Junior problems

J79. Find all integers that can be represented as $a^3+b^3+c^3-3abc$ for some positive integers a, b, and c.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Salem Malikic Sarajevo, Bosnia and Herzegovina

We have

$$w = a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca) =$$

$$= \frac{(a+b+c)}{2}[(a-b)^2+(b-c)^2+(c-a)^2].$$

Let $a=b=n+1$ and $c=n$, then $w=3n+2$ so each positive integer of the form $3n+2$ can be written as $a^3+b^3+c^3-3abc$. Taking $a=n+1$, $b=c=n$ we can represent all positive integers of the form $3n+1$.

If we take $a=n-1$, $b=n$, and $c=n+1$ we find that $w=9n$, $n \geq 2$. Thus each positive integer divisible by 9 except 9 can be written in the given form. Now we will prove that integers of the form $9k+3$ and $9k+6$ cannot be written as

$$\frac{(a+b+c)}{2}[(a-b)^2+(b-c)^2+(c-a)^2].$$

Since $x^2 \equiv 0, 1 \pmod 3$ the number $(a-b)^2+(b-c)^2+(c-a)^2$ is divisible by three if and only if $(a-b)^2 \equiv (b-c)^2 \equiv (c-a)^2 \equiv 0 \pmod 3$ or $(a-b)^2 \equiv (b-c)^2 \equiv (c-a)^2 \equiv 1 \pmod 3$.

If $(a-b)^2 \equiv (b-c)^2 \equiv (c-a)^2 \equiv 0 \pmod 3$ then $a \equiv b \equiv c \equiv 0 \pmod 3$. Thus $3|(a+b+c)$ so $9|\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$ and so $w \neq 9k \pm 3$.

If $(a-b)^2 \equiv (b-c)^2 \equiv (c-a)^2 \equiv 1 \pmod 3$ then no two of a, b, c give same residue mod 3 and we can assume without loss of generality that $a \equiv 1, b \equiv 2$, and $c \equiv 0 \pmod 3$ and so $a+b+c \equiv 0 \pmod 3$ and $9 \mid \frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$ which means $w \neq 9k \pm 3$. From these, we conclude that, if w is of the form $9k \pm 3$ then $3 \mid a+b+c$, while 3 does not divide $(a-b)^2+(b-c)^2+(c-a)^2$.

It is easy to check that $3 \mid a+b+c$ if one of the following residue situation occurs:

$$(a, b, c) = (0,0,0), (1,1,1), (2,2,2), (2,1,0).$$

In each case $(a-b)^2+(b-c)^2+(c-a)^2$ is divisible by three so if $\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$ is divisible by 3, then it is divisible by 9 and therefore it cannot be of the form $9k \pm 3$.

Second solution by Daniel Campos Salas, Costa Rica

Let us say an integer is nice if it can be represented as $a^3+b^3+c^3-3abc$ for some positive integers a, b, c. Assume without loss of generality that $b = a+x$ and $c = a+x+y$, for some nonnegative integers x, y. Therefore,

$$a^3+b^3+c^3-3abc = (3a+2x+y)(x^2+xy+y^2).$$

For $x = y = 0$ it follows that 0 is nice. Suppose that x, y are not both zero. Since $(3a+2x+y)(x^2+xy+y^2) > 0$ we have that any nonzero nice integer is nonnegative. Let us prove first that 1 and 2 are not nice. We have that

$$(3a+2x+y)(x^2+xy+y^2) > 3a(x^2+xy+y^2) \geq 3,$$

from where it follows the claim. Let us prove also that any nice integer divisible by 3 must be divisible by 9. We have that

$$0 \equiv (3a+2x+y)(x^2+xy+y^2) \equiv (y-x)\cdot(x-y)^2 \equiv (y-x)^3 \pmod 3,$$

from where it follows that $x \equiv y \pmod 3$. Therefore,

$$3a+2x+y \equiv x^2+xy+y^2 \equiv 0 \pmod 3,$$

which implies the claim. Let us prove that 9 is not nice. From the previous result we have that $x \equiv y \pmod 3$, from where it follows that

$$(3a+2x+y)(x^2+xy+y^2) \geq (3a+3)\cdot 3 > 9.$$

Let us proceed to find which integers are nice. Taking $x = 0, y = 1$ it follows that any positive integer of the form $3a+1$ is nice. Taking $x = 1, y = 0$ it follows that any positive integer of the form $3a+2$ is nice. Taking $x = y = 1$ it follows that any positive integer of the form $9(a+1)$ is nice. From these we conclude that all the nice integers are 0, any positive integer greater than 3 of the form $3a+1$ or $3a+2$, and the integers greater than 9 of the form $9a$, and we are done.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; John T. Robinson, Yorktown Heights, NY, USA; Roberto Bosch Cabrera, Faculty of Mathematics, University of Havana, Cuba.

J80. Characterize triangles with sidelengths in arithmetical progression and lengths of medians also in arithmetical progression.

Proposed by Daniel Lasaosa, Universidad Publica de Navarra, Spain

First solution by Magkos Athanasios, Kozani, Greece

We prove that only equilateral triangles have the desired property. Let ABC be a triangle with sides a, b, c and the corresponding medians m_a, m_b, m_c. Assume that $a \geq b \geq c$. Then we have $m_a \leq m_b \leq m_c$. Since the sides and the medians form arithmetic progressions we have

$$2b = a + c, \quad 2m_b = m_a + m_c. \qquad (1)$$

It is a known fact that $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$. and we also know that for reals x, y, z we have the inequality $3(x^2 + y^2 + z^2) \geq (x + y + z)^2$. Hence, we have

$$\frac{3}{4}(a^2 + b^2 + c^2) = m_a^2 + m_b^2 + m_c^2 \geq \frac{1}{3}(m_a + m_b + m_c)^2.$$

From this, using (1) and the relation $4m_b^2 = 2a^2 + 2c^2 - b^2$, we obtain the following chain of inequalities

$$\frac{9}{4}(a^2+b^2+c^2) \geq (3m_b)^2 \;\Leftrightarrow\; a^2+b^2+c^2 \geq 4m_b^2 \;\Leftrightarrow\; a^2+b^2+c^2 \geq 2a^2+2c^2-b^2.$$

Hence, $2b^2 \geq a^2 + c^2 \;\Leftrightarrow\; (a + c)^2 \geq 2a^2 + 2c^2 \;\Leftrightarrow\; (a - c)^2 \leq 0$. This means that $a = c$. Therefore, we have $a = b = c$.

Second solution by Vicente Vicario Garcia, Huelva, Spain

We use the habitual notation in a triangle. Without loss of generality $a \leq b \leq c$. By the well known Apollonius-formulas (application of Stewart's theorem) for the medians of a triangle we have

$$m_A = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

and the analoguous ones. We can then deduce that $m_C \leq m_B \leq m_A$. By the properties of arithmetic progression we have that

$$a + c = 2b \quad [1]$$

$$m_A + m_C = 2m_B \quad [2].$$

By the relations [1] and [2] we have

$$\sqrt{2b^2+2c^2-a^2}+\sqrt{2a^2+2b^2-c^2}=2\sqrt{2a^2+2c^2-b^2}\Leftrightarrow$$

$$a^2+4b^2+2\sqrt{(2b^2+2c^2-a^2)(2a^2+2b^2-c^2)}=4(2a^2+2c^2-b^2).$$

Doing all the subsequent calculations will yeild

$$5\left(\frac{c}{a}\right)^4-8\left(\frac{c}{a}\right)^3+6\left(\frac{c}{a}\right)^2-8\left(\frac{c}{a}\right)+5=0.$$

Then the polynomial $P(x)=5x^4-8x^3+6x^2-8x+5=(x-1)^2(5x^2+2x+5)$ and the quadratic equation $5x^2+2x+5$ has no real roots. Then $\frac{c}{a}=1$ which means $a=b=c$ and the triangle is equilateral. Finally, it is clear that equilateral triangle satisfies the problem and we are done.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; John T. Robinson, Yorktown Heights, NY, USA; Roberto Bosch Cabrera, Faculty of Mathematics, University of Havana, Cuba.

J81. Let a, b, c be positive real numbers such that

$$\frac{1}{a^2+b^2+1}+\frac{1}{b^2+c^2+1}+\frac{1}{c^2+a^2+1}\geq 1.$$

Prove that $ab+bc+ca\leq 3$.

Proposed by Alex Anderson, New Trier High School, Winnetka, USA

First solution by Dinh Cao Phan, Pleiku, GiaLai, Vietnam

By applying the Cauchy-Scharz inequlity, we have

$$(a^2+b^2+1)(1+1+c^2)\geq(a+b+c)^2$$

or

$$\frac{1}{a^2+b^2+1}\leq\frac{2+c^2}{(a+b+c)^2}.$$

Similarily, we obtain

$$\frac{1}{b^2+c^2+1}\leq\frac{2+a^2}{(a+b+c)^2},\frac{1}{c^2+a^2+1}\leq\frac{2+b^2}{(a+b+c)^2}.$$

Therefore

$$1\leq\frac{1}{a^2+b^2+1}+\frac{1}{b^2+c^2+1}+\frac{1}{c^2+a^2+1}\leq\frac{6+a^2+b^2+c^2}{(a+b+c)^2}$$

or

$$(a+b+c)^2\leq 6+a^2+b^2+c^2$$

equivalent to

$$a^2+b^2+c^2+2(ab+bc+ca)\leq 6+a^2+b^2+c^2.$$

Thus $ab+bc+ca\leq 3$ and we are done.

Also solved by Oleh Faynshteyn, Leipzig, Germany; Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Nguyen Manh Dung, Hanoi, Vietnam; Perfetti Paolo, Dipartimento di matematica Universita degli studi di Tor Vergata, Italy.

J82. Let $ABCD$ be a quadrilateral whose diagonals are perpendicular. Denote by $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ the centers of the nine-point circles of triangles ABC, BCD, CDA, DAB, respectively. Prove that the diagonals of $\Omega_1\Omega_2\Omega_3\Omega_4$ intersect at the centroid of $ABCD$.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

First solution by G.R.A.20 Math Problems Group, Roma, Italy

Let M_{AB}, M_{BC}, M_{CD}, M_{AD} be the midpoints of the sides AB, BC, CD, DA, respectively. Since the nine-point circle of the triangle ABC passes through the midpoints of its sides we have Ω_1 belongs to the perpedicular bisector of $M_{AB}M_{BC}$. Similarly, Ω_3 belongs to the perpedicular bisector of $M_{CD}M_{DA}$. Since AC and BD are perpendicular we get that $M_{AB}M_{BC}M_{CD}M_{AD}$ is a rectangle. This implies that the line $\Omega_1\Omega_3$ is the midline of opposite sides of this rectangle: $M_{AB}M_{BC}$ and $M_{CD}M_{DA}$. Finally, the intersection of the lines $\Omega_1\Omega_3$ and $\Omega_2\Omega_4$ coincides with the intersection of the diagonals of the rectangle $M_{AB}M_{BC}M_{CD}M_{AD}$ which is the centroid of $ABCD$.

Second solution by Roberto Bosch Cabrera, University of Havana, Cuba

We proceed by coordinate geometry. Let the point of intersection of the diagonals be $(0,0)$. Let $A = (a,0)$, $B = (0,b)$, $C = (c,0)$, $D = (0,d)$, then the centroid of $ABCD$, $G = (\frac{a+c}{4}, \frac{b+d}{4})$.

Now we will find the coordinates of Ω_1. Let H and O be the orthocenter and the circumcenter of triangle ABC. Then $H = (0, \frac{-ac}{b})$ and using that $BH = 2OF$ where F is the feet of the perpendicular to AC from O we obtain that $O = (\frac{a+c}{2}, \frac{b^2+ac}{2b})$. It is well known that Ω_1 is the midpoint of HO, hence $\Omega_1 = (\frac{a+c}{4}, \frac{b^2-ac}{4b})$.

Analogously, $\Omega_2 = (\frac{c^2-bd}{4c}, \frac{b+d}{4})$, $\Omega_3 = (\frac{a+c}{4}, \frac{d^2-ac}{4d})$, $\Omega_4 = (\frac{a^2-bd}{4a}, \frac{b+d}{4})$. Clearly, $\Omega_1\Omega_3$ and $\Omega_2\Omega_4$ pass through G, and we are done.

Third solution by Mihai Miculita, Oradea, Romania

Let M_{ac} and M_{bd} be the midpoints of the diagonals AC and BD. It is known that the centroid G of $ABCD$ coincides with the midpoint of $M_{ac}M_{bd}$. Let O_x and H_x be the cicrumcenter and the orthocenter of triangle YZT, $\{X,Y,Z,T\} = \{A,B,C,D\}$. Points O_d and O_c being cicrumcenters of triangle ABC and ACD, are on the same perpendicular bisector of AC, yielding

$$M_{ac} \in O_dO_b \perp AC. \quad (1)$$

The quadrilateral $ABCD$ having the diagonals intersecting at a right angle implies that BO and DO are heights in the triangles ABC and ACD. Thus $H_d, H_b \in BD$ and

$$H_d H_B \perp AC. \quad (2).$$

From (1) and (2) it follows that $O_d O_b \parallel H_d H_b$. Last relation proves that $O_d O_b H_d H_b$ is a trapezoid. The Euler circle's center in a triangle is the midpoint of the segment determined by the circumcenter and the orthocenter. It follows that Ω_d and Ω_b are the midpoints of $O_d H_d$ and $O_b H_b$. Thus the line $\Omega_d \Omega_b$ is the midline of this trapezoid and so passes through G, the midpoint of $M_{ac} M_{bd}$. Analogoulsly, we prove that $G \in \Omega_a \Omega_c$.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Vicente Vicario Garcia, Huelva, Spain.

J83. Find all positive integers n such that a divides n for all odd positive integers a not exceeding $\sqrt{n}$.

Proposed by Dorin Andrica, Babes-Bolyai University, Romania

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain;

If 1 is the largest odd integer not exceeding $\sqrt{n}$, the result is trivially true, and $\sqrt{n} < 3$, or $n \leq 8$. Assume now that $m \geq 1$ is an integer such that $2m+1$ is the largest odd integer not exceeding $\sqrt{n}$. Then, $2m+3 > \sqrt{n} \geq 2m+1$, or $4m^2+12m+9 > n \geq 4m^2+4m+1$. Since $2m+1$ and $2m-1$ are positive odd integers with difference 2, they are coprime, and if both divide n, then their product $4m^2-1$ must also divide n, which is larger than $4m^2-1$. Therefore, $n \geq 2(4m^2-1)$, and $4m^2+12m+9 > 8m^2-2$, or $4m^2-12m-11 < 0$. Now, if $m \geq 4$, then $4m^2-12m-11 = (m^2-11)+3(m-4)m > 0$, and necessarily $m \leq 3$. Assume that $m = 3$. Then $2m+3 = 9 > \sqrt{n}$, and $n < 81$, but n must be divisible by 3, 5 and 7, which are coprime. Therefore, n must be divisible by 105, which is absurd, and $m \leq 2$. Assume now $m = 2$. Then, $2m+3 = 7 > \sqrt{n} \geq 5 = 2m+1$, and $25 \leq n < 49$, but n must be divisible by 3 and 5, which are coprime. Therefore, n must be divisible by 15, or $n = 30, 45$. Assume next that $m = 1$. Then, $9 \leq n < 25$ and n must be divisible by 3, or $n = 9, 12, 15, 18, 21, 24$. The integers that we are looking for are then 1 through 9, 12, 15, 18, 21, 24, 30 and 45.

Second solution by John T. Robinson, Yorktown Heights, NY, USA

For $n \leq 25$ such integers can be computed as 1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 15, 18, 21, and 24. In order to get some intuition, consider what happens for $n \geq 25$: up to $n = 7^2 = 49$, these are the integers divisible by 3 and 5, that is multiples of 15, which are 30 and 45 in this range. After $n = 49$, up to $n = 9^2 = 81$, these are the integers divisible by $3, 5$, and 7, that is multiples of 105, a contradiction. Recall Bertrand's postulate, that is there is always a prime between m and $2m$, where m is any integer with $m > 1$. Using induction we can see that every time we jump from n to $4n$ we get at least one more prime in the range $[\sqrt{n}, 2\sqrt{n}]$. This prime is greater than 4 for $n \geq 9$, so the product of primes that must divide n grows faster than n. In summary, the only positive integers n such that a divides n for all odd positive integers a not exceeding $\sqrt{n}$ are

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 15, 18, 21, 24, 30, 45.$$

Third solution by Roberto Bosch Cabrera, University of Havana, Cuba

Let $\lfloor\sqrt{n}\rfloor = m$, then $n \in \left\{m^2, m^2+1, ..., m^2+2m\right\}$. Now we have two cases m is odd and m is even.

Assume m is odd and $m \geq 3$, then

m odd $\Rightarrow m \mid n \Rightarrow n \in \left\{m^2, m^2+m, m^2+2m\right\}$
m odd $\Rightarrow m-2 \mid n$. Now we have three sub-cases:

- $m-2 \mid m^2 \Rightarrow m-2 \mid (m-2)^2 + 4(m-2) + 4 \Rightarrow m-2 \mid 4 \Rightarrow m = 3$.
- $m-2 \mid m^2+m \Rightarrow m-2 \mid (m-2)^2 + 5(m-2) + 6 \Rightarrow m-2 \mid 6 \Rightarrow m = 3$ or $m = 5$.
- $m-2 \mid m^2+2m \Rightarrow m-2 \mid (m-2)^2 + 6(m-2) + 8 \Rightarrow m-2 \mid 8 \Rightarrow m = 3$.

From $m = 3$ we obtain $n = 9,\ n = 12,\ n = 15$.
From $m = 5$ we obtain $n = 30$ since 3 not divide 25 neither 35.

Assume m is even and $m \geq 6$, then

m even $\Rightarrow m-1 \mid n \Rightarrow n \in \left\{m^2+m-2, m^2+2m-3\right\}$
m even $\Rightarrow m-3 \mid n$. Now we have two sub-cases:

- $m-3 \mid m^2+m-2 \Rightarrow m-3 \mid (m-3)^2+7(m-3)+10 \Rightarrow m-3 \mid 10 \Rightarrow m = 8$.
- $m-3 \mid m^2+2m-3 \Rightarrow m-3 \mid (m-3)^2+8(m-3)+12 \Rightarrow m-3 \mid 12 \Rightarrow m = 6$.

From $m = 6$ we obtain $n = 45$ since 3 not divide 40.
From $m = 8$ we not obtain n since 3 not divide 70 neither 77.

Now just we need consider the trivial cases $m = 1,\ m = 2,\ m = 4$.
From $m = 1$ we obtain $n = 1,\ n = 2,\ n = 3$.
From $m = 2$ we obtain $n = 4,\ n = 5,\ n = 6,\ n = 7,\ n = 8$.
From $m = 4$ we obtain $n = 18,\ n = 21,\ n = 24$.

Finally, $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 15, 18, 21, 24, 30, 45\}$

Also solved by G.R.A.20 Math Problems Group, Roma, Italy.

J84. Al and Bo play the following game: there are 22 cards labeled 1 through 22. Al chooses one of them and places it on a table. Bo then places one of the remaining cards at the right of the one placed by Al such that the sum of the two numbers on the cards is a perfect square. Al then places one of the remaining cards such that the sum of the numbers on the last two cards played is a perfect square, and so on. The game ends when all the cards were played or no more card can be placed on the table. The winner is the one who played the last card. Does Al have a winning strategy?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Roberto Bosch Cabrera, University of Havana, Cuba

The winning strategy for Al is to choose the card labeled 2 in the first step. Note that the perfect squares in the play are: $4, 9, 16, 25, 36$. We consider the equation $2 + m = n^2$ to obtain that Bo just can choose 7 or 14 in the second step.

If *Bo* chosses 7, then Al chosses 18. Bo cannot play since the equation $18 + m = n^2$ is not solvable, because 7 was chosen before. Thus Al is winner $(2 - 7 - 18)$.

If Bo chosses 14, then Al have another winning chain $(2 - 14 - 11 - 5 - 20 - 16 - 9 - 7 - 18)$. At each step Bo have no choice other than choosing the card shown in the chain. So Al has a winning strategy and we are done.

Second solution by Ganesh Ajjanagadde, Acharya Vidya Kula, Mysore, India

We claim that Al has a winning strategy.

On his first move, let Al choose 22. On his subsequent moves, let Al choose the maximum number available to him, such that the sum of his number and the previous number is a perfect square. It is clear that Bo has two choices for her first move, namely 3 and 14. Let us consider these two cases separately.

Case 1. Bo chooses 14. Thus Bo has only one choice in each of her subsequent moves if Al sticks to his strategy. The sequence of moves are the following: 22, 14, 11, 5, 20, 16, 9, 7, 18. Once Al places 18, Bo has to either place 7, or 18, both of which are impossible.

Case 2. Bo chooses 3. The sequence of moves runs 22, 3, 13, 12, 4. Bo can now either place 5 or 21.

> **Case 2 a.** Bo plays 5. Then the sequence of moves continues as follows: 5, 20, 16, 9, 7, 18. Once again we reach a state when Bo can make no further move.

Case 2 b Bo plays 21. Then the sequence of moves continues as follows: 21, 15, and Bo can now play either 1 or 10.

> **Case 2 b i** Bo plays 1. The sequence continues thus: 1, 8, 17, 19, 6, 10. Now Bo can play either 15 or 6, both of which are not possible as they have been played earlier.
>
> **Case 2 b ii** Bo plays 10. The sequence continues as follows: 10, 6, 19, 17, 8, 1. Now Bo has to play either 3, 8, or 15, none of which is possible as they have been played earlier.

Thus Al can always force a win by sticking to this strategy.

Also solved by Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Publica de Navarra, Spain; John T. Robinson, Yorktown Heights, NY, USA; Salem Malikic Sarajevo, Bosnia and Herzegovina.

Senior problems

S79. Let $a_n = \sqrt[4]{2} + \sqrt[n]{4}$, $n = 2, 3, 4, \ldots$ Prove that $\frac{1}{a_5} + \frac{1}{a_6} + \frac{1}{a_{12}} + \frac{1}{a_{20}} = \sqrt[4]{8}$.

Proposed by Titu Andreescu, University of Texas at Dallas

First solution by Simon Morris, Guernsey

$$\begin{aligned}
\frac{1}{a_5} + \frac{1}{a_6} + \frac{1}{a_{12}} + \frac{1}{a_{20}} &= \frac{1}{2^{\frac{1}{4}} + 2^{\frac{2}{5}}} + \frac{1}{2^{\frac{1}{4}} + 2^{\frac{2}{6}}} + \frac{1}{2^{\frac{1}{4}} + 2^{\frac{2}{12}}} + \frac{1}{2^{\frac{1}{4}} + 2^{\frac{2}{20}}} \\
&= \frac{1}{2^{\frac{15}{60}} + 2^{\frac{24}{60}}} + \frac{1}{2^{\frac{15}{60}} + 2^{\frac{20}{60}}} + \frac{1}{2^{\frac{15}{60}} + 2^{\frac{10}{60}}} + \frac{1}{2^{\frac{15}{60}} + 2^{\frac{6}{60}}} \\
&= \frac{1}{2^{\frac{15}{60}}(2^{\frac{9}{60}} + 1)} + \frac{1}{2^{\frac{15}{60}}(2^{\frac{5}{60}} + 1)} + \frac{1}{2^{\frac{10}{60}}(2^{\frac{5}{60}} + 1)} + \frac{1}{2^{\frac{6}{60}}(2^{\frac{9}{60}} + 1)} \\
&= \frac{2^{-\frac{15}{60}}}{2^{\frac{9}{60}} + 1} + \frac{2^{-\frac{6}{60}}}{2^{\frac{9}{60}} + 1} + \frac{2^{-\frac{15}{60}}}{2^{\frac{5}{60}} + 1} + \frac{2^{-\frac{10}{60}}}{2^{\frac{5}{60}} + 1} \\
&= \frac{2^{-\frac{15}{60}} + 2^{-\frac{6}{60}}}{2^{\frac{9}{60}} + 1} + \frac{2^{-\frac{15}{60}} + 2^{-\frac{10}{60}}}{2^{\frac{5}{60}} + 1} \\
&= \frac{2^{\frac{3}{4}}(2^{-1} + 2^{-\frac{51}{60}})}{2(2^{-\frac{51}{60}} + 2^{-1})} + \frac{2^{\frac{3}{4}}(2^{-1} + 2^{-\frac{55}{60}})}{2(2^{-\frac{55}{60}} + 2^{-1})} \\
&= \frac{2^{\frac{3}{4}}}{2} + \frac{2^{\frac{3}{4}}}{2} = 2^{\frac{3}{4}}.
\end{aligned}$$

Second solution by Johnathon M. Ashcraft, Auburn Montgomery, USA

Given the above information, we find

$$a_5 = \sqrt[4]{2} + \sqrt[5]{4} = 2^{\frac{1}{4}} + 2^{\frac{2}{5}} = 2^{\frac{5}{20}} + 2^{\frac{8}{20}} = 2^{\frac{1}{4}}\left(1 + 2^{\frac{3}{20}}\right)$$

$$a_6 = \sqrt[4]{2} + \sqrt[6]{4} = 2^{\frac{1}{4}} + 2^{\frac{1}{3}} = 2^{\frac{15}{60}} + 2^{\frac{20}{60}} = 2^{\frac{1}{4}}\left(1 + 2^{\frac{1}{12}}\right)$$

$$a_{12} = \sqrt[4]{2} + \sqrt[12]{4} = 2^{\frac{1}{4}} + 2^{\frac{1}{6}} = 2^{\frac{3}{12}} + 2^{\frac{2}{12}} = 2^{\frac{1}{4}}\left(1 + 2^{-\frac{1}{12}}\right)$$

$$a_{20} = \sqrt[4]{2} + \sqrt[20]{4} = 2^{\frac{1}{4}} + 2^{\frac{1}{10}} = 2^{\frac{10}{40}} + 2^{\frac{4}{40}} = 2^{\frac{1}{4}}\left(1 + 2^{-\frac{3}{20}}\right).$$

Therefore

$\frac{1}{a_5} + \frac{1}{a_6} + \frac{1}{a_{12}} + \frac{1}{a_{20}} =$

$$= \frac{1}{2^{\frac14}\left(1+2^{\frac{3}{20}}\right)} + \frac{1}{2^{\frac14}\left(1+2^{\frac{1}{12}}\right)} + \frac{1}{2^{\frac14}\left(1+2^{-\frac{1}{12}}\right)} + \frac{1}{2^{\frac14}\left(1+2^{-\frac{3}{20}}\right)}$$

$$= \frac{1}{2^{\frac14}}\left(\frac{1}{1+2^{\frac{3}{20}}} + \frac{1}{1+2^{-\frac{3}{20}}} + \frac{1}{1+2^{\frac{1}{12}}} + \frac{1}{1+2^{-\frac{1}{12}}}\right)$$

$$= \frac{1}{2^{\frac14}}\left(\frac{2+2^{\frac{3}{20}}+2^{-\frac{3}{20}}}{2+2^{\frac{3}{20}}+2^{-\frac{3}{20}}} + \frac{2+2^{\frac{1}{12}}+2^{-\frac{1}{12}}}{2+2^{\frac{1}{12}}+2^{-\frac{1}{12}}}\right)$$

$$= \frac{2}{2^{\frac14}} = 2^{\frac34} = \sqrt[4]{2^3} = \sqrt[4]{8}.$$

Third solution by Prithwijit De, Kolkata, India

We observe that $a_5 = 2^{\frac14} + 2^{\frac25}; a_6 = 2^{\frac14} + 2^{\frac13}; a_{12} = 2^{\frac14} + 2^{\frac13}; a_{20} = 2^{\frac14} + 2^{\frac{1}{10}}$.

Let $u = 2^{\frac{1}{60}} = u$ and observe that

$$a_5 = u^{15} + u^{24} = u^{15}(u^9+1);$$

$$a_6 = u^{15} + u^{20} = u^{15}(u^5+1);$$

$$a_{12} = u^{15} + u^{10} = u^{10}(u^5+1);$$

$$a_{20} = u^{15} + u^{6} = u^{6}(u^9+1).$$

Now,

$$\frac{1}{a_5} + \frac{1}{a_6} + \frac{1}{a_{12}} + \frac{1}{a_{20}} =$$

$$= \frac{1}{u^{15}(u^9+1)} + \frac{1}{u^{15}(u^5+1)} + \frac{1}{u^{10}(u^5+1)} + \frac{1}{u^{6}(u^9+1)}$$

$$= \frac{2}{u^{15}} = \sqrt[4]{8}, \text{ as desired.}$$

Also solved by Arkady Alt, San Jose, California, USA; Brian Bradie, Christopher Newport University, USA; Daniel Lasaosa, Universidad Publica de Navarra, Spain; John T. Robinson, Yorktown Heights, NY, USA; Vicente Vicario Garcia, Huelva, Spain.

S80. Let ABC be a triangle and let M_a, M_b, M_c be the midpoints of sides BC, CA, AB, respectively. Let the feet of the perpendiculars from vertices M_b, M_c in triangle AM_bM_C be C_2 and B_1; the feet of the perpendiculars from vertices M_a, M_b in triangle CM_aM_b be B_2 and A_1; the feet of the perpendiculars from vertices M_c, M_a in triangle BM_aM_c be A_2 and C_1. Prove that the perpendicular bisectors of B_1C_2, C_1A_2, and A_1B_2 are concurrent.

Proposed by Vinoth Nandakumar, Sydney University, Australia

Solution by Mihai Miculita, Oradea, Romania

Let A_0, B_0, C_0 be the midpoints of M_bM_c, M_aM_c, M_aM_b and let a, b, c be the perpendicular bisectors of B_1C_1, C_1A_2, A_1B_2, respectively. Since triangle $A_0B_0C_0$ is the complementary triangle of triangle $M_aM_bM_c$ and triangle $M_aM_bM_c$ is the complementray triangle of ABC, triangles $A_0B_0C_0$ and ABC are homothetic. They have the same centroid G and at the same time G is the center of homothety with the ratio $\frac{1}{4}$. Let O and O_0 be the centers of their circumcircles. Thus

$$\overline{GO_0} = \frac{1}{4}\overline{GO}. \quad (1)$$

Quadrilateral $M_cM_bB_1C_2$ is cyclic, having center at A_0, the midpoint of M_bM_c. Thus $A_0B_1 = A_0C_2$, and triangle $A_0B_1C_2$ is isosceles. This implies that $A_0 \in a$.

On the other hand $M_bM_c \| BC$, thus $OA \perp B_1C_2$ and since $a \perp B_1C_2$ we get $OA \| a$. This means that line a in triangle $A_0B_0C_0$ is homologous to the radius OA of triangle ABC and so $O_0 \in a$. Analoguously it can be proved that b and c pass through O_0, the circumcenter of triangle $A_0B_0C_0$.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Oleh Faynshteyn, Leipzig, Germany; Ricardo Barroso, University of Sevilla, Spain.

S81. Consider the polynomial

$$P(x)=\sum_{k=0}^{n}\frac{1}{n+k+1}x^k,$$

with $n\geq 1$. Prove that the equation $P(x^2)=(P(x))^2$ has no real roots.

Proposed by Dorin Andrica, Babes-Bolyai University, Romania

First solution by Daniel Campos Salas, Costa Rica

Suppose there exist a real root t to the equation. Since

$$P(t^2)\geq\frac{1}{n+1}>0,$$

it follows that $P(t^2)=(P(t))^2>0$. From Cauchy-Schwarz we get

$$\left(\sum_{k=0}^{n}\frac{1}{n+k+1}\right)\left(\sum_{k=0}^{n}\frac{1}{n+k+1}t^{2k}\right)\geq\left(\sum_{k=0}^{n}\frac{1}{n+k+1}t^k\right)^2,$$

which implies that

$$\sum_{k=0}^{n}\frac{1}{n+k+1}\geq 1.$$

However, we have

$$\sum_{k=0}^{n}\frac{1}{n+k+1}<(n+1)\frac{1}{n+1}=1,$$

a contradiction. It follows that the equation $P(x^2)=(P(x))^2$ has no real roots.

Second solution by John T. Robinson, Yorktown Heights, NY, USA

Consider the polynomial $Q(x)=P(x^2)-P(x)^2$. Since $Q(0)=\frac{1}{(n+1)}-\frac{1}{(n+1)^2}>0$, the problem is equivalent to proving the inequality $P(x^2)>P(x)^2$.

Define the vector $V(x)$ as

$$V(x)=\left(\frac{x^n}{\sqrt{2n+1}},\frac{x^{n-1}}{\sqrt{2n}},\dots,\frac{1}{\sqrt{n+1}}\right).$$

Using vector dot product we have $P(x^2)=V(x)\cdot V(x)=|V(x)|^2$ and

$$P(x)=V(x)\cdot V(1)=|V(x)|\cdot|V(1)|\cdot\cos\alpha,$$

where α is the angle between $V(x)$ and $V(1)$. Therefore

$$P(x)^2 = |V(x)|^2 \cdot |V(1)|^2 \cos^2 \alpha.$$

Since $\cos^2(a) \leq 1$, if we can show $|V(1)|^2 < 1$ this will establish the inequality $P(x^2) > P(x)^2$. Note that

$$|V(1)|^2 = V(1) \cdot V(1) = \frac{1}{2n+1} + \frac{1}{2n} + \ldots + \frac{1}{n+1}.$$

It can be shown that $|V(1)|^2 < 1$ for $n \geq 1$ by induction on n. For $n = 1$, $|V(1)|^2 = 1/3 + 1/2 < 1$. Next suppose that for some n:

$$|V(1)|^2 = \frac{1}{2n+1} + \frac{1}{2n} + \ldots + \frac{1}{n+1} < 1.$$

Moving to $n+1$:

$$|V(1)|^2 = \frac{1}{2n+3} + \frac{1}{2n+2} + \ldots + \frac{1}{n+2}.$$

We see that we subtracted $\frac{1}{n+1}$, and added $\frac{1}{2n+3}$ and $\frac{1}{2n+2}$. But since $\frac{1}{n+1} > \frac{1}{2n+3} + \frac{1}{2n+2}$, the value subtracted was larger than the values added, so $|V(1)|^2$ is decreasing for increasing n. Therefore $P(x^2) > P(x)^2$ for all $n \geq 1$, and $Q(x)$ has no real roots.

Third solution by Perfetti Paolo, Dipartimento di Matematica Tor Vergata Roma, Italy

Since

$$\frac{x^k}{n+k+1} = \int_0^x \frac{t^{k+n}}{x^{n+1}} dt$$

the equation $(P(x))^2 = P(x^2)$ becomes

$$\frac{1}{x^{2n+2}} \int_0^{x^2} t^n \frac{t^{n+1}-1}{t-1} dt - \frac{1}{x^{2n+2}} \Big(\int_0^x t^n \frac{t^{n+1}-1}{t-1} dt \Big)^2 \doteq \frac{R(x)}{x^{2n+2}}$$

$x = 0$ is not a solution of the equation for any n by $\frac{1}{n+1} \neq \frac{1}{(n+1)^2}$ for any $n \geq 1$. We show that the derivative of $R(x) \doteq \int_0^{x^2} Q(t)dt - \big(\int_0^x Q(t)dt\big)^2$ is positive for $x > 0$ and negative for $x < 0$. This is enough together with $R(0) = 0$.

$$R'(x) = 2xQ(x^2) - 2Q(x) \int_0^x Q(t)dt =$$

$$= 2x^{2n+1} \Big(\sum_{k=0}^{n} x^{2k} - \sum_{q=0}^{2n} x^q \sum_{r=0}^{q} \frac{1}{n+1+r} \cdot \frac{1}{n+2+q-r} \Big) =$$

$$= 2x^{2n+1}\sum_{k=0}^{n} x^{2k}\Big(1-\sum_{r=0}^{q}\frac{1}{n+1+r}\cdot\frac{1}{n+2+q-r}\Big)-$$

$$-x^{2n+1}\sum_{q=0}^{n-1} x^{2q+1}\sum_{r=0}^{q}\frac{1}{n+1+r}\cdot\frac{1}{n+2+q-r}$$

We observe that

$$\sum_{r=0}^{q}\frac{1}{n+1+r}\frac{1}{n+2+q-r}\leq\sum_{r=0}^{n}\frac{1}{n+1+r}\cdot\frac{1}{n+2+q-r}\leq\frac{n+1}{(n+1)^2}\leq\frac{1}{2} \tag{1}$$

For $x<0$ the conclusion $R'(x)<0$ immediately follows.

As for $x>0$, rewrite $R'(x)$ as

$$2x^{2n+1}\sum_{k=0}^{n} x^{2k}\Big(\frac{1}{2}-\sum_{r=0}^{q}\frac{1}{n+1+r}\cdot\frac{1}{n+2+q-r}\Big)+$$

$$+2x^{2n+1}\sum_{k=0}^{n}\frac{x^{2k}}{2}-2x^{2n+1}\sum_{q=0}^{n-1} x^{2q+1}\sum_{r=0}^{q}\frac{1}{n+1+r}\cdot\frac{1}{n+2+q-r}.$$

The first sum over k is positive by (1). Apart from the factor $2x^{2n+1}$ we rewrite the second and the third sums as

$$\sum_{k=0}^{n}\frac{x^{2k}}{4}+\sum_{k=0}^{n}\frac{x^{2k}}{4}-\sum_{q=0}^{n-1} x^{2q+1}\sum_{r=0}^{q}\frac{1}{n+1+r}\cdot\frac{1}{n+2+q-r}$$

and by means of the AM-GM inequality we have

$$\frac{1}{4}x^{2q}+\frac{1}{4}x^{2q+2}\geq\frac{1}{2}x^{2q+1}\geq x^{2q+1}\sum_{r=0}^{q}\frac{1}{n+1+r}\cdot\frac{1}{n+2+q-r}$$

which again in implied by (1). The proof is completed.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Nguyen Manh Dung, Hanoi, Vietnam.

S82. Let a and b be positive real numbers with $a \geq 1$. Further, let s_1, s_2, s_3 be nonnegative real numbers for which there is a real number x such that

$$s_1 \geq x^2, \;\; as_2 + s_3 \geq 1 - bx.$$

What is the least possible value of $s_1 + s_2 + s_3$ in terms of a and b (the minimum is taken over all possible values of x)?

Proposed by Zoran Sunic, Texas A&M University, USA

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Clearly, $s_2 + s_3 \geq s_2 + \frac{s_3}{a} = \frac{as_2 + s_3}{a} \geq \frac{1-bx}{a}$, where the first inequality reaches equality iff $a = 1$ or $s_3 = 0$. Therefore, $s_1 + s_2 + s_3 \geq \frac{ax^2 - bx + 1}{a} \geq \frac{4a - b^2}{4a^2}$, where the second inequality reaches equality iff $x = \frac{b}{2a}$. We may consider two cases:

(1) $b^2 \leq 2a$. Then, we may indeed take $x = \frac{b}{2a}$, $s_1 = x^2$, $s_2 = \frac{2a-b^2}{2a^2} \geq 0$ and $s_3 = 0$, yielding $s_1 + s_2 + s_3 = \frac{4a-b^2}{4a^2}$.

(2) $b^2 \geq 2a$. Take then $x = \frac{1}{b}$, $s_1 = \frac{1}{b^2}$, $s_2 = s_3 = 0$, for $s_1 + s_2 + s_3 = \frac{1}{b^2}$. If a lower value were possible, then an $x < \frac{1}{b}$ would exist such that $\frac{1-bx}{a} \leq s_2 + s_3 < \frac{1}{b^2} - s_1 \leq \frac{1}{b^2} - x^2$, or x would exist such that $0 > ab^2x^2 - b^3x + b^2 - a = (bx-1)(abx - b^2 + a)$. Note that, if $x < \frac{1}{b}$, the first factor is negative, or the second factor must be positive, which is absurd, since $abx - b^2 + a < ab\frac{1}{b} - b^2 + a = 2a - b^2 \leq 0$. Therefore, no value of $s_1 + s_2 + s_3$ exists lower than the proposed one in this case.

Finally note that, if $b^2 > 2a$, then $\frac{1}{b^2} < \frac{2a}{4a^2} < \frac{4a-b^2}{4a^2}$, whereas if $b^2 < 2a$, $\frac{1}{b^2} > \frac{2a}{4a^2} > \frac{4a-b^2}{4a^2}$, and the least value that $s_1 + s_2 + s_3$ may take is $\min\left\{\frac{1}{b^2}, \frac{4a-b^2}{4a^2}\right\}$, where $x = \min\left\{\frac{1}{b}, \frac{b}{2a}\right\}$, $s_1 = \min\left\{\frac{1}{b^2}, \frac{b^2}{4a^2}\right\}$, $s_2 = \max\left\{0, \frac{2a-b^2}{2a^2}\right\}$, and $s_3 = 0$.

Second solution by John T. Robinson, Yorktown Heights, NY, USA

Since $s_1 \geq x^2 \geq 0$ we see that $\sqrt{s_1} \geq |x|$. We may assume that x is nonnegative, since we are minimizing $s_1 + s_2 + s_3$ over all x, and for $x \geq 0$, $as_2 + s_3 \geq 1 - bx$ is a less restrictive constraint than $as_2 + s_3 \geq 1 + bx$ (which is the constraint we would have if we replaced x with $-x$). Therefore

$$0 \leq x \leq \sqrt{s_1}, \;\; -x \geq -\sqrt{s_1}, \;\; as_2 + s_3 \geq 1 - bx \geq 1 - b\sqrt{s_1}$$

and

$$as_2 + s_3 + b\sqrt{s_1} \geq 1.$$

Since we are trying to minimize $s_1+s_2+s_3$, and the LHS is the sum of terms involving s_1, s_2, and s_3 each of which increases for increasing s_1, s_2, or s_3, the minimum will be on the surface $as_2+s_3+b\sqrt{s_1}=1$ in the octant $s_1, s_2, s_3 \geq 0$.

Note that since as_2, s_3, and $b\sqrt{s_1}$ are all nonnegative, we require as_2, s_3, and $b\sqrt{s_1} \leq 1$. Next consider the s_2+s_3 component of $s_1+s_2+s_3$ - for any given value of s_1, we have $as_2+s_3 = 1-b\sqrt{s_1}$, and since $a \geq 1, s_2+s_3$ is minimized by choosing $s_3=0$. So the problem is now simplified to minimizing s_1+s_2 subject to the constraint $as_2+b\sqrt{s_1}=1$.

Since $\sqrt{s_1}=(1-as_2)/b$, we have $s_1+s_2 = \frac{(1-as_2)^2}{b^2}+s_2$. Taking the derivative of the RHS, we find the minimum when $\frac{-2a(1-as_2)}{b^2}+1=0$, $\frac{b^2}{2a}=1-as_2$, $s_2=\frac{1}{a}-\frac{b^2}{2a^2}$. So if $\frac{1}{a} \geq \frac{b^2}{2a^2}$, or $2a \geq b^2$, then $s_1+s_2+s_3$ is minimized by choosing

$$s_2 = \frac{1}{a}-\frac{b^2}{2a^2},$$

$$s_1 = \left(\frac{1-as_2}{b}\right)^2 = \frac{b^2}{4a^2},$$

$$s_3 = 0.$$

Otherwise, if $b^2 > 2a$, the minimum of the quadratic $\frac{(1-as_2)^2}{b^2}+s_2$ occurs for a negative value of s_2, and is at its minimum value for nonnegative s_2 when $s_2=0$. Therefore $s_1+s_2+s_3$ is minimized by choosing

$$s_1 = \frac{1}{b^2},\ s_2 = s_3 = 0.$$

S83. Find all complex numbers x, y, z of modulus 1, satisfying

$$\frac{y^2+z^2}{x}+\frac{x^2+z^2}{y}+\frac{x^2+y^2}{z}=2(x+y+z).$$

Proposed by Cosmin Pohoata, Bucharest, Romania

First solution by Magkos Athanasios, Kozani, Greece

ewrite the given relation as

$$(x^2+y^2+z^2)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=3(x+y+z).$$

Since the numbers are of unit modulus we obtain $(x^2+y^2+z^2)(\bar{x}+\bar{y}+\bar{z}) = 3(x+y+z)$. Passing to the moduli we get either $|x^2+y^2+z^2|=3$ or $x+y+z=0$. If $x+y+z=0$, because $|x|=|y|=|z|=1$, the images of the numbers are the vertices of an equilateral triangle inscribed in the unit circle. Set $x = \cos a + i\sin a, a \in [0, 2\pi)$. Then

$$y = x(\cos 120^\circ + i\sin 120^\circ) = x\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right),$$

$$z = x(\cos 240^\circ + i\sin 240^\circ) = x\left(\frac{-1}{2}+i\frac{-\sqrt{3}}{2}\right).$$

If $|x^2+y^2+z^2|=3$, we have

$$|x^2+y^2+z^2| = |x^2|+|y^2|+|z^2|.$$

In this case, it is known that there exist positive reals A, B such that $x^2 = Ay^2$ and $x^2 = Bz^2$. Since $|x|=|y|=|z|=1$ we get $A=B=1$, hence, $x^2=y^2=z^2$. This means that we have one of the following possibilities:

$$x=y=z, x=y=-z, x=-y=z, x=-y=-z,$$

where $|x|=1$. It is easy to verify that all of of the above triplets satisfy the initial condition.

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Clearly, $x+\frac{y^2+z^2}{x}=\frac{x^2+y^2+z^2}{x}=\overline{x}\left(x^2+y^2+z^2\right)$, where $\overline{x}$ denotes the complex conjugate of x, and we have used that the modulus of x is $1 = x\overline{x}$. Therefore, the given equation is equivalent to

$$3\left(x+y+z\right) = \left(\overline{x+y+z}\right)\left(x^2+y^2+z^2\right).$$

Taking the complex conjugate of both sides of this last equation and using it for simplification yields

$$9(x+y+z) = 3\left(\overline{x+y+z}\right)\left(x^2+y^2+z^2\right) = (x+y+z)\left|x^2+y^2+z^2\right|^2.$$

The given equation may have a solution then in one of the following two cases: 1) $\left|x^2+y^2+z^2\right| = 3 = \left|x^2\right| + \left|y^2\right| + \left|z^2\right|$. Now, equality in the triangular inequality forces x^2, y^2, z^2 to define collinear vectors in the complex plane, or $x^2 = y^2 = z^2$. Note that any such triplet yields in fact a solution to the given equation, since $\frac{y^2+z^2}{x} = \frac{2x^2}{x} = 2x$, and similarly for the other two fractions.

2) $x+y+z=0$. Then, $\frac{y^2+z^2}{x} = \frac{(y+z)^2-2yz}{x} = x - \frac{2yz}{x}$, and similarly for the other two fractions. Substitution in the original equation yields

$$0 = \frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z} = xyz\left(\overline{x^2+y^2+z^2}\right),$$

and since $xyz \neq 0$, then $x^2+y^2+z^2=0$. Now, $x^2 = -y^2-z^2 = -(y+z)^2+2yz = -x^2+2yz$, and $x^2 = yz$, and similarly $y^2 = zx$, $z^2 = xy$. Substitution in the given equation yields $\frac{y^2+z^2}{x} = z+y$, and similarly for the other two fractions, or all triplets x, y, z such that $x+y+z=0$ and $x^2+y^2+z^2=0$ are solutions of the given equation. Now, $xy+yz+zx = \frac{(x+y+z)^2-x^2-y^2-z^2}{2} = 0$, and thus by Cardano-Vieta relations, x, y, z are the three roots of an equation of the form $r^3 - a = 0$, where $a = xyz$ is a complex number with modulus 1, ie, x, y, z are the three cubic roots of a complex number of modulus 1.

Therefore, all the solutions of the given equation either satisfy $x^2 = y^2 = z^2$ (ie, $x = \pm y$ and $x = \pm z$), or x, y, z are the three cubic roots of any complex number of modulus 1.

Third solution by Perfetti Paolo, Dipartimento di matematica Universita degli studi di Tor Vergata, Italy

Clearing the denominators we obtain

$$\frac{(xy+yz+zx)(x^2+y^2+z^2) - 3(xyz)(x+y+z)}{xyx} = 0$$

and then

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)(x^2+y^2+z^2) = 3(x+y+z)$$

Taking the absolute value and observing that for complex numbers of modulus one the following holds

$$|x+y+z| = \left|\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right|,$$

we get $|x^2+y^2+z^2| = 3$ or $x+y+z=0$ and this is possible only when the three complex numbers are collinear or they are vertices of a equilateral triangle inscribed in the unit circle.

S84. Let ABC be an acute triangle and let ω and Ω be its incircle and circumcircle, respectively. Circle ω_A is tangent internally to Ω at A and tangent externally to ω. Circle Ω_A is tangent internally to Ω at A and tangent internally to ω. Denote by P_A and Q_A the centers of ω_A and Ω_A, respectively. Define the points P_B, Q_B, P_C, Q_C analogously. Prove that

$$\frac{P_AQ_A}{BC} + \frac{P_BQ_B}{CA} + \frac{P_CQ_C}{AB} \geq \frac{\sqrt{3}}{2}.$$

Proposed by Cezar Lupu, Univeristy of Bucharest, Romania

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Denote by R, r, R_A, r_A the respective radii of $\Omega, \omega, \Omega_A, \omega_A$. It is clear by construction that P_A, Q_A are on segment OA, and $AP_A = r_A$, $AQ_A = R_A$, yielding $P_AQ_A = R_A - r_A$. Denote by A', A'' the respective points where ω_A, Ω_A touch ω, it is clear that A', A'' are respectively on lines IP_A, IQ_A, and $IP_A = IA' + A'P_A = r + r_A$, $IQ_A = Q_AA'' - IA'' = R_A - r$. Furthermore, $\angle IAP_A = \angle IAQ_A = \angle IAO = \angle IAC - \angle OAC = \frac{A}{2} - \frac{\pi}{2} + B = \frac{B-C}{2}$. Therefore, using the theorem of the cosine,

$$r^2 + r_A^2 + 2rr_A = IP_A^2 = IA^2 + AP_A^2 - 2IA \cdot AP_A \cos\angle IAP_A$$

$$= \frac{r^2}{\sin^2\frac{A}{2}} + r_A^2 - 2\frac{rr_A\cos\frac{B-C}{2}}{\sin\frac{A}{2}},$$

$$r\cos^2\frac{A}{2} = 2r_A\sin\frac{A}{2}\left(\sin\frac{A}{2} + \cos\frac{B-C}{2}\right) = \frac{rr_A}{R}\frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}};$$

$$r^2 + R_A^2 - 2rR_A = IQ_A^2 = IA^2 + AQ_A^2 - 2IA \cdot AQ_A\cos\angle IAQ_A$$

$$= \frac{r^2}{\sin^2\frac{A}{2}} + R_A^2 - 2\frac{rR_A\cos\frac{B-C}{2}}{\sin\frac{A}{2}},$$

$$r\cos^2\frac{A}{2} = 2R_A\sin\frac{A}{2}\left(\cos\frac{B-C}{2} - \sin\frac{A}{2}\right) = \frac{rR_A}{R}.$$

We have used that $\sin\frac{A}{2} = \cos\frac{B+C}{2}$ and $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$. Then,

$$P_AQ_A = R_A - r_A = R\cos^2\frac{A}{2}\left(1 - \frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}\right) = \frac{R\sin\frac{A}{2}\cos^2\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}};$$

$$\frac{P_AQ_A}{BC} = \frac{\cos\frac{A}{2}}{4\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{\tan\frac{B}{2} + \tan\frac{C}{2}}{4},$$

and similarly $\frac{P_B Q_B}{CA} = \frac{\tan\frac{C}{2}+\tan\frac{A}{2}}{4}$ and $\frac{P_C Q_C}{AB} = \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{4}$.

It suffices therefore to prove that $\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \geq \sqrt{3}$. But it is well known (or easily provable using that $A + B + C = \pi$) that

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1,$$

while application of the inequality between arithmetic and quadratic means for any positive real numbers u, v, w yields

$$(u + v + w)^2 \geq \frac{(u + v + w)^2}{3} + 2(uv + vw + wu),$$

with equality iff $u = v = w$, or $u + v + w \geq \sqrt{3(uv + vw + wu)}$. The result follows, with equality holding if and only if $A = B = C$, i.e. if and only if triangle ABC is equilateral.

Also solved by Oleh Faynshteyn, Leipzig, Germany.

Undergraduate problems

U79. Let $a_1 = 1$ and $a_n = a_{n-1} + \ln n$. Prove that the sequence $\sum_{i=1}^{n} \frac{1}{a_i}$ is divergent.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

First solution by Arin Chaudhuri, North Carolina State University, NC, USA

Note $a_n = a_1 + (a_2 - a_1) + \cdots + (a_n - a_{n-1}) = 1 + \ln(2) + \cdots + \ln(n) = 1 + \ln(n!)$.
From Stirling's approximation we have

$$\ln n! = \ln(\sqrt{2\pi}) - n + n \ln n + \frac{1}{2} \ln n + c_n$$

where $c_n \to 0$. Hence,

$$a_n = C - n + n \ln n + \frac{1}{2} \ln n + c_n$$

where $C = 1 + \ln(\sqrt{2\pi})$.
If $n \geq 2$ then diving throughout by $n \ln n$ we have

$$\frac{a_n}{n \ln n} = \frac{C}{n \ln n} - \frac{1}{\ln n} + 1 + \frac{1}{2n} + \frac{c_n}{n \ln n}$$

Note all terms above vanish as $n \to \infty$ except 1. Hence

$$\lim_{n\to\infty} \frac{a_n}{n \ln n} = 1 \tag{1}$$

Hence we can find an N such that for all $n \geq N$

$$\frac{a_n}{n \ln n} \leq 2$$

Hence for all $n \geq N$

$$\frac{1}{2n \ln n} \leq \frac{1}{a_n}$$

Using the well known result that $\sum_{k=2}^{\infty} \frac{1}{k \ln k} = +\infty$, we have $\sum_{k=1}^{\infty} \frac{1}{a_k} = +\infty$.

Second solution by Arkady Alt, San Jose ,California, USA

Since $n! < \left(\frac{n}{2}\right)^n$ and $a_n - a_1 = \sum\limits_{k=2}^{n} (a_k - a_{k-1}) = \sum\limits_{k=2}^{n} \ln k = \ln n!$ we get

$$a_n = 1 + \ln n!.$$

Note that $a_n < 1 + n \ln\left(\frac{n}{2}\right) < n \ln n, n \geq 2 \iff \frac{1}{a_n} > \frac{1}{n \ln n}, n \geq 2.$

Moreover, $\frac{1}{a_n} > \ln\ln(n+1) - \ln\ln n$, for $n \geq 2$ because by the Mean Value Theorem for some $c_n \in (n, n+1)$ we have

$$\ln\ln(n+1) - \ln\ln n = \frac{1}{c_n \ln c_n} < \frac{1}{n \ln n}.$$

Hence,

$$\sum_{k=1}^{n} \frac{1}{a_n} = 1 + \sum_{k=2}^{n} \frac{1}{a_n} > 1 + \sum_{i=2}^{n} \left(\ln\ln(k+1) - \ln\ln k\right) =$$

$$= 1 + \ln\ln(n+1) - \ln\ln 2,$$

and, therefore, sequence $\sum_{k=1}^{n} \frac{1}{a_n}$ is divergent.

Third solution by Jean Mathieux, Senegal

We have that $a_3 \leq 3 \ln 3$. Suppose that for $n > 3, a_n \leq n \ln n$, then

$$a_{n+1} \leq n \ln n + \ln(n+1) \leq (n+1)\ln(n+1).$$

So for all $i > 2$, $\frac{1}{a_i} \geq \frac{1}{i \ln 1}$. Also, since $t \to \frac{1}{t \ln t}$ is decreasing, $\int_i^{i+1} \frac{1}{t \ln t} dt \leq \frac{1}{i \ln i.}$
Thus

$$\sum_{i=3}^{n} \frac{1}{a_i} \geq \int_3^{n+1} \frac{1}{t \ln t} dt = \ln(\ln(n+1)) - \ln(\ln(3)).$$

Hence the given sequence is divergent.

Also solved by Magkos Athanasios, Kozani, Greece; Brian Bradie, Christopher Newport University, USA; Daniel Lasaosa, Universidad Publica de Navarra, Spain; John T. Robinson, Yorktown Heights, NY, USA; Perfetti Paolo, Dipartimento di matematica Universita degli studi di Tor Vergata, Italy; Vicente Vicario Garcia, Huelva, Spain; Roberto Bosch Cabrera, Faculty of Mathematics, University of Havana, Cuba.

U80. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function at the origin satisfying $f(0) = 0$ and $f'(0) = 1$. Evaluate

$$\lim_{x\to 0} \frac{1}{x} \sum_{n=1}^{\infty} (-1)^n f\left(\frac{x}{n}\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Brian Bradie, Christopher Newport University, USA

Because $f(0) = 0$ and $f'(0) = 1$, we have

$$\begin{aligned} \lim_{x\to 0} \frac{1}{x} f\left(\frac{x}{n}\right) &= \lim_{x\to 0} \frac{f(x/n) - f(0)}{x} \\ &= \lim_{w\to 0} \frac{f(w) - f(0)}{nw} \\ &= \frac{1}{n} f'(0) = \frac{1}{n}. \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{x\to 0} \frac{1}{x} \sum_{n=1}^{\infty} (-1)^n f\left(\frac{x}{n}\right) &= \sum_{n=1}^{\infty} (-1)^n \lim_{x\to 0} \frac{1}{x} f\left(\frac{x}{n}\right) \\ &= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \\ &= -\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = -\ln 2. \end{aligned}$$

Second solution by John T. Robinson, Yorktown Heights, NY, USA

By definition, the limit as x goes to 0 of $(f(x) - f(0))/x = f(x)/x$ is $f'(0) = 1$. Consider $f(x/n)/x$: substituting $y = x/n$,

$$\frac{f(x/n)}{x} = \frac{f(y)}{n \cdot y} = \frac{1}{n} \cdot \frac{f(y)}{y},$$

therefore the limit as x goes to 0 of $\frac{f(x/n)}{x} = \frac{1}{n} \cdot f'(0) = \frac{1}{n}$. It follows that the limit being asked for is

$$-1 + 1/2 - 1/3 + 1/4 - 1/5 + \cdots = -\ln 2.$$

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Perfetti Paolo, Dipartimento di matematica Universita degli studi di Tor Vergata, Italy.

U81. The sequence $(x_n)_{n\geq 1}$ is defined by

$$x_1 < 0, \quad x_{n+1} = e^{x_n} - 1, \quad n \geq 1.$$

Prove that $\lim_{n\to\infty} nx_n = -2$.

Proposed by Dorin Andrica, Babes-Bolyai University, Romania

First solution by Ovidiu Furdui, The University of Toledo, OH

By induction it can be proved that $x_n < 0$ for all $n \geq 1$. On the other hand, since $e^x - 1 \geq x$ for all $x \in \mathbb{R}$, we get that $x_{n+1} = e^{x_n} - 1 \geq x_n$, and hence, the sequence increases. It follows that $x_1 < x_n < 0$ for all $n \geq 1$, and hence, the sequence converges. If $l = \lim x_n$, then passing to the limit as $n \to \infty$ in the recurrence relation we obtain that $l = e^l - 1$ from which it follows that $l = 0$. We calculate $\lim\limits_{n\to\infty} nx_n$ by using Cesaro-Stolz lemma. We have, since $\lim\limits_{x\to 0} \frac{e^x-1}{x} = 1$ and $\lim\limits_{x\to 0} \frac{x^2}{e^x-1-x} = 2$, that

$$\begin{aligned}
\lim_{n\to\infty} nx_n &= -\lim_{n\to\infty} \frac{n}{\frac{-1}{x_n}} = -\lim_{n\to\infty} \frac{1}{\frac{1}{x_n} - \frac{1}{x_{n+1}}} = -\lim_{n\to\infty} \frac{x_{n+1}\cdot x_n}{x_{n+1} - x_n} \\
&= -\lim_{n\to\infty} \frac{e^{x_n} - 1}{x_n} \cdot \lim_{n\to\infty} \frac{x_n^2}{e^{x_n} - 1 - x_n} = -2,
\end{aligned}$$

and the problem is solved.

Second solution by Brian Bradie, Christopher Newport University, USA

If $x_n < 0$, then $x_{n+1} = e^{x_n} - 1 < 1 - 1 = 0$. As we are given that $x_1 < 0$, it follows by induction on n that $x_n < 0$ for all n. Moreover, it is clear that $\lim_{n\to infty} x_n = 0$ for any $x_1 < 0$. Now, let $y_n = -\frac{1}{2}x_n$. Then, $y_n > 0$ for all n, $y_n \to 0$ and

$$y_{n+1} = -\frac{1}{2}\left(e^{-2y_n} - 1\right) = y_n - y_n^2 + O(y_n^3).$$

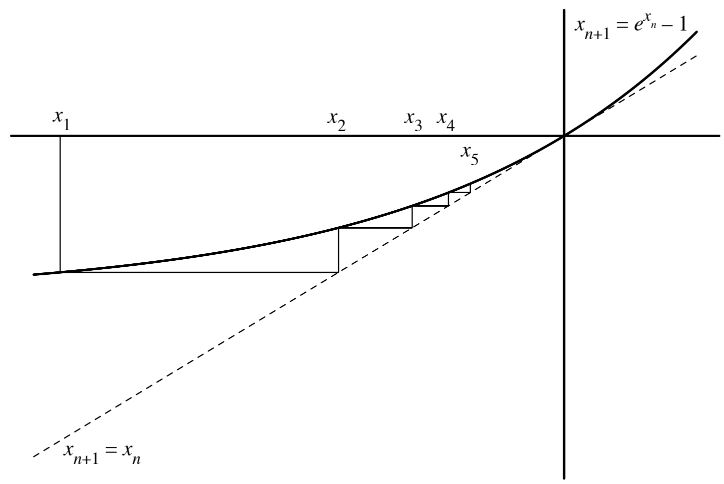
According to formula (8.5.3) (N. G. de Bruijn, *Asymptotic Methods in Analysis*, Dover Publications Inc., New York, 1981, page 155) it follows that

$$y_n = \frac{1}{n} + O(n^{-2}\ln n)$$

as $n \to \infty$. Therefore,

$$x_n = -\frac{2}{n} + O(n^{-2}\ln n)$$

as $n \to \infty$, and $\lim_{n\to\infty} nx_n = -2$.



Also solved by Arin Chaudhuri, North Carolina State University, NC, USA; Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Publica de Navarra, Spain; G.R.A.20 Math Problems Group, Roma, Italy; John T. Robinson, Yorktown Heights, NY, USA; Perfetti Paolo, Dipartimento di matematica Universita degli studi di Tor Vergata, Italy.

U82. Evaluate

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)^{\frac{n}{k^3}}.$$

Proposed by Cezar Lupu, Univeristy of Bucharest, Romania

First solution by Brian Bradie, Christopher Newport University, USA

Let

$$y=\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)^{n/k^3}.$$

Then,

$$\begin{aligned}\ln y &= \sum_{k=1}^{n}\frac{n}{k^3}\ln\left(1+\frac{k}{n}\right)=\sum_{k=1}^{n}\frac{n}{k^3}\left(\sum_{\ell=1}^{\infty}(-1)^{\ell-1}\frac{(k/n)^\ell}{\ell}\right)\\ &= \sum_{\ell=1}^{\infty}(-1)^{\ell-1}\frac{n^{1-\ell}}{\ell}\left(\sum_{k=1}^{n}k^{\ell-3}\right)\\ &= \sum_{k=1}^{n}\frac{1}{k^2}-\frac{1}{2n}\sum_{k=1}^{n}\frac{1}{k}+\sum_{\ell=3}^{\infty}(-1)^{\ell-1}\frac{n^{1-\ell}}{\ell}\left(\sum_{k=1}^{n}k^{\ell-3}\right).\end{aligned}$$

Now, as $n\to\infty$, $\sum_{k=1}^{n}k^{\ell-3}=O(n^{\ell-2})$, so

$$\sum_{\ell=3}^{\infty}(-1)^{\ell-1}\frac{n^{1-\ell}}{\ell}\left(\sum_{k=1}^{n}k^{\ell-3}\right)=O(n^{-1})\sum_{\ell=3}^{\infty}\frac{(-1)^{\ell-1}}{\ell}\to 0.$$

Additionally, as $n\to\infty$, $\sum_{k=1}^{n}\frac{1}{k}=O(\ln n)$, so

$$\frac{1}{2n}\sum_{k=1}^{n}\frac{1}{k}=O\left(\frac{\ln n}{n}\right)\to 0.$$

Finally, as $n\to\infty$,

$$\sum_{k=1}^{n}\frac{1}{k^2}\to\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}.$$

Thus,

$$\lim_{n\to\infty}\ln y=\frac{\pi^2}{6}\quad\text{and}\quad\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)^{n/k^3}=e^{\pi^2/6}.$$

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Call $P_n = \prod_{k=1}^{n} \left(1+\frac{k}{n}\right)^{\frac{n}{k^3}}$. Then,

$$\ln P_n = \sum_{k=1}^{n} \frac{n}{k^3} \ln\left(1+\frac{k}{n}\right).$$

But since $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$, we may write

$$\frac{n}{k^3} \ln\left(1+\frac{k}{n}\right) = \frac{1}{k^2} - \frac{1}{2nk} + \frac{1}{3n^2} - ...$$

The terms in the sum in the RHS have alternating signs and decreasing absolute value, or

$$\frac{1}{k^2} - \frac{1}{2nk} + \frac{1}{3n^2} > \frac{n}{k^3} \ln\left(1+\frac{k}{n}\right) > \frac{1}{k^2} - \frac{1}{2nk}.$$

Adding over k,

$$\frac{1}{3n} > \ln P_n - \sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{2n} \sum_{k=1}^{n} \frac{1}{k} > 0.$$

As n tends to infinity, the upper bound tends to 0, or the middle term has limit 0. Now,

$$\ln(k) - \ln(k-1) = \int_{k-1}^{k} \frac{1}{x} dx > \frac{1}{k} > \int_{k}^{k+1} \frac{1}{x} dx = \ln(k+1) - \ln(k),$$

and $\frac{1+\ln(n)}{2n} > \frac{1}{2n} \sum_{k=1}^{n} \frac{1}{k} > \frac{\ln(n+1)}{2n}$, and since the upper and lower bounds tend to 0 as n grows, we conclude that $\lim_{n\to\infty} P_n = \sum_{k=1}^{\infty} \frac{1}{k^2}$, which we may recognize as $\zeta(2) = \frac{\pi^2}{6}$. It follows that

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1+\frac{k}{n}\right)^{\frac{n}{k^3}} = e^{\zeta(2)} = e^{\frac{\pi^2}{6}}.$$

Third solution by John Mangual, New York, USA

The limit is $e^{\frac{\pi^2}{6}}$. Let S denote the limit:

$$S = \lim_{n\to\infty} \prod_{k=1}^{n} \left(1+\frac{k}{n}\right)^{\frac{n}{k^3}} \tag{2}$$

Instead of evluating S directly, let's examine the logarithm. By continuity of the logarithm we can write

$$\ln S = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^3} \ln\left(1+\frac{k}{n}\right) \tag{3}$$

Huristically speaking each term behaves like $1/k^2$ therefore we will subtract our guess from the summation:

$$\ln S - \frac{\pi^2}{6} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2}\left[\frac{n}{k}\ln\left(1+\frac{k}{n}\right) - 1\right] \tag{4}$$

By Taylor's theorem we can estimate the logarithm. For any $x \in [0,1]$ there exists $\xi \in [0,x]$ such that:

$$|\ln(1+x) - (x + x^2/2)| < \frac{x^3}{(1+\xi)^2} \le x^3 \tag{5}$$

This is a uniform bound on the difference. Therefore we can rewrite (3):

$$\sum_{k=1}^{n} \frac{1}{k^2}\left[\frac{n}{k}\left(\frac{k}{n} - \frac{k^2}{n^2}\right) - 1\right] + \sum_{k=1}^{n} \frac{1}{k^2}\left[\frac{n}{k}\left[\ln\left(1+\frac{k}{n}\right) - \left(\frac{k}{n} - \frac{k^2}{n^2}\right)\right]\right] \tag{6}$$

By triangle inequality the second term is bounded by:

$$\sum_{k=1}^{n} \frac{1}{k^2}\left(\frac{k}{n}\right)^2 = \frac{1}{n} \tag{7}$$

The second term also converges, despite the harmonic series:

$$\sum_{k=1}^{n} \frac{1}{k^2}\left[\frac{n}{k}\left(\frac{k}{n} - \frac{k^2}{n^2}\right) - 1\right] = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{k} = o\left(\frac{\ln n}{n}\right) \tag{8}$$

Both (6) and (7) are bounded as $n \to \infty$ so $\ln S - \pi^2/6 = 0$.

Also solved by Perfetti Paolo, Dipartimento di matematica Universita degli studi di Tor Vergata, Italy; Arin Chaudhuri, North Carolina State University, Raleigh, NC; Arkady Alt , San Jose ,California, USA; G.R.A.20 Math Problems Group, Roma, Italy; John T. Robinson, Yorktown Heights, NY, USA; Vicente Vicario Garcia, Huelva, Spain.

U83. Find all functions $f : [0,2] \to (0,1]$ that are differentiable at the origin and satisfy $f(2x) = 2f^2(x) - 1$, for all $x \in [0,1]$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Li Zhou, Polk Community College, USA

Let $g(x) = \arccos f(x)$ for all $x \in [0,2]$. Then for $x \in [0,1]$,

$$\cos g(2x) = f(2x) = 2\cos^2 g(x) - 1 = \cos(2g(x)),$$

thus $g(2x) = 2g(x)$. Hence, for any $x \in [0,2]$, $g(x) = 2g(x/2) = 4g(x/4) = \cdots = 2^n g(x/2^n)$ for all $n \geq 1$. Since g is differentiable at 0, $\lim_{n\to\infty} \frac{g(x/2^n)}{x/2^n} = k$ for some constant k. Therefore, $g(x) = kx$ for all $x \in [0,2]$. Considering the range of f, we conclude that $f(x) = \cos(kx)$ with $-\pi/4 < k < \pi/4$. Finally, it is easy to verify that all such functions f do satisfy the conditions.

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Taking $x = 0$ yields $f(0) = 2f^2(0) - 1$, or $f(0)$ is a root of $0 = 2r^2 - r - 1 = (2r+1)(r-1)$. Since $f(0) \in (0,1]$, then $f(0) = 1$. Take now any $\epsilon > 0$. Obviously,

$$\frac{f(2\epsilon) - f(0)}{2\epsilon} = \frac{f^2(\epsilon) - f^2(0)}{\epsilon} = (f(\epsilon) + f(0))\,\frac{f(\epsilon) - f(0)}{\epsilon}.$$

If f is differentiable at the origin, it is obviously continuous at the origin, and when $\epsilon \to 0$, the previous equality becomes $f'(0) = 2f(0)f'(0) = 2f'(0)$, or $f'(0) = 0$. This relation is obviously equivalent to $\lim_{x\to 0} \frac{f(x)}{x} = 0$.

The interval $(0,2]$ may be defined as the disjoint union of sets of the form $A_y = \left\{y, \frac{y}{2}, \frac{y}{4}, \frac{y}{8}, \ldots\right\}$, where y takes on all possible real values in $(1,2]$. Note that this is true since (1) $A_y \cap A_{y'} = \emptyset$ iff $y \neq y'$, since if an element belongs to both A_y and $A_{y'}$, then non-positive integers a, a' exist such that $x = 2^a y = 2^{a'} y'$, and since $2y > 2 \geq y'$, and $2y' > 2 \geq y$, then $a = a'$ and $y = y'$, and (2) for any $x \in (0,2]$, sufficiently high integral exponents a yield $2^a x > 1$. The minimum of all such exponents will obviously yield $2^a x = y$ for some $y \in (1,2]$, and $x \in A_y$.

Take now any function (not necessarily continuous) $g : (1,2] \to (0,1]$. We will now construct a function $f : [0,2] \to (0,1]$ such that $f(0) = 1$, which satisfies the given functional equation, and such that

(1) if $x \in (1,2]$, then $f(x) = g(x)$,

(2) $\lim_{x\to 0} f(x) = 1$ and

(3) $\lim_{x\to 0} \frac{f(x)-1}{x} = 0$.

The last two conditions are equivalent to f being differentiable at the origin with value 1 and derivative 0. In order to construct f, for any $y \in (1,2]$, take $f(y) = g(y)$; we may write $f(y) = \cos(\alpha_y y)$ for some $\alpha_y \in \left[0, \frac{\pi}{2}\right)$ since $f(y) \in (0,1]$. Now,

$$f\left(\frac{y}{2}\right) = \sqrt{\frac{1+\cos(\alpha_y y)}{2}} = \sqrt{\cos^2\left(\frac{\alpha_y y}{2}\right)} = \cos\left(\alpha_y \frac{y}{2}\right),$$

where we have selected the positive root since $f(x) \in (0,1]$ for all x, and a trivial exercise in induction yields $f(x) = \cos(\alpha_y x)$ for all $x \in A_y$. Repeat the same procedure for all $y \in (1,2]$. It is obvious that f thus constructed satisfies the given functional equation, and condition (1). Conditions (2) and (3) are also easily checked, since, as it is well known, $\cos(\beta) \to 1$ and $\frac{\cos(\beta)-1}{\beta} \to 0$ for any $\beta \to 0$. Therefore, for any $g(x)$ defined over $(1,2]$, a function f has been constructed that satisfies the requirements of the problem, and given any function f that satisfies the requirements of the problem, its values in $(1,2]$ biunivocally determine all values of f in $(0,1]$. Therefore, for any such g an f may be constructed, and no other solutions may exist.

U84. Let f be a three times differentiable function on an interval I, and let $a, b, c \in I$. Prove that there exists $\xi \in I$ such that

$$f\left(\frac{a+2b}{3}\right)+f\left(\frac{b+2c}{3}\right)+f\left(\frac{c+2a}{3}\right)-f\left(\frac{2a+b}{3}\right)-f\left(\frac{2b+c}{3}\right)-f\left(\frac{2c+a}{3}\right)=$$

$$=\frac{1}{27}(a-b)(b-c)(c-a)f'''(\xi).$$

Proposed by Vasile Cirtoaje, University of Ploiesti, Romania

First solution by Arkady Alt, San Jose, California, USA

Let $g(t) := f\left(\dfrac{a+b+c}{3}+t\right)-f\left(\dfrac{a+b+c}{3}-t\right)$ and let $x=\dfrac{b-c}{3}$, $y=\dfrac{c-a}{3}$,

$z=\dfrac{a-b}{3}$ then $x+y+z=0$ and $\delta(a,b,c) := f\left(\dfrac{a+2b}{3}\right)+f\left(\dfrac{b+2c}{3}\right)+f\left(\dfrac{c+2a}{3}\right)-f\left(\dfrac{2a+b}{3}\right)-f\left(\dfrac{2b+c}{3}\right)-f\left(\dfrac{2c+a}{3}\right)=g(x)+g(y)+g(z)$.

We will consider non-trivial case where $x, y, z \neq 0$.

Note that if $I=(p,q)$ then $x,y,z\in(p_1,q_1)$ where $p_1 := p-\dfrac{a+b+c}{3}$ and $q_1 := q-\dfrac{a+b+c}{3}$ and g is three times differentiable function on the interval (p_1,q_1).

Since $g(0)=0$ and $g''(0)=0$ then by Maclaurin's Theorem

(1) $\quad g(t)=g'(0)\,t+\dfrac{g'''(\theta)\,t^3}{6}$ fore some $\theta\in(p_1,q_1)$.

Applying (1) to $t=x,y,z$ we obtain

$g(x)+g(y)+g(z)=\dfrac{g'''(\theta_x)\,x^3+g'''(\theta_z)\,y^3+g'''(\theta_z)\,z^3}{6}$ (because $x+y+z=0$).

Since $g'''(t) := f'''\left(\dfrac{a+b+c}{3}+t\right)+f'''\left(\dfrac{a+b+c}{3}-t\right)$ then

$\delta(a,b,c)=\dfrac{1}{6}\sum_{cyc}x^3\left(f'''\left(\dfrac{a+b+c}{3}+x\right)+f'''\left(\dfrac{a+b+c}{3}-x\right)\right)$ and by Darboux's Theorem about intermediate values of derivative for differentiable function f'' we there is such $\xi\in I$ such that

$$\sum_{cyc} x^3\left(f'''\left(\frac{a+b+c}{3}+x\right)+f'''\left(\frac{a+b+c}{3}-x\right)\right)=2\left(x^3+y^3+z^3\right)f'''(\xi).$$

Thus $\delta(a,b,c)=\dfrac{\left(x^3+y^3+z^3\right)f'''(\xi)}{3}$ and, because $x^3+y^3+z^3=3xyz,$ we finally

obtain $\delta(a,b,c)=\dfrac{1}{27}(a-b)(b-c)(c-a)f'''(\xi).$

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Note first of all that we may choose wlog $c>b>a$, since exchanging any two of these values, inverts the sign of both sides of the given equation. Define now, for $m\neq 0$, and suitable parameters to be defined later $\Delta_1,\Delta_2>0$, functions $f_3(x), g_3(x)$:

$$f_3(x)=\frac{f(x+\Delta_1+\Delta_2)-f(x-\Delta_1+\Delta_2)-f(x+\Delta_1-\Delta_2)+f(x-\Delta_1-\Delta_2)}{4\Delta_1\Delta_2},$$

$$g_3(x)=m(x-x_3)+h_3.$$

Assume now that x_3 and $\Delta_3>0$ are chosen in a way such that

$$H_3=(x_3-\Delta_3,x_3+\Delta_3)\subset I.$$

Obviously, $f_3(x)$ and $g_3(x)$ are differentiable in the interval H_3. Therefore, by Cauchy's generalization of the mean value theorem, $x_2\in H_3$ exists such that

$$f_3'(x_2)=\frac{f_3(x_3+\Delta_3)-f_3(x_3-\Delta_3)}{g_3(x_3+\Delta_3)-g_3(x_3-\Delta_3)}g_3'(x_2)=\frac{f_3(x_3+\Delta_3)-f_3(x_3-\Delta_3)}{2\Delta_3}.$$

Using now this value of x_2, define functions $f_2(x), g_2(x)$:

$$f_2(x)=\frac{f'(x+\Delta_1)-f'(x-\Delta_1)}{2\Delta_1},$$

$$g_2(x)=m(x-x_2)+h_2.$$

Note that

$$f_3'(x)=\frac{f_2(x+\Delta_2)-f_2(x-\Delta_2)}{2\Delta_2}$$

Assume again that Δ_2 is chosen such that

$$H_2=(x_2-\Delta_2,x_2+\Delta_2)\subset I.$$

Again, $f_2(x)$ and $g_2(x)$ are differentiable in H_2, and $x_1\in H_2$ exists such that

$$f_2'(x_1)=\frac{f_2(x_2+\Delta_2)-f_2(x_2-\Delta_2)}{g_2(x_2+\Delta_2)-g_2(x_2-\Delta_2)}g_2'(x_1)=\frac{f_2(x_2+\Delta_2)-f_2(x_2-\Delta_2)}{2\Delta_2}=$$

$$= f_3'(x_2).$$

Using now this value of x_1, define finally functions $f_1(x) = f''(x)$ and $g_1(x) = m(x - x_1) + h_1$. Note that

$$f_2'(x) = \frac{f_1(x+\Delta_1) - f_1(x-\Delta_1)}{2\Delta_1}.$$

If once more Δ_1 is chosen so that

$$H_1 = (x_1 - \Delta_1, x_1 + \Delta_1) \subset I,$$

then $f_1(x)$ and $g_1(x)$ are differentiable in H_1, and $\xi \in H_1$ exists such that

$$f_1'(\xi) = \frac{f_1(x_1+\Delta_1) - f_1(x_1+\Delta_1)}{g_1(x_1+\Delta_1) - g_1(x_1-\Delta_1)} g_1'(\xi) = \frac{f_1(x_1+\Delta_1) - f_1(x_1-\Delta_1)}{2\Delta_1} =$$

$$= f_2'(x_1).$$

Therefore, we have proved that $\xi \in H_1 \subset I$ exists such that

$$f'''(\xi) = f_1'(\xi) = f_2'(x_1) = f_3'(x_2) = \frac{f_3(x_3+\Delta_3) - f_3(x_3-\Delta_3)}{2\Delta_3},$$

for suitably defined $x_3, \Delta_1, \Delta_2, \Delta_3$. Taking $\Delta_1 = \frac{c-b}{6}$, $\Delta_2 = \frac{b-a}{6}$, $\Delta_3 = \frac{c-a}{6}$, $x_3 = \frac{a+b+c}{3}$, it follows that

$$H_1, H_2, H_3 \subset \left(\frac{2a+b}{3}, \frac{2c+b}{3}\right) \subset I,$$

and inserting these very values into the form of $f_3(x)$ substituted in the expression for $f'''(\xi)$, the conclusion follows. Note that this general process may be used to find other possible values of $f'''(x)$ in I by selecting other values for $x_3, \Delta_1, \Delta_2, \Delta_3$ (always, of course, values such that $H_1, H_2, H_3 \in I$), and that for functions differentiable more than three times, the process may be carried on in the same way.

Olympiad problems

O79. Let $a_1, a_2, \ldots, a_n$ be integer numbers, not all zero, such that $a_1+a_2+\ldots+a_n = 0$. Prove that

$$|a_1 + 2a_2 + \ldots + 2^{k-1}a_k| > \frac{2^k}{3},$$

for some $k \in \{1, 2, \ldots n\}$.

Proposed by Bogdan Enescu, "B.P.Hasdeu" National College, Romania

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Assume that $\left|a_1 + 2a_2 + ... + 2^{k-1}a_k\right| \leq \frac{2^k}{3}$ for all $k \in \{1, 2, ..., n\}$, the a_i being integers. We shall prove by induction that $a_1 = a_2 = ... = a_n = 0$. For $k = 1$, the result is trivial, since $|a_1| \leq \frac{2}{3} < 1$ directly results in $a_1 = 0$. If the result is true for $i = 1, 2, ..., k-1$, then

$$\frac{2^k}{3} \geq \left|a_1 + 2a_2 + ... + 2^{k-1}a_k\right| = 2^{k-1}|a_k|,$$

yielding $|a_k| \leq \frac{2}{3} < 1$, and again $a_k = 0$. All the a_i are then zero, which is not true. The result follows.

Second solution by John T. Robinson, Yorktown Heights, NY, USA

Let k be the smallest integer such that a_k is non-zero, that is, $a_1 = a_2 = \ldots = a_{k-1} = 0$ (if $k > 1$) and $|a_k| > 0$. Since a_k is an integer, $|a_k| \geq 1$. Therefore $|a_1 + 2a_2 + \ldots + 2^{k-1}a_k| = 2^{k-1}|a_k|$, and $2^{k-1}|a_k| \geq 2^{k-1} > \frac{2}{3} \cdot 2^{k-1} = \frac{2^k}{3}$.

O80. Let n be an integer greater than 1. Find the least number of rooks such that no matter how they are placed on an $n \times n$ chessboard there are two rooks that do not attack each other, but at the same time they are under attack by third rook.

Proposed by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

First solution by Kee-Wai Lau, Hong Kong, China

We show that the least number of rooks is $2n-1$. The standard algebraic notation of the $n \times n$ chessboard is used. By placing the rooks on a_{12}, a_{13}, …, a_{1n}, a_{21}, a_{31}, …, a_{n1}, we see that $2n-2$ rooks are not sufficient. We will prove by induction that $2n-1$ rooks are sufficient. For $n=2$, the result is clear. We now suppose that the result is true for $n=k \geq 1$. By placing the $2k+1$ rooks on the $(k+1)\times(k+1)$ chessboard, there is at least one row containing one rook or no rooks. Otherwise the total number of rooks is greater than or equal to $2k+2$, which is not true. Similarly there is at least one column containing one rook or no rooks. Select any such row and any such column and delete them from the $(k+1)\times(k+1)$ chessboard. We combine the undeleted parts of the $(k+1)\times(k+1)$ chessboard to obtain a $k \times k$ chessboard which contains at least $2k-1$ rooks. Select any $2k-1$ rooks. By the induction assumption, they are sufficient. It follows that $2k-1$ rooks are sufficient for the $(k+1)\times(k+1)$ chessboard. This completes the solution.

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

We will show by induction that $2n-1$ rooks are enough to guarantee that one rook attacks two rooks that do not attack each other. For $n=2$, the result is obvious, since 3 rooks can only be placed in such a way that one row contains 2 rooks and one column contains 2 rooks, and the rook in the intersection of this row and this column attacks the other two, which do not attack each other. Assume the result is true for $n = 2, 3, ..., m$, and consider $2m+1$ rooks in an $(m+1)\times(m+1)$ board. Clearly, there is at least one row with at most one rook (otherwise there would be at least $2m+2$ rooks), and analogously at least one column with at most one rook. Eliminate one such row and one such column, and the rooks contained therein, resulting in an $m \times m$ board with at least $2m+1-2 = 2m-1$ rooks. Now, if two rooks attacked each other before the elimination, they were either in the same row or in the same column, so if the survived the elimination, they still attack each other. Conversely, if they attack each other after the elimination, they also attacked each other before the elimination. But by hypothesis of induction, there is after the elimination one rook that attacks two which do not attack each other, and the situation was exactly the same before the elimination, completing the proof.

Let us now see that $2n-2$ rooks are not enough to guarantee that one rook attacks two rooks which do not attack each other: consider all squares in the first row and first column filled, except for the intersection of the first row and first column. Obviously, two rooks attack each other iff either they are both in the first row, or they are both in the first column, and if one rook attacks two, then these two also attack each other, since the intersection of the first row and the first column is empty in the proposed distribution. So $2n-2$ rooks do not guarantee the desired arrangement, and $2n-1$ is the number that we are looking for.

Third solution by G.R.A.20 Math Problems Group, Roma, Italy

We will show that the least number of rooks such that the property holds in a $m \times n$ chessboard is $m+n-1$.

If the rooks are less than $m+n-1$, we can place them along the first column and along the first row but not at the top left corner (there are $m+n-2$ places). The property does not hold for this displacement.

The thesis holds trivially when $m+n \leq 6$, $n > 1$ and $m > 1$. Now we consider a $m \times n$ chessboard with $m+n > 6$, $n > 1$ and $m > 1$. and we assume that our thesis holds for any $m' \times n'$ chessboard such that $m'+n' < m+n$ and $m' > 1$, $n' > 1$. We can assume without loss of generality that $n \geq m$ and therefore $n > 3$. Since we have at least $m+n-1 \geq m+1$ rooks, then there is a row with at least two rooks. If there is at least another rook in the corresponding colums then the property holds. Otherwise we can cancel these two columns obtaining a $m \times (n-2)$ chessboard with at least $m+(n-2)-1$ rooks. Since $m+n > m+(n-2)$, $m > 1$ and $n-2 > 1$, by the inductive hypothesis, the property holds in this smaller checkboard and therefore it holds also in the initial one.

Also solved by John T. Robinson, Yorktown Heights, NY, USA.

O81. Let $a, b, c, x, y, z \geq 0$. Prove that

$$(a^2 + x^2)(b^2 + y^2)(c^2 + z^2) \geq (ayz + bzx + cxy - xyz)^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas

First solution by Daniel Campos Salas, Costa Rica

The inequality is trivial if any of x, y, z equals 0. Suppose that $xyz \neq 0$. Therefore, dividing by $(xyz)^2$ it follows that the inequality is equivalent to

$$(m^2 + 1)(n^2 + 1)(p^2 + 1) \geq (m + n + p - 1)^2,$$

where $(m, n, p) = \left(\frac{a}{x}, \frac{b}{y}, \frac{c}{z}\right)$. After expanding and rearranging some terms it follows that this inequality is equivalent to

$$m^2n^2p^2 + (m^2n^2 + m + n) + (n^2p^2 + n + p) + (p^2m^2 + p + m) \geq 2mn + 2np + 2pm.$$

From AM-GM it follows that $m^2n^2 + m + n \geq 3mn \geq 2mn$, from where it is easy to conclude the result.

Second solution by Nguyen Manh Dung, HUS, Hanoi, Vietnam

By expanding, we have

$LHS : a^2b^2c^2 + x^2b^2c^2 + y^2c^2a^2 + z^2a^2b^2 + a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2 + x^2y^2z^2,$
$RHS : x^2y^2z^2 + 2xyz(abz + bcx + cay - ayz - bzx - cxy).$

The inequality becomes

$$a^2b^2c^2 + x^2b^2c^2 + y^2c^2a^2 + z^2a^2b^2 + 2xyz(ayz + bzx + cxy) \geq 2xyz(abz + bcx + cay)$$

By the AM-GM inequality, we have

$$x^2b^2c^2 + xyz \cdot bzx + xyz \cdot cxy \geq 3xyz \cdot xbc.$$

Adding two similar inequalities, we obtain

$$x^2b^2c^2 + y^2c^2a^2 + z^2a^2b^2 + 2xyz(ayz + bzx + cxy) \geq 2xyz(abz + bcx + cay),$$

and we are done. Equality holds if and only if $a = b = c = x = y = z = 0$.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Nguyen Manh Dung, Hanoi, Vietnam; John T. Robinson, Yorktown Heights, NY, USA; Perfetti Paolo, Dipartimento di matematica Universita degli studi di Tor Vergata, Italy; Salem Malikic Sarajevo, Bosnia and Herzegovina.

O82. Let $ABCD$ be a cyclic quadrilateral inscribed in the circle $C(O, R)$ and let E be the intersection of its diagonals. Suppose P is the point inside $ABCD$ such that triangle ABP is directly similar to triangle CDP. Prove that $OP \perp PE$.

Proposed by Alex Anderson, New Trier High School, Winnetka, USA

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Since $\frac{PA}{PC} = \frac{PB}{PD} = \frac{AB}{CD}$, point P is the intersection of two distinct Apollonius' circles constructed by taking the ratios of the distances to A and C for one, B and D for the other. If $AB = CD$, the circles degenerate to straight lines that meet only at one point. Otherwise, assuming wlog that $AB < CD$, then the centers of both circles are on the rays CA, DB from C, B, but not on segments CA or DB respectively. Since both points where the circles meet are symmetric with respect to the line joining the centers of both circles, and this line is outside $ABCD$, then both points cannot be in $ABCD$ simultaneously, and P is therefore unique.

If $AB \parallel CD$, then $ABCD$ is an isosceles trapezoid, and $P = E$, or line PE cannot be defined. Assume henceforth then that AB and CD are not parallel, and call $F = AB \cap CD$. Assume furthermore wlog that $BC < DA$ (if $BC = DA$ then $ABCD$ would be an isosceles trapezium and $AB \parallel CD$, which we are assuming not to be true). Obviously, line EF contains all points Q such that the distances from Q to lines AB and CD, respectively $d(Q, AB)$ and $d(Q, CD)$, satisfy $\frac{d(Q,AB)}{d(Q,CD)} = \frac{AB}{CD}$, since it contains E and passes through the intersection of both lines, and trivially AEB and DEC are similar, or the altitudes from E to AB and CD are proportional to the lengths of the sides AB and CD. Therefore, since APB and CPD are similar, the altitudes from D to AB and CD are also proportional to AB and CD, or $P \in EF$.

We will now show that P is the point the circumcircles of ABE and CDE, and line EF, meet. Call first P the second point where the circumcircle of ABE and line EF meet. The power of F with respect to the circumcircle of ABE (which is also the power of F with respect to the circumcircle of $ABCD$ is then $FE \cdot FP = FA \cdot FB = FC \cdot FD$. Therefore, $CDPE$ is also cyclic, and P is also on the circumcircle of CDE. Now, since $ABEP$ and $CDPE$ are cyclic, then $\angle PAB = \angle BEF = \angle PED = \angle PCD$, and similarly $\angle ABP = \angle AEP = \angle CEF = \angle CDP$, or indeed PAB and PCD are similar. Note finally that, if the circumcircles of ABE and CDE where tangent, then $\angle ABE = \angle BEF = \pi - \angle DEF = \angle DCE = \angle ABE$, and ABE and CDE are isosceles and similar, or $AB \parallel CD$.

Call now A', B', C', D' the second points where PD, PC, PB, PA meet the circumcircle of $ABCD$. Trivially, $\angle ACB' = \angle ECP = \angle EDP = \angle BDA'$, or $AB' = BA'$, and similarly $AC' = CA'$, $AD' = DA'$, or $AA'BB'$, $AA'CC'$ and $AA'D'D$ are isosceles trapezii, and $AA' \parallel BB' \parallel CC' \parallel DD'$. Trivially, the diagonals AD' and DA' of $AA'D'D$ meet at P, which is then in the common perpendicular bisector of AA', BB', CC', DD', which trivially passes also through O, and $A'B'C'D'$ is the result of taking the reflection of $ABCD$ with respect to OP. Therefore, OP is the internal bisector of angles $\angle APA'$, $\angle BPB'$, $\angle CPC'$ and DPD'. Now, $\angle BPE = \angle BAE = \angle CDE = \angle CPE$, and PE is the internal bisector of angle $\angle BPC = \pi - \angle BPB'$, or PE is the external bisector of angles $\angle BPB'$ and $\angle CPC'$, and hence perpendicular to their internal bisectors, ie, to OP. The proof is completed.

Remark. Note that this solution includes also the way to construct point P, i.e., the second point where the circumcircles of ABE and CDE meet. If both circles are tangent, then as shown $P = E$, and $ABCD$ is an isosceles trapezium with $AB \parallel CD$.

Also solved by Salem Malikic Sarajevo, Bosnia and Herzegovina.

O83. Let $P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$, $a_n \neq 0$, be a polynomial with complex coefficients such that there is an m with

$$\left|\frac{a_m}{a_n}\right| > \binom{n}{m}.$$

Prove that the polynomial P has at least a zero with the absolute value less than 1.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Perfetti Paolo, Dipartimento di Matematica Tor Vergata Roma, Italy

If $x_1, x_2, \ldots, x_n$ are the roots of $P(x) = 0$, by the Viète's formulae

$$\frac{a_m}{a_0} = (-1)^m \sum x_1x_2 \ldots x_m, \qquad \frac{a_n}{a_0} = (-1)^n x_1x_2 \ldots x_n$$

hence

$$\sum \frac{1}{|x_1|\,|x_2|\,\ldots|x_{n-m}|} \geq \left|\sum \frac{1}{x_1x_2\ldots x_{n-m}}\right| = \left|\frac{a_m}{a_n}\right| > \binom{n}{m}$$

If $\varepsilon = \min_{1\leq k\leq n}\{|x_i|\}$ we have

$$\frac{1}{\varepsilon^{n-m}}\binom{n}{n-m} \geq \sum \frac{1}{|x_1|\,|x_2|\,\ldots|x_{n-m}|} > \binom{n}{m}$$

and then $\varepsilon < 1$. The proof is completed.

Second solution by G.R.A.20 Math Problems Group, Roma, Italy

The zeros of the polynomial

$$Q(x) = a_nx^n + a_{n-1}x_{n-1} + \cdots + a_0$$

are $\{1/w_k, k = 1, \ldots, n\}$ (note that $w_k \neq 0$ because $a_n \neq 0$).

By Vieta's formula

$$\left|\frac{a_m}{a_n}\right| = \sum_{I\in\mathcal{I}_{n-m}} \prod_{k\in I} \frac{1}{|w_k|}$$

where $\mathcal{I}_{n-m}$ is the set of all subsets of $\{1, 2, \ldots, n\}$ such that $|\mathcal{I}_{n-m}| = n - m$. If all zeros of P has the absolute value greater or equal than 1 then $1/|w_k| \leq 1$ and for any integer $m \in [0, n-1]$

$$\left|\frac{a_m}{a_n}\right| \leq \sum_{I\in\mathcal{I}_{n-m}} 1 = \binom{n}{n-m} = \binom{n}{m}$$

and this contradicts the hypothesis.

Third solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Denote by $r_1, r_2, \ldots, r_n$ zeros of $P(x)$, and assume without out loss of generality that $a_0 = 1$, since we may divide all coefficients of $P(x)$ by a_0 without changing the conditions of the problem. For the value of m such that $\left|\frac{a_m}{a_n}\right| > \binom{n}{m}$, call $s_1, s_2, \ldots, s_{\binom{n}{m}}$ the products of each subset of exactly m roots in some order. Clearly, a_m is the sum of the s_k, and using the triangular inequality,

$$|a_n| < \left|\frac{\sum_{k=1}^{\binom{n}{m}} s_k}{\binom{n}{m}}\right| \leq \frac{\sum_{k=1}^{\binom{n}{m}} |s_k|}{\binom{n}{m}},$$

and without loss of generality $\max |s_k| = |s_1| > |a_n|$, where again without loss of generality $s_1 = r_1 r_2 \cdots r_m$. Therefore,

$$|r_{m+1} r_{m+2} \cdot s r_n| = \frac{|r_1 r_2 \ldots r_n|}{|s_1|} < 1,$$

and at least one of $|r_{m+1}|, |r_{m+2}|, \ldots, |r_n|$ is less than 1.

O84. Let $ABCD$ be a cyclic quadrilateral and let P be the intersection of its diagonals. Consider the angle bisectors of the angles $\angle APB$, $\angle BPC$, $\angle CPD$, $\angle DPA$. They intersect the sides AB, BC, CD, DA at $P_{ab}, P_{bc}, P_{cd}, P_{da}$, respectively and the extensions of the same sides at $Q_{ab}, Q_{bc}, Q_{cd}, Q_{da}$, respectively. Prove that the midpoints of $P_{ab}Q_{ab}, P_{bc}Q_{bc}, P_{cd}Q_{cd}, P_{da}Q_{da}$ are collinear.

Proposed by Mihai Miculita, Oradea, Romania

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

The internal bisector of angles $\angle APB$ and $\angle CPD$ is the same straight line, which passes through $P, P_{ab}, P_{cd}, Q_{bc}, Q_{da}$. Clearly, the internal bisector of $\angle BPC$ and $\angle DPA$ passes through $P, P_{bc}, P_{da}, Q_{ab}, Q_{cd}$. Trivially, both internal bisectors meet perpendicularly at P since $\angle APB + \angle BPC = \pi$. Therefore, triangles $P_{ab}PQ_{ab}$, $P_{bc}PQ_{bc}$, $P_{cd}PQ_{cd}$ and $P_{da}PQ_{da}$ are right triangles, and the respective midpoints $O_{ab}, O_{bc}, O_{cd}, O_{da}$ of $P_{ab}Q_{ab}, P_{bc}Q_{bc}, P_{cd}Q_{cd}, P_{da}Q_{da}$, are their respective circumcenters.

Now, without loss of generality $\angle BPQ_{ab} = \frac{\pi - \angle APB}{2}$ and $\angle APQ_{ab} = \frac{\pi + \angle APB}{2}$, or

$$\frac{AQ_{ab} \sin \angle AQ_{ab}P}{AP} = \sin \angle APQ_{ab} = \sin \angle BPQ_{ab} = \frac{BQ_{ab} \sin \angle BQ_{ab}P}{BP},$$

and $\frac{AQ_{ab}}{BQ_{ab}} = \frac{AP}{BP} = \frac{AP_{ab}}{BP_{ab}}$, and the circumcircle of $P_{ab}PQ_{ab}$ is defined as the Apollonius circle such that $\frac{AX}{BX} = \frac{AP}{BP}$.

Note that, denoting by r_{ab} the circumradius of $P_{ab}PQ_{ab}$, we find

$$\frac{O_{ab}A - r_{ab}}{r_{ab} - O_{ab}B} = \frac{P_{ab}A}{P_{ab}B} = \frac{Q_{ab}A}{Q_{ab}B} = \frac{O_{ab}A + r_{ab}}{O_{ab}B + r_{ab}},$$

leading to $O_{ab}A \cdot O_{ab}B = r_{ab}^2$, and A is the result of performing the inversion of B with respect to the circumcircle of $P_{ab}PQ_{ab}$.

Denote by O and R the circumcenter and thecircumradius of $ABCD$. Then the power of O_{ab} with respect to the circumcircle of $ABCD$ is $OO_{ab}^2 - R^2 = O_{ab}A \cdot O_{ab}B = r_{ab}^2$, and the circumcircles of $ABCD$ and $P_{ab}PQ_{ab}$ are orthogonal. In an entirely analogous manner, we deduce that the circumcircle of $ABCD$ is orthogonal to the circumcircles of $P_{ab}PQ_{ab}$, $P_{bc}PQ_{bc}$, $P_{cd}PQ_{cd}$ and $P_{da}PQ_{da}$. For each pair of these circles, O is then in their radical axis, and since each pair of circles meet at P, they either meet at a second point P' also on OP, or they are pairwise tangent at P. In the first case, the centers $O_{ab}, O_{bc}, O_{cd}, O_{da}$ of the four circles are on the perpendicular bisector of PP', and in the second case, on the perpendicular to OP through P. The conlusion follows.