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The Reisner–Stanley system and equivariant cohomology for a class of wonderful varieties

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ABSTRACT

In this paper we are going to determine the equivariant cohomology of the wonderful compactification of a symmetric variety G/H and its equivariant ring of conditions under the assumption that $\mathrm{rk}(H) + \mathrm{rk}(G/H) = \mathrm{rk}(G)$.

1. Introduction

Let G be a connected affine algebraic group over the complex numbers $\mathbb{C}$. In [BDP] one introduces the notion of a regular embedding X of a G-homogeneous space and gives a recipe to compute the (rational) G-equivariant cohomology of X in terms of the (rational) G-equivariant cohomology of each of the G-orbits in X and of some combinatorial data associated to the incidence structure of orbit closures. A special case of this situation is given by the so-called symmetric varieties, as explained in [BDP].

In this paper we are going to apply this recipe in the case in which G is semisimple adjoint, and X is the wonderful compactification (see [DP1]) of a symmetric variety G/H, with H the subgroup of elements fixed by an involution σ, under the further assumption that the sum of the ranks of H and of G/H equals the rank of G. This is equivalent to the fact that there is a single orbit of maximal tori preserved by σ under conjugation by elements in H.

This is a rather strong assumption and indeed all such symmetric varieties can be written as a product of those associated to the following involutions (G, σ):

(1) $G = H \times H$, H is simple and σ is the involution given by $\sigma(h_1, h_2) = (h_2, h_1)$, so that the fixed subgroup is the diagonal and G/H is just H with the action of $H \times H$ given by left and right multiplication. One usually refers to this case as to the case of a group.

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(2) $G = PGL(2n)$, σ is the symplectic involution, so that $G/H = PGL(2n)/PSp(2n)$.
(3) G is of type D_n and the involution is of type DII (notations as in [H, Ch. X, §6]) so H is of type B_{n-1}.
(4) G is of type E_6 and the involution is of type EIV (notations as in [H, Ch. X, §6]) so that H is of type F_4.

The main reason why our assumption makes the computation possible is that, if we fix a maximal torus $T \subset G$ (which we are always going to assume to be σ-stable), then all the T fix-points in X lie in the unique closed orbit X_0. Using the localization theorem, this implies that $H_G(X, \mathbb{Q})$ embeds into $H_G(X_0, \mathbb{Q})$. Our main result identifies the image of this embedding in terms of invariants under the action of various Weyl groups.

These results are known in the case of a group, see [St2], and both the formulation of the result and its proof are strongly inspired by that paper, although our treatment will have those results as a consequence. A different approach to the study of the cohomology of wonderful compactifications has been developed in [DP2,LP] and, in the case treated here, in [BJ], using the rational equivariant Chow ring which in our situation turns out to be isomorphic to the rational equivariant cohomology ring.

More generally we are going to explain how to compute the equivariant cohomology of any so-called regular embedding of G/H and, as a consequence, we are going to determine, using the results of [DP3], the so-called equivariant ring of conditions $R_G(G/H)$ of our symmetric variety. This ring has been introduced in order to study some classical problems in enumerative geometry and has been recently used to give explicit formulae for intersection indices and Euler characteristic of hypersurfaces in G (see [K1,K2]).

The paper is organized as follows. Section 2 contains a brief digest of the main definitions and properties related to equivariant cohomology which we are going to use in our work.

In Section 3 we are going to give a few recollections on the construction and properties of the wonderful compactification X of G/H.

In Section 4 we are going to explain how to deduce from this the computation of the equivariant cohomology ring of X.

In Sections 5 we are going to work out in detail all the examples to which our general result applies.

Finally in Section 6 we are going to deduce rather easily, as a consequence of our previous work, the description of the equivariant ring of conditions of G/H.

2. A brief digest of equivariant cohomology

In this section we are going to recall the definition of equivariant cohomology and a few of its properties [Bo2].

Let K be a topological group. Consider the universal fibration $p: EK \to BK$, where EK is contractible with free K-action and $BK = EK/K$ is the classifying space of K.

Consider a K-space X. The equivariant cohomology of X with coefficients in a commutative ring A is the cohomology ring $H^*_K(X, A) := H^*(X_K, A)$ where $X_K := EK \times_K X$. We denote by $\pi : X_K \to BK$ the fibration over BK with fiber X.

In what follows K will always be a complex algebraic group and X an algebraic variety. A will be the field $\mathbb{Q}$ of rational numbers.

By functoriality, the projection to a point $q: X \to \mathrm{pt}$ induces on $H^*_K(X, \mathbb{Q})$ the structure of an algebra over $H^*_K(\mathrm{pt}, \mathbb{Q}) = H^*(BK, \mathbb{Q})$.

Let U be the unipotent radical of K. Using the fact that U is contractible, we deduce that $H^*_K(\mathrm{pt}) \simeq H^*_{K/U}(\mathrm{pt})$. Thus we can assume that K is reductive. If this is the case, choose a maximal torus T in K. Set $\mathfrak{t} = \mathrm{Lie}\, T$ and $W = N(T)/T$ the Weyl group, Λ the character group of T which we consider as a lattice in $\mathfrak{t}^*$, $\mathfrak{t}^*_{\mathbb{Q}}$ the rational vector space spanned by Λ. Then one knows [Bo1, §27], that $H^*_K(\mathrm{pt}, \mathbb{Q}) \simeq \mathbb{Q}[\mathfrak{t}]^W$, where $\mathbb{Q}[\mathfrak{t}] := S[\mathfrak{t}^*_{\mathbb{Q}}]$, the symmetric algebra of $\mathfrak{t}^*_{\mathbb{Q}}$ and the elements of $\mathfrak{t}^*_{\mathbb{Q}}$ have degree 2. This is well known to be a polynomial ring, since W is generated by reflections. In what follows we shall usually omit the subscript $\mathbb{Q}$ in $\mathfrak{t}^*_{\mathbb{Q}}$ and related spaces since the rational structure will be evident from the contest.

Following for example [GKM], we define a space X to be equivariantly formal if, setting $I \subset H^*_K(\mathrm{pt})$ equal to the ideal of elements of positive degree, then

(1) $H^*_K(X)$ is a free $H^*_K(\mathrm{pt})$-module of finite rank,
(2) $H^*(X) = H^*_K(X)/IH^*_K(X)$.

It is known, see [Bo2], that if X has only cohomology in even degrees, then X is equivariantly formal. Now if X is a smooth projective K-variety with finitely many K-orbits, one knows as a consequence of a result by Białynicki-Birula [Bia] (see also [BDP] for a discussion), that X can be paved by a finite number of locally closed affine spaces and hence has only cohomology in even degrees. It follows that X is equivariantly formal. As a consequence of these considerations, it will be clear that all the spaces considered in this paper are indeed equivariantly formal.

3. Recollections on the wonderful compactification

Here and in what follows, we consider a semisimple simply connected algebraic group $\tilde{G}$ together with an involution $\sigma : \tilde{G} \to \tilde{G}$ and we let $\tilde{H} \subset \tilde{G}$ be the subgroup of elements fixed by σ. We also denote by G the adjoint quotient of $\tilde{G}$. σ induces an involution on G and we denote by $H \subset G$ the subgroup of elements fixed by σ. H is the image of the normalizer of $\tilde{H}$ in $\tilde{G}$ under the quotient homomorphism.

We choose a maximal σ-split torus T_1 in G, that is a torus such that, for each $t \in T_1$, $\sigma(t) = t^{-1}$ and also T_1 is maximal with this property. We know that any maximal torus $T \supset T_1$ is automatically σ-stable. We choose one such maximal torus and we set $T_0 = T \cap H$.

Our basic assumption in this paper is:

Assumption 3.1. T_0 is a maximal torus in H.

We now set $\mathfrak{t} = \mathrm{Lie}\, T$, $\mathfrak{t}_\varepsilon = \mathrm{Lie}\, T_\varepsilon$, $\varepsilon = 0, 1$, so that $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1$. So clearly, our assumption means that $\mathrm{rk}(H) + \mathrm{rk}(G/H) = \mathrm{rk}(G)$.

We let $\Phi \subset \mathfrak{t}^*$ denote the root system. σ induces an involution on $\mathfrak{t}$ and its dual, preserving Φ and we may choose the set of positive roots Φ^+ in such a way that, writing $\Phi = \Phi_0 \cup \Phi_1$, where Φ_0 is the set of roots fixed by σ, $\Phi_1 = \Phi \setminus \Phi_0$, if $\alpha \in \Phi_1^+ = \Phi^+ \cap \Phi_1$, $\sigma(\alpha) \in -\Phi_1^+$. We write the corresponding set of simple roots Δ as $\Delta_0 \cup \Delta_1$, with $\Delta_\varepsilon = \Delta \cap \Phi_\varepsilon$, $\varepsilon = 0, 1$. We know that we can define an involution $\alpha \mapsto \hat{\alpha}$ on Δ_1 in such a way that if $\alpha \in \Delta_1$, $\sigma(\alpha) = -\hat{\alpha} + \gamma$, where γ is a linear combination of the roots in Δ_0 with negative coefficients. Let us now totally order

$$\Delta_1 = \{\alpha_1, \dots, \alpha_h, \alpha_{h+1}, \dots \alpha_k, \alpha_{k+1}, \dots, \alpha_s\}$$

in such a way that if $1 \leqslant j \leqslant h$, $\hat{\alpha}_j = \alpha_{k+j}$, while if $h + 1 \leqslant j \leqslant k$, $\hat{\alpha}_j = \alpha_j$. Moreover $\Delta_0 = \{\alpha_{s+1}, \dots, \alpha_r\}$. Finally for each $j = 1, \dots, k$, we set $\overline{\alpha}_j = \alpha_j - \sigma(\alpha_j)$ and set $\overline{\Delta} = \{\overline{\alpha}_1, \dots, \overline{\alpha}_k\}$ ($\overline{\alpha}_j$ is a simple restricted root if $1 \leqslant j \leqslant h$ and is twice a simple restricted root if $h < j \leqslant k$).

Consequently, let us divide the set of fundamental weights Ω as $\Omega_0 \cup \Omega_1$. If $1 \leqslant j \leqslant h$, then $\sigma(\omega_j) = -\omega_{k+j}$, while if $h + 1 \leqslant j \leqslant k$, $\sigma(\omega_j) = -\omega_j$. Clearly the elements $\omega_j - \sigma(\omega_j)$, for $1 \leqslant j \leqslant k$, form a basis for $\mathfrak{t}_1^*$.

Given any subset $\Gamma \subset \{1, \dots, k\}$, set $\Delta_{1,\Gamma} = \{\alpha_j \in \Delta_1 \mid j, k - j \notin \Gamma\}$ and $\Delta_\Gamma = \Delta_0 \cup \Delta_{1,\Gamma}$. Δ_Γ is the basis of a root system Φ_Γ which is stable under σ. Correspondingly, we get a Levi factor $\tilde{L}_\Gamma$ stable under σ and a parabolic subgroup P_Γ. Set L_Γ equal to the adjoint quotient of $\tilde{L}_\Gamma$. σ induces an involution on L_Γ and we set $H_\Gamma \subset L_\Gamma$ equal to the subgroup of elements fixed by σ.

With these notations in place, we recall some facts about the structure of X and its G-orbits.

As we know, see [DP1], there is a one to one correspondence between the subsets $\Gamma \subset \{1, \dots, k\}$ and the orbits in X. Let us denote by $\mathcal{O}_\Gamma$ the orbit associated to such a subset Γ and by D_Γ its closure. Then the following facts hold:

(1) $\mathcal{O}_\Gamma$ has codimension $|\Gamma|$ in X. In particular the open orbit $G/H = \mathcal{O}_\emptyset$.
(2) $D_\Gamma \subset D_{\Gamma'}$ if and only if $\Gamma' \subset \Gamma$. In particular $\mathcal{O}_{\{1,\dots,k\}}$ is the unique closed orbit in X.
(3) D_Γ is the transversal intersection of the divisors $D_{\{i\}}$, for $i \in \Gamma$.
(4) Consider the quotient homomorphism $p_\Gamma : P_\Gamma \to L_\Gamma$ and set $\tilde{H}_\Gamma := p_\Gamma^{-1}(H_\Gamma)$. Then, as a G-homogeneous space,

$$\mathcal{O}_\Gamma \simeq G/\tilde{H}_\Gamma.$$

(5) There is G-equivariant fibration

$$\mathcal{O}_\Gamma \to G/P_\Gamma$$

whose fiber is isomorphic to L_Γ/H_Γ.

Our main observation is:

Lemma 3.2. *For each subset $\Gamma \subset \{1,\dots,k\}$, the connected component of the identity S_Γ of the intersection $\tilde{H}_\Gamma \cap T$ is a maximal torus in $\tilde{H}_\Gamma$ which contains T_0.*

Proof. Set $\tilde{H}'_\Gamma$ equal to the intersection $\tilde{H}_\Gamma \cap \tilde{L}_\Gamma$. It clearly suffices to show that, if we consider the Lie algebra $\tilde{\mathfrak{l}}_\Gamma$ of $\tilde{L}_\Gamma$, which is a Levi subalgebra of $\operatorname{Lie} P_\Gamma$, and $\tilde{\mathfrak{h}}'_\Gamma$ of $\tilde{H}'_\Gamma$, then $\tilde{\mathfrak{h}}_\Gamma \cap \mathfrak{t}$ is a Cartan subalgebra in $\tilde{\mathfrak{l}}_\Gamma$ containing $\mathfrak{t}_0$.

Let $\mathfrak{l}_\Gamma$ be the Lie algebra of L_Γ. We have a decomposition

$$\tilde{\mathfrak{l}}_\Gamma = \mathfrak{l}_\Gamma \oplus \mathfrak{t}_\Gamma$$

where $\mathfrak{t}_\Gamma$ is the central subalgebra consisting of those elements in $\mathfrak{t}$ on which the simple roots in Δ_Γ vanish. Since $\mathfrak{t}_\Gamma$ is σ-stable, we can write $\mathfrak{t}_0 = \mathfrak{t}'_0 \oplus \mathfrak{t}''_0$ with $\mathfrak{t}'_0 = \mathfrak{t}_0 \cap \mathfrak{l}_\Gamma$ and $\mathfrak{t}''_0 = \mathfrak{t}_0 \cap \mathfrak{t}_\Gamma$.

Now $\mathfrak{h}_\Gamma := \operatorname{Lie} H_\Gamma$ is the subalgebra of $\mathfrak{l}_\Gamma$ of elements fixed by σ. We deduce that

$$\tilde{\mathfrak{h}}'_\Gamma = \mathfrak{h}_\Gamma \oplus \mathfrak{t}_\Gamma$$

and that a Cartan subalgebra in $\tilde{\mathfrak{h}}'_\Gamma$ is given by the direct sum of $\mathfrak{t}_\Gamma$ and a Cartan subalgebra in $\mathfrak{h}_\Gamma$.

We claim that $\mathfrak{t}'_0$ is a Cartan subalgebra in $\mathfrak{h}_\Gamma$. Otherwise, taking a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{h}_\Gamma$, $\dim \mathfrak{c} > \dim \mathfrak{t}'_0$, so that $\mathfrak{c} \oplus \mathfrak{t}''_0$ would be a toral subalgebra in $\mathfrak{h}$ of dimension larger than that of $\mathfrak{t}_0$, contradicting our Assumption 3.1.

It then follows that $\mathfrak{t}'_0 \oplus \mathfrak{t}_\Gamma$, which contains $\mathfrak{t}_0$, is a Cartan subalgebra in $\tilde{\mathfrak{h}}'_\Gamma$, proving both our claims. $\square$

The above lemma gives us a way of computing $H^*_G(\mathcal{O}_\Gamma)$. First of all notice that

$$H^*_G(\mathcal{O}_\Gamma) \simeq H^*_G(G/\tilde{H}_\Gamma) \simeq H^*_{\tilde{H}_\Gamma}(pt). \tag{1}$$

Now by Lemma 3.2, S_Γ is a maximal torus in $\tilde{H}_\Gamma$, so we get that

$$H^*_G(\mathcal{O}_\Gamma) \simeq H^*_{\tilde{H}_\Gamma}(pt) \simeq \big(H^*_{S_\Gamma}(pt)\big)^{W_\Gamma} \simeq S\big[\mathfrak{s}^*_\Gamma\big]^{W_\Gamma} \tag{2}$$

where $W_\Gamma = N_{\tilde{H}_\Gamma}(S_\Gamma)/S_\Gamma$ and $\mathfrak{s}_\Gamma = \operatorname{Lie} S_\Gamma$.

By what we have seen in Lemma 3.2, we get that $\mathfrak{s}^*_\Gamma$ decomposes as the direct sum of $\mathfrak{t}^*_0$ and the space $\mathfrak{s}_{1,\Gamma}$ quotient of $\mathfrak{t}^*_1$ modulo the subspace having as basis the elements $\overline{\alpha}_j$, with $1 \leqslant j \leqslant k$ and $j \notin \Gamma$. Notice that the images of the elements $\overline{\alpha}_j$ with $j \in \Gamma$ are a basis of $\mathfrak{s}_{1,\Gamma}$.

As for the group W_Γ, notice that $W_\Gamma \simeq N_{H_\Gamma}(S'_\Gamma)/S'_\Gamma$, S'_Γ being the image of S_Γ in L_Γ, in particular it acts trivially on $\mathfrak{s}_{1,\Gamma}$.

Let us now write $\mathfrak{t}^* = \mathfrak{t}_0^* \oplus \mathfrak{t}_1^*$, so that we have $S[\mathfrak{t}^*] = S[\mathfrak{t}_0^*] \otimes S[\mathfrak{t}_1^*]$. Recall that $\overline{\Delta}$ is a basis of $\mathfrak{t}_1^*$. We thus can identify $S[\mathfrak{t}_1^*]$ with $\mathbb{Q}[\overline{\alpha}_1, \ldots, \overline{\alpha}_k]$. In conclusion we get the identification

$$S\big[\mathfrak{t}^*\big] \simeq S\big[\mathfrak{t}_0^*\big][\overline{\alpha}_1, \ldots, \overline{\alpha}_k]. \tag{3}$$

Similarly, if for any $\Gamma \subset \{1, \ldots, k\}$ we set I_Γ equal to the ideal in $S[\mathfrak{t}^*]$ generated by the $\overline{\alpha}_j$ with $j \notin \Gamma$, we get

$$S\big[\mathfrak{s}_\Gamma^*\big] \simeq S\big[\mathfrak{t}^*\big]/I_\Gamma \simeq S\big[\mathfrak{t}_0^*\big][\overline{\alpha}_j,]_{j\in\Gamma}.$$

Going back to the cohomology of $\mathcal{O}_\Gamma$, we then deduce the following:

Proposition 3.3. *For any $\Gamma \subset \{1, \ldots, k\}$, we have that*

$$H_G(\mathcal{O}_\Gamma, \mathbb{Q}) \simeq S\big[\mathfrak{t}_0^*\big]^{W_\Gamma}[\overline{\alpha}_j,]_{j\in\Gamma}.$$

Remark 3.4. Notice that, by our description of the group $\tilde{H}_\Gamma$, it follows immediately that, if $\Gamma' \supset \Gamma$, then $W_{\Gamma'}$ is a subgroup of W_Γ. In particular we get an injective homomorphism

$$\mu^\Gamma_{\Gamma'} : S\big[\mathfrak{t}_0^*\big]^{W_\Gamma} \to S\big[\mathfrak{t}_0^*\big]^{W_{\Gamma'}}.$$

We shall denote by the same letter its extension

$$\mu^\Gamma_{\Gamma'} : S\big[\mathfrak{t}_0^*\big]^{W_\Gamma}[\overline{\alpha}_j,]_{j\in\Gamma} \to S\big[\mathfrak{t}_0^*\big]^{W_{\Gamma'}}[\overline{\alpha}_j,]_{j\in\Gamma} \simeq S\big[\mathfrak{t}_0^*\big]^{W_{\Gamma'}}[\overline{\alpha}_j,]_{j\in\Gamma'}/(\overline{\alpha}_r)_{r\in\Gamma'\setminus\Gamma}.$$

Notice that if $\Gamma'' \supset \Gamma' \supset \Gamma$, then

$$\mu^\Gamma_{\Gamma''} = \mu^{\Gamma'}_{\Gamma''} \circ \mu^\Gamma_{\Gamma'}. \tag{4}$$

4. The R–S system associated to the wonderful embedding

Since we are going to apply the results of [BDP] only in the case in which the relevant regular fan is the positive quadrant $C = \{(a_1, \ldots, a_k) \in \mathbb{R}^k \mid a_i \geqslant 0,\ \forall i = 1, \ldots, k\}$, we shall directly assume that we are in this case and hence we shall not recall the definition of a regular fan here.

Definition 1. A Reisner–Stanley (R–S)-system $\mathfrak{A}$ on C is the following set of data:

(1) For any subset $\Gamma \subset \{1, \ldots, k\}$ or equivalently for the face C_Γ defined by $C_\Gamma = \{(a_1, \ldots, a_k) \in C \mid a_i = 0,\ \forall i \notin \Gamma\}$, a graded commutative $\mathbb{Q}$-algebra with identity, A_Γ, together with a regular sequence of homogeneous elements $\underline{x}^\Gamma = x^\Gamma_{i_1}, \ldots, x^\Gamma_{i_h}$.

(2) For all $j \in \Gamma$, setting $\Gamma_j := \Gamma - \{j\}$, a homomorphism of graded algebras

$$\phi^{\Gamma_j}_\Gamma : A_{\Gamma_j} \to A_\Gamma/\big(x^\Gamma_j\big)$$

such that

$$\phi^{\Gamma_j}_\Gamma\big(x^{\Gamma_j}_i\big) \equiv x^\Gamma_i \mod \big(x^\Gamma_j\big), \quad \forall i \in \Gamma_j.$$

Given such an (R–S)-system $\mathfrak{A}$, we associate to it an algebra A, called the (R–S)-algebra of $\mathfrak{A}$. This algebra is defined as the subalgebra $A \subset \bigoplus_\Gamma A_\Gamma$ consisting of the sequences (a_Γ), $a_\Gamma \in A_\Gamma$ such that

$$\phi_\Gamma^{\Gamma_j}(a_{\Gamma_j}) \equiv a_\Gamma \mod \left(x_j^\Gamma\right)$$

for all $\Gamma \subset \{1,\dots,k\}$ and for all $j \in \Gamma$.

We now want to recall how one can associate such an (R–S)-system to the wonderful compactification X.

One knows, [DP1], that every line bundle on X admits a canonical $\tilde{G}$-linearization. Denoting by X_0 the unique closed orbit $\mathcal{O}_{\{1,\dots,k\}}$ in X, we also have that the homomorphism

$$i^*: H_G^2(X) \to H_G^2(X_0)$$

induced by inclusion is injective. Since $H_G^2(X_0)$ is a sublattice of the weight lattice Λ, we get an inclusion of $H_G^2(X)$ into Λ. Under this inclusion, one knows that the equivariant Chern class of the divisor $\mathcal{O}(D_{\{i\}})$ is given by $\overline{\alpha}_i$. Using this and following the recipe given in [BDP], we get that if we identify by Proposition 3.3 $H_G(\mathcal{O}_\Gamma,\mathbb{Q})$ with $S[\mathfrak{t}_0^*]^{W_\Gamma}[\overline{\alpha}_j,]_{j\in\Gamma}$, we obtain the following:

Proposition 4.1. *The system $\mathfrak{A}$ on C given by*

(1) *for any subset $\Gamma \subset \{1,\dots,k\}$, the algebra $A_\Gamma := S[\mathfrak{t}_0^*]^{W_\Gamma}[\overline{\alpha}_j]_{j\in\Gamma}$, with regular sequence $\underline{\bar{x}}^\Gamma = (\overline{\alpha}_j)_{j\in\Gamma}$,*
(2) *for all $j \in \Gamma$, setting $\Gamma_j := \Gamma - \{j\}$, the homomorphism of graded algebras*

$$\mu_\Gamma^{\Gamma_j}: A_{\Gamma_j} \to A_\Gamma/(\overline{\alpha}_j),$$

is a Stanley–Reisner system whose associated (R–S)-algebra A is the equivalent cohomology algebra $H_G^*(X)$.

Let us now write the polynomial $S[\mathfrak{t}^*]$ as $S[\mathfrak{t}_0^*][\overline{\alpha}_1,\dots,\overline{\alpha}_k]$. Accordingly, we write an element $F \in S[\mathfrak{t}^*]$ as a polynomial

$$F(\overline{\alpha}_1,\dots,\overline{\alpha}_k) = \sum_{I=(i_1,\dots,i_k)} F_I m_I$$

with $F_I \in S[\mathfrak{t}_0^*]$ and $m_I = \overline{\alpha}_1^{i_1}\cdots\overline{\alpha}_k^{i_k}$. Given a sequence $I = (i_1,\dots,i_k)$ we take its support, $\Gamma_I = \{j \mid i_j \neq 0\}$.

We now define the subalgebra $B \subset S[\mathfrak{t}^*]$ as the span of all elements $F_I m_I$ with the property that $F_I \in S[\mathfrak{t}_0^*]^{W_{\Gamma_I}}$. Remark that the fact that B is a subalgebra follows immediately from the fact that if we consider two monomials m_I, m_J, then $m_I m_J = m_{I+J}$. Since $\Gamma_{I+J} = \Gamma_I \cup \Gamma_J$ then $S[\mathfrak{t}_0^*]^{W_{\Gamma_I \cup \Gamma_J}} \supset S[\mathfrak{t}_0^*]^{W_{\Gamma_I}}, S[\mathfrak{t}_0^*]^{W_{\Gamma_J}}$.

We can finally state the following:

Theorem 4.2. *As a graded algebra, the algebra $H_G^*(X)$ is isomorphic to the subalgebra $B \subset S[\mathfrak{t}^*]$.*

Proof. Let us first define a homomorphism $\psi: A \to S[\mathfrak{t}_0^*][\overline{\alpha}_1,\dots,\overline{\alpha}_k]$ by projecting $A \subset \bigoplus_\Gamma A_\Gamma$ to the summand $A_{\{1,\dots,k\}} = S[\mathfrak{t}_0^*]^{W_{\{1,\dots,k\}}}[\overline{\alpha}_1,\dots,\overline{\alpha}_k] \subset S[\mathfrak{t}_0^*][\overline{\alpha}_1,\dots,\overline{\alpha}_k]$.

Let us show that p is injective. For this, let $a = (a_\Gamma) \in A$ with $a_\Gamma \in A_\Gamma$. Assume $p(a) = 0$, that is $a_{\{1,\dots,k\}} = 0$. Notice that by the identity (4), we have

$$\mu_\Gamma^{\{1,\dots,k\}}(a_\Gamma) \equiv a_{\{1,\dots,k\}} \mod I_\Gamma = 0. \tag{5}$$

Since $\mu_{\Gamma}^{\{1,\ldots,k\}}$ is injective, it follows that $a_\Gamma = 0$ for each Γ and p is injective. Let us now show that p maps A onto B. Let us consider $a = (a_\Gamma) \in A$. Write $F = p(a)$ as

$$F(\overline{\alpha}_1, \ldots, \overline{\alpha}_k) = \sum_{I=(i_1,\ldots,i_k)} F_I m_I.$$

Then from the equality (5), we get that, for each $\Gamma \subset \{1, \ldots, k\}$,

$$a_\Gamma = \sum_{I \mid \Gamma_I \subseteq \Gamma} F_I m_I.$$

Since $a_\Gamma \in S[\mathfrak{t}_0^*]^{W_{\Gamma_I}}[\overline{\alpha}_j]_{j\in\Gamma}$, we deduce that $F_I \in S[\mathfrak{t}_0^*]^{W_{\Gamma_I}}$ for each I and $p(a) \in B$.

On the other hand given

$$F(\overline{\alpha}_1, \ldots, \overline{\alpha}_k) = \sum_{I=(i_1,\ldots,i_k)} F_I m_I \in B,$$

we define for each Γ, $a_\Gamma = \sum_{I \mid \Gamma_I \subseteq \Gamma} F_I m_I$. We then see immediately that $a = (a_\Gamma) \in A$ and $p(a) = F(\overline{\alpha}_1, \ldots, \overline{\alpha}_k)$. So everything follows. $\square$

5. Examples

5.1. The group compactification

In this case we have $G = H \times H$ and the involution $\sigma : H \times H \to H \times H$ is given by $\sigma((h_1, h_2)) = (h_2, h_1)$ for $h_1, h_2 \in H$. It follows that G/H is just H considered as an $H \times H$-homogeneous space with respect to the action given by left and right multiplication. This case has been treated extensively already in [St2] (see also [St1,U]). Here, for completeness, we recall the final result, referring the reader to the paper [St2] for details. Fix a Cartan subalgebra $\mathfrak{h}$ in the Lie algebra of H. Let $R \subset \mathfrak{h}^*$ be the root system and W be the Weyl group. Choose a set of simple roots $\{\alpha_1, \ldots, \alpha_\ell\}$. If we consider $\mathfrak{h} \oplus \mathfrak{h}$, then a basis of linear functions on this space is given by $x_1, \ldots, x_\ell, y_1, \ldots, y_\ell$, with $x_i((a_1, a_2)) = \alpha_i(a_1)$ and $y_i((a_1, a_2)) = \alpha_i(a_2)$. For any subset $\Gamma \subset \{1, \ldots, \ell\}$ consider the subgroup W_Γ generated by the simple reflections s_i, $i \notin \Gamma$. Set S^Γ equal to the subring of $\mathbb{Q}[x_1, \ldots, x_\ell]$ of polynomials invariant under W_Γ. Finally, given a monomial $m_I = y_1^{i_1} \cdots y_\ell^{i_\ell}$, define the support of $I = (i_1, \ldots, i_\ell)$ as $\Gamma_I = \{j \mid i_j \neq 0\}$. One then gets:

Theorem 5.1. *The ring $H_G^*(X)$ is isomorphic to the subring*

$$B \subset \mathbb{Q}[x_1, \ldots, x_\ell, y_1, \ldots, y_\ell]$$

which is the linear span of the polynomials

$$q(x_1, \ldots, x_\ell) y_1^{i_1} \cdots y_\ell^{i_\ell}$$

with $q(x_1, \ldots, x_\ell) \in S^{\Gamma_I}$.

5.2. The symplectic involution and the corresponding wonderful embedding

Our next example is that of the symplectic involution. Let us recall a few facts about it. Consider the 2×2 matrix

$$J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and consider the $2n \times 2n$ matrix

$$J_n = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_1 \end{pmatrix}.$$

We define the symplectic involution σ on $\tilde{G} = SL(2n)$ by setting, for each $A \in \tilde{G}$,

$$\sigma(A) = J_n\left({}^t A^{-1}\right) J_n^{-1}.$$

Notice that for $n = 1$, σ is the identity (so this allows us, from now on, to assume $n \geqslant 2$) while in general the subgroup $\tilde{H}$ of fix-points equals the symplectic group $Sp(2n)$ of isometries with respect to the antisymmetric bilinear form defined by J_n.

The maximal torus $\tilde{T}$ of diagonal matrices is stable under σ, so that we get an induced involution on its Lie algebra and on the root system Φ which we are going to denote with the same letter. Let us consider the set of simple roots Δ associated to the choice of the Borel subgroup $\tilde{B} \supset \tilde{T}$ of upper triangular matrices. We get that

$$\sigma(\alpha_i) = \alpha_i \quad \text{if } i \text{ is odd}, \qquad \sigma(\alpha_i) = -\alpha_i - \alpha_{i-1} - \alpha_{i+1} \quad \text{if } i \text{ is even}.$$

We set $\Delta_{odd} = \{\alpha_i \mid i \text{ is odd}\}$, $\Delta_{even} = \Delta \setminus \Delta_{odd}$.

$\tilde{T}_1$ is a maximal σ-split torus and the corresponding restricted root system is of type A_{n-1}. Notice that since $\mathrm{rk}(Sp(2n)) = n$, our Assumption 3.1 is satisfied. We can choose as simple restricted roots the restrictions $\overline{\Delta}_{even} = \{\overline{\alpha}_2, \ldots, \overline{\alpha}_{2n-2}\}$ of the simple roots moved by σ. Finally for each $1 \leqslant i \leqslant n-1$ we set $\beta_i := 2\alpha_{2i} + \alpha_{2i-1} + \alpha_{2i+1}$. Remark that $\sigma(\beta_i) = -\beta_i$.

Let us now consider $G = PGL(2n)$. σ induces an involution of G and we let H denote the fix-points subgroup in G which is just the image of the normalizer of $Sp(2n)$ in G. We set T (resp. B) equal to the image of $\tilde{T}$ (resp. $\tilde{B}$) in G. We denote by $P \supset B$ the parabolic subgroup associated to the simple roots fixed by σ. Thus G/P is the variety of partial flags

$$\left(V_2 \subset V_4 \subset \cdots \subset V_{2n-2} \subset \mathbb{C}^{2n}\right), \quad \text{with } \dim V_j = j.$$

The already recalled structure of the wonderful compactification X of G/H in this case gives that the divisor with normal crossings $D = X - G/H$ has $n-1$ irreducible components $D_1, \ldots, D_{n-1}$.

For each subset $\Gamma \subset \{1, \ldots, n-1\}$, denote by $\mathcal{O}_\Gamma$ the unique G-orbit whose closure is

$$D_\Gamma = \bigcap_{j \in \Gamma} D_j.$$

In particular the orbit corresponding to $\{1, \ldots, n-1\}$ is the unique closed orbit in X, which is isomorphic to the variety G/P considered above, and we have that $\Gamma \subset \Gamma'$ if and only if $\overline{\mathcal{O}_\Gamma} \supset \mathcal{O}_{\Gamma'}$.

Let us start describing line bundles on X, following [DP1]. Recall that every line bundle on X admits a canonical $\tilde{G}$-linearization. This implies that if $\mathrm{Pic}(X)$ is the Picard group of X, then, taking the equivariant Chern classes, we get an isomorphism between $\mathrm{Pic}(X)$ and $H^2_G(X, \mathbb{Z})$. Denote by Λ the weight lattice, i.e. the character group of the maximal torus $\tilde{T} \subset \tilde{G}$ and set

$$M = \left\{\lambda \in \Lambda \;\middle|\; \sigma(\lambda) = -\lambda\right\}.$$

Notice that M is just the sublattice spanned by the fundamental weights $\{\omega_2, \ldots, \omega_{2n-2}\}$. We have a commutative diagram

$$\begin{tikzcd} \mathrm{Pic}(X) \arrow[rr, "h^*"] \arrow[dr] & & \mathrm{Pic}(G/P) \arrow[dl] \\ & M & \end{tikzcd} \tag{6}$$

where h^* is induced by inclusion. By [DP1] we know that the homomorphism $\mathrm{Pic}(X) \to M$ is injective. Let us explain why it is also surjective.

We associate to a fundamental weight $\omega_{2i} \in M$ a line bundle on X as follows. First we consider the fundamental representation $V_{\omega_i} = \bigwedge^i V_{\omega_1}$ where V_{ω_1} is the tautological $2n$-dimensional representation of $SL(2n)$. We then remark that $V_{\omega_{2n-2}}$, which is isomorphic to $\bigwedge^2 V^*_{\omega_1}$, contains a vector invariant under $Sp(2n)$, namely the antisymmetric bilinear form Ω whose matrix is J_n. Thus $V_{\omega_{2n-2i}} \simeq \bigwedge^{2i} V_{\omega_{2n-2}}$ contains the invariant vector $\bigwedge^i \Omega$.

It then follows that, if we set X_i equal to the closure of the G-orbit of the class of $\bigwedge^i \Omega$ in $\mathbb{P}(V_{\omega_{2n-2i}})$, we get an embedding of G/H together with a G-equivariant morphism $X \to X_i$ (see [DP1] for details). The G-linearized line bundle $L_{\omega_{2i}}$ is the pull back of the ample generator of the Picard group of $\mathbb{P}(V_{\omega_{2n-2i}})$ with its canonical linearization.

With this in mind, given $\mu \in M$, we denote by L_μ the corresponding G-linearized line bundle.

We are now going to recall the geometric structure of each orbit $\mathcal{O}_\Gamma$. First of all we identify the two sets $\{\beta_1, \ldots, \beta_{n-1}\}$ and $\{1, \ldots, n-1\}$ in such a way that, for the irreducible divisor D_i, one has that the line bundle associated to $\mathcal{O}(D_i)$ is L_{β_i}.

Now notice that for any $\Gamma \subset \Delta_{even}$ (or equivalently $\overline{\Delta}_{even}$), we may consider the subset $\Delta_\Gamma = \Delta \setminus \Gamma$, the corresponding Levi factor $L_\Gamma \subset G$ and parabolic subgroup P_Γ generated by L_Γ and B. Notice that clearly L_Γ is preserved by σ.

Now let us write Δ_Γ as a union of the following disjoint segments:

$$\{\alpha_1, \ldots, \alpha_{2h_1-1}\} \cup \{\alpha_{2h_1+1}, \ldots, \alpha_{2h_2-1}\} \cup \cdots \cup \{\alpha_{2h_{s-1}+1}, \ldots, \alpha_{2n-1}\}.$$

We get that L_Γ is isogenous to the product $\prod_{r=1}^s SL(2(h_r - h_{r-1})) \times T_\Gamma$, where $h_0 = 0$, $h_s = n$ and $T_\Gamma \subset T$ is the subtorus whose Lie algebra is the subspace of the Lie algebra of T where all the simple roots in Δ_Γ vanish. It follows that, if R_Γ is the solvable radical of P_Γ, then the quotient $P_\Gamma/R_\Gamma = \prod_{r=1}^s PGL(2(h_r - h_{r-1}))$ and σ induces an involution on this quotient which coincides with the symplectic involution on each factor. Set $\overline{H}_\Gamma \subset P_\Gamma/R_\Gamma$ equal to subgroup of elements fixed by this involution and define H_Γ as $\pi_\Gamma^{-1}(\overline{H}_\Gamma) \subset P_\Gamma$, $\pi_\Gamma : P_\Gamma \to P_\Gamma/R_\Gamma$ being the quotient homomorphism. Thus to every subset Γ of Δ_{even} we have associated a subgroup $H_\Gamma \subset P_\Gamma$.

The following proposition is immediate from the results in [DP1]:

Proposition 5.2. *As a G-variety, the orbit $\mathcal{O}_\Gamma$ is isomorphic to G/H_Γ.*

Furthermore, given $\Gamma' = \Gamma \setminus \{i\}$, the line bundle on $D_{\Gamma'}$ associated to $\mathcal{O}(D_\Gamma)$ is the restriction of L_{β_i} to $D_{\Gamma'}$.

At this point we can perform the computation of the ring A_Γ. First of all remark that Δ_{odd} is a basis of $\mathfrak{t}_0^*$. So, setting $x_i = \alpha_{2i-1}$, for $i = 1, \ldots, n$, we identify $S[\mathfrak{t}_0^*]$ with the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$. Now the Weyl group $W_\emptyset$ is just the hyperoctahedral group $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ with the group $(\mathbb{Z}/2\mathbb{Z})^n$ acting on the x_i by changing signs, while the symmetric group $S_n := S_{1,\ldots,n}$ acts by permutations. More generally, for a general $\Gamma = \{\overline{\alpha}_{2i_1}, \ldots, \overline{\alpha}_{2i_h}\}$, $i_1 < \cdots < i_h$, W_Γ is the subgroup $S_\Gamma \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, where $S_\Gamma \subset S_n$ is the parabolic subgroup $S_{1,\ldots,i_1} \times \cdots \times S_{i_h+1,\ldots,n}$. To simplify notations, let us also set $y_i = \overline{\alpha}_{2i}$ for $i = 1, \ldots, n-1$. Thus, setting $z_i = x_i^2$, $i = 1, \ldots, n$, we deduce the following:

Lemma 5.3. *The graded ring A_Γ is isomorphic to the subring of the polynomial ring $\mathbb{Q}[z_1, \dots, z_n, y_j]_{|\overline{\alpha}_{2j} \in \Gamma}$, with $\deg z_i = 4$, $\deg y_j = 2$, consisting of those polynomials which, as polynomials in the variables $z_1, \dots, z_n$, are invariant under the group S_Γ, i.e. they are symmetric in each of the groups of variables $z_{i_s+1}, \dots, z_{i_{s+1}}$, $s = 0, \dots, h$, with $i_0 = 0$, $i_{h+1} = n$.*

It is then immediate to deduce that, given $\overline{\alpha}_{2j} \in \Gamma$, the homomorphism

$$\mu_\Gamma^{\Gamma_j} : A_{\Gamma_j} \to A_\Gamma/(y_j),$$

is given by the inclusion

$$A_{\Gamma_j} = \mathbb{Q}[z_1, \dots, z_n, y_s]_{|\overline{\alpha}_{2s} \in \Gamma_j}^{W_{\Gamma_j}} \subset \mathbb{Q}[z_1, \dots, z_n, y_s]_{|\overline{\alpha}_{2s} \in \Gamma}^{W_{\Gamma_j}} \simeq A_\Gamma/(y_j).$$

As in the case of the group, given a monomial $m_I = y_1^{i_1} \cdots y_\ell^{i_\ell}$, define the support of $I = (i_1, \dots, i_\ell)$ as $\Gamma_I = \{j \mid i_j \neq 0\}$. By taking $\overline{\alpha}_{2i}$ for any $i \in \Gamma_I$, we are going to consider Γ_I as a subset of $\overline{\Delta}_{even}$. Then from Theorem 4.2 we get:

Theorem 5.4. *The ring $H_G^*(X)$ is isomorphic to the subring*

$$B \subset \mathbb{Q}[z_1, \dots, z_n, y_1, \dots, y_{n-1}]$$

which is the linear span of the polynomials

$$q(z_1, \dots, z_n) y_1^{i_1} \cdots y_\ell^{i_\ell}$$

with $q(z_1, \dots, z_n)$ invariant under S_{Γ_I}.

5.3. Case D_n

Let us now take G to be the adjoint quotient of $SO(2n)$, which we consider as the group of $2n \times 2n$ matrices X of determinant 1 and such that $X^{-1} = {}^tX$. Let σ be the involution on G induced by the involution on $SO(2n)$ given by conjugation by the matrix

$$J := \begin{pmatrix} -1 & 0 \\ 0 & I_{2n-1} \end{pmatrix}$$

where I_{2n-1} is the $(2n-1) \times (2n-1)$ identity matrix (notice that J does not lie in $SO(2n)$). H is the of type B_{n-1}.

Then it is clear that G/H is the open set in $\mathbb{P}^{2n-1}$ of non-isotropic lines with respect to the form $x_1^2 + x_2^2 + \cdots + x_{2n}^2$. The wonderful compactification X is nothing else that $\mathbb{P}^{2n-1}$ itself and has two orbits, G/H and the closed orbit of isotropic lines, that is the quadric of equation $x_1^2 + x_2^2 + \cdots + x_{2n}^2 = 0$.

In this case G/H has rank 1, so again Assumption 3.1 is fulfilled.

At this point the two rings $A_\emptyset$ and $A_{\{1\}}$ are easy to determine and we leave the details to the reader. One gets

$$A_{\{1\}} = \mathbb{Q}[y, x_2, x_3, \dots, x_{n-2}, z], \quad \text{with } \deg y = 2,\ \deg x_i = 4i,\ \deg z = 2n-2,$$

$$A_\emptyset = \mathbb{Q}\big[x_2, x_3, \dots, x_{n-2}, z^2\big].$$

Moreover the map

$$\mu^{\emptyset}_{\{1\}}: A_{\emptyset} \to A_{\{1\}}/(y) \simeq \mathbb{Q}[x_2, x_3, \dots, x_{n-2}, z]$$

is the obvious inclusion. Summarizing we get:

Theorem 5.5. *The ring $H^*_G(X)$ is isomorphic to the subring*

$$B \subset \mathbb{Q}[y, x_2, x_3, \dots, x_{n-2}, z]$$

consisting of the polynomials of the form

$$q\big(x_2, x_3, \dots, x_{n-2}, z^2\big) + yp(x_2, x_3, \dots, x_{n-2}, z).$$

Equivalently

$$B \simeq \mathbb{Q}[y, x_2, x_3, \dots, x_{n-2}, t, u]/\big(u^2 - y^2 t\big).$$

5.4. An exceptional involution

As we have already mentioned, the only case for which G is exceptional satisfying our Assumption 3.1 is when G is of type E_6, the involution if of type EIV, so that H is of type F_4. Let us describe the involution on the root system. Consider an Euclidean space E of dimension 6 with orthonormal basis $x_1, \dots, x_6$. We let σ be the involution whose matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

In E we consider a root system of type E_6 with simple roots the vectors:

$$\alpha_1 = \frac{1}{2}(x_1 - x_2 - x_3 - x_4 - x_5 - \sqrt{3}x_6); \qquad \alpha_2 = x_2 - x_1; \qquad \alpha_3 = x_3 - x_2;$$
$$\alpha_4 = x_4 - x_3; \qquad \alpha_5 = x_5 - x_4; \qquad \alpha_6 = x_1 + x_2.$$

Then σ acts as follows on the simple roots. The roots α_2, α_3, α_4, α_6 are fixed and span a root system of type D_4. We denote the corresponding Weyl group by $W_{\{1,2\}}$.

$$\sigma(\alpha_1) = -\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_6; \qquad \sigma(\alpha_5) = -\alpha_5 - 2\alpha_4 - 2\alpha_3 - \alpha_2 - \alpha_6.$$

Let us now consider the vectors

$$\beta = \frac{1}{2}\big(\alpha_1 + \sigma(\alpha_1)\big) = \frac{1}{2}(x_1 - x_2 - x_3 - x_4) \quad \text{and} \quad \gamma = \frac{1}{2}\big(\alpha_1 + \sigma(\alpha_1)\big) = -x_4.$$

Then $\{\beta, \alpha_2, \alpha_3, \alpha_4\}$ and $\{\alpha_2, \alpha_3, \alpha_4, \gamma\}$ are both simple roots for two systems of type B_4. We denote the corresponding Weyl groups by $W_{\{1\}}$ and $W_{\{2\}}$.

Finally, the vectors $\{\alpha_3, \alpha_4, \gamma, -\beta\}$ are simple roots for the system of type F_4, which is the root system associated to H. Of course the same is true for the vectors $-\gamma$, β, α_2, α_3. We denote the corresponding Weyl group by $W_{\emptyset}$.

Clearly, $\mathfrak{t}_0^*$ is the $\mathbb{Q}$-vector space spanned by the basis $\{x_1, x_2, x_3, x_4\}$, so we can identify $S[\mathfrak{t}_0^*]$ with $\mathbb{Q}[x_1, x_2, x_3, x_4]$. Moreover, $\mathfrak{t}_0^*$ has another basis given by the vectors

$$z_i = \frac{1}{2}\left(x_i - \sum_{j \neq i} x_j\right),$$

so that we also have an identification of $S[\mathfrak{t}_0^*]$ with $\mathbb{Q}[z_1, z_2, z_3, z_4]$.

Now notice that $W_\emptyset$ contains both $W_{\{1\}}$ and $W_{\{2\}}$, which both contain $W_{\{12\}}$. So, setting $C_\Gamma := S[\mathfrak{t}_0^*]^{W_\Gamma}$, for $\Gamma \subseteq \{1, 2\}$, we have that $C_\emptyset$ is contained in both $C_{\{1\}}$ and $C_{\{2\}}$, which are both contained in $C_{\{12\}}$.

Let us describe these four rings. $C_{\{1\}}$ is the ring of symmetric functions in the z_i^2, $C_{\{2\}}$ is the ring of symmetric functions in the x_i^2. Indeed the Weyl group of type B_4, as we have already seen, is the semidirect product of S_4 and $(\mathbb{Z}/2\mathbb{Z})^4$ and in the first case this last group changes the signs of the z_i, in the second those of the x_i, while the symmetric group acts by permutations of the z_i and the x_i respectively. $C_{\{12\}} = C_{\{1\}}[z_1 z_2 z_3 z_4] = C_{\{2\}}[x_1 x_2 x_3 x_4]$. Finally $W_\emptyset$ is the Weyl group of F_4 and, using [Me], we know that if we consider the polynomials

$$f_{2k} := \sum_{1 \leqslant i < j \leqslant 4} \left((x_1 + x_j)^{2k} + (x_1 - x_j)^{2k}\right),$$

then $C_{\{12\}}$ is the polynomial ring $\mathbb{Q}[f_2, f_6, f_8, f_{12}]$. Consider now the vectors

$$y_1 = \alpha_1 - \sigma(\alpha_1) = \sqrt{3}x_6 - x_5 \quad \text{and} \quad y_2 = \alpha_5 - \sigma(\alpha_5) = 2x_5.$$

These are the fundamental weights relative to the simple roots α_1 and α_5. Then clearly

$$A_{\{12\}} = C_{\{12\}}[y_1, y_2]; \qquad A_{\{1\}} = C_{\{1\}}[y_2]; \qquad A_{\{2\}} = C_{\{2\}}[y_1]; \qquad A_\emptyset = C_\emptyset.$$

Given a sequence $I = (i_1, i_2)$, we define $\Gamma_I = \{j \mid i_j \neq 0\}$, as we have already done in the cases considered above. We get the following:

Theorem 5.6. *The ring $H_G^*(X)$ is isomorphic to the subring*

$$B \subset A_{\{12\}}$$

which is the linear span of the polynomials

$$q(x_1, \ldots, x_4) y_1^{i_1} y_2^{i_2}$$

with $q(z_1, \ldots, z_n) \in C_{\Gamma_I}$ where $I = (i_1, i_2)$.

As a consequence, we can determine very easily both the equivariant Poincaré series and the Poincaré polynomial of X. We get:

Proposition 5.7.

(1) *The equivariant Poincaré series $\sum_n \dim H_G^{2n}(X) t^n$ equals*

$$\frac{1 + 2t^5 + 2t^9 + t^{14}}{(1-t)^2(1-t^2)(1-t^6)(1-t^8)(1-t^{12})}.$$

(2) *The Poincaré polynomial* $\sum_n \dim H^{2n}(X)t^n$ *equals*

$$\left(1+2t^5+2t^9+t^{14}\right)\left(1+t+t^2+t^3+t^4\right)\left(1+t+t^2+t^3+t^4+t^5+t^6+t^7+t^8\right).$$

Proof. The first statement follows readily from Theorem 5.6, summing up the various contributions. As for the second, recall that $H^*_G(X)$ is a free module on $H^*_G(pt)$ and that one knows that

$$\sum_n \dim H^{2n}_G(pt)t^n = \frac{1}{(1-t^2)(1-t^5)(1-t^6)(1-t^8)(1-t^9)(1-t^{12})}. \qquad \square$$

6. The equivariant ring of conditions

In this section we are going to determine the equivariant ring of conditions $R_G(G/H)$, introduced in [DP3], of the symmetric variety G/H under Assumption 3.1. The proofs are exactly parallel to the ones given in [St3] in the case of the group, so we shall omit them. In order to state our result, we need some facts. Recall that we have introduced in Section 3 the set

$$\overline{\Delta} = \left\{\overline{\alpha}_1, \dots, \overline{\alpha}_h, \frac{1}{2}\overline{\alpha}_{h+1}, \dots, \frac{1}{2}\overline{\alpha}_k\right\} \subset \mathfrak{t}_1^*$$

of simple restricted roots.

Correspondingly, we are going to consider the fundamental Weyl chamber $C \subset \mathfrak{t}_1$ of positive linear combinations of fundamental coweights associated to these simple restricted roots and the lattice $\Lambda_1^\vee$ in $\mathfrak{t}_1$ that they span.

We are going to consider the set of rational regular fans $\mathcal{F}$ giving a polyhedral decomposition of C. By this we mean a finite collection $\mathcal{F} = \{F_1, \dots, F_m\}$ of rational polyhedral cones, called faces, each contained in C and such that

(1) for each $F_i, F_j \in \mathcal{F}$, $F_i \cap F_j$ is a face of both F_i and F_j,
(2) each face of a face in $\mathcal{F}$ also lies in $\mathcal{F}$,
(3) $C = F_1 \cup \cdots \cup F_m$,
(4) each face F_i is spanned by part of a basis of the lattice $\Lambda_1^\vee$.

We now give the basic definition of this section:

Definition 2. A function f on the space $\mathfrak{t}_0 \times C$ is admissible relative to a rational regular fan $\mathcal{F}$ giving a polyhedral decomposition of C if:

(1) For every (closed) cone F of $\mathcal{F}$ the restriction of f to $\mathfrak{t}_0 \times F$ is a polynomial function.
(2) Let Γ be a subset of the set of $\overline{\Delta}$ and let C_Γ be the face of C defined by the vanishing of the roots in Γ. Then the restriction of f to $\mathfrak{t}_0 \times C_\Gamma$ is invariant under the action of W_Γ on $\mathfrak{t}_0$.

A function f on the space $\mathfrak{t}_0 \times C$ is admissible if there exists a rational regular fan $\mathcal{F}$ giving a polyhedral decomposition of C, such f is admissible relative to $\mathcal{F}$.

Notice that given $\mathcal{F}$, the space of admissible functions relative to $\mathcal{F}$ is clearly a ring under multiplication of functions. Furthermore, since two fans $\mathcal{F}$ and $\mathcal{F}'$ with the above properties have a common decomposition (see for example [Tor] or [O]), it immediately follows that also the space $\mathcal{R}$ of admissible functions is a ring.

Now we know that to every regular fan $\mathcal{F}$ which gives a polyhedral decomposition of C, it is associated a regular compactification $Y_{\mathcal{F}}$ of the symmetric variety G/H, see [DP3]. By our previous considerations we then have, repeating verbatim the proofs of Theorem 4.5 and Theorem 5.1 in [St3]:

Theorem 6.1.

(1) *The equivariant cohomology ring* $H^*_G(Y_{\mathcal{F}}, \mathbb{Q})$ *is naturally isomorphic to the ring of admissible functions relative to* $\mathcal{F}$.
(2) *The equivariant ring of conditions* $R_{G\times G}(G)$ *is naturally isomorphic to the ring of admissible functions.*

Remark 6.2. The reader might have noticed that all our rings are graded and that all our isomorphisms are isomorphisms of graded rings.

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