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CHARACTERIZATION OF PROBABILITY MEASURES THROUGH THE CANONICALLY ASSOCIATED INTERACTING FOCK SPACES

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We continue our program of coding the whole information of a probability measure into a set of commutation relations canonically associated to it by presenting some characterization theorems for the symmetry and factorizability of a probability measure on $\mathbb{R}^d$ in terms of the canonically associated interacting creation, annihilation and number operators.

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1. Introduction

The notion of interacting Fock space (IFS) was introduced in Ref. 2 and axiomatized in Ref. 3 where it was conjectured that the category of IFS could play for general probability measures the same role as the usual Fock space for the Gaussian measures. The first confirmation of this conjecture came from the paper[1] of Accardi and Bożejko who showed that the theory of one-mode interacting Fock spaces is canonically isomorphic to the theory of orthogonal polynomials in one variable. The isomorphism is canonical in the sense that it carries the multiplication operator by the independent variable to a linear combination of creation, annihilation, and number operators on the corresponding interacting Fock space.

The problem to extend the Accardi–Bożejko isomorphism to the case of several variables has been recently solved by Accardi and Nahni[4] and extended to the infinite dimensional case by Accardi, Kuo and Stan. A new feature in the multi-mode case is that not all interacting Fock spaces are canonically isomorphic to spaces of orthogonal polynomials. Those being so are characterized in terms of a sequence of quadratic commutation relations among finite dimensional matrices. Moreover, the quantum decomposition of an arbitrary vector-valued random variable with finite moments of any order can be easily written down as a sum of creation, annihilation and number operators.

This result opens the way to the program of coding the whole information of a probability measure into a set of commutation relations canonically associated to it in full analogy to what happens with the codification of the properties of a Gaussian measure into the Heisenberg commutation relations plus the Fock property.

In this paper we begin to realize this program by showing how some properties of a probability measure on $\mathbb{R}^d$ are reflected by the actions of its creation, annihilation, and number operators.

2. Fundamental Identities

Let $d \in \mathbb{N}$ be fixed. Let μ be a probability measure on the Borel subsets of $\mathbb{R}^d$. Throughout this paper we fix the canonical basis in $\mathbb{R}^d$ and identify vectors $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ with ordered d-tuples of real numbers. However, all the results below can be formulated in an intrinsic, i.e. coordinate independent way and this will be discussed in a future paper. We assume that μ has finite moments of all orders, i.e. for all $1 \le p < \infty$ and $j \in \{1, 2, \ldots, d\}$, $\int_{\mathbb{R}^d} |x_j|^p \mu(dx) < \infty$. We denote the inner product on $L^2(\mathbb{R}^d, \mu)$ by $\langle , \rangle$.

Let $F_0 = \mathbb{C} \cdot 1$ be the complex multiples of the constant function equal to 1 in $L^2(\mathbb{R}^d, \mu)$, and for $n \ge 1$ let F_n be the complex vector space of all polynomial functions of variables $x_1, x_2, \ldots, x_d$ of degree less than or equal to n. We have:

$$F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots \subset L^2(\mathbb{R}^d, \mu) .$$

Let $G_0 = \mathbb{C} \cdot 1$ and for $n \ge 1$ let G_n be the orthogonal complement of F_{n-1} in F_n. Notice that F_{n-1} and F_n are finite dimensional (therefore closed) subspaces of $L^2(\mathbb{R}^d, \mu)$. Then the Hilbert spaces G_n, $n \ge 0$, are orthogonal subspaces of $L^2(\mathbb{R}^d, \mu)$. Let $\mathcal{H}$ denote the orthogonal direct sum of G_n, $n \ge 0$:

$$\mathcal{H} = \bigoplus_{n \ge 0} G_n \qquad \text{(Hilbert space sense)} . \tag{2.1}$$

For any $j \in \{1, 2, \ldots, d\}$, we denote by X_j the multiplication by x_j operator. This operator is densely defined on $\mathcal{H}$. Its domain contains F_n, $\forall\, n \ge 0$, since μ has finite moments of any order. Note that, for every $n \geq 0$, X_j maps F_n into F_{n+1} and is a symmetric operator.

Lemma 2.1. *For any* $j \in \{1, 2, \ldots, d\}$ *and* $n \geq 0$, *we have*

$$X_j G_n \perp G_k , \qquad \forall\ k \neq n-1, n, n+1 .$$

Proof. Let $\phi \in G_n$. Then $X_j\phi \in F_{n+1}$. Hence $X_j\phi \perp G_k$ for all $k \geq n+2$. On the other hand, for any $\psi \in G_k$ with $k \leq n-2$, we have $X_j\psi \in F_{n-1}$ hence, by the symmetry of X_j

$$\langle X_j\phi, \psi\rangle = \langle \phi, X_j\psi\rangle = 0 .$$

Thus $X_j\phi \perp G_k$ for all $k \leq n-2$. □

For any $n \geq 0$, let P_n denote the orthogonal projection of $\mathcal{H}$ onto G_n.

Theorem 2.1. (Recurrence relations) *For any* $j \in \{1, 2, \ldots, d\}$ *and* $n \geq 0$, *the following equality holds*:

$$X_j P_n = P_{n+1} X_j P_n + P_n X_j P_n + P_{n-1} X_j P_n , \tag{2.2}$$

where $P_{-1} = 0$ *by convention.*

Proof. Equation (2.2) is equivalent to

$$X_j\phi = P_{n+1}X_j\phi + P_nX_j\phi + P_{n-1}X_j\phi , \qquad \forall\ \phi \in G_n . \tag{2.3}$$

Let $\phi \in G_n$. By Lemma 2.1, $X_j\phi$ can be written as

$$X_j\phi = u + v + w , \tag{2.4}$$

where $u \in G_{n+1}$, $v \in G_n$ and $w \in G_{n-1}$. Apply P_{n+1} to both sides of Eq. (2.4). Since $P_{n+1}u = u$ and $P_{n+1}v = P_{n+1}w = 0$, we get $u = P_{n+1}X_j\phi$. Similarly, we can apply P_n and P_{n-1} to both sides of Eq. (2.4) to get $v = P_nX_j\phi$ and $w = P_{n-1}X_j\phi$. Thus Eq. (2.3) is proved. □

Now for each $j \in \{1, 2, \ldots, d\}$ and $n \geq 0$ we define the following operators:

$$D_n^+(j) = P_{n+1}X_jP_n : G_n \to G_{n+1} , \tag{2.5}$$

$$D_n^0(j) = P_nX_jP_n : G_n \to G_n , \tag{2.6}$$

$$D_n^-(j) = P_{n-1}X_jP_n : G_n \to G_{n-1} . \tag{2.7}$$

We define F_{-1} and G_{-1} to be the null space $\{0\}$.

Theorem 2.2. *For any* i, $j \in \{1, 2, \ldots, d\}$ *and* $n \geq 0$ *the following identities hold*:

- $$D_{n+1}^+(i)D_n^+(j) = D_{n+1}^+(j)D_n^+(i) , \tag{2.8}$$
- $$D_{n+1}^0(i)D_n^+(j) + D_n^+(i)D_n^0(j) = D_{n+1}^0(j)D_n^+(i) + D_n^+(j)D_n^0(i) , \tag{2.9}$$

$$\bullet \quad D^-_{n+1}(i)D^+_n(j) + D^0_n(i)D^0_n(j) + D^+_{n-1}(i)D^-_n(j) = D^-_{n+1}(j)D^+_n(i) + D^0_n(j)D^0_n(i) + D^+_{n-1}(j)D^-_n(i)\,. \tag{2.10}$$

Proof. Apply $P_{n+2}X_i$ to both sides of Eq. (2.2) to obtain

$$P_{n+2}X_iX_jP_n = P_{n+2}X_iP_{n+1}X_jP_n + P_{n+2}X_iP_nX_jP_n + P_{n+2}X_iP_{n-1}X_jP_n\,.$$

Observe that

$$P_{n+2}X_iP_{n+1}X_jP_n = D^+_{n+1}(i)D^+_n(j)\,,$$
$$P_{n+2}X_iP_nX_jP_n = P_{n+2}X_iP_{n-1}X_jP_n = 0\,.$$

Therefore, we obtain the equality:

$$P_{n+2}X_iX_jP_n = D^+_{n+1}(i)D^+_n(j)\,. \tag{2.11}$$

Interchange the role of i and j to get

$$P_{n+2}X_jX_iP_n = D^+_{n+1}(j)D^+_n(i)\,. \tag{2.12}$$

Since $P_{n+2}X_iX_jP_n = P_{n+2}X_jX_iP_n$, Eqs. (2.11) and (2.12) yield the identity in Eq. (2.8).

Similarly, apply $P_{n+1}X_i$ to both sides of Eq. (2.2) to get

$$P_{n+1}X_iX_jP_n = P_{n+1}X_iP_{n+1}X_jP_n + P_{n+1}X_iP_nX_jP_n + P_{n+1}X_iP_{n-1}X_jP_n\,.$$

Observe that

$$P_{n+1}X_iP_{n+1}X_jP_n = D^0_{n+1}(i)D^+_n(j)\,,$$
$$P_{n+1}X_iP_nX_jP_n = D^+_n(i)D^0_n(j)$$
$$P_{n+1}X_iP_{n-1}X_jP_n = 0\,.$$

Therefore, we obtain the equality:

$$P_{n+1}X_iX_jP_n = D^0_{n+1}(i)D^+_n(j) + D^+_n(i)D^0_n(j)\,.$$

Interchange the role of i and j. Since $P_{n+1}X_iX_jP_n = P_{n+1}X_jX_iP_n$, we obtain the identity in Eq. (2.9).

Finally, apply P_nX_i to both sides of Eq. (2.2) and interchange the role of i and j to obtain the identity in Eq. (2.10). □

Proposition 2.1. *For any $j \in \{1, 2, \ldots, d\}$ and $n \geq 0$ the operators $D^+_n(j)$, $D^0_n(j)$ and $D^-_n(j)$ satisfy the identities*:

$$(D^+_n(j))^* = D^-_{n+1}(j)\,, \qquad (D^0_n(j))^* = D^0_n(j)\,.$$

Proof. Since $X_j^* = X_j$ we have the equalities

$$(D_n^+(j))^* = (P_{n+1}X_jP_n)^* = P_nX_jP_{n+1} = D_{n+1}^-(j)\,,$$
$$(D_n^0(j))^* = (P_nX_jP_n)^* = P_nX_jP_n = D_n^0(j)\,,$$

which prove the proposition. □

For each $j \in \{1, 2, \ldots, d\}$, we define the densely defined linear operators $a^+(j)$, $a^0(j)$ and $a^-(j)$, on $\mathcal{H}$, by

$$a^+(j)|_{G_n} = D_n^+(j)\,, \qquad a^0(j)|_{G_n} = D_n^0(j)\,, \qquad a^-(j)|_{G_n} = D_n^-(j)\,, \tag{2.13}$$

for all $n \geq 0$.

Equation (2.2) becomes now:

$$X_j = a^+(j) + a^0(j) + a^-(j)\,, \qquad \forall\, j \in \{1, 2, \ldots, d\}\,. \tag{2.14}$$

Theorem 2.3. *If μ is any probability measure on $\mathbb{R}^d$ having finite moments of any orders, then $\forall\ i,\, j \in \{1, 2, \ldots, d\}$*

$$a^-(i)a^-(j) = a^-(j)a^-(i)\,, \qquad a^+(i)a^+(j) = a^+(j)a^+(i)\,. \tag{2.15}$$

Proof. The equality $a^+(i)a^+(j) = a^+(j)a^+(i)$ is a restatement of Eq. (2.8). Taking the adjoint on both sides of this relation we obtain $a^-(j)a^-(i) = a^-(i)a^-(j)$. □

For all $j \in \{1, 2, \ldots, d\}$ and $n \geq 1$, we have $a^-(j) : G_n \to G_{n-1}$. The constant polynomial 1 is called the *vacuum vector*. We have $a^-(j)1 = 0$, $\forall\ j \in \{1, 2, \ldots, d\}$.

Lemma 2.2. *Let $j \in \{1, 2, \ldots, d\}$. We have $a^0(j)1 = 0$ if and only if*

$$\langle x_j \rangle := \int_{\mathbb{R}^d} x_j \mu(dx) = 0\,. \tag{2.16}$$

Proof. We have

$$D_0^0(j)1 = P_0X_jP_01 = \langle x_j, 1\rangle 1 = \left(\int_{\mathbb{R}^d} x_j\mu(dx)\right) 1\,.$$

Thus $a^0(j)1 = 0$ if and only if $\int_{\mathbb{R}^d} x_j\mu(dx) = 0$. □

3. Polynomially Symmetric Measures

For any monomial $x_1^{i_1}x_2^{i_2}\cdots x_d^{i_d}$, we define its degree to be $i_1 + i_2 + \cdots + i_d$.

Definition 3.1. A measure μ on $\mathbb{R}^d$ is called *polynomially symmetric* if all of its mixed moments of odd order vanish, i.e. for all monomials $x_1^{i_1}x_2^{i_2}\cdots x_d^{i_d}$ of odd degree, we have $\int_{\mathbb{R}^d} x_1^{i_1}x_2^{i_2}\cdots x_d^{i_d}\mu(dx) = 0$.

Definition 3.2. A measure μ on $\mathbb{R}^d$ is called *symmetric* if for any Borel subset A of $\mathbb{R}^d$ we have $\mu(A) = \mu(-A)$, where $-A := \{-x \,|\, x \in A\}$.

Observe that if μ is symmetric, then μ is also polynomially symmetric. The converse is not true. In Example 5.1 of Sec. 5 we present a polynomially symmetric measure that is not symmetric.

Let $\mathcal{P}$ be the set of all polynomials in the variables $x_1, x_2, \ldots, x_d$. Let W_{even} be the vector subspace of $\mathcal{P}$ spanned by the set of all monomials of even degree and W_{odd} the vector subspace of $\mathcal{P}$ spanned by the set of all monomials of odd degrees.

Let us assume that μ is polynomially symmetric. If $f \in W_{\text{even}}$ and $g \in W_{\text{odd}}$, then, since μ is polynomially symmetric, $E[f\bar{g}] = 0$, where E denotes the expectation with respect to μ, i.e. the polynomials f and g are orthogonal. Thus $W_{\text{even}} \perp W_{\text{odd}}$. We may apply the Grahm–Schmidt orthogonalization procedure first to W_{even} and obtain a complete orthonormal set S_1 for W_{even}. After this we may apply the Grahm–Schmidt orthogonalization procedure to W_{odd} and obtain a complete orthonormal set S_2 for W_{odd}. Since $S_1 \subset W_{\text{even}}$, $S_2 \subset W_{\text{odd}}$, and $W_{\text{even}} \perp W_{\text{odd}}$, we have $S_1 \perp S_2$. Thus $S := S_1 \cup S_2$ is a complete orthonormal set for the space $\mathcal{P}$ of all polynomial functions. Using this complete orthonormal set S for $\mathcal{P}$, we can see that all polynomials in G_n, when n is even, are linear combinations of monomials of even degree. When n is odd, all polynomials in G_n are linear combinations of monomials of odd degree.

Theorem 3.1. *If μ is a probability measure on $\mathbb{R}^d$, having finite moments of all orders, then μ is polynomially symmetric if and only if for all $j \in \{1, 2, \ldots, d\}$,*

$$X_j = a^+(j) + a^-(j)\,.$$

This means that for all $j \in \{1, 2, \ldots, d\}$, $a^0(j) = 0$.

Proof. ($\Rightarrow$) Let us assume that μ is polynomially symmetric. Let $j \in \{1, 2, \ldots, d\}$. To show that $a^0(j) = 0$, we must prove that for any $n \geq 0$, $P_n X_j P_n = 0$. To prove this, we will show that for any polynomials f and g, we have $\langle P_n X_j P_n f, g\rangle = 0$.

Let $n \geq 0$ be fixed. Let f and g be two polynomials. Since $P_n f \in G_n$ and $P_n g \in G_n$, $P_n f$ and $P_n g$ are linear combinations of monomials that are either all of even degree if n is even or all of odd degree if n is odd. We have:

$$\begin{aligned}
\langle P_n X_j P_n f, g\rangle &= \langle X_j P_n f, P_n g\rangle \\
&= E[X_j (P_n f)\overline{P_n g}] \\
&= E\left[\left(x_j \sum a_{k_1\cdots k_d} x_1^{k_1} \cdots x_d^{k_d}\right)\left(\sum \overline{b_{l_1\cdots l_d}} x_1^{l_1} \cdots x_d^{l_d}\right)\right] \\
&= E\left[\sum a_{k_1\cdots k_d} \overline{b_{l_1\cdots l_d}} x_1^{k_1+l_1} \cdots x_j^{k_j+l_j+1} \cdots x_d^{k_d+l_d}\right] \\
&= \sum a_{k_1\cdots k_d} \overline{b_{l_1\cdots l_d}} E[x_1^{k_1+l_1} \cdots x_j^{k_j+l_j+1} \cdots x_d^{k_d+l_d}]\,.
\end{aligned}$$

Observe that:

$$\begin{aligned}
&(k_1 + l_1) + \cdots + (k_j + l_j + 1) + \cdots + (k_d + l_d) \\
&\quad = (k_1 + \cdots + k_d) + (l_1 + \cdots + l_d) + 1
\end{aligned}$$

$$\equiv n(\text{mod } 2) + n(\text{mod } 2) + 1(\text{mod } 2)$$
$$\equiv (2n+1)(\text{mod } 2)$$
$$\equiv 1(\text{mod } 2)\,.$$

Since μ is polynomially symmetric, we have:

$$E[x_1^{k_1+l_1} \cdots x_j^{k_j+l_j+1} \cdots x_d^{k_d+l_d}] = 0\,.$$

($\Leftarrow$) Let μ be a probability measure on $\mathbb{R}^d$ having finite moments of all orders, such that for all $1 \leq j \leq d$, $a^0(j) = 0$. We will prove by induction on k that for all monomials $m(x) = x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d}$, of degree $2k+1$, we have $\int_{\mathbb{R}^d} m(x)\mu(dx) = 0$.

For $k = 0$, the only monomials having degree 1 are $m_j(x) = x_j$, for $1 \leq j \leq d$. Since $a^0(j)1 = 0$, it follows that $P_0 X_j 1 = 0$. Since P_0 is the projection on the one-dimensional vector space $\mathbb{C}$ for which 1 is an orthonormal basis, we have $P_0 X_j 1 = \langle x_j, 1\rangle 1$. Thus $\langle x_j, 1\rangle = 0$, which means $\int_{\mathbb{R}^d} x_j \mu(dx) = 0$, for all $1 \leq j \leq d$.

Let us assume now that the expectation of all monomials of odd degree less than or equal to $2k-1$ is zero, where $k \geq 1$. We want to prove that the expectation of all monomials of degree $2k+1$ is 0. Let $m(x) = x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d}$ be a monomial such that $i_1 + i_2 + \cdots + i_d = 2k+1$. Let us a choose a number $j \in \{1, 2, \ldots, d\}$ such that $i_j > 0$. We can write $m(x) = x_j q(x) r(x)$, where $q(x)$ and $r(x)$ are monomials of degree equal to k. Let F_k^{e} be the vector space spanned by the monomials of even degree less than or equal to k. Let F_k^{o} be the vector space spanned by the monomials of odd degree less than or equal to k. According to our induction hypothesis F_k^{e} and F_k^{o} are orthogonal subspaces of $L^2(\mathbb{R}^d, \mu)$. Thus if we choose an orthonormal basis $\{e_i\}_{i \in I}$ of F_k^{e} and an orthonormal basis $\{f_j\}_{j \in J}$ of F_k^{o}, then $\{e_i\}_{i \in I} \cup \{f_j\}_{j \in J}$ is an orthonormal basis of the space F_k of all polynomials of degree less than or equal to k. Using this particular basis for F_k, according to our induction hypothesis, we can see that: $P_k q(x) = q(x) - u_{k-2}(x) - u_{k-4}(x) - \cdots$ and $P_k r(x) = r(x) - v_{k-2}(x) - v_{k-4}(x) - \cdots$, where u_i and v_i are linear combinations of monomials of the same degree i.

Since $a^0(j) = 0$, we have $P_k X_j P_k q(x) = 0$. Thus we obtain:

$$\begin{aligned} 0 &= \langle P_k X_j P_k q(x), r(x)\rangle \\ &= \langle X_j P_k q(x), P_k r(x)\rangle \\ &= \langle x_j(q(x) - u_{k-2}(x) - u_{k-4}(x) - \cdots), r(x) - v_{k-2}(x) - v_{k-4}(x) - \cdots\rangle \\ &= \langle x_j q(x), r(x)\rangle - \sum E[w(x)]\,, \end{aligned}$$

where all the $w(x)$ are monomials of odd degree less than or equal to $2k-1$. By the induction hypothesis $E[w(x)] = 0$. From the last equality we obtain $\langle x_j q(x), r(x)\rangle = 0$. This means $E[m(x)] = 0$.

Thus we have proved by induction that the expectation of all monomials of odd degree is zero. Hence μ is polynomially symmetric. $\square$

Definition 3.3. Let μ be a probability measure on $\mathbb{R}^d$ and $c \in \mathbb{R}^d$. We say that μ is *polynomially symmetric about* c, if for any monomial $m(x)$ of odd degree, we have:

$$\int_{\mathbb{R}^d} m(x-c)\mu(dx) = 0\,.$$

Let us observe that if μ is a polynomially symmetric probability measure on $\mathbb{R}^d$, then for any $c = (c_1, c_2, \ldots, c_d) \in \mathbb{R}^d$, the probability measure μ_c, defined by $\mu_c(B) := \mu(B-c)$, where B is a Borel subset of $\mathbb{R}^d$ and $B - c := \{x - c \,|\, x \in B\}$, is polynomially symmetric about c. We do the same construction for μ_c as we did for μ, and call the corresponding spaces $\tilde{G}_n$, $n \geq 0$, and the corresponding operators $\tilde{P}_n$, $n \geq 0$, $\tilde{a}^-$, $\tilde{a}^0$, and $\tilde{a}^+$. Since for any integrable function f with respect to μ, we have $\int_{\mathbb{R}^d} f(x)\mu(dx) = \int_{\mathbb{R}^d} f(x-c)\mu_c(dx)$, we can see that a polynomial $Q(x)$ belongs to G_n if and only if $Q(x-c)$ belongs to $\tilde{G}_n$. Moreover, $\{e_i(x)\}_{i\in I}$ is an orthonormal basis for G_n if and only if $\{e_i(x-c)\}_{i\in I}$ is an orthonormal basis for $\tilde{G}_n$. Since $P_n X_j P_n = 0$, $\forall\, 1 \leq j \leq d$, we conclude that $\tilde{P}_n(X_j - c_j)\tilde{P}_n = 0$, $\forall\, 1 \leq j \leq d$. Thus, for all $n \geq 0$, we have:

$$\begin{aligned}\tilde{a}^0(j)|_{G_n} &= \tilde{P}_n X_j \tilde{P}_n \\ &= \tilde{P}_n(X_j - c_j)\tilde{P}_n + c_j\tilde{P}_n \\ &= c_j\tilde{P}_n\,.\end{aligned}$$

Hence $\tilde{a}^0(j) = c_j I$. Therefore, we obtain the following:

Theorem 3.2. *If μ is a probability measure on $\mathbb{R}^d$, having finite moments of all orders, then μ is polynomially symmetric about the point $c = (c_1, c_2, \ldots, c_d) \in \mathbb{R}^d$ if and only if for all $j \in \{1, 2, \ldots, d\}$, we have*:

$$a^0(j) = c_j I\,. \tag{3.1}$$

4. Polynomially Factorizable Measures

Definition 4.1. If μ is a probability measure on the Borel subsets of $\mathbb{R}^d$, having finite moments of any order, then we say that μ is *polynomially factorizable* if for any non-negative integers $i_1, i_2, \ldots, i_d$ we have:

$$E[x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d}] = E[x_1^{i_1}]E[x_2^{i_2}] \cdots E[x_d^{i_d}]\,.$$

In the above definition E denotes the expectation with respect to μ.

Observe that if μ is polynomially factorizable, then for any polynomial functions $f_1(x_1)$, $f_2(x_2), \ldots, f_d(x_d)$, we have:

$$E[f_1(x_1)f_2(x_2)\cdots f_d(x_d)] = E[f_1(x_1)]E[f_2(x_2)]\cdots E[f_d(x_d)]\,.$$

If there exist d probability measures $\mu_1, \mu_2, \ldots, \mu_d$ on $\mathbb{R}$, such that for all B_1, $B_2, \ldots, B_d$ Borel subsets of $\mathbb{R}$,

$$\mu(B_1 \times B_2 \times \cdots \times B_d) = \mu_1(B_1)\mu_2(B_2)\cdots\mu_d(B_d)\,,$$

then μ is polynomially factorizable. The converse is not true. In Example 5.2 of Sec. 5 we present a polynomially factorizable measure that is not a product measure.

Let μ be a polynomially factorizable measure on the Borel subsets of $\mathbb{R}^d$. For any $i \in \{1, 2, \ldots, d\}$, let $\mathcal{H}_i$ be the closure of the space $\mathcal{P}_i$, of all polynomial functions of the variable x_i, in the space $L^2(\mathbb{R}^d, \mu)$. Every function in $\mathcal{P}_i$ is a polynomial function depending only on the variable x_i. Let $\mathcal{H}$ be the closure of the space $\mathcal{P}$, of all polynomial functions of d variables: $x_1, x_2, \ldots, x_d$, in the space $L^2(\mathbb{R}^d, \mu)$. The multilinear function $T : \mathcal{P}_1 \times \mathcal{P}_2 \times \cdots \times \mathcal{P}_d \to \mathcal{P}$, defined by $T(f_1, f_2, \ldots, f_d) = f$, where $f(x_1, x_2, \ldots, x_d) = f_1(x_1)f_2(x_2) \cdots f_d(x_d)$, generates a linear map U between the algebraic tensor product $\mathcal{P}_1 \otimes \mathcal{P}_2 \otimes \cdots \otimes \mathcal{P}_d$ and the space $\mathcal{P}$. Because μ is polynomially factorizable, U preserves the inner product. Indeed, for any $f_1, g_1 \in \mathcal{P}_1$, $f_2, g_2 \in \mathcal{P}_2, \ldots, f_d, g_d \in \mathcal{P}_d$, we have:

$$\begin{aligned}
&\langle U(f_1 \otimes f_2 \otimes \cdots \otimes f_d), U(g_1 \otimes g_2 \otimes \cdots \otimes g_d)\rangle \\
&\quad = \langle f_1(x_1)f_2(x_2) \cdots f_d(x_d), g_1(x_1)g_2(x_2) \cdots g_d(x_d)\rangle \\
&\quad = E[f_1(x_1)\bar{g}_1(x_1)f_2(x_2)\bar{g}_2(x_2) \cdots f_d(x_d)\bar{g}_d(x_d)] \\
&\quad = E[f_1(x_1)\bar{g}_1(x_1)]E[f_2(x_2)\bar{g}_2(x_2)] \cdots E[f_d(x_d)\bar{g}_d(x_d)] \\
&\quad = \langle f_1, g_1\rangle\langle f_2, g_2\rangle \cdots \langle f_d, g_d\rangle \\
&\quad = \langle f_1 \otimes f_2 \otimes \cdots \otimes f_d, g_1 \otimes g_2 \otimes \cdots \otimes g_d\rangle\,.
\end{aligned}$$

U is onto since any monomial belongs to its range. Since $\mathcal{P}_1 \otimes \mathcal{P}_2 \otimes \cdots \otimes \mathcal{P}_d$ is dense in the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_d$ and $\mathcal{P}$ is dense in $\mathcal{H}$, the operator U can be uniquely extended to a unitary operator $\tilde{U}$ from $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_d$ onto $\mathcal{H}$. Thus we may identify the Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_d$ and $\mathcal{H}$.

For any $i \in \{1, 2, \ldots, d\}$ and k a non-negative integer, let $F_k^{(i)}$ be the vector space spanned by the polynomial functions $1, x_i, x_i^2, \ldots, x_i^k$. Let $G_k^{(i)}$ be the orthogonal complement of $F_{k-1}^{(i)}$ into $F_k^{(i)}$, where $F_{-1}^{(i)} := \{0\}$. To compute this orthogonal complement we consider $F_{k-1}^{(i)}$ and $F_k^{(i)}$ as being subspaces of the Hilbert space $\mathcal{H}_i$. We denote by $P_k^{(i)}$ the projection operator from $\mathcal{H}_i$ onto $G_k^{(i)}$ and by $X_i^{(i)}$ the densely defined operator on $\mathcal{H}_i$ given by the multiplication by the polynomial function x_i.

For any $i \in \{1, 2, \ldots, d\}$ and n a non-negative integer, we define the operators $D_{n,i}^-$, $D_{n,i}^0$ and $D_{n,i}^+$, from $\mathcal{H}_i$ into $\mathcal{H}_i$, in the following way:

$$D_{n,i}^- = P_{n-1}^{(i)}X_i^{(i)}P_n^{(i)}\,, \qquad D_{n,i}^0 = P_n^{(i)}X_i^{(i)}P_n^{(i)} \qquad \text{and} \qquad D_{n,i}^+ = P_{n+1}^{(i)}X_i^{(i)}P_n^{(i)}\,.$$

Lemma 4.1. *Let μ be a polynomially factorizable probability measure on the Borel subsets of $\mathbb{R}^d$. If we identify the space $\mathcal{H}$ with the space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_d$, then for any $n \geq 0$, we have*:

$$P_n = \sum_{i_1+i_2+\cdots+i_d=n} P_{i_1}^{(1)} \otimes P_{i_2}^{(2)} \otimes \cdots \otimes P_{i_d}^{(d)}\,. \tag{4.1}$$

In the above sum all the indices i_j are considered to be non-negative.

Proof. Since the tensor product of orthogonal projections is an orthogonal projection, it follows that each term $P_{i_1}^{(1)} \otimes P_{i_2}^{(2)} \otimes \cdots \otimes P_{i_d}^{(d)}$ is an orthogonal projection. If $i_j \neq k_j$, then $P_{i_j}^{(j)} P_{k_j}^{(j)} = 0$ and thus the terms in the sum from the right-hand side of (4.1) are orthogonal (i.e. the composition of any two different terms is zero). Thus the right-hand side of (4.1) is an orthogonal projection. If P and Q are two orthogonal projections of the same Hilbert space H, then we say that $P \geq Q$ if the range of P contains the range of Q. If $i_1 + i_2 + \cdots + i_d = n$ and $f(x) \in F_{n-1}$, then $f(x) = \sum a_{j_1, j_2, \ldots, j_d} x_1^{j_1} x_2^{j_2} \cdots x_d^{j_d}$ with $j_1 + j_2 + \cdots + j_d \leq n - 1$. Thus there exists $k \in \{1, 2, \ldots, d\}$ such that $i_k > j_k$ and so $P_{i_1}^{(1)} \otimes \cdots \otimes P_{i_d}^{(d)}(x_1^{j_1} \cdots x_d^{j_d}) = 0$. Hence $P_{i_1}^{(1)} \otimes \cdots \otimes P_{i_d}^{(d)} f(x) = 0$. Therefore the range of $P_{i_1}^{(1)} \otimes \cdots \otimes P_{i_d}^{(d)}$ is orthogonal to the space F_{n-1}. It is obvious that the range of $P_{i_1}^{(1)} \otimes \cdots \otimes P_{i_d}^{(d)}$ is contained in F_n. Thus $P_n \geq P_{i_1}^{(1)} \otimes \cdots \otimes P_{i_d}^{(d)}$. Therefore $P_n \geq Q_n$, where Q_n denotes the right-hand side of (4.1).

We prove now by induction on n, that $P_n = Q_n$. For $n = 0$, this is obvious since both P_0 and Q_0 are projections on the one-dimensional vector space spanned by the vacuum vector 1. Let us assume that $P_k = Q_k$, for all $k \leq n - 1$, and prove that $P_n = Q_n$. To see this, we must show that for any monomial $m(x) = x^{i_1} \cdots x_d^{i_d}$, such that $i_1 + \cdots + i_d = n$, we have $P_n m(x) = Q_n m(x)$. We can write:

$$x_1^{i_1} = P_{i_1}^{(1)} x_1^{i_1} + \sum_{j=0}^{i_1 - 1} P_j^{(1)} x_1^{i_1} ,$$

$$\vdots$$

$$x_d^{i_d} = P_{i_d}^{(d)} x_d^{i_d} + \sum_{j=0}^{i_d - 1} P_j^{(d)} x_d^{i_d} .$$

Multiplying these relations and keeping in mind that we have identified $\mathcal{H}$ with $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_d$, according to our induction bypothesis, we can see that:

$$\begin{aligned} P_n m(x) &= P_{i_1}^{(1)} x_1^{i_1} P_{i_2}^{(2)} x_2^{i_2} \cdots P_{i_d}^{(d)} x_d^{i_d} \\ &= (P_{i_1}^{(1)} \otimes P_{i_2}^{(2)} \otimes \cdots \otimes P_{i_d}^{(d)})(x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d}) \\ &= Q_n m(x) . \end{aligned}$$

□

Let μ be a polynomially factorizable measure on $\mathbb{R}^d$. If we identify the space $\mathcal{H}$ with the space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_d$, then for any $1 \leq j \leq d$, we have the operatorial relation:

$$X_j = I_1 \otimes \cdots \otimes I_{j-1} \otimes X_j^{(j)} \otimes I_{j+1} \otimes \cdots \otimes I_d , \tag{4.2}$$

where I_k is the identity operator of the space $\mathcal{H}_k$, for any $k \neq j$.

Theorem 4.1. *Let μ be a polynomially factorizable measure on $\mathbb{R}^d$. If we identify the space $\mathcal{H}$ with the space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_d$, then for all $1 \leq j \leq d$ and $n \geq 0$*

the following three operatorial relations hold:

$$D_n^-(j) = \sum_{i_1+\cdots+i_d=n} P_{i_1}^{(1)} \otimes \cdots \otimes P_{i_{j-1}}^{(j-1)} \otimes D_{i_j,j}^- \otimes P_{i_{j+1}}^{(j+1)} \otimes \cdots \otimes P_{i_d}^{(d)},$$

$$D_n^0(j) = \sum_{i_1+\cdots+i_d=n} P_{i_1}^{(1)} \otimes \cdots \otimes P_{i_{j-1}}^{(j-1)} \otimes D_{i_j,j}^0 \otimes P_{i_{j+1}}^{(j+1)} \otimes \cdots \otimes P_{i_d}^{(d)},$$

$$D_n^+(j) = \sum_{i_1+\cdots+i_d=n} P_{i_1}^{(1)} \otimes \cdots \otimes P_{i_{j-1}}^{(j-1)} \otimes D_{i_j,j}^+ \otimes P_{i_{j+1}}^{(j+1)} \otimes \cdots \otimes P_{i_d}^{(d)}.$$

Proof. We will check only the first relation. The other two will be proved similarly. We have:

$$\begin{aligned}
&D_n^-(j)\\
&\quad = P_{n-1}X_jP_n\\
&\quad = P_{n-1}(I_1 \otimes \cdots \otimes X_j^{(j)} \otimes \cdots \otimes I_d)\\
&\qquad \times \sum_{i_1+\cdots+i_d=n} P_{i_1}^{(1)} \otimes \cdots \otimes P_{i_j}^{(j)} \otimes \cdots \otimes P_{i_d}^{(d)}\\
&\quad = P_{n-1} \sum_{i_1+\cdots+i_d=n} (I_1P_{i_1}^{(1)}) \otimes \cdots \otimes (X_j^{(j)}P_{i_j}^{(j)}) \otimes \cdots \otimes (I_dP_{i_d}^{(d)})\\
&\quad = \sum_{k_1+\cdots+k_d=n-1} P_{k_1}^{(1)} \otimes \cdots \otimes P_{k_j}^{(j)} \otimes \cdots \otimes P_{k_d}^{(d)}\\
&\qquad \times \sum_{i_1+\cdots+i_d=n} P_{i_1}^{(1)} \otimes \cdots \otimes (X_j^{(j)}P_{i_j}^{(j)}) \otimes \cdots \otimes P_{i_d}^{(d)}\\
&\quad = \sum_{k_1+\cdots+k_d=n-1} \sum_{i_1+\cdots+i_d=n} [(P_{k_1}^{(1)}P_{i_1}^{(1)}) \otimes \cdots \otimes (P_{k_j}^{(j)}X_j^{(j)}P_{i_j}^{(j)})\\
&\qquad \otimes \cdots \otimes (P_{k_d}^{(d)}P_{i_d}^{(d)})],
\end{aligned}$$

where "×" means composition of operators.

Observe that if

$$(P_{k_1}^{(1)}P_{i_1}^{(1)}) \otimes \cdots \otimes (P_{k_j}^{(j)}X_j^{(j)}P_{i_j}^{(j)}) \otimes \cdots \otimes (P_{k_d}^{(d)}P_{i_d}^{(d)}) \neq 0,$$

then $i_1 = k_1, \ldots, i_{j-1} = k_{j-1}$, $i_{j+1} = k_{j+1}, \ldots, i_d = k_d$. Since $i_1 + \cdots + i_d = n$ and $k_1 + \cdots + k_d = n - 1$, it follows that $i_j = k_j + 1$. Thus we obtain:

$$\begin{aligned}
D_n^-(j) &= \sum_{i_1+\cdots+i_d=n} P_{i_1}^{(1)} \otimes \cdots \otimes (P_{i_j-1}^{(j)}X_j^{(j)}P_{i_j}^{(j)}) \otimes \cdots \otimes P_{i_d}^{(d)}\\
&= \sum_{i_1+\cdots+i_d=n} P_{i_1}^{(1)} \otimes \cdots \otimes D_{i_j,j}^- \otimes \cdots \otimes P_{i_d}^{(d)}.
\end{aligned}$$

□

Theorem 4.2. *Let μ be a polynomially factorizable probability measure on the Borel subsets of $\mathbb{R}^d$. Then, for all $j \neq k$, we have:*

$$a^-(j)a^+(k) = a^+(k)a^-(j)\,, \tag{4.3}$$

$$a^0(j)a^+(k) = a^+(k)a^0(j)\,, \tag{4.4}$$

$$a^0(j)a^-(k) = a^-(k)a^0(j)\,. \tag{4.5}$$

Proof. To check the commutation relation $a^-(j)a^+(k) = a^+(k)a^-(j)$, we need to check that for all $n \geq 0$, we have $D^-_{n+1}(j)D^+_n(k) = D^+_{n-1}(k)D^-_n(j)$. Let us assume that $j < k$. The case $j > k$ can be treated similarly or by duality. From the previous theorem we know that:

$$\begin{aligned}
&D^-_{n+1}(j)D^+_n(k) \\
&\quad = \sum_{r_1+\cdots+r_d=n+1} P^{(1)}_{r_1} \otimes \cdots \otimes P^{(j-1)}_{r_{j-1}} \otimes D^-_{r_j,j} \otimes P^{(j+1)}_{r_{j+1}} \otimes \cdots \otimes P^{(d)}_{r_d} \\
&\sum_{s_1+\cdots+s_d=n} P^{(1)}_{s_1} \otimes \cdots \otimes P^{(k-1)}_{s_{k-1}} \otimes D^+_{s_k,k} \otimes P^{(k+1)}_{s_{k+1}} \otimes \cdots \otimes P^{(d)}_{s_d} \\
&\quad = \sum_{r_1+\cdots+r_d=n+1} \sum_{s_1+\cdots+s_d=n} [(P^{(1)}_{r_1}P^{(1)}_{s_1}) \otimes \cdots \otimes (D^-_{r_j,j}P^{(j)}_{s_j}) \\
&\qquad \otimes \cdots \otimes (P^{(k)}_{r_k}D^+_{s_k,k}) \otimes \cdots \otimes (P^{(d)}_{r_d}P^{(d)}_{s_d})]\,.
\end{aligned}$$

To have $P^{(l)}_{r_l}P^{(l)}_{s_l} \neq 0$, for all $l \in \{1,\ldots,d\} \setminus \{j,k\}$, we must have $r_l = s_l$, for all $l \in \{1,\ldots,d\} \setminus \{j,k\}$.

Since $D^-_{r_j,j}P^{(j)}_{s_j} = P^{(j)}_{r_j-1}X^{(j)}_j P^{(j)}_{r_j}P^{(j)}_{s_j}$, if $D^-_{r_j,j}P^{(j)}_{s_j} \neq 0$, then $r_j = s_j$. If $r_j = s_j$, then $D^-_{r_j,j}P^{(j)}_{s_j} = D^-_{s_j,j}$.

Since $P^{(k)}_{r_k}D^+_{s_k,k} = P^{(k)}_{r_k}P^{(k)}_{s_k+1}X^{(k)}_k P^{(k)}_{s_k}$, if $P^{(k)}_{r_k}D^+_{s_k,k} \neq 0$, then $r_k = s_k + 1$. If $r_k = s_k + 1$, then $P^{(k)}_{r_k}D^+_{s_k,k} = D^+_{s_k,k}$.

Thus we obtain:

$$\begin{aligned}
&D^-_{n+1}(j)D^+_n(k) \\
&\quad = \sum_{s_1+\cdots+s_d=n} P^{(1)}_{s_1} \otimes \ldots \otimes D^-_{s_j,j} \otimes \cdots \otimes D^+_{s_k,k} \otimes \cdots \otimes P^{(d)}_{s_d}\,.
\end{aligned}$$

In the same way, we can show that:

$$\begin{aligned}
&D^+_{n-1}(k)D^-_n(j) \\
&\quad = \sum_{s_1+\cdots+s_d=n} P^{(1)}_{s_1} \otimes \cdots \otimes D^-_{s_j,j} \otimes \cdots \otimes D^+_{s_k,k} \otimes \cdots \otimes P^{(d)}_{s_d}\,.
\end{aligned}$$

Thus $D^-_{n+1}(j)D^+_n(k) = D^+_{n-1}(k)D^-_n(j)$.

The commutation relationships (4.4) and (4.5) are proved similarly. □

Proposition 4.1. *If μ is a probability measure on $\mathbb{R}^d$ such that for any $j \neq k$, the operators $a^-(j)$ and $a^+(k)$ commute, then for any j, $k \in \{1, 2, \ldots, d\}$ we have*:

$$a^0(j)a^0(k) = a^0(k)a^0(j)\,. \tag{4.6}$$

Proof. Let j, $k \in \{1, 2, \ldots, d\}$ be fixed.

If $j = k$, then it is obvious that $a^0(j)$ and $a^0(k)$ commute.

If $j \neq k$, then, according to Eq. (2.10), for any $n \geq 0$, we have:

$$\begin{aligned} &D^-_{n+1}(j)D^+_n(k) + D^0_n(j)D^0_n(k) + D^+_{n-1}(j)D^-_n(k) \\ &\quad = D^-_{n+1}(k)D^+_n(j) + D^0_n(k)D^0_n(j) + D^+_{n-1}(k)D^-_n(j)\,. \end{aligned}$$

Since $a^-(j)a^+(k) = a^+(k)a^-(j)$ it follows that

$$D^-_{n+1}(j)D^+_n(k) = D^+_{n-1}(k)D^-_n(j)\,.$$

Since $a^+(j)a^-(k) = a^-(k)a^+(j)$ it follows that

$$D^+_{n-1}(j)D^-_n(k) = D^-_{n+1}(k)D^+_n(j)\,.$$

Thus we obtain

$$D^0_n(j)D^0_n(k) = D^0_n(k)D^0_n(j)\,.$$

Since this is true for all $n \geq 1$, we obtain $a^0(j)a^0(k) = a^0(k)a^0(j)$. □

Proposition 4.2. *Let μ be a probability measure on $\mathbb{R}^d$ having finite moments of any order. Let j, $k \in \{1, 2, \ldots, d\}$. Then the following two statements are equivalent*:

(1) $a^0(j)a^+(k) = a^+(k)a^0(j)$ *and* $a^0(j)a^0(k) = a^0(k)a^0(j)$.
(2) $a^0(j)X_k = X_k a^0(j)$.

Proof. (1) ⇒ (2) Let us assume that $a^0(j)a^+(k) = a^+(k)a^0(j)$ and $a^0(j)a^0(k) = a^0(k)a^0(j)$. Taking the adjoints on both sides of the first equality we obtain $a^-(k)a^0(j) = a^0(j)a^-(k)$. Thus:

$$\begin{aligned} a^0(j)X_k &= a^0(j)[a^+(k) + a^0(k) + a^-(k)] \\ &= a^0(j)a^+(k) + a^0(j)a^0(k) + a^0(j)a^-(k) \\ &= a^+(k)a^0(j) + a^0(k)a^0(j) + a^-(k)a^0(j) \\ &= [a^+(k) + a^0(k) + a^-(k)]a^0(j) \\ &= X_k a^0(j)\,. \end{aligned}$$

(2) ⇒ (1) Let us assume now that $a^0(j)X_k = X_k a^0(j)$. Let $n \geq 0$ be fixed. For any $\varphi \in G_n$ we have $a^0(j)X_k\varphi = X_k a^0(j)\varphi$. This means

$$\begin{aligned} &a^0(j)a^+(k)\varphi + a^0(j)a^0(k)\varphi + a^0(j)a^-(k)\varphi \\ &\quad = a^+(k)a^0(j)\varphi + a^0(k)a^0(j)\varphi + a^-(k)a^0(j)\varphi\,. \end{aligned}$$

Since $a^0(j)a^+(k)\varphi \in G_{n+1}$ and $a^+(k)a^0(j)\varphi \in G_{n+1}$, $a^0(j)a^0(k)\varphi \in G_n$ $a^0(k)a^0(j)\varphi \in G_n$, $a^0(j)a^-(k)\varphi \in G_{n-1}$ and $a^-(k)a^0(j)\varphi \in G_{n-1}$, and the spaces G_{n+1}, G_n and G_{n-1} are orthogonal, we obtain: $a^0(j)a^+(k)\varphi = a^+(k)a^0(j)\varphi$, $a^0(j)a^0(k)\varphi = a^0(k)a^0(j)\varphi$ and $a^0(j)a^-(k)\varphi = a^-(k)a^0(j)\varphi$. □

Combining Theorem 4.2, Proposition 4.1, and Theorem 2.3 we obtain the following:

Corollary 4.1. *Let μ be a polynomially factorizable probability measure on the Borel subsets of $\mathbb{R}^d$. If j, $k \in \{1, 2, \ldots, d\}$ and $j \neq k$, then for any $Y \in \{a^-(j), a^0(j), a^+(j)\}$ and $Z \in \{a^-(k), a^0(k), a^+(k)\}$, we have:*

$$YZ = ZY\,.$$

Theorem 4.3. *If μ is a polynomially factorizable probability measure on the Borel subsets of $\mathbb{R}^d$, then for all i, j, $k \in \{1, 2, \ldots, d\}$, we have:*

$$[a^0(i), [a^-(j), a^+(k)]] = 0\,, \tag{4.7}$$

where $[A, B] := AB - BA$ for any two operators A and B.

Proof. We analyze three cases:

Case I. If $j \neq k$, then Theorem 4.2 implies $[a^-(j), a^+(k)] = 0$. Thus $[a^0(i), [a^-(j), a^+(k)]] = 0$.

Case II. If $j = k$ and $i \neq j$, then according to Theorem 4.2, $a^0(i)$ commutes with both operators $a^-(j)$ and $a^+(j)$. Thus $a^0(i)$ commutes with the commutator of $a^-(j)$ and $a^+(j)$.

Case III. If $i = j = k$, then according to Theorem 4.1, for any $n \geq 0$, we have:

$$a^0(j)[a^-(j), a^+(j)]_{|G_n}$$
$$= \sum_{l_1+\cdots+l_d=n} P_{l_1}^{(1)} \otimes \cdots \otimes D_{l_j,j}^0 (D_{l_j+1,j}^- D_{l_j,j}^+ - D_{l_j-1,j}^+ D_{l_j,j}^-) \otimes \cdots \otimes P_{l_d}^{(d)}\,.$$

We also have:

$$[a^-(j), a^+(j)]a^0(j)_{|G_n}$$
$$= \sum_{l_1+\cdots+l_d=n} P_{l_1}^{(1)} \otimes \cdots \otimes (D_{l_j+1,j}^- D_{l_j,j}^+ - D_{l_j-1,j}^+ D_{l_j,j}^-) D_{l_j,j}^0 \otimes \cdots \otimes P_{l_d}^{(d)}\,.$$

$D_{l_j,j}^0$ and $D_{l_j+1,j}^- D_{l_j,j}^+ - D_{l_j-1,j}^+ D_{l_j,j}^-$ are linear operators from the space $G_n^{(j)}$ into itself. Since the vector space $G_n^{(j)}$ has dimension at most 1, any two linear operators from $G_n^{(j)}$ into $G_n^{(j)}$ commute. Thus, we have:

$$D_{l_j,j}^0 (D_{l_j+1,j}^- D_{l_j,j}^+ - D_{l_j-1,j}^+ D_{l_j,j}^-) = (D_{l_j+1,j}^- D_{l_j,j}^+ - D_{l_j-1,j}^+ D_{l_j,j}^-) D_{l_j,j}^0\,.$$

Therefore, we obtain:

$$a^0(j)[a^-(j), a^+(j)]_{|G_n} = [a^-(j), a^+(j)]a^0(j)_{|G_n}\,,$$

for all $n \geq 0$. Thus $[a^0(j), [a^-(j), a^+(j)]] = 0$. □

We prove now the converse of Theorem 4.2.

Theorem 4.4. *Let μ be a probability measure on the Borel subsets of $\mathbb{R}^d$ such that, for all $j \neq k$, we have*:

$$a^-(j)a^+(k) = a^+(k)a^-(j)\,,$$
$$a^0(j)a^+(k) = a^+(k)a^0(j)\,.$$

Then μ is polynomially factorizable.

Proof. Since for all $j \neq k$, $a^0(j)a^+(k) = a^+(k)a^0(j)$, taking the adjoints on both sides of this equation, we obtain: $a^-(k)a^0(j) = a^0(j)a^-(k)$. Because for all $j \neq k$, the operators $a^-(j)$ and $a^+(k)$ commute, according to Proposition 4.1, for any r, $s \in \{1, 2, \ldots, d\}$, the operators $a^0(r)$ and $a^0(s)$ also commute. Using Theorem 2.3, we conclude that, for all $j \neq k$, any operator from the set $\{a^-(j), a^0(j), a^+(j)\}$ commutes with any operator from the set $\{a^-(k), a^0(k), a^+(k)\}$.

For any $1 \leq j \leq d$ and $n \geq 1$, we define a j-word of length n, to be any operator of the form $w = a^{\varepsilon_1}(j)a^{\varepsilon_2}(j)\cdots a^{\varepsilon_n}(j)$, where $\varepsilon_i \in \{-, 0, +\}$, $\forall\ 1 \leq i \leq n$. The operators $a^{\varepsilon_1}(j)$, $a^{\varepsilon_2}(j), \ldots, a^{\varepsilon_n}(j)$ are called the letters of the word w.

The identity operator I, of $L^2(\mathbb{R}^d, \mu)$, is considered to be a j-word of length 0, for any $1 \leq j \leq d$.

Let $m(x) = x_1^{i_1}x_2^{i_2}\cdots x_d^{i_d}$ be a monomial. Let $\phi = 1$ be the vacuum vector (the constant polynomial function 1). We have

$$\begin{aligned} m(x) &= X_1^{i_1}X_2^{i_2}\cdots X_d^{i_d}\phi \\ &= (a^-(1) + a^0(1) + a^+(1))^{i_1}\cdots(a^-(d) + a^0(d) + a^+(d))^{i_d}\phi \\ &= \sum (a^{\varepsilon_1^1}(1)a^{\varepsilon_2^1}(1)\cdots a^{\varepsilon_{i_1}^1}(1))\cdots(a^{\varepsilon_1^d}(d)a^{\varepsilon_2^d}(d)\cdots a^{\varepsilon_{i_d}^d}(d))\phi\,, \end{aligned}$$

where $\varepsilon_r^s \in \{-, 0, +\}$, for all $1 \leq s \leq d$ and $1 \leq r \leq i_s$. Thus

$$m(x) = \sum w_1 w_2 \cdots w_d \phi\,,$$

where w_j is a j-word of length i_j, $\forall\ 1 \leq j \leq d$. According to our hypothesis, if $j \neq k$, then any j-word commutes with any k-word.

Since for any $1 \leq j \leq d$ and any integer n, $a^-(j)_{|F_n} : F_n \to F_{n-1}$, we say that $a^-(j)$ represents one step backward. Similarly we say that $a^+(j)$ represents one step forward, while $a^0(j)$ represents a neutral step. Thus $a^-(j)$ is considered to be a negative letter, $a^+(j)$ a positive letter, and $a^0(j)$ a neutral letter. If n is a negative integer, then we declare F_n to be the null space $\{0\}$. We define the signum $s(w)$ of a word w, to be the number of positive letters of w minus the number of negative letters of w. If w^* denotes the adjoint of w, then $s(w^*) = -s(w)$.

All the terms $w_1 w_2 \cdots w_d \phi$, containing at least one word w_{j_0} that has more negative letters than positive letters, are equal to 0. Indeed, for such a term,

$$w_1 w_2 \cdots w_d \phi = w_1 \cdots w_{j_0-1} w_{j_0+1} \cdots w_d (w_{j_0}\phi) = 0\,,$$

since $w_{j_0}\phi = 0$, because we start from the vacuum space and do more steps backward than we do forward. Thus $m(x) = \sum w_1 w_2 \cdots w_d \phi$, where each term contains only words having the number of negative letters less than or equal to the number of positive letters. Hence:

$$\begin{aligned} E[m(x)] &= \langle m(x), \phi \rangle \\ &= \sum_{s(w_1)\geq 0, \ldots, s(w_d)\geq 0} \langle w_1 \cdots w_d \phi, \phi \rangle \\ &= \sum_{s(w_1)\geq 0, \ldots, s(w_d)\geq 0} \langle \phi, w_d^* \cdots w_1^* \phi \rangle \,. \end{aligned}$$

Observe that in the last sum all the terms, for which at least one of the words $\{w_j\}_{1\leq j\leq d}$ has a positive signum, are equal to zero. This is true, because if there exists $j \in \{1, 2, \ldots, d\}$ such that $s(w_j) > 0$, then $s(w_j^*) < 0$ and, it follows, as before, that $w_d^* \cdots w_1^* \phi = 0$. Therefore, we have:

$$E[m(x)] = \sum_{s(w_1)=0, \ldots, s(w_d)=0} \langle w_1 \cdots w_d \phi, \phi \rangle \,.$$

Observe that, since $s(w_d) = 0$, $w_d\phi \in F_0$, and thus $w_d\phi = P_0(w_d\phi) = \langle w_d\phi, \phi\rangle\phi = E[w_d\phi]\phi$. Applying w_{d-1} to both sides of the equality: $w_d\phi = E[w_d\phi]\phi$, we get $w_{d-1}w_d\phi = E[w_d\phi]E[w_{d-1}\phi]\phi$. Iterating this process we obtain finally:

$$w_1 \cdots w_d \phi = E[w_d\phi] \cdots E[w_1\phi]\phi \,.$$

Thus we obtain:

$$\begin{aligned} E[m(x)] &= \sum_{s(w_1)=0, s(w_2)=0, \ldots, s(w_d)=0} \langle w_1 w_2 \cdots w_d \phi, \phi \rangle \\ &= \sum_{s(w_1)=0, s(w_2)=0, \ldots, s(w_d)=0} \langle E[w_d\phi]E[w_{d-1}\phi] \cdots E[w_1\phi]\phi, \phi \rangle \\ &= \sum_{s(w_1)=0, s(w_2)=0, \ldots, s(w_d)=0} E[w_d\phi]E[w_{d-1}\phi] \cdots E[w_1\phi]\langle \phi, \phi \rangle \\ &= \sum_{s(w_1)=0, s(w_2)=0, \ldots, s(w_d)=0} E[w_1\phi]E[w_2\phi] \cdots E[w_d\phi] \\ &= \sum_{s(w_1)=0} E[w_1\phi] \sum_{s(w_2)=0} E[w_2\phi] \cdots \sum_{s(w_d)=0} E[w_d\phi] \,. \end{aligned}$$

Applying the last equality to the particular monomials $m_1(x) = x_1^{i_1}$, $m_2(x) = x_2^{i_2}, \ldots, m_d(x) = x_d^{i_d}$, we can see that: $E[x_1^{i_1}] = \sum_{s(w_1)=0} E[w_1\phi]$, $E[x_2^{i_2}] = \sum_{s(w_2)=0} E[w_2\phi], \ldots, E[x_d^{i_d}] = \sum_{s(w_d)=0} E[w_d\phi]$. Thus:

$$E[m(x)] = E[x_1^{i_1}]E[x_2^{i_2}] \cdots E[x_d^{i_d}] \,.$$

Therefore μ is polynomially factorizable. $\square$

From Proposition 4.2, Theorem 4.2 and Theorem 4.4 we obtain the following:

Theorem 4.5. *A probability measure μ on $\mathbb{R}^d$, having finite moments of any order, is polynomially factorizable if and only if for all $j, k \in \{1, 2, \ldots, d\}$, such that $j \neq k$, the commutators $[a^-(j), a^+(k)]$ and $[a^0(j), X_k]$ are both equal to zero.*

Corollary 4.2. *A polynomially symmetric about a point probability measure on $\mathbb{R}^d$ is polynomially factorizable if and only if for all $j \neq k$ we have $a^-(j)a^+(k) = a^+(k)a^-(j)$.*

5. Examples

In this section we will give an example of a polynomially symmetric probability measure that is not symmetric and an example of a polynomially factorizable probability measure that is not a product measure.

Let us consider the following function, introduced by Stieltjes in Ref. 12: $f : (0, \infty) \to \mathbb{R}$, $f(t) = t^{-\ln t} \sin(2\pi \ln t)$.

Claim 5.1. *For all non-negative integer n, we have*:

$$\int_0^\infty t^n f(t)dt = 0\,. \tag{5.1}$$

We present below the proof of this claim as it appears in Ref. 11. Indeed, for any non-negative integer n, making the change of variable $t = e^{u+\frac{n+1}{2}}$, we have:

$$\begin{aligned}
&\int_0^\infty t^n f(t)dt \\
&\quad = \int_0^\infty t^n t^{-\ln t} \sin(2\pi \ln t)dt \\
&\quad = \int_{\mathbb{R}} e^{n(u+\frac{n+1}{2})} e^{-(u+\frac{n+1}{2})^2} \sin\left(2\pi\left(u + \frac{n+1}{2}\right)\right) e^{u+\frac{n+1}{2}} du \\
&\quad = e^{(\frac{n+1}{2})^2} \int_{\mathbb{R}} e^{-u^2} [\sin(2\pi u)\cos((n+1)\pi) + \cos(2\pi u)\sin((n+1)\pi)]du \\
&\quad = (-1)^{n+1} e^{(\frac{n+1}{2})^2} \int_{\mathbb{R}} e^{-u^2} \sin(2\pi u)du \\
&\quad = 0\,.
\end{aligned}$$

The last integral is zero since the integrand is an odd function.

Example 5.1. Let μ be the probability measure on $\mathbb{R}$ given by the density function $g : \mathbb{R} \to [0, \infty)$, defined by:

$$g(x) = \begin{cases} cf^+(x) & \text{if } x > 0\,, \\ 0 & \text{if } x = 0\,, \\ cf^-(-x) & \text{if } x < 0\,, \end{cases}$$

where $a^+ = \max(a, 0)$, $a^- = -\min(a, 0)$, $\forall\, a \in \mathbb{R}$, and c is a positive constant chosen such that $\int_{\mathbb{R}} g(x)dx = 1$.

It is easy to see that μ has finite moments of any order.

Claim 5.2. *μ is polynomially symmetric.*

Indeed, if n is an odd natural number, then we have:

$$\begin{aligned}\int_{\mathbb{R}} x^n \mu(dx) &= \int_{\mathbb{R}} x^n g(x)dx \\ &= c\int_{-\infty}^{0} x^n f^-(-x)dx + c\int_0^\infty x^n f^+(x)dx \\ &= c\int_0^\infty (-t)^n f^-(t)dt + c\int_0^\infty t^n f^+(t)dt \\ &= -c\int_0^\infty t^n f^-(t)dt + c\int_0^\infty t^n f^+(t)dt \\ &= c\int_0^\infty t^n [f^+(t) - f^-(t)]dt \\ &= c\int_0^\infty t^n f(t)dt \\ &= 0\,.\end{aligned}$$

Claim 5.3. *μ is not symmetric.*

Indeed, for any interval $[a, b]$ contained in the set: $\{x > 0 | f(x) > 0\}$ we have $\int_a^b g(x)dx = c\int_a^b f^+(x)dx > 0$, but $\int_{-b}^{-a} g(x)dx = c\int_a^b f^-(x)dx = 0$. Thus $\mu([a, b]) \neq \mu([-b, -a])$.

Let $f_1 : (0,\infty) \to [0,\infty)$, $f_1(t) = t^{-\ln t}[1+\sin(2\pi \ln t)]$ and $f_2 : (0,\infty) \to [0,\infty)$, $f_2(t) = t^{-\ln t}[1 - \sin(2\pi \ln t)]$. Since, according to (5.1), $\int_0^\infty t^n t^{-\ln t} \sin(2\pi \ln t)dt = 0$, for any non-negative integer n, we have:

$$\int_0^\infty t^n f_1(t)dt = \int_0^\infty t^n f_2(t)dt\,, \qquad \forall\, n \in \mathbb{N} \cup \{0\}\,. \tag{5.2}$$

In particular for $n = 0$, let $k := \int_0^\infty f_1(t)dt = \int_0^\infty f_2(t)dt$.

Example 5.2. Let μ be the probability measure on $\mathbb{R}^2$ given by the density function $h : \mathbb{R}^2 \to [0, \infty)$, defined by:

$$h(x, y) = \begin{cases} \dfrac{1}{k}[f_1(x)\sin^2 y + f_2(x)\cos^2 y]e^{-y} & \text{if } x > 0 \text{ and } y > 0 \\ 0 & \text{otherwise}\,. \end{cases}$$

For any non-negative integers m and n, we have:

$$\begin{aligned}
&\int_{\mathbb{R}^2} x^n y^m \mu(dxdy) \\
&\quad = \frac{1}{k}\int_0^\infty y^m e^{-y}\left[\sin^2 y \int_0^\infty x^n f_1(x)dx + \cos^2 y \int_0^\infty x^n f_2(x)dx\right] dy \\
&\quad = \frac{1}{k}\int_0^\infty y^m e^{-y}\left[\int_0^\infty x^n f_1(x)dx\right](\sin^2 y + \cos^2 y)dy \\
&\quad = \frac{1}{k}\left[\int_0^\infty x^n f_1(x)dx\right]\int_0^\infty y^m e^{-y}dy\,.
\end{aligned}$$

Thus we obtain:

$$\int_{\mathbb{R}^2} x^n y^m \mu(dxdy) = \frac{1}{k}\left[\int_0^\infty x^n f_1(x)dx\right]\int_0^\infty y^m e^{-y}dy\,. \tag{5.3}$$

In particular for $m = n = 0$, we obtain:

$$\begin{aligned}
\int_{\mathbb{R}^2} 1\mu(dxdy) &= \frac{1}{k}\left[\int_0^\infty f_1(x)dx\right]\int_0^\infty e^{-y}dy \\
&= \frac{1}{k}\cdot k \cdot 1 \\
&= 1\,.
\end{aligned}$$

Hence μ is a probability measure on $\mathbb{R}^2$.

Taking $m = 0$ in formula (5.3), we can see that:

$$\begin{aligned}
\int_{\mathbb{R}^2} x^n \mu(dxdy) &= \frac{1}{k}\int_0^\infty x^n f_1(x)dx \int_0^\infty e^{-y}dy \\
&= \frac{1}{k}\int_0^\infty x^n f_1(x)dx\,.
\end{aligned}$$

Hence

$$\int_{\mathbb{R}^2} x^n \mu(dxdy) = \frac{1}{k}\int_0^\infty x^n f_1(x)dx\,. \tag{5.4}$$

Taking $n = 0$ in formula (5.3), we can see that:

$$\begin{aligned}
\int_{\mathbb{R}^2} y^m \mu(dxdy) &= \frac{1}{k}\int_0^\infty f_1(x)dx \int_0^\infty y^m e^{-y}dy \\
&= \frac{1}{k}\cdot k \cdot \int_0^\infty y^m e^{-y}dy \\
&= \int_0^\infty y^m e^{-y}dy\,.
\end{aligned}$$

Hence

$$\int_{\mathbb{R}^2} y^m \mu(dxdy) = \int_0^\infty y^m e^{-y} dy\,. \tag{5.5}$$

From formulas (5.3)–(5.5), we can see that:

$$\int_{\mathbb{R}^2} x^n y^m \mu(dxdy) = \int_0^\infty x^n \mu(dxdy) \int_0^\infty y^m \mu(dxdy)\,. \tag{5.6}$$

Thus μ is polynomially factorizable.

Claim 5.4. *μ is not a product measure.*

Let us assume that two probability measures, μ_1 and μ_2, on $\mathbb{R}$, exist, such that for any two Borel subsets B_1 and B_2 of $\mathbb{R}$, we have $\mu(B_1 \times B_2) = \mu_1(B_1)\mu_2(B_2)$. Since μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$, it follows that μ_1 and μ_2 are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. If u and v are the density functions of μ_1 and μ_2, respectively, then we must have for almost all $(x, y) \in \mathbb{R}^2$, $h(x, y) = u(x)v(y)$, which is impossible since $h(x, y)$ cannot be written as a function of x times a function of y.

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