g})'\big)_0$ *in the left hand side, and* $\hbar^2U_\hbar(\mathfrak{g}) \subset \hbar U_\hbar(\mathfrak{g})$ *in the right hand side,*

(c) *is the map*

$$\begin{gathered} \hbar U_\hbar(\mathfrak{g})/\hbar^2U_\hbar(\mathfrak{g}) \to U_\hbar(\mathfrak{g})/\hbar U_\hbar(\mathfrak{g}) \simeq U(\mathfrak{g}) \\ x \mapsto \hbar^{-1}x, \end{gathered}$$

(d) *is induced by the pairing* $\mathfrak{g}^*\otimes\big(U(\mathfrak{g}^*)^*\big)_0\big/\big(U(\mathfrak{g}^*)^*\big)_0^2 \to \mathbb{K}$*, quotient of the pairing* $\mathfrak{g}^*\otimes\big(U(\mathfrak{g}^*)^*\big)_0 \to \mathbb{K}$*, given by* $\xi\otimes T \mapsto \langle T, \mathrm{can}_{\mathfrak{g}^*}(\xi)\rangle$ *(*$\mathrm{can}_{\mathfrak{g}^*}: \mathfrak{g}^* \hookrightarrow U(\mathfrak{g}^*)$ *is the canonical injection).*

Proof. Let φ' be an element of $\big(U_\hbar(\mathfrak{g})'\big)_0$. Recall that $[\varphi']$ denotes its class in $\big(U_\hbar(\mathfrak{g})'\big)_0\big/\big(\hbar\big(U_\hbar(\mathfrak{g})'\big)_0+\big(U_\hbar(\mathfrak{g})'\big)_0^2\big)$. Thanks to Lemma 5.2, we have $\frac{\varphi'}{\hbar}\Big|_{U_\hbar\to U} \in \mathfrak{g}$ so (c) $\circ$ (b) $([\varphi']) = \mathrm{can}_\mathfrak{g}\left(\frac{\varphi'}{\hbar}\Big|_{U_\hbar\to U}\right)$. Thus we should show that

$$\text{(d)}\circ\text{(a)}\big([\varphi']\big) = \frac{\varphi'}{\hbar}\bigg|_{U_\hbar\to U}. \tag{5.20}$$

Let S be a supplementary of $\mathfrak{g}^*$ in $U(\mathfrak{g}^*)_0$ (e.g., the image of $\oplus_{i\geq 2}S^i(\mathfrak{g}^*)$ under the symmetrization map). For $x \in \mathfrak{g}$, let $f_x \in U(\mathfrak{g}^*)_0^*$ be defined by $f_x(\xi) = \langle \xi, x \rangle$ for $\xi \in \mathfrak{g}^*$ and $f_x|_S = 0$. Then the class $[f_x]_0$ of f_x in $U(\mathfrak{g}^*)_0^* / \big(U(\mathfrak{g}^*)_0^*\big)^2$ is independent of S. Moreover, it is clear that we have the identity (d)$(f_x) = x$ for any $x \in \mathfrak{g}$, so

$$\text{(d)} \left(\left[f_{\frac{\varphi'}{\hbar}|_{U_\hbar \to U}} \right]_0 \right) = \frac{\varphi'}{\hbar}\Big|_{U_\hbar \to U}.$$

So the identity (5.20) (and thus the proposition) will be true if

$$\text{(a)}([\varphi']) = \left[f_{\frac{\varphi'}{\hbar}|_{U_\hbar \to U}} \right]_0. \tag{5.21}$$

Both sides belong to $U(\mathfrak{g}^*)_0^* / \big(U(\mathfrak{g}^*)_0^*\big)^2 = \mathfrak{g}$, so it is enough to show that their pairings with any element $\xi \in \mathfrak{g}^*$ coincide. We have

$$\left\langle \xi, \left[f_{\frac{\varphi'}{\hbar}|_{U_\hbar \to U}} \right]_0 \right\rangle = \left\langle \xi, \frac{\varphi'}{\hbar}\Big|_{U_\hbar \to U} \right\rangle,$$

and

$$\big\langle \xi, \text{(a)}([\varphi']) \big\rangle = \left\langle \xi, \frac{\varphi'}{\hbar}\Big|_{U_\hbar \to U} \right\rangle \text{ by construction of (a),}$$

which proves identity (5.21). □

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