Mathematical Reflections

Junior problems

J73. Let

$$a_n = \begin{cases} n^2 - n, & \text{if } 4 \text{ divides } n^2 - n \\ n - n^2, & \text{otherwise.} \end{cases}$$

Evaluate $a_1 + a_2 + \ldots + a_{2008}$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Andrea Munaro, Italy

It is well known that 4 divides $n^2 - n$ if and only if $n \equiv 0 (\text{mod } 4)$ or $n \equiv 1 (\text{mod } 4)$. Hence the sum is

$$S = ((1^2-1)+(2-2^2))+((3-3^2)+(4^2-4))+\cdots+((2007-2007^2)+(2008^2-2008)).$$

But $n^2 - n + (n+1) - (n+1)^2 = -2n$ and so

$$S = -2\cdot 1+2\cdot 3+\cdots+2\cdot 2007 = 2((-1+3)+(-5+7)+\cdots+(-2005+2007)) = 2008.$$

Second solution by Ganesh Ajjanagadde, Acharya Vidya Kula, Mysore, India

By examining the residues modulo 4, we see that 4 divides $n^2 - n$ if and only if $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$. We know a_0 to be 0, by substituting $n = 0$. $\sum_{i=0}^{2008} a_i$ can be written as

$$\begin{aligned} & a_{2008} + \sum_{k=0}^{501} (a_{4k} + a_{4k+1} + a_{4k+2} + a_{4k+3}) \\ & = a_{2008} + \sum_{k=0}^{501} [((4k)^2 - 4k) + ((4k+1)^2 - (4k+1)) \\ & - ((4k+2)^2 - (4k+2)) - ((4k+3)^2 - (4k+3))] \\ & = a_{2008} + \sum_{k=0}^{501} [-4k + 4k - 12k - 20k - 2 - 6] \\ & = a_{2008} + \sum_{k=0}^{501} (-32k - 8) \\ & = 2008^2 - 2008 - 32 \times \frac{501 \times 502}{2} - 8 \times 502 = 2008. \end{aligned}$$

Third solution by Brian Bradie, Newport News, VA

$$\begin{aligned}\sum_{n=1}^{2008} a_n &= \sum_{k=0}^{501} (a_{4k+1} + a_{4k+2} + a_{4k+3} + a_{4k+4}) \\ &= \sum_{k=0}^{501} \big[(4k+1)^2 - (4k+1) + 4k+2 - (4k+2)^2 + 4k+3 - (4k+3)^2 \\ &\qquad\qquad +(4k+4)^2 - (4k+4)\big] \\ &= \sum_{k=0}^{501} \big(16k^2 + 4k - 16k^2 - 12k - 2 - 16k^2 - 20k - 6 + 16k^2 + 28k + 12\big) \\ &= \sum_{k=0}^{501} 4 = 4(502) = 2008.\end{aligned}$$

Also solved by Arkady Alt, San Jose, California, USA; Athanasios Magkos, Kozani, Greece; Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Publica de Navarra, Spain; G.R.A.20 Math Problems Group, Roma, Italy; John Mangual, New York, USA; John T. Robinson, Yorktown Heights, NY, USA; Navid Safaei, Tehran, Iran; Salem Malikic, Sarajevo, Bosnia and Herzegovina; Vicente Vicario Garcia, Huelva, Spain; Vinoth Nandakumar, Sydney University, Australia; Sarah Burnham, Auburn University Montgomery; Jose Hernandez Santiago, Oaxaca, Mexico; Raul A. Simon, Chile; Roberto Bosch Cabrera, Cuba.

J74. A triangle has altitudes h_a, h_b, h_c and inradius r. Prove that

$$\frac{3}{5} \leq \frac{h_a - 2r}{h_a + 2r} + \frac{h_b - 2r}{h_b + 2r} + \frac{h_c - 2r}{h_c + 2r} < \frac{3}{2}.$$

Proposed by Oleh Faynshteyn, Leipzig, Germany

First solution by Arkady Alt, San Jose, California, USA

Let s be the semiperimeter in our triangle. Since $2rs = ah_a$ then $\frac{2r}{h_a} = \frac{a}{s}$ and

$$\sum_{cyc} \frac{h_a - 2r}{h_a + 2r} = \sum_{cyc} \frac{s-a}{s+a} = 2s \sum_{cyc} \frac{1}{s+a} - 3.$$

Thus

$$\frac{3}{5} \leq \sum_{cyc} \frac{h_a - 2r}{h_a + 2r} \iff 9 \leq 5s \sum_{cyc} \frac{1}{s+a} = \sum_{cyc} (s+a) \cdot \sum_{cyc} \frac{1}{s+a},$$

where the latter inequality is an application of the Cauchy Inequality to triples

$$\left(\sqrt{s+a}, \sqrt{s+b}, \sqrt{s+c}\right), \left(\frac{1}{\sqrt{s+a}}, \frac{1}{\sqrt{s+b}}, \frac{1}{\sqrt{s+c}}\right).$$

Instead of proving that $\sum_{cyc} \dfrac{h_a - 2r}{h_a + 2r} < \dfrac{3}{2}$, we prove that $\sum_{cyc} \dfrac{h_a - 2r}{h_a + 2r} < 1$.

Let $x = s - a, y = s - b, z = s - c$, $x, y, z > 0$. Due to homogeneity we can assume that $s = x + y + z = 1$, then $a = 1 - x, b = 1 - y, c = 1 - z$. Thus,

$$\begin{aligned}
\sum_{cyc} \frac{h_a - 2r}{h_a + 2r} < 1 &\iff \sum_{cyc} \frac{s-a}{s+a} < 1 \\
&\iff s \sum_{cyc} \frac{1}{s+a} < 2 \\
&\iff \sum_{cyc} \frac{1}{2-x} < 2 \\
&\iff \sum_{cyc} (2-y)(2-z) < 2(2-x)(2-y)(2-z).
\end{aligned}$$

The last expression is equivalent to

$$\sum_{cyc} (4 - 2(y+z) + yz) < 2(8 - 4(x+y+z) + 2(xy+yz+zx) - xyz),$$

$$8 + xy + yz + zx < 8 + 4(xy+yz+zx) - 2xyz \iff 2xyz < 3(xy+yz+zx),$$

and the latter inequality holds since

$$3(xy+yz+zx)=3\,(x+y+z)\,(xy+yz+zx)\geq 27xyz>2xyz.$$

Second solution by G.R.A.20 Math Problems Group, Roma, Italy

We know that

$$2A=h_a a=h_b b=h_c c=sr$$

where s is the semiperimeter and A is the area of the triangle. By replacing h_a, h_b, h_c, r in the original equation we obtain

$$\frac{3}{5}\leq\frac{s-a}{s+a}+\frac{s-b}{s+b}+\frac{s-c}{s+c}<\frac{3}{2}$$

that is

$$\frac{3}{5}\leq\frac{x}{x+2y+2z}+\frac{y}{y+2z+2z}+\frac{z}{z+2x+2y}<\frac{3}{2}$$

where $x=s-a$, $y=s-b$, $z=s-c$ are non-negative (and not all zero otherwise the triangle becomes a point).

The inequality on the left is equivalent to

$$2\sum_{\text{sym}}x^2y+2\sum_{\text{sym}}xyz\leq 4\sum_{\text{sym}}x^3$$

which holds by Muirhead's inequality (the equality holds when $x=y=z$, that is when the triangle is equilateral).

The inequality on the right is equivalent to

$$0<4\sum_{\text{sym}}x^3+52\sum_{\text{sym}}x^2y+19\sum_{\text{sym}}xyz$$

which holds because x, y and z are non-negative and not all zero.

Third solution by Salem Malikic, Sarajevo, Bosnia and Herzegovina

Let P denote the area of a triangle. Then

$$\frac{h_a-2r}{h_a+2r}=1-\frac{4r}{h_a+2r}=1-\frac{\frac{8P}{a+b+c}}{\frac{2P}{a}+\frac{4P}{a+b+c}}=1-\frac{\frac{4}{a+b+c}}{\frac{1}{a}+\frac{2}{a+b+c}}=1-\frac{4a}{3a+b+c}.$$

Let us prove the right side of the inequality first.

$$\sum\frac{h_a-2r}{h_a+2r}=3-\sum\frac{4a}{3a+b+c}<\frac{3}{2},$$

which is equivalent to
$$\sum \frac{4a}{3a+b+c} > \frac{3}{2}.$$
From the Triangle Inequality we have that $a < b+c$ thus
$$\sum \frac{4a}{3a+b+c} > \sum \frac{4a}{2a+b+c+b+c} = \sum \frac{4a}{2(a+b+c)} = 2 > \frac{3}{2}.$$
Let us prove the left hand side next. We have
$$\frac{h_a-2r}{h_a+2r} = 1 - \frac{4a}{3a+b+c} = \frac{b+c-a}{3a+b+c} = \frac{x+z+x+y-y-z}{3(y+z)+x+z+x+y} = \frac{x}{x+2y+2z}$$
where we used the substitution $a = x+y$, $b = y+z$, and $c + x + y$. Thus we have to prove
$$\sum \frac{x}{x+2y+2z} \geq \frac{3}{5}.$$
By the Cauchy Shwartz inequality we have
$$\left(\sum \frac{x}{x+2y+2z}\right)\left(\sum x(x+2y+2z)\right) \geq (x+y+z)^2$$
so it is enough to prove that
$$\sum \frac{x}{x+2y+2z} \geq \frac{(x+y+z)^2}{x^2+y^2+z^2+4(xy+yz+zx)}$$
or
$$\frac{(x+y+z)^2}{x^2+y^2+z^2+4(xy+yz+zx)} \geq \frac{3}{5},$$
which is equivalent to
$$x^2+y^2+z^2 \geq xy+yz+zx,$$
and we are done.

Fourth solution by Vicente Vicario Garcia, Huelva, Spain

Recall the following facts:

Lemma 1. In a triangle ABC we have: $ab+bc+ca = r^2+4Rr+s^2$.

Lemma 2. In a triangle ABC we have: $\dfrac{s^2}{27} \geq \dfrac{Rr}{2} \geq r^2$.

Returing back to our problem we have
$$\Delta = \frac{1}{2}ah_a \to h_a = \frac{2\Delta}{a} = \frac{2rs}{a} \quad \text{etc.}$$

Thus

$$\begin{aligned}\frac{h_a-2r}{h_a+2r}+\frac{h_b-2r}{h_b+2r}+\frac{h_c-2r}{h_c+2r} &= \frac{\frac{2rs}{a}-2r}{\frac{2rs}{a}+2r}+\frac{\frac{2rs}{b}-2r}{\frac{2rs}{b}+2r}+\frac{\frac{2rs}{c}-2r}{\frac{2rs}{c}+2r}\\ &= \frac{s-a}{s+a}+\frac{s-b}{s+b}+\frac{s-c}{s+c}.\end{aligned}$$

Let

$$\frac{s-a}{s+a}+\frac{s-b}{s+b}+\frac{s-c}{s+c}=\frac{\Omega(a,b,c,s)}{\Psi(a,b,c,s)}$$

where

$$\begin{aligned}\Omega(a,b,c,s) &= (s-a)(s+b)(s+c)+(s-b)(s+a)(s+c)+(s-a)(s+b)(s+c)\\ &= 3s^3+(a+b+c)s^2-(ab+bc+ca)s-3abc\\ &= 3s^3+2s^3-(r^2+4Rr+s^2)s-3\cdot 4Rrs\\ &= s(4s^2-16Rr-r^2),\end{aligned}$$

and

$$\begin{aligned}\Psi(a,b,c,s) &= (s+a)(s+b)(s+c)\\ &= s^3+(a+b+c)s^2+(ab+bc+ca)s+abc\\ &= s^3+2s^3+(r^2+4Rr+s^2)s+4Rrs = s(4s^2+r^2+8Rr).\end{aligned}$$

Then our inequality becomes

$$\frac{3}{5}\le\frac{4s^2-16Rr-r^2}{4s^2+r^2+8Rr}<\frac{3}{2}.$$

The left hand side of the inequality is equivalent to

$$13Rr+r^2\le s^2,$$

and the right hand side is equivalent to

$$-56Rr-5r^2<4s^2.$$

Using our lemmas we can conclude the result.

Also solved by Andrea Munaro, Italy; Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Vinoth Nandakumar, Sydney University, Australia; Samin Riasat, Notre Dame College, Dhaka, Bangladesh; Esteban Jose Arreaga Ambeliz, Guatemala City, Guatemala; Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; Roberto Bosch Cabrera, Cuba.

J75. Jimmy has a box with n not necessarily equal matches. He is able to construct with them a cyclic n-gon. Jimmy then constructs other cyclic n-gons with these matches. Prove that all of them have the same area.

Proposed by Ivan Borsenco, University of Texas at Dallas

Second solution by G.R.A.20 Math Problems Group, Roma, Italy

Two cyclic n-gons P and P' with the same sides, but not necessarily in the same order, are inscribed in circles with the same radius and therefore they have the same area (because it is equal to the sum of the areas of the n isosceles triangles whose bases are the sides and the two other sides are equal to the radius).

Assume by contradiction that the radius R' of the circle of P' is larger than the radius R of the circle of P. On the sides of P' we construct the arcs of the circle in which the P is inscribed (the arcs do not overlap because P' is convex). We get a closed curve of lenght $2\pi R$ which contains an area larger than πR^2. Since among all closed curves of the same lenght, the circle is the one with the largest area, we have a contradiction. Note that when Jimmy constructs a cyclic n-gon, then keeping the same circle, he can swap any two adjacent sides and therefore he can obtain any permutation of the sides.

Second solution by Mario Garcia Armas, Roberto Bosch Cabrera, Cuba

Let $a_1, a_2, \dots, a_n$ be the n matches. We consider an arbitrary cyclic n-gon of radius R constructed with these matches. Let β_i be the central angle that corresponds to the a_i. By the Law of Cosines we have that $a_i^2 = 2R^2 - 2R^2\cos\beta_i$. Thus $\cos\beta_i = 1 - \frac{a_i^2}{2R^2}$ and we obtain $\beta_i = \arccos\left(1 - \frac{a_i^2}{2R^2}\right)$. Let $f_i(x) = \arccos\left(1 - \frac{a_i^2}{2x^2}\right) \Rightarrow \beta_i = f_i(R)$ and let $g(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$, we will prove that $g(x)$ is strictly decreasing.

$$\begin{aligned} x < y, \quad &\Rightarrow 1 - \frac{a_i^2}{2x^2} < 1 - \frac{a_i^2}{2y^2} \\ &\Rightarrow \arccos\left(1 - \frac{a_i^2}{2x^2}\right) > \arccos\left(1 - \frac{a_i^2}{2y^2}\right) \\ &\Rightarrow f_i(x) > f_i(y) \end{aligned}$$

It suffices to add over all of the i. The following equation has an unique solution:

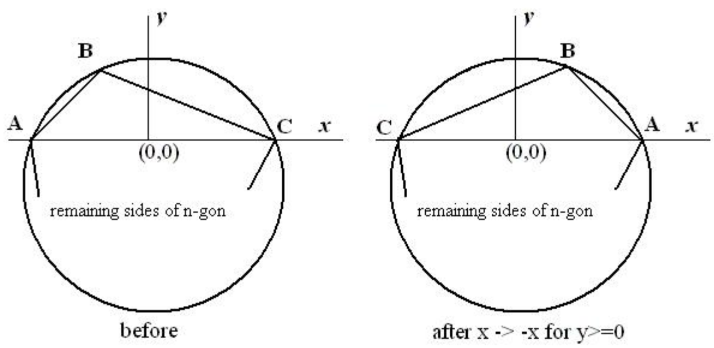
$$g(x) = 2\pi,$$

since $g(x)$ is continuos and strictly decreasing, the solution is $x = R$. It follows that all cyclic n-gons constructed by Jimmy have the same radius and the area of them is

$$S = \frac{1}{2}R^2\sin\beta_1 + \frac{1}{2}R^2\sin\beta_2 + \cdots + \frac{1}{2}R^2\sin\beta_n.$$

Third solution by John T. Robinson, Yorktown Heights, NY, USA

Note that the triangle flipping operation illustrated by the figure is area preserving (since that part of the n-gon below AC is unchanged, and the that part above, namely the triangle ABC, still has the same area after being flipped). Because with this operation we can construct all possible cyclic n-gons, we get that all of them have the same area.



Also solved by Vinoth Nandakumar, Sydney University, Australia; Samin Riasat, Notre Dame College, Dhaka, Bangladesh, Daniel Lasaosa, Universidad Publica de Navarra, Spain

J76. Let $a, b, c \geq 1$ be real numbers such that $a+b+c=2abc$. Prove that

$$\sqrt[3]{(a+b+c)^2} \geq \sqrt[3]{ab-1}+\sqrt[3]{bc-1}+\sqrt[3]{ca-1}.$$

Proposed by Bruno de Lima Holanda, Fortaleza, Brazil

First solution by Athanasios Magkos, Kozani, Greece

We write the given equality as follows

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=2 \Leftrightarrow \frac{ab-1}{ab}+\frac{bc-1}{bc}+\frac{ca-1}{ca}=1.$$

Applying Hölder's inequality to the triples $(a, b, c), (b, c, a), (\frac{ab-1}{ab}, \frac{bc-1}{bc}, \frac{ca-1}{ca})$, we get

$$(a+b+c)^{1/3} \cdot (b+c+a)^{1/3} \cdot \left(\frac{ab-1}{ab}+\frac{bc-1}{bc}+\frac{ca-1}{ca}\right)^{1/3} \geq$$

$$\left(ab\frac{ab-1}{ab}\right)^{1/3}+\left(bc\frac{bc-1}{bc}\right)^{1/3}+\left(ca\frac{ca-1}{ca}\right)^{1/3}=$$

$$\sqrt[3]{ab-1}+\sqrt[3]{bc-1}+\sqrt[3]{ca-1}$$

and we are done.

Second solution by Vardan Verdiyan, Yerevan, Armenia

We are given that

$$0 \leq (ab-1)=\frac{a+b-c}{2c},$$

thus by the AM-GM inequality

$$a+b+c=\sum \frac{(a+b-c)+a+b}{3} \geq \sum \sqrt[3]{ab(a+b-c)}$$

$$=\sum \sqrt[3]{2abc(ab-1)}=\sum \sqrt[3]{(a+b+c)(ab-1)}.$$

Therefore $\sqrt[3]{(a+b+c)^2} \geq \sum \sqrt[3]{ab-1}$ and we are done.

Third solution by Navid Safaei, Teheran, Iran

We have that $\sum a=2abc$ and $\sum \frac{1}{ab}=2 \Rightarrow 3-\sum \frac{1}{ab}=1$ or $\sum\left(1-\frac{1}{ab}=1\right)$. We also know that $a, b, c>1$. Thus each of the terms in the above sum is positive and we can apply Holder's Inequality to get

$$(a+b+c)^{\frac{1}{3}}(b+c+a)^{\frac{1}{3}}\left(\left(1-\frac{1}{ab}\right)+\left(1-\frac{1}{bc}\right)+\left(1-\frac{1}{ca}\right)\right)^{\frac{1}{3}}$$

is greater or equal to

$$\sum \left(ab\left(1-\frac{1}{ab}\right)\right)^{\frac{1}{3}} = \sum \sqrt[3]{ab-1}$$

and we are done.

Fourth solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

The given inequality is equivalent to

$$\sqrt[3]{(a+b+c)^2} \geq \sqrt[3]{\frac{a+b-c}{2c}} + \sqrt[3]{\frac{b+c-a}{2a}} + \sqrt[3]{\frac{a+c-b}{2b}}$$

By Hölder

$$\sqrt[3]{\frac{a+b-c}{2c}} + \sqrt[3]{\frac{b+c-a}{2a}} + \sqrt[3]{\frac{a+c-b}{2b}} \leq$$

$$\big((a+b-c)+(b+c-a)+(a+c-b)\big)^{1/3}\left(\left(\frac{1}{\sqrt[3]{2a}}\right)^{3/2}+\left(\frac{1}{\sqrt[3]{2b}}\right)^{3/2}+\left(\frac{1}{\sqrt[3]{2c}}\right)^{3/2}\right)^{2/3}.$$

Hence

$$(a+b+c)^2 \geq (a+b+c)\left(\frac{1}{\sqrt{2a}}+\frac{1}{\sqrt{2b}}+\frac{1}{\sqrt{2c}}\right)^2,$$

or (using $a+b+c=2abc$)

$$a+b+c \geq \sqrt{ab}+\sqrt{bc}+\sqrt{ca},$$

which clearly holds.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Oleh Faynshteyn, Leipzig, Germany; Salem Malikic, Sarajevo, Bosnia and Herzegovina; Samin Riasat, Notre Dame College, Dhaka, Bangladesh; Vinoth Nandakumar, Sydney University, Australia; Orif Olimovich Ibrogimov, Uzbekistan; Roberto Bosch Cabrera, Cuba.

J77. Prove that in each triangle

$$\frac{1}{r}\left(\frac{b^2}{r_b}+\frac{c^2}{r_c}\right)-\frac{a^2}{r_b r_c}=4\left(\frac{R}{r_a}+1\right).$$

Proposed by Dorin Andrica, Babes-Bolyai University and Khoa Lu Nguyen, MIT

First solution by Prithwijit De, Calcutta, India

We know the follwing facts:

$$r_a=\frac{\Delta}{s-a},\quad r_b=\frac{\Delta}{s-b},\quad r_c=\frac{\Delta}{s-c},\quad r=\frac{\Delta}{s},\quad R=\frac{abc}{4\Delta}.$$

The left hand side of the equality may be simplified by using the above facts to obtain

$$\begin{aligned}
&\frac{b^2s(s-b)+c^2s(s-c)-a^2s(s-b)(s-c)}{\Delta^2}\\
&=\frac{(b^2+c^2-a^2)s^2+s(b+c)(b^2-bc-c^2+a^2(b+c)s-a^2bc)}{\Delta^2}\\
&=\frac{s(b^2+c^2-a^2)(s-b-c)+bc((b+c)s-a^2)}{\Delta^2}\\
&=\frac{2a^2(b^2+c^2+bc)-a^4-(b^4+c^4-2b^2c^2)+2abc(b+c)-4a^2bc}{4\Delta^2}\\
&=\frac{2(a^2b^2+b^2c^2+c^2a^2)-a^4-b^4-c^4+2abc(b+c-a)}{4\Delta^2}\\
&=\frac{16\Delta^2+4abc(s-a)}{4\Delta^2}=4+\frac{4\Delta R(s-a)}{\Delta^2}=4+\frac{4R}{r_a}=4\left(\frac{R}{r_a}+1\right)=\text{R.H.S.}
\end{aligned}$$

and we are done.

Second solution by Andrea Munaro, Italy

Let S and p be the area and semiperimeter of the triangle, respectively. Then we have the well-known identities

$$S=\frac{abc}{4R}=rp=\sqrt{p(p-a)(p-b)(p-c)},\ r_a=\frac{S}{p-a}$$

and the respective cyclic ones. Using these the equality becomes

$$b^2p(p-b)+c^2p(p-c)-a^2(p-b)(p-c)-abc(p-a)=4S^2$$

$$\Leftrightarrow \frac{p}{2}(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2-a^3-b^3-c^3-2abc)=\frac{p}{2}(b+c-a)(c+a-b)(a+b-c),$$

which is true.

Third solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

It is well known that the semiperimeter $s = 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$, then $r_a = s\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$, and similarly for the other two exinradii. It is also well known that $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, and that $\frac{b+c-a}{2} = s-a = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$. It follows that $r_b r_c = bc\cos^2\frac{A}{2}$, $rr_br_c = r_a(s-a)^2$, $r_b - r = 4R\sin^2\frac{B}{2}$ and $r_c - r = 4R\sin^2\frac{C}{2}$. Therefore, using the Cosine Law

$$a^2 = b^2 + c^2 - 2bc\cos A = (b+c)^2 - 4r_br_c,$$

and the given equality is equivalent to

$$\frac{b^2(r_c - r) + c^2(r_b - r) - 2bcr}{rr_br_c} = \frac{4R}{r_a} = \frac{4R(s-a)^2}{rr_br_c},$$

$$b^2\sin^2\frac{C}{2} + c^2\sin^2\frac{B}{2} - 2bc\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = (s-a)^2.$$

But

$$b^2\sin^2\frac{C}{2}\cos^2\frac{A}{2} = (s-a)^2\cos^2\frac{B}{2},$$

$$bc\sin\frac{B}{2}\sin\frac{C}{2}\cos^2\frac{A}{2} = (s-a)^2\cos\frac{B}{2}\cos\frac{C}{2},$$

and the proposed equality is equivalent to

$$\cos^2\frac{B}{2} + \cos^2\frac{C}{2} - 2\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \cos^2\frac{A}{2}.$$

Since

$$\cos^2\frac{A}{2} + 2\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 1 + \sin\frac{A}{2}\cos\frac{B-C}{2}$$

$$= 1 + \cos\frac{B+C}{2}\cos\frac{B-C}{2} = 1 + \frac{\cos B + \cos C}{2} = \cos^2\frac{B}{2} + \cos^2\frac{C}{2},$$

our proof is thus complete.

Also solved by Arkady Alt, San Jose, California, USA; Oleh Faynshteyn, Leipzig, Germany; G.R.A.20 Math Problems Group, Roma; Salem Malikic, Sarajevo, Bosnia and Herzegovina; Vicente Vicario Garcia, Huelva, Spain; Vinoth Nandakumar, Sydney University, Australia; Roberto Bosch Cabrera, Cuba.

J78. Let p and q be odd primes. Prove that for any odd integer $d > 0$ there is an integer r such that the numerator of the rational number

$$\sum_{n=1}^{p-1} \frac{[n \equiv r(\text{mod } q)]}{n^d}$$

is divisible by p, where $[Q]$ is equal to 1 or 0 as the proposition Q is true or false.

Proposed by Robert Tauraso, Roma, Italy

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

If $p < q$, chose $r = p$. Then, $n \equiv r \pmod q$ is always false for $n = 1, 2, ..., p-1$, and the rational number has numerator 0 divisible by p.

If $p > q$, positive integers a, b exist such that $p = aq + b$ with $b < q$. If b is even, take $r = \frac{b}{2}$, and if b is odd, take $r = \frac{q+b}{2}$. In either case, we have that $b - 2r \equiv 0 \pmod q$, or if $n \equiv r \pmod q$, then $p - n \equiv b - r \equiv r \pmod q$. Since n and $p - n$ cannot be equal because p is odd, all numbers $n \in \{1, 2, ..., p-1\}$ such that $n \equiv r \pmod q$ may be grouped up in distinct pairs of the form $(n, p-n)$. For each one of these pairs, their contribution to the total sum is

$$\frac{1}{n^d} + \frac{1}{(p-n)^d} = \frac{n^d + (p-n)^d}{n^d(p-n)^d}.$$

But for odd d,

$$n^d + (p-n)^d = p\left(n^{d-1} - n^{d-2}(p-n) + n^{d-3}(p-n)^2 - ... + (p-n)^{d-1}\right),$$

and the numerator of this fraction is divisible by p, but not its denominator, since n and $p - n$ are both smaller than prime p. Adding any number of such fractions, a common factor p will always appear in the numerator, but never in the denominator, qed.

Second solution by John T. Robinson, Yorktown Heights, NY, USA

First note that if $q \geq p$, by choosing $r \equiv 0, n \equiv 0 \pmod q$ is false for $1 < n \leq p-1$, so the numerator is 0 (which is divisible by p). Therefore in the following we only consider the case $q < p$.

Lemma 1. If $a + b \equiv 0 \pmod m$, then $a^d + b^d = 0$ modulo m, where d is a positive odd integer.

Proof. Since $a \equiv -b \pmod m$, by multiplying the LHS by a $(d-1)$ times and the RHS by $-b(d-1)$ times we have $a^d \equiv (-b)^d \equiv -(b^d) \pmod m$ (since d is odd), from which the result follows.

Lemma 2. If a prime p divides the numerators of the fractions a/b and c/d, then p divides the numerator of $a/b + c/d$.

Proof. Since $a/b + c/d = (a * d + b * c)/(b * d)$, we see that the numerator is divisible by p (by assumption, p does not divide b or d).

Claim. If r is chosen as $r = (p - q)/2$ (p, q odd primes; $p > q$), then the numerator of sum for $1 \leq n \leq p - 1$ of $[n \equiv r \pmod q)]/n^d$ is divisible by p.

Proof. We are summing terms for which n takes on values $(p-q)/2, (p-q)/2+q = (p+q)/2$ $(p-q)/2-q, (p+q)/2+q$ $(p-q)/2-2*q, (p+q)/2+2*q$ etc., where this list of pairs continues for $(p-q)/2-k*q > 0[a]$ $(p+q)/2+k*q < p[b]$ up to some non-negative integer k. Note that for $[a]$ we have $k*q < (p-q)/2$ $k < (p-q)/(2*q)$, and for $[b]$ we have $k*q < p-(p+q)/2 = (p-q)/2$ $k < (p-q)/(2*q)$. Therefore since the constraint on k is the same for the decreasing and increasing terms in each pair in the list, every pair occurs up to the maximum value of k. (Note that k could be 0, for example if $p = 11, q = 7$ then $k < 2/7$ which means there is only one pair $2, 9$ for this case; also note that $(p-q)/2 \geq 1$ and $(p+q)/2 \leq p-1$, which means there is always at least one pair above). Next note that for each pair, the sum of the two values of n, say n_1 and n_2, is p. Therefore for a given pair (n_1, n_2), we have for this partial sum $1/n_1^d + 1/n_2^d = (n_1^d + n_2^d)/((n_1^d) * (n_2^d))$. By Lemma 1 the numerator is divisible by p. Finally, when we add the fractions $(1/n_1^d + 1/n_2^d)$ together for all pairs (n_1, n_2), by Lemma 2 (noting that p cannot divide any denominator since all $n_1, n_2 < p$) the numerator is divisible by p.

Third solution by Vinoth Nandakumar, Sydney University, Australia

In the case where $p \leq q$, we may simply choose $r = 0$: then, among the numbers $(1, 2, 3..., p-1)$ none of them are equal to 0 in $\mathbb{Z}_q$, so the sum is simply 0, and 0 is divisible by p. In the case where $q \leq p$, we shall choose r so that $q|p - 2r$. We can do this, since we select r to be the solution of the modular equation $2r \equiv p(mod\ q)$, which has a solution since p and q are odd primes (we choose r so that $1 \leq r \leq q$). So suppose the set of numbers such that $n \equiv r(modq)$, $n \leq p-1$, is $r, r+q, r+2q, ..., r+kq$ (here $r+kq$ is the last term, that is at most $p-1$). Since $q|p-2r$, $q|(p-r)-r$, so for some value, k', we have $p-r = k'q+r$. Then $(k'+1)q + r = p - r + q > p$ (since $q > p \geq r$). So that means not only does $p - r$ occur in the sequence $r, r+q, r+2q...$, but it is also the last term in the sequence - so that implies $k' = k$ and $r + kq = p - r$. Since $kq = p - 2r$, and p is odd, this means kq is odd, so k is odd. That means there are an even number of terms in the sequence $r, r+q, r+2q, ...r+kq$. Thus we can pair the first one with the last one, the second with the second-last and so-on; and the sum of the numbers in each pair will be equal to $r + (r + kq) = p$. Working in

$\mathbb{Z}_p$, our sum:

$$\sum_{n=1}^{p-1} \frac{[n \equiv r(mod\ q)]}{n^d}$$

will be equal to the sum, of pairs of terms of the form $\frac{1}{t^d} + \frac{1}{(p-t)^d}$. But in $\mathbb{Z}_p$, we have: $\frac{1}{t^d} + \frac{1}{(p-t)^d} = \frac{t^d+(p-t)^d}{(t(p-t))^d}$. Since d is odd, we have $t^d + (p-t)^d = 0$. This means our sum, is equal to the sum of a string of 0s, and is thus 0 - so since a fraction is only 0 in $\mathbb{Z}_p$ when its numerator is divisible by p, it follows that the numerator of our required sum is divisible by p, as required.

Senior problems

S73. The zeros of the polynomial $P(x) = x^3 + x^2 + ax + b$ are all real and negative. Prove that $4a - 9b \leq 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Brian Bradie, VA, USA

Let r_1, r_2, and r_3 be the zeros of $P(x) = x^3 + x^2 + ax + b$. Then

$$\begin{aligned} -1 &= r_1 + r_2 + r_3; \\ a &= r_1 r_2 + r_1 r_3 + r_2 r_3; \\ -b &= r_1 r_2 r_3; \end{aligned}$$

and $4a - 9b = 4(r_1r_2 + r_1r_3 + r_2r_3) + 9r_1r_2r_3$. For each j, let $R_j = -r_j$. Then R_1, R_2, R_3 are positive, with $R_1 + R_2 + R_3 = 1$ and

$$4a - 9b = 4(R_1R_2 + R_1R_3 + R_2R_3) - 9R_1R_2R_3.$$

If we eliminate $R_3 = 1 - R_1 - R_2$ and then examine $4a - 9b$ along lines of constant $R_1 + R_2$ – that is, $R_1 + R_2 = k$ for $0 < k < 1$ – we find

$$\begin{aligned} 4a - 9b &= 4\left[R_1(k - R_1) + R_1(1 - k) + (1 - k)(k - R_1)\right] - 9R_1(k - R_1)(1 - k) \\ &= (5 - 9k)R_1^2 + (9k^2 - 5k)R_1 + 4(k - k^2), \end{aligned}$$

where, for each fixed k, $0 < R_1 < k$. Note that for each k, $4a - 9b$ is a parabola with vertex located at

$$\left(\frac{k}{2}, \frac{9k^3}{4} - \frac{21k^2}{4} + 4k\right).$$

For $0 < k < \frac{5}{9}$, the parabolas open upward, while for $\frac{5}{9} < k < 1$, the parabolas open downward. Examining the upward opening parabolas, we find that $0 < 4a - 9b < 1$ when $0 < k < \frac{5}{9}$. When $\frac{5}{9} < k < 1$, we find $0 < 4a - 9b \leq 1$, with equality for $k = \frac{2}{3}$, $R_1 = \frac{1}{3}$; that is, for $r_1 = r_2 = r_3 = -\frac{1}{3}$. Hence, when the zeros of the polynomial $P(x) = x^3 + x^2 + ax + b$ are all real and negative, $0 < 4a - 9b \leq 1$.

Second solution by Ovidiu Furdui, Ohio, USA

We need the following lemma.

Lemma (Schur's inequality). For any nonnegative real numbers a, b, and c and any positive

real number r the following inequality holds $\sum_{cyclic} a^r(a - b)(a - c) \geq 0$.

When $r = 1$ we obtain that the following third degree Schur's inequality holds

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca). \tag{1}$$

Let x_1, x_2 and x_3 be the roots of P and let $\alpha = -x_1$, $\beta = -x_2$ and $\gamma = -x_3$. It follows,

based on Viete's formulae, that $\alpha+\beta+\gamma = 1$, $\alpha\beta+\beta\gamma+\gamma\alpha = a$, and $\alpha\beta\gamma = b$. On the other hand, inequality (1) implies that

$$(\alpha+\beta+\gamma)^3 + 9\alpha\beta\gamma \geq 4(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha),$$

from which it follows that $1 + 9b \geq 4a$, and the problem is solved.

Third solution by Salem Malikic, Sarajevo, Bosnia and Herzegovina

Let $x_1, x_2, x_3 < 0$ be the roots of polynomial P. We can write $P(x) = (x - x_1)(x - x_2)(x - x_3)$ and using Viete's formula we find that

$$x_1 + x_2 + x_3 = -1 \; x_1x_2 + x_2x_3 + x_3x_1 = a \; -x_1x_2x_3 = b.$$

Let us make the following substitutions $-x_1 = x \; -x_2 = y \; -x_3 = z$. Then x, y, z are positive reals and we have

$$x + y + z = 1 \; xy + yz + zx = a \; xyz = b$$

and our inequality is equivalent to

$$x^3 + y^3 + z^3 + 3xyz \geq x^2y + xy^2 + x^2z + xz^2 + y^2z + zy^2$$

which is true by Schur's inequality.

Also solved by Abhishek Deshpande, Mumbai, India; Andrea Munaro, Italy; Andrei Iliasenco, Chisinau, Moldova; Andrei Frimu, Chisinau, Moldova; Arkady Alt, San Jose, California, USA;Athanasios Magkos, Kozani, Greece; Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Publica de Navarra, Spain; G.R.A.20 Math Problems Group, Roma, Italy; John Mangual, New York, USA; John T. Robinson, Yorktown Heights, NY, USA; Navid Safaei, Teheran, Iran; Oleh Faynshteyn, Leipzig, Germany; Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; Samin Riasat, Notre Dame College, Dhaka, Bangladesh; Vardan, Verdiyan, Yerevan, Armenia; Vinoth Nandakumar, Sydney University, Australia; Orif Olimovich Ibrogimov, Uzbekistan; Roberto Bosch Cabrera, Cuba.

S74. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$(a^a+b^a+c^a)\left(a^b+b^b+c^b\right)(a^c+b^c+c^c) \geq \left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^3.$$

Proposed by Jose Luis-Diaz Barrero, Spain

First solution by Navid Safaei, Teheran, Iran

By Holder's Inequality

$$(a^a+b^a+c^a)^{\frac{1}{3}}(a^b+b^b+c^b)^{\frac{1}{3}}(a^c+b^c+c^c)^{\frac{1}{3}} \geq \left(a^{\frac{a+b+c}{3}}+b^{\frac{a+b+c}{3}}+c^{\frac{a+b+c}{3}}\right) = \sum \sqrt[3]{a}$$

and by cubing both sides we get the desired result.

Second solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

By the symmetry of both the L.H.S. and R.H.S. we can assume $a \geq b \geq c$ so that (a, b, c) majorizes $(1/3, 1/3, 1/3)$:

$$a \geq 1/3, \quad a+b \geq 2/3, \quad a+b+c=1.$$

The inequality is

$$\ln(a^a+b^a+c^a)+\ln(a^b+b^b+c^b)+\ln(a^c+b^c+c^c) \geq 3\ln(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c})$$

and if we show that $f''(x) > 0$, $f(x) = \ln(a^x+b^x+c^x)$, it follows by the majorization inequality for convex functions that $f(a)+f(b)+f(c) \geq f(1/3)+f(1/3)+f(1/3)$ which is our inequality.

$$f''(x) = \Big((ab)^x(\ln a-\ln b)^2+(ac)^x(\ln a-\ln c)^2+(bc)^x(\ln b-\ln c)^2\Big)(a^x+b^x+c^x)^{-2} > 0.$$

The proof is completed.

Also solved by Andrei Frimu, Chisinau, Moldova; Arkady Alt, San Jose, California, USA; Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Nguyen Manh Dung, Hanoi University of Science, Vietnam; Andrei Iliasenco, Chisinau, Moldova; Samin Riasat, Notre Dame College, Dhaka, Bangladesh; Vinoth Nandakumar, Sydney University, Australia; Orif Olimovich Ibrogimov, Uzbekistan; Roberto Bosch Cabrera, Cuba.

S75. Let ABC be a right triangle with $\angle A = 90°$. Let D be an arbitrary point on BC and let E be its reflection in the side AB. Denote by F and G the intersections of AB with lines DE and CE, respectively. Let H be the projection of G onto BC and let I be the intersection of HF and CE. Prove that G is the incenter of triangle AHI.

Proposed by Son Hong Ta, Ha Noi University, Vietnam

First solution by Daniel Campos Salas, Costa Rica

We will use the trivial fact that if a point P in the interior of XYZ is such that P lies on the bisector of angle YXZ and $\angle YPZ = 90 + \frac{\angle YXZ}{2}$ then P is the incenter. Note that G lies on segment AF and that $AGHC$ is cyclic, so

$$\angle AHG = \angle ACG = \angle ACE = \angle CED = \angle EDG = \angle FDG.$$

If H lies between C and D then $DHGF$ is cyclic, which implies that $\angle FDG = \angle FHG$. If D lies between C and H, then $HDGF$ is cyclic, which implies that $\angle FDG = \angle FHG$. In both cases, $\angle FDG = \angle FHG$, so $\angle AHG = \angle FHG$, and this proves that G lies on the internal bisector of angle AHF.

In order to prove that G is in the interior of triangle AHI it is enough to show that F lies between H and I. If D lies between C and H, then angles GFH and CGF are both obtuse, which implies that rays CE and HF intersect beyond E and F, respectively, as we wanted to prove. If H lies between C and D, then $DHGF$ is cyclic, so

$$\angle GFH = \angle GDH = 180 - \angle FDG - \angle C = 180 - \angle ACG - \angle C,$$

where we have used the fact that $\angle FDG = \angle ACG$, as we proved it before. Therefore,

$$\angle GFH + \angle CGF = (180 - \angle ACG - \angle C) + (90 + \angle ACG) > 180,$$

which implies that rays CE and HF intersect beyond E and F, respectively, as we wanted to prove.

This result and the fact that G lies in the internal bisector of angle AHF implies that G lies on the internal bisector of angle AHI. Note that $\angle AGI = 90 + \angle ACG = 90 + \frac{\angle AHI}{2}$, and this completes the proof.

Second solution by Dinh Cao Phan, Vietnam

We have $\angle GAC = \angle GHC = 90°$. Thus $AGHC$ is a cyclic quadrilateral, so $\angle ACG = \angle AHG(1)$ and $\angle GHD = \angle GFD = 90°$. Quadrilateral $FHDG$ is cyclic, so $\angle GDF = \angle GHF(2)$. Point E is the reflection of D through AB thus EGD is isosceles, so $\angle GDE = \angle GED(3)$. From (2) and (3) we conclude that $\angle GED = \angle GHF$. Because $DE \perp AB$ and $AC \perp AB$ we have that $DE \| AC$, therefore, $\angle GED = \angle ACG$ and so $\angle ACG = \angle GHF(4)$. From (1) and (4) we have that $\angle AHG = \angle IHG$ meaning HG is the bisector of $\angle AHI$. Triangles EGK and HGI are similar thus $\angle GIH = \angle GKE$. Since $BFKH$ is a cyclic quadrilateral $\angle GKE = \angle GBH$, therefore, $\angle GIH = \angle GBH$, hence, $BIGH$ is a cyclic quadrilateral and so $\angle BIC = 90°$. Quadrilateral $BIAC$ is cyclic, hence, $\angle IAB = \angle ICB$ because $\angle GAH = \angle GCH$ thus $\angle IAG = \angle GAH$. GA is thus the bisector of $\angle HAI$. G is the intersection of the angle bisector of $\angle HAI$ and $\angle AHI$ or G is the incenter of triangle AHI, and we are done.

Third solution by Ricardo Barroso Campos, Spain

The quadrilateral $GHDF$ is cyclic so $\angle GHF = \angle GDF = \alpha$. GF is the perpendicular bisector of triangle GDE so $\angle GDF = \angle GEF = \alpha$. Line CA is pararell to DE so $\angle GEF = \angle GCA = \alpha$. The quadrilateral $CAGH$ is also cyclic so $\angle GCA = \angle GHA = \alpha$. HG is the bisector of $\angle AHF = \angle AHI$. Let M be the circle circumscribed to ABC. $M \cap CG = \{C, J\}$, thus $\angle ABJ = \alpha$, and $\angle BJA = 90°$. Quadrilateral $BJGH$, is cyclic, and $\angle GHJ = \alpha$, and $J = I$. $ACBI$ is cyclic too thus $\angle BAI = \angle BCI = \beta$, $AGHC$ is cyclic so $\angle HAG = \angle HCG = \beta$. We conclude that AG bisects $\angle HAI$ and thus G is on the biectors of $\angle HAI$ and $\angle AHI$ meaning G is the incenter of AHI.

Also solved by Salem Malikic, Sarajevo, Bosnia and Herzegovina; Andrei Frimu, Chisinau, Moldova; Vicente Vicario Garca, Huelva, Spain; Universidad Publica de Navarra, Spain; Andrea Munaro, Italy; Vinoth Nandakumar, Sydney University, Australia; Roberto Bosch Cabrera, Cuba.

S76. Let x, y, and z be complex numbers such that

$$(y+z)(x-y)(x-z) = (z+x)(y-z)(y-x) = (x+y)(z-x)(z-y) = 1.$$

Determine all possible values of $(y+z)(z+x)(x+y)$.

Proposed by Alex Anderson, New Trier High School, Winnetka, USA

First solution by Vinoth Nandakumar, Sydney University, Australia

First, we note that $x-y \neq 0$, since if $x-y=0$, then $(y+z)(x-y)(x-z)=0$. Thus we can divide both sides of the equation $(y+z)(x-y)(x-z) = (z+x)(y-z)(y-x)$ by $x-y$:

$$(y+z)(x-z) + (z+x)(y-z) = 0$$

$$xy + xz - zy - z^2 + yz + yx - z^2 - zx = 0$$

$$xy = z^2.$$

If $x = 0$, then since $x-y, x-z \neq 0$, y, z are non-zero. Then $yz(y-z) = (z+x)(y-z)(y-x) = (x+y)(z-x)(z-y) = yz(z-y)$, so since $yz \neq 0$, then $y-z=0$, which is a contradiction. Thus x, y, z are non-zero. Since $xy = z^2$, we obtain $\frac{x}{z} = \frac{z}{y}$. Similarly, $zx = y^2$, and $\frac{z}{y} = \frac{y}{x}$. Let $\omega = \frac{y}{x} = \frac{z}{y} = \frac{x}{z}$. Clearly $\omega^3 = (\frac{y}{x})(\frac{z}{y})(\frac{x}{z}) = 1$, but since $\omega \neq 1$ (if $\omega = 1$, then $x-y=0$), it follows that

$$\omega^2 + \omega + 1 = \frac{\omega^3 - 1}{\omega - 1} = 0.$$

Thus we have $y = x\omega$, $z = y\omega = x\omega^2$. Substituting back into the equation $(y+z)(x-y)(x-z) = 1$:

$$(\omega x + \omega^2 x)(x - \omega x)(x - \omega^2 x) = 1$$

$$\Leftrightarrow x^3(\omega + \omega^2)(1-\omega)^2(1+\omega) = 1.$$

Using the identity $\omega^2 + \omega + 1 = 0$, we obtain $\omega + \omega^2 = -1$, $(1-\omega)^2 = -3\omega$, $1 + \omega = -\omega^2$, so:

$$x^3(-1)(-3\omega)(-\omega^2) = 1,$$

$$x^3(-3) = 1,$$

$$x^3 = \frac{-1}{3}.$$

Now, we can compute $(x+y)(y+z)(z+x)$:

$$\begin{aligned}(x+y)(y+z)(x+z) &= (x+\omega x)(\omega x+\omega^2 x)(x+\omega^2 x)\\ &= x^3(1+\omega)(\omega)(1+\omega)(1+\omega^2)\\ &= \frac{-1}{3}(-\omega^2)(\omega)(-\omega^2)(-\omega)\\ &= \frac{-1}{3}(-\omega^6)\\ &= \frac{1}{3}.\end{aligned}$$

Thus, the only possible value of $(x+y)(y+z)(z+x)$ is $\frac{1}{3}$.

Second solution by Daniel Campos Salas, Costa Rica

Note that x, y, z are all distinct. Dividing the first equation by $x-y$ we have that

$$(y+z)(x-z) = (z+x)(z-y),$$

which implies that $xy = z^2$. Analogously, $yz = x^2$ and $zx = y^2$. Note that if $x = 0$ then $z = 0$, which is a contradiction. Thus, x, y, z are all nonzero. It follows that

$$xy \cdot x^2 = z^2 \cdot yz,$$

which implies $x^3 = z^3$. It follows analogously that $x^3 = y^3 = z^3$. Since x, y, z are all distinct we have that $(x,y,z) = (k, \omega k, \omega^2 k)$, or $(x,y,z) = (k, \omega^2 k, \omega k)$, where $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ and k is a complex number. In both cases we have that

$$(y+z)(x-y)(x-z) = (z+x)(y-z)(y-x) = (x+y)(z-x)(z-y) = -3k^3,$$

and

$$(y+z)(z+x)(x+y) = -k^3.$$

From these relations we conclude that the only possible value of $(y+z)(z+x)(x+y)$ is $\frac{1}{3}$.

Third solution by John T. Robinson, Yorktown Heights, NY, USA

Let $a = x+y, b = y+z$, and $c = z+x$. Then since $c-b = x-y, a-c = y-z$, and $b-a = z-x$, the three equations are equivalent to $a \cdot (b-a) \cdot (c-a) = 1$ $b \cdot (a-b) \cdot (c-b) = 1$ $c \cdot (a-c) \cdot (b-c) = 1$ and we are asked to find all possible values for abc (note that neither a, b, nor c can be zero; therefore abc is non-zero which means we can divide by it later).

Multiplying the three equations out, we have
$a^3 - (b+c)a^2 + abc = 1[1]$
$b^3 - (a+c)b^2 + abc = 1[2]$
$c^3 - (a+b)c^2 + abc = 1[3]$

Rearranging terms and factoring out a^2, b^2, and c^2 from [1], [2], and [3] respectively we then have
$a^2(a-b-c) = 1 - abc[4]$
$b^2(b-a-c) = 1 - abc[5]$
$c^2(c-a-b) = 1 - abc[6]$

The following identity [A] will be useful:

$$(a-b-c)(b-a-c)(c-a-b) = a^3+b^3+c^3-a^2(b+c)-b^2(a+c)-c^2(a+b)+2abc.$$

If we add [1], [2], and [3] together we have

$$a^3 - (b+c)a^2 + b^3 - (a+c)b^2 + c^3 - (a+b)c^2 + 3abc = 3.$$

Using [A] this becomes $(a-b-c)(b-a-c)(c-a-b) + abc = 3[\mathrm{B}]$

If we now multiply [4], [5], and [6] together we have

$$(abc)^2(a-b-c)(b-a-c)(c-a-b) = (1-abc)^3 \quad [\mathrm{C}]$$

Let $X = abc$. Combining [B] and [C] we have

$$(a-b-c)(b-a-c)(c-a-b) = 3 - X = (1-X)^3/X^2.$$

Simplifying $3 - X = (1-X)^3/X^2 : \quad 3X^2 - X^3 = 1 - 3X + 3X^2 - X^3$. Thus $1 = 3X$ and so $X = 1/3$.

Thus $abc = 1/3$, that is, the only possible value for $(y+z)(z+x)(x+y)$ is $1/3$.

Also solved by Vardan Verdiyan, Yerevan, Armenia; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; Roberto Bosch Cabrera, Cuba.

S77. Let ABC be a triangle and let X be the projection of A onto BC. The circle with center A and radius AX intersects line AB at P and R and line AC at Q and S such that $P \in AB$ and $Q \in AC$. Let $U = AB \cap XS$ and $V = AC \cap XR$. Prove that lines BC, PQ, UV are concurrent.

Proposed by F. J. Garcia Capitan, Spain and J. B. Romero Marquez, Spain

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Since X, U, R, and X, V, S, are triplets of collinear points, Menelaus' theorem ensures that $\frac{BX}{XC} \cdot \frac{CU}{UA} \cdot \frac{AR}{RB} = 1$ and $\frac{CX}{XB} \cdot \frac{BV}{VA} \cdot \frac{AS}{SC} = 1$. Call $Y = BC \cap UV$. Therefore, again by Menelaus' theorem,

$$\frac{BY}{YC} = \frac{UA}{CU} \cdot \frac{BV}{VA} = \frac{BX^2}{CX^2} \cdot \frac{AR}{BR} \cdot \frac{CS}{AS} = \frac{c\cos^2 B}{b\cos^2 C} \cdot \frac{1+\sin C}{1+\sin B}$$

$$= \frac{c - c\sin B}{b - b\sin C} = \frac{PB}{CQ} = \frac{QA}{CQ} \cdot \frac{PB}{AP},$$

where we have applied the Sine Law, and used that $AX = AP = AQ = AR = AS = b\sin C = c\sin B$, $BX = c|\cos B|$, and $CX = b|\cos C|$, where a, b, c are obviously the lengths of the sides opposing vertices A, B, C. Therefore, by the reciprocal of Menelaus' theorem, Y, P, Q are collinear, or BC, PQ, UV meet at Y.

Second solution by Andrei Iliasenco, Chisinau, Moldova

Denote by Y the intersection of BC and PQ and by Z the intersection of BC and UV. It would suffice to prove that $\frac{BY}{YC} = \frac{BZ}{ZC}$.

From Menelaus' theorem for triangle ABC and secants PQ, UV, SX, RX we get:

1)
$$\frac{BY}{YC} = \frac{BP}{AP} \cdot \frac{AQ}{CQ} = \frac{BP}{CQ}$$

2)
$$\frac{BZ}{ZC} = \frac{BU}{AU} \cdot \frac{AV}{CV}$$

3)
$$\frac{BU}{AU} = \frac{SC}{AS} \cdot \frac{BX}{CX}$$

4)
$$\frac{AV}{CV} = \frac{AR}{BR} \cdot \frac{BX}{CX}$$

Because $AX \perp BC$, BC is tangent to the circle with radius AX, so:

5) $BX^2 = BP \cdot BR$ and $CX^2 = CQ \cdot CS$.

Multiplying (3) by (4), using $(1), (2)$, and (5) we get:

$$\frac{BZ}{ZC} = \frac{SC}{AS} \cdot \frac{BX}{CX} \cdot \frac{AR}{BR} \cdot \frac{BX}{CX} = \frac{BX^2}{CX^2} \cdot \frac{SC}{BR} = \frac{BP \cdot BR \cdot SC}{CQ \cdot CS \cdot BR} = \frac{BP}{CQ} = \frac{BY}{YC}.$$

Also solved by Andrea Munaro, Italy; Salem Malikic, Sarajevo, Bosnia and Herzegovina; Daniel Campos Salas, Costa Rica; Roberto Bosch Cabrera, Cuba; Vicente Vicario Garca, Huelva, Spain; Vinoth Nandakumar, Sydney University, Australia.

S78. Let $ABCD$ be a quadrilateral inscribed into a circle $C(O,R)$ and let (O_{ab}), (O_{bc}), (O_{cd}), (O_{ad}) be the symmetric circles to $C(O)$ with respect to AB, BC, CD, DA, respectively. The pairs of circles $(O_{ab}),(O_{ad})$; $(O_{ab}),(O_{bc})$; $(O_{bc}),(O_{cd})$; $(O_{cd}),(O_{ad})$ intersect again at A',B',C',D'. Prove that A',B',C',D' lie on a circle of radius R.

Proposed by Mihai Miculita, Oradea, Romania

First solution by Andrei Frimu, Chisinau, Moldova

We prove that the quadrilateral $A'B'C'D'$ is congruent to $ABCD$ and the conclusion follows. Let M,N,P,Q be the reflections of O with respect to BC,AB,DA, and CD, respectively. Then it is easy to see that A',B',C',D' are reflections of A,B,C,D with respect to NP,MN,QM and PQ respectively. We are going to show that $AD'QO$ is a rhombus. Indeed, let X, Y be the midpoints of OP and QP respectively. Then X is the midpoint of AD and Y is the midpoint of PQ. Since XY is median line in $\triangle OPQ$ and $\triangle ADD'$, we get $AD' \| XY \| OQ$ and $AD' = 2XY = OQ = R$. Hence $AD'QO$ is a parallelogram with $AD' = OQ = R$. Since $OA = R$, $AD'QO$ is a rhombus. Similarly, $BC'QO$ is a rhombus, hence $BC' = AD' = R$ and $BC' \| OQ \| AD'$. It follows that $ABC'D'$ is a parallelogram, so $C'D' = AB$ and $C'D' \| AB$. Working analogously for other sides of the quadrilateral, we obatin the conclusion.

Second solution by Andrei Iliasenco, Chisinau, Moldova

Suppose the circumcircle of the quadrilateral is the unit circle, and the complex coordinates of it are a,b,c,d. Because (O_{AB}) is simmetric to $C(O)$ with respect to AB, the coordinate of the center of (O_{AB}) is $a+b$. Similarly we can find the coordinates of the other centers.

The equations of the circles (O_{AB}) and (O_{AD}) will be following:

$(a+b-x)(\overline{a}+\overline{b}-\overline{x}) = (a+b-b)(\overline{a}+\overline{b}-\overline{b}) = a*\overline{a} = 1$

$(a+d-x)(\overline{a}+\overline{d}-\overline{x}) = (a+d-d)(\overline{a}+\overline{d}-\overline{d}) = a*\overline{a} = 1.$

Solving these equations we derive that two intersections of circles (O_{AB}) and O_{AD} will have the coordinates a and $a+b+d$. Because $\acute{A}$ must be different from A the complex coordinate of $\acute{A}$ is $a+b+d$. Similarly we derive the coordinates of the other points of intersections of the circles. Then

$\acute{A}-\acute{B} = (a+b+d)-(a+b+c) = d-c.$

Therefore $\acute{A}\acute{B}$ is equal and parallel to the CD. By analogy we derive similar relations between all other sides of quadrilateral $ABCD$ and $\acute{C}\acute{D}\acute{A}\acute{B}$. Therefore quadrilateral $\acute{C}\acute{D}\acute{A}\acute{B}$ is congruent to $ABCD$ and has the same circumradius.

Third solution by Roberto Bosch Cabrera, Cuba

We proceed by coordinate geometry. Setting $O = (0,0)$, $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$, $D = (d_1, d_2)$ we have that the midpoint of AD is $(\frac{a_1+d_1}{2}, \frac{a_2+d_2}{2})$ and the center of the circle (O_{ad}) is $(a_1 + d_1, a_2 + d_2)$, the midpoint of AB is $(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2})$ and the center of the circle (O_{ab}) is $(a_1 + b_1, a_2 + b_2)$, now the midpoint of the segment that joins both centers is $(\frac{a_1+d_1+a_1+b_1}{2}, \frac{a_2+d_2+a_2+b_2}{2})$.

Let $A' = (x, y)$ we obtain:

$$\frac{a_1 + x}{2} = \frac{a_1 + d_1 + a_1 + b_1}{2}$$
$$\frac{a_2 + y}{2} = \frac{a_2 + d_2 + a_2 + b_2}{2}$$

and so $A' = (a_1 + b_1 + d_1, a_2 + b_2 + d_2)$.
Analogously $B' = (a_1 + b_1 + c_1, a_2 + b_2 + c_2)$, $C' = (b_1 + c_1 + d_1, b_2 + c_2 + d_2)$, $D' = (a_1 + c_1 + d_1, a_2 + c_2 + d_2)$.

$$\begin{aligned}
A'B' &= \sqrt{(c_1 - d_1)^2 + (c_2 - d_2)^2} = CD \\
B'C' &= \sqrt{(a_1 - d_1)^2 + (a_2 - d_2)^2} = AD \\
C'D' &= \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} = AB \\
D'A' &= \sqrt{(b_1 - c_1)^2 + (b_2 - c_2)^2} = BC \\
A'C' &= \sqrt{(a_1 - c_1)^2 + (a_2 - c_2)^2} = AC \\
B'D' &= \sqrt{(b_1 - d_1)^2 + (b_2 - d_2)^2} = BD.
\end{aligned}$$

By Ptolemy's theorem in the quadrilateral $ABCD$ and the latter equations we have:

$$A'B' \cdot C'D' + D'A' \cdot B'C' = CD \cdot AB + BC \cdot AD = AC \cdot BD = A'C' \cdot B'D'$$

and so $A'B'C'D'$ is cyclic by Ptolemy's theorem. Note that $\triangle ACD = \triangle A'B'C'$. It follows that the radius of the circle is R, and we are done.

Also solved by Raul A. Simon, Chile; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Vinoth Nandakumar, Sydney University, Australia.

Undergraduate problems

U73. Prove that there is no polynomial $P \in \mathbb{R}[X]$ of degree $n \geq 1$ such that $P(x) \in \mathbb{Q}$ for all $x \in \mathbb{R} \backslash \mathbb{Q}$.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

First solution by Andrei Frimu, Chisinau, Moldova

If there were such q polynomial than we could build an injection $f : \mathbb{R} \setminus \mathbb{Q} \to \{0, 1, 2, \ldots, n\} \times \mathbb{Q}$ in the following way: take some $t \in \mathbb{R} \backslash \mathbb{Q}$. Let $P(t) = y \in \mathbb{Q}$. The equation $P(x) = y$ has $k \leq n$ solutions. Let them be $t_1 < t_2 < \ldots < t_k$. Clearly $t = t_i$ for some $1 \leq i \leq k \leq n$. Define $f(t) = (i, y)$. It is clear why this function is injective. The set $\{0, 1, 2, \ldots, n\} \times \mathbb{Q}$ is countable, hence Imf must be countable too. Then $g : \mathbb{R} \setminus \mathbb{Q} \to \text{Im}f$, $g(x) = f(x)$ is a bijection, so g^{-1} exists, hence $\mathbb{R} \setminus \mathbb{Q}$ is countable, impossible.

Second solution by Arkady Alt, San Jose, California, USA

We will prove the statement of problem using induction on the degree $n \geq 1$.

Suppose that $P(x) = ax + b$, where $a, b \in \mathbb{R}$and $a \neq 0$, such that $P(x) \in \mathbb{Q}$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$. Since $x + 1, \dfrac{x}{2} \in \mathbb{R} \setminus \mathbb{Q}$ and $P(x+1), P\left(\dfrac{x}{2}\right) \in \mathbb{Q}$ then

$a = P(x+1) - P(x) \in \mathbb{Q}$and $b = 2P\left(\dfrac{x}{2}\right) - P(x) \in \mathbb{Q}$.

Hence, $x = \dfrac{P(x) - b}{a} \in \mathbb{Q}$ and that contradicts that $x \in \mathbb{R} \setminus \mathbb{Q}$.

Let $n \geq 2$. Suppose that the statement of problem holds for polynomials of degree

$m \in \{1, 2, ..., n-1\}$ we should to prove that there is no polynomial $P \in \mathbb{R}[X]$ of degree n such that $P(x) \in \mathbb{Q}$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$. Suppose the opposite

$P(x) = a_0 x^n + a_1 x^{n-1} + \cdot + a_{n-1} x + a_n$, where $a_0 \neq 0$, holds $P(x) \in \mathbb{Q}$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$.

Since $x + 1 \in \mathbb{R} \setminus \mathbb{Q}$ then $P(x+1) \in \mathbb{Q}$ and for $P_1(x) := P(x+1) - P(x)$ holds

$1 \leq \deg P_1(x) < n$, $P_1(x) \in \mathbb{Q}$ for any $x \in \mathbb{R} \setminus \mathbb{Q}$. Thus we get a contradiction with earlier asumption of the induction, and so we are done.

Third solution by G.R.A.20 Math Problems Group, Roma, Italy

Since $\mathbb{R} \setminus \mathbb{Q}$ is uncountable and $\mathbb{Q}$ is countable there is a rational number $q \in \mathbb{Q}$ such that $P(x) = q$ for an infinite number of $x \in \mathbb{R} \setminus \mathbb{Q}$. This contradicts the fact that the polynomial P, which is not constant, has at most $n \geq 1$ real solutions.

Also solved by John T. Robinson, Yorktown Heights, NY, USA; Orif Olimovich Ibrogimov, Uzbekistan; Vinoth Nandakumar, Sydney University, Australia; Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; Roberto Bosch Cabrera, Cuba

U74. Prove that there is no differentiable function $f:(0,1)\to\mathbb{R}$ for which $\sup\limits_{x\in E}|f'(x)| = M\in\mathbb{R}$, where E is a dense subset of the domain, and $|f|$ is nowhere differentiable on $(0,1)$.

Proposed by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Assume that a non-empty interval $(a,b)\subset(0,1)$ exists such that $f(x)\neq 0$ for all $x\in(a,b)$. Since f is differentiable on (a,b), thus continuous, f does not change signs on (a,b), and $|f| = f$ on all of (a,b) or $|f| = -f$ on all of (a,b), hence $|f|$ is differenciable on (a,b), reaching a contradiction. Therefore, in every non-empty subinterval of $(0,1)$, at least one point x has zero image. Since f is differentiable, hence continuous, then f is identically zero on $(0,1)$, hence $|f|$ is constant (zero) and thus differentiable, reaching a new contradiction. There is therefore no function $f:(0,1)\to\mathbb{R}$, differentiable on $(0,1)$, such that $|f|$ is nowhere differentiable in $(0,1)$.

U75. Let P be a complex polynomial of degree $n > 2$ and let A and B be 2×2 complex matrices such that $AB \neq BA$ and $P(AB) = P(BA)$. Prove that $P(AB) = cI_2$ for some complex number c.

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, Babes-Bolyai University, Romania

First solution by G.R.A.20 Math Problems Group, Roma, Italy

Let $t = \operatorname{tr}(AB) = \operatorname{tr}(BA)$ and $d = \det(AB) = \det(BA)$ then the characteristic polynomial of AB and BA are the same and

$$(AB)^2 = tAB - dI_2 \quad \text{and} \quad (BA)^2 = tBA - dI_2.$$

This means that any power of $(AB)^k$ with $k \geq 2$ can be reduced to a linear combination of AB and I_2 and that $(BA)^k$ can be reduced to a linear combination of BA and I_2 with the same coefficients. Hence there are complex numbers c_1 and c_2 such that

$$P(AB) = c_1 AB + c_2 I_2 \quad , \quad P(BA) = c_1 BA + c_2 I_2,$$

and

$$P(AB) - P(BA) = c_1(AB - BA) = 0$$

which means that $c_1 = 0$ because $AB \neq BA$ and finally

$$P(AB) = P(BA) = c_2 I_2.$$

Second solution by Jean-Charles Mathieux, Dakar University, Senegal

Denote by χ_M the characteric polynomial of a matrix M. We know that $\chi_{AB} = \chi_{BA}$. By Euclid's division, there is a complex polynomial R of degree at most 1, and a complex polynomial Q, such that $P = Q\chi_{AB} + R$. $R(X) = bX + c$, for some $b, c \in \mathbb{C}$, and $\chi_{AB}(AB) = \chi_{AB}(BA) = 0$. So $P(AB) = bAB + cI_2 = P(BA) = bBA + cI_2$. Since $AB \neq BA$, we have $b = 0$ and $P(AB) = cI_2$. $\square$

Third solution by Orif Olimovich Ibrogimov, Uzbekistan

Lemma. For any $n \times n$ matrices A and B

$$P_{AB}(\lambda) = P_{BA}(\lambda)$$

where $P_c(\lambda)$ is the characteristic polynomial of C. *Proof.* 1-case: Assume that at least one of A and B is nonsingular. Without loss of generality we may assume that B is nonsingular. Then since $AB = B^{-1}BAB$ we have

$\det(AB - \lambda E) = \det(B^{-1}BAB - \lambda E) = \det(BA - \lambda E)$ i.e. $P_{AB}(\lambda) = P_{BA}(\lambda)$.

2-case: Both of A and B are singular. Since $\det(B) = 0$ for any $0 < \epsilon << 1$ we have $\det(B + \epsilon E) \neq 0$ and $P_{A(B+\epsilon E)}(\lambda) = P_{(B+\epsilon E)A}(\lambda)$. Then

$$\lim_{\epsilon \longrightarrow 0} P_{A(B+\epsilon E)}(\lambda) = \lim_{\epsilon \longrightarrow 0} P_{(B+\epsilon E)A}(\lambda)$$

i.e. $P_{AB}(\lambda) = P_{BA}(\lambda)$ and poof is completed. Let P be a complex polynomial of degree $n > 2$. Then we have $P(\lambda) = r(\lambda) P_{AB}\lambda + c_1\lambda + c_2$. Using Hamilton-Cayley theorem we have $P(AB) = c_1 AB + c_2 E$ and $P(BA) = c_1 BA + c_2 E$. Then $c_1 AB + c_2 E = c_1 BA + c_2 E$ therefore $c_1(AB - BA) = 0$ and this means that $c_1 = 0$ i.e. $P(AB) = c_2 E$.

Fourth solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

It is well known that the trace of the product of two square matrices is independent on the multiplication order, so let us call t the trace of AB and BA. Furthermore, $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$, so the characteristic equation of AB and BA is the same, $\lambda^2 - t\lambda + \det(A)\det(B) = 0$, and the eigenvalues λ_1, λ_2 of AB and BA are the same. Let

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Invertible matrices Q, R exist such that $AB = QDQ^{-1}$ and $BA = RDR^{-1}$, and therefore $QP(D)Q^{-1} = P(AB) = P(BA) = RP(D)R^{-1}$, where

$$P(D) = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} = \begin{pmatrix} P(\lambda_1) & 0 \\ 0 & P(\lambda_2) \end{pmatrix}.$$

Thus, $\left(R^{-1}Q\right) P(D) = P(D) \left(R^{-1}Q\right)$. We notice that the diagonal terms of these products are always equal, but the off-diagonal terms are equal if and only if either $P(\lambda_1) = P(\lambda_2)$ or the off-diagonal terms of $R^{-1}Q$ are zero. Now, if $R^{-1}Q = E$ is a diagonal matrix (ie, if a diagonal matrix E exists such that $Q = RE$), and since the product of diagonal matrices is conmutative, then $D = EDE^{-1}$, and $BA = REDE^{-1}R^{-1} = QDQ^{-1} = AB$, which is absurd. We may then call $c = P(\lambda_1) = P(\lambda_2)$, and $P(D) = cI_2$. It trivially follows that $P(AB) = P(BA) = cI_2$.

Also solved by Roberto Bosch Cabrera, Cuba

U76. Let $f:[0,1]\to\mathbb{R}$ be an integrable function such that $\int_0^1 xf(x)dx = 0$. Prove that $\int_0^1 f^2(x)dx \geq 4\left(\int_0^1 f(x)dx\right)^2$.

Proposed by Cezar Lupu, University of Bucharest, Romania and Tudorel Lupu, Constanza, Romania

First solution by Arkady Alt, San Jose, California, USA

Since

$$\begin{aligned}\int_0^1 f(x)\,dx &= \int_0^1 \left(1-\frac{3x}{2}+\frac{3x}{2}\right) f(x)\,dx\\ &= \int_0^1 \left(1-\frac{3x}{2}\right) f(x)\,dx + \frac{3}{2}\int_0^1 xf(x)\,dx\\ &= \int_0^1 \left(1-\frac{3x}{2}\right) f(x)\,dx\end{aligned}$$

and $\int_0^1 \left(1-\frac{3x}{2}\right)^2 dx = \frac{1}{4}$. Then by the Cauchy Inequality

$\left(\int_0^1 f(x)\,dx\right)^2 = \left(\int_0^1 \left(1-\frac{3x}{2}\right) f(x)\,dx\right)^2 \leq \int_0^1 \left(1-\frac{3x}{2}\right)^2 dx \cdot \int_0^1 f^2(x)\,dx = \frac{1}{4}\int_0^1 f^2(x)\,dx.$

Equality occurs if $f(x) = 1-\frac{3x}{2}$. Next we see that

$$\int_0^1 f(x)\,dx = \int_0^1 \left(1-\frac{3x}{2}\right) dx = \frac{1}{4},\quad \int_0^1 xf(x)\,dx = \int_0^1 x\left(1-\frac{3x}{2}\right) dx = 0$$

and $\int_0^1 f^2(x)\,dx = \int_0^1 \left(1-\frac{3x}{2}\right)^2 = \frac{1}{4}$.

Second solution by Li Zhou, Florida, USA

Let $p(x) = 6x-4$, then $\int_0^1 xp(x)dx = 0$ and $\int_0^1 p(x)^2dx = 4$. Let $a = \int_0^1 f(x)dx$. Then

$$\begin{aligned} 0 &\leq \int_0^1 [f(x) + ap(x)]^2 dx \\ &= \int_0^1 f(x)^2dx + 2a\int_0^1 (6x-4)f(x)dx + a^2\int_0^1 p(x)^2dx \\ &= \int_0^1 f(x)^2dx - 8a\int_0^1 f(x)dx + 4a^2 \\ &= \int_0^1 f(x)^2dx - 4a^2, \end{aligned}$$

completing the proof.

Third solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

If $\int_0^1 f(x)dx = 0$ there is nothing to prove so we assume $\int_0^1 f(x)dx \neq 0$. The inequality is invariant respect to the multiplication by a constant and then we take $F(x) = f(x)\Big(\int_0^1 f(x)dx\Big)^{-1}$. By observing that any function of the form $p_c(x) = cx - \frac{2}{3}c$ satisfies $\int_0^1 xp_c(x)dx = 0$, we consider the function $F(x) - p_c(x)$ and the quantity $\int_0^1 (F(x) - p_c(x))^2dx$ yielding (use repeatedly $\int_0^1 xF(x) = \int_0^1 xp_c(x)dx = 0$ and $\int_0^1 F(x)dx = 1$)

$$\begin{aligned} 0 \leq \int_0^1 (F-p_c)^2dx &= \int_0^1 F(F-p_c) - \int_0^1 p_c(F-p_c)dx \\ &= \int_0^1 F(F - cx + \frac{2}{3}c)dx + \frac{2}{3}c\int_0^1 (F-p_c)dx \\ &= \int_0^1 F^2dx + \frac{2}{3}c + \frac{2}{3}c - \frac{2}{3}c(\frac{c}{2} - \frac{2}{3}c) \\ &= \int_0^1 F^2dx + \frac{4}{3}c + \frac{c^2}{9} \end{aligned}$$

and we look for c such that $-\frac{4}{3}c - \frac{c^2}{9} \geq 4$. It is evident that only $c = -6$ satisfies the request and we are done.

Fourth solution by G.R.A.20 Math Problems Group,Roma, Italy

Let $a \in \mathbb{R}$, then by Cauchy-Schwartz inequality

$$\int_0^1 1^2\,dx \cdot \int_0^1 (f(x)+ax)^2\,dx \geq \left(\int_0^1 1\cdot(f(x)+ax)\,dx\right)^2.$$

that is

$$\int_0^1 f^2(x)\,dx + 2a\int_0^1 xf(x)\,dx + \frac{a^2}{3} \geq \left(\mu + \frac{a}{2}\right)^2.$$

where $\mu = \int_0^1 f(x)\,dx$. Since $\int_0^1 xf(x)\,dx = 0$

$$\int_0^1 f^2(x)\,dx \geq \sup_{a\in\mathbb{R}}\left\{-\frac{a^2}{12} + \mu a + \mu^2\right\} = 4\mu^2 = 4\left(\int_0^1 f(x)\,dx\right)^2.$$

The constant 4 is the best one since for $f(x) = 3x - 2$ the equality holds.

Also solved by John T. Robinson, Yorktown Heights, NY, USA; Orif Olimovich Ibrogimov, Uzbekistan; Roberto Bosch Cabrera, Cuba.

U77. Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^2. Prove that if the function $\sqrt{f(x)}$ is differentiable, then its derivative is a continuous function.

Suggested by Gabriel Dospinescu, Ecole Normale Superieure, France

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Note that taking $x' = x - a$ for any real-valued a, the problem is not changed, and we need only to prove the continuity of the derivative of $\sqrt{f(x)}$ at $x = 0$ for all functions satisfying the given conditions. We may consider three cases:

1) $f(0) \neq 0$. By continuity of $f(x)$, we have that $\frac{f'(x)}{2\sqrt{f(x)}}$ is a well-defined and continuous function at $x = 0$, and it is the derivative of $\sqrt{f(x)}$.

2) $f(0) = 0$ in an open interval that contains 0. Then $\sqrt{f(x)}$ is zero (therefore constant) in a non-empty open interval that contains 0, and the derivative of $\sqrt{f(x)}$ is zero in that given interval, hence continuous at $x = 0$.

3) $f(0) = 0$ but no open interval exists that contains 0 and where $f(x)$ is identically 0. Since $f(x)$ is non-negative (otherwise $\sqrt{f(x)}$ would not be defined as a real-valued function) and $\sqrt{f(x)}$ is non-negative by definition, then $f(x)$ is tangent to the horizontal axis at $x = 0$. Note that we need only to prove that $\sqrt{f(x)}$ is also tangent to the horizontal axis at $x = 0$, ie, that $\lim_{x\to 0} \frac{\sqrt{f(x)}}{x} = 0$. Assume that it is not so. Therefore, a non-zero real value a exists such that $\lim_{x\to 0} \frac{\sqrt{f(x)}}{x} = a$, or equivalently, $\lim_{x\to 0} \frac{f(x)}{x^2} = a$. Therefore, for any given $\epsilon > 0$, a sufficiently small $\delta > 0$ exists such that for all $0 < |x| < \delta$, it holds that $\left|f(x) - ax^2\right| < \epsilon^2$, which leads to $\left|\sqrt{f(x)} - \sqrt{|a|}|x|\right| < \epsilon$. In other words, $\sqrt{f(x)}$ behaves in a sufficiently small, but nonzero, interval around $x = 0$ like $|x|$, and it is therefore not differentiable at $x = 0$. Contradiction, or $a = 0$ and $\sqrt{f(x)}$ is tangent to the horizontal axis at $x = 0$. The result follows.

Second solution by John Mangual, New York, USA

Since $f(x)$ is differentiable and $g(x) = \sqrt{f(x)}$ is differentiable we get

$$\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{\sqrt{f(x)}}. \tag{1}$$

Now we wish to establish continuity of the expression on the right. In other words, we wish to establish the following limit:

$$\lim_{\epsilon\to 0} \frac{f'(x+\epsilon)}{\sqrt{f(x+\epsilon)}} = \frac{f'(x)}{\sqrt{f(x)}}.$$

We do not know *a priori* this limit should hold. Instead we can write:

$$L = \lim_{\epsilon \to 0} \left[\frac{f'(x+\epsilon)}{\sqrt{f(x+\epsilon)}} - \frac{f'(x)}{\sqrt{f(x)}} \right] = \lim_{\epsilon \to 0} \frac{f'(x+\epsilon)\sqrt{f(x)} - f'(x)\sqrt{f(x+\epsilon)}}{\sqrt{f(x+\epsilon)f(x)}}.$$

Because $f(x)$ is C^2 and because $\sqrt{f(x)}$ is differentiable we can make the following substitutions in the limit:

- $f'(x+\epsilon) \approx f'(x) + \epsilon f''(x)$
- $\sqrt{f(x+\epsilon)} \approx \sqrt{f(x)} + \epsilon f'(x)/\sqrt{f(x)}$.

These both fall out of Taylor's theorem. The error term vanishes as ϵ goes to zero. Finally,

$$L = \lim_{\epsilon \to 0} \epsilon \cdot \frac{f''(x)\sqrt{f(x)} - f'(x)^2}{\sqrt{f(x+\epsilon)f(x)}} = 0.$$

This limit is 0 because $f(x)$ is continuous.

Third solution by Roberto Bosch Cabrera, Cuba

Let $g(x) = \sqrt{f(x)}$. We will first find $g'(x)$. If $f(x) \neq 0$ we have:

$$g'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$$

Now we consider x such that $f(x) = 0$.

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{g(x+h)}{h} = \lim_{h \to 0} g'(x+h)$$

by L'Hopital's Rule.
So

$$g'(x) = \lim_{h \to 0} \frac{f'(x+h)}{2\sqrt{f(x+h)}} = \lim_{h \to 0} \frac{f''(x+h)}{\frac{f'(x+h)}{\sqrt{f(x+h)}}} = \frac{f''(x)}{2g'(x)}$$

by L'Hopital's Rule.
We have

$$2g'(x)^2 = f"(x) \Rightarrow g'(x) = \frac{\sqrt{f''(x)}}{\sqrt{2}}.$$

We need to prove that $g'(x)$ is continuos, it suffices prove

$$\lim_{h \to 0} \big(g'(x+h) - g'(x)\big) = 0 \quad x : f(x) = 0$$

since $\frac{f'(x)}{2\sqrt{f(x)}}$ and $\frac{\sqrt{f''(x)}}{\sqrt{2}}$ are continuous functions

$$\lim_{h \to 0} \big(g'(x+h) - g'(x)\big) = 0 \Leftrightarrow \lim_{h \to 0} g'(x+h) = g'(x)$$

and we are done.

U78. Let $n = \prod_{i=1}^{k} p_i$, where $p_1, p_2, \ldots, p_k$ are distinct odd primes. Prove that there is a $A \in M_n(\mathbb{Z})$ with $A^m = I_n$ if and only if the symmetric group S_{n+k} has an element of order m.

Proposed by Jean-Charles Mathieux, Dakar University, Senegal

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

We obvserve first that an additional necessary condition for the problem is that $A^j \neq I_n$ for $j = 1, 2, ..., m-1$, since otherwise we could take $A = I_n$, and $A^m = I_n$ for all integer m, the number of elements of S_{n+k} are finite, and so is the number of values that their orders may take.

Consider a matrix $A \in M_n(\mathbb{Z})$ such that $A^m = I_n$. Obviously, A is diagonalizable, and the eigenvalues of A are λ such that $\lambda^n = 1$, ie, the eigenvalues of A are roots of unity. Assume now that a prime p exists such that $n+1 < p \leq n+k$. Obviously, an element $\gamma \in S_{n+k}$ exists that may be decomposed in one p-cycle and $n+k-p$ 1-cycles, its order being p. If the proposed statement were true, then $A^p = I_n$, or the eigenvalues of A are p-th roots of unity, ie, roots of the polynomial $x^p - 1$. Therefore, the characteristic polynomial of A is an n-degree integer polynomial with $n < p-1$, whose roots are p-th roots of unity. If all the eigenvalues are 1, an invertible matrix exists such that $A = P^{-1} I_n P = I_n$, so not all of them are 1. Therefore, some eigenvalue of A is a root of $\frac{x^p-1}{x-1}$, which is well-known to be an integer polynomial, irreducible in $\mathbb{Z}[X]$, and by Abel's irreducibility theorem, the characteristic polynomial of A of degree n is divisible by $\frac{x^p-1}{x-1}$ of degree $p-1 > n$, which is absurd. The proposed statement is then false whenever a prime p exists such that $n+1 < p \leq n+k$. Two particular counterexamples are found for $p_1 = 3$, $p_2 = 5$, $n = 15$, and $n+2 = 17$ is prime, or for $p_1 = 3$, $p_2 = 5$, $p_3 = 7$, and $n+2 = 107$ is prime.

Assume now that for some integer l, n may be expressed as $q_1 + q_2 + ... + q_l - l + 1$ for distinct primes $q_1, q_2, ..., q_l$, ie, that $n+l-1$ may be decomposed as the sum of l primes (for example, for $n = 7$, $7 = 3+5-2+1$, and for $n = 13$, $13 = 3+5+7-3+1$). Assume now that integer matrices A_j may be found with characteristic equations $\frac{x^{q_j}-1}{x-1}$ for $j = 1, 2, ..., l$. For example,for $q_1 = 3$ and $q_2 = 5$ and $q_3 = 7$ we may find

$$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Construct then a matrix A as a diagonal of boxes A_j,

$$A = \begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & A_1 & 0 & ... & 0 \\ 0 & 0 & A_2 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & A_l \end{pmatrix}.$$

A^m is then a diagonal of boxes $1, A_1^m, A_2^m, ..., A_l^m$ with zeros elsewhere, and it will be equal to I_n iff each one of the A_j^m is I_{q_j-1}, ie, iff m is a common multiple of $q_1, q_2, ..., q_l$, since the eigenvalues of A_j are q_j-th roots of unity. Therefore, $A^m = I_n$ iff $q_1 q_2 ... q_l$ divides m. But this means that, if an element $\gamma \in S_{n+k}$ exists of order m, it must contain at least one q_1-cycle, at least one q_2-cycle, and so on, or $n+k \geq q_1 + q_2 + ... + q_l = n + l - 1$, or $k+1 \geq l$. It is now obvious that, if n may be expressed as the product of k primes, it will most probably be expressable as the sum of more than $k+1$ primes, and in none of these cases the proposed statement will be true, as shown above. For example, for $n = 13$, we may take $q_1 = 3$, $q_2 = 5$, $q_3 = 7$, for which the proposed statement is false.

We have therefore shown that: 1) The existance of an element $\gamma \in S_{n+k}$ of order m does not necessarily mean that a matrix $A \in M_n(\mathbb{Z})$ will exist such that $A^m = I_n$ with $A^j \neq I_n$ for $j = 1, 2, ..., n$, and 2) the existance of a matrix $A \in M_n(\mathbb{Z})$ such that $A^m = I_n$ with $A^j \neq I_n$ for $j = 1, 2, ..., n$ does not necessarily mean that an element $\gamma \in S_{n+k}$ of order m will exist.

What is true, however, is that if an element $\gamma \in S_n$ of order m exists, then a matrix $A \in M_n(\mathbb{Z})$ exists such that $A^m = I_n$. Since any element of S_n may be decomposed in a number of disjoint cycles, if an element of order m exists, then $n = n_1 + n_2 + ... + n_C$, where γ is decomposed in one n_1-cycle, one n_2-cycle, and so on, and m is the least common multiple of $n_1, n_2, ..., n_C$. We will show that, for any n_c with $c = 1, 2, ..., C$, it is possible to find an $n_c \times n_c$ matrix A_c such that $A_c^{n_c} = I_{n_c}$. Since the n_c cycle in γ is equivalent of a cycle γ' of the first n_c positive integers, construct matrix A_c as follows: $A_c(i,j) = 1$ if the position of 1 after performing the cycle γ' i times is the j-th, 0 otherwise. Note that multiplying by A_c to the left an $n_c \times n_c$ matrix is equivalent to cycling its rows by γ'. Therefore, multiplying by $A_c^{n_c}$ is equivalent to performing the cycle n_c times, ie, $A_c^{n_c} = I_{n_c}$. Note also that, for any $j = 1, 2, ..., n_c - 1$, $A^j \neq I_{n_c}$, since otherwise the n_c-cycle would contain a j-cycle, $j < n_c$, which is absurd.

Construct now the box matrix

$$A = \begin{pmatrix} A_{n_1} & 0 & 0 & ... & 0 \\ 0 & A_{n_2} & 0 & ... & 0 \\ 0 & 0 & A_{n_3} & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & A_{n_C} \end{pmatrix}.$$

Obviously, A^j is the box matrix whose diagonal is $A^j_{n_1}, A^j_{n_2}, ..., A^j_{n_C}$, and each box will be the identity box iff j is a multiple of the corresponding n_c, or $A^j = I_n$ iff j is a multiple of each one of the n_c, ie, a multiple of m. The result follows.

Olympiad problems

O73. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+a+b+c \geq \frac{2(a+b+c)^3}{3(ab+bc+ca)}.$$

Proposed by Pham Huu Duc, Ballajura, Australia

First solution by Navid Safaei, Tehran, Iran

$$\begin{aligned}
&3\left(\sum ab\right)(LHS) = 3\left(\sum ab\right)\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+a+b+c\right)\\
&= 3\left(\sum a^3 + \frac{ab^3}{c}+\frac{ca^3}{b}+\frac{bc^3}{a}+a^2c+c^2b+b^2a+\sum a^2b+3abc\right)\\
&= 3\sum a^3 + 3\left(\frac{ab^3}{c}+\frac{ca^3}{b}+\frac{bc^3}{a}\right)+3(a^2c+c^2b+b^2a)+3\sum a^2b+9abc.
\end{aligned}$$

Thus

$$\begin{aligned}
&3\left(\sum ab\right)(LHS) - 2\left(\sum a\right)^3\\
&= \sum a^3 - 3abc + 3\left(\frac{ab^3}{c}+\frac{ca^3}{b}+\frac{bc^3}{a}\right) - 3(a^2c+c^2b+b^2a).
\end{aligned}$$

Now by the AM-GM inequality we have

$$\frac{ab^3}{c}+\frac{ca^3}{b} \geq 2\sqrt{\frac{ab^3}{c}\times\frac{ca^3}{b}} = 2a^2b$$

etc,. By adding these inequalities we have

$$\frac{ab^3}{c}+\frac{ca^3}{b}+\frac{bc^3}{a} \geq a^2c+c^2b+b^2a.$$

And it remains to prove that $\sum a^3 \geq 3abc$ which is known to be true.

Second solution by Nguyen Manh Dung, Vietnam

Write the inequality as follows

$$\begin{aligned}
&\sum_{cyc}\left(\frac{a^2}{b}-2a+b\right) \geq \left(\frac{2(a+b+c)^3}{3(ab+bc+ca)}-2(a+b+c)\right)\\
&\Leftrightarrow \sum_{cyc}\frac{(a-b)^2}{b} \geq \frac{a+b+c}{3(ab+bc+ca)}\sum_{cyc}(a-b)^2\\
&\Leftrightarrow (b-c)^2A+(c-a)^2B+(a-cb^2C \geq 0,
\end{aligned}$$

where

$$A = 2(a+b) - c + \frac{3ab}{c},\ B = 2(b+c) - a + \frac{3bc}{a},\ C = 2(c+a) - b + \frac{3ca}{b}.$$

Taking into account the identity

$$(b-c)^2A+(c-a)^2B+(a-cb^2C = \frac{[(a-c)B+(a-b)C]^2+(b-c)^2(AB+BC+CA)}{B+C},$$

it suffieces to show that $B+C>0$ and $AB+BC+CA>0$. We have

$$B+C = 4c+a+b+\frac{3bc}{a}+\frac{3ca}{b}>0$$

and

$$AB+BC+CA = 9(a^2+b^2+c^2)+21(ab+bc+ca)+3\sum_{cyc}\frac{ab(a+b)}{c}>0.$$

Equality holds if and only if $a=b=c$.

Third solution by Daniel Campos Salas, Costa Rica

Rewrite the inequality as

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}-(a+b+c) \geq \frac{2(a+b+c)^3}{3(ab+bc+ca)}-2(a+b+c).$$

The left-hand side expression equals

$$\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-a)^2}{a},$$

and the right-hand side equals

$$\frac{(a+b+c)\left((a-b)^2+(b-c)^2+(c-a)^2\right)}{3(ab+bc+ca)}.$$

We have to prove that

$$\sum_{cyc}\left(\frac{3(ab+bc+ca)}{b}-(a+b+c)\right)(a-b)^2 =$$
$$\sum_{cyc}\left(2a-b+2c+\frac{3ac}{b}\right)(a-b)^2 \geq 0.$$

We will prove the stronger inequality

$$\sum_{cyc} (2a - b + 2c)(a - b)^2 \geq 0.$$

Since the inequality is cyclic with respect to a, b, c, we can assume without loss of generality that $a \geq b, c$. Note that $2a - b + 2c, 2b - c + 2a \geq 0$, so, if $2c - a + 2b \geq 0$, we are done. Suppose that $2c - a + 2b < 0$. From the inequality

$$2\left((a - b)^2 + (b - c)^2\right) \geq (a - c)^2,$$

we have that

$$(2c - a + 2b)(c - a)^2 \geq 2(2c - a + 2b)(a - b)^2 + 2(2c - a + 2b)(b - c)^2.$$

Thus,

$$\begin{aligned} & \sum_{cyc} (2a - b + 2c)(a - b)^2 \\ \geq \;& (2a - b + 2c + 2(2c - a + 2b))(a - b)^2 \\ & +(2b - c + 2a + 2(2c - a + 2b))(b - c)^2 \\ = \;& (3b + 6c)(a - b)^2 + (6b + 3c)(b - c)^2 \geq 0, \end{aligned}$$

as we wanted to prove.

Also solved by Andrea Munaro, Italy; Arkady Alt, San Jose, California, USA; Andrei Frimu, Chisinau, Moldova; Athanasios Magkos, Kozani, Greece; Daniel Lasaosa, Universidad Publica de Navarra, Spain; G.R.A.20 Math Problems Group,Roma, Italy; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; Oleh Faynstein, Leipzig, Germany; Roberto Bosch Cabrera, Cuba; Samin Riasat, Notre Dame College, Dhaka, Bangladesh; Vinoth Nandakumar, Sydney University, Australia; Salem Malikic, Sarajevo, Bosnia and Herzegovina.

O74. Consider a non-isosceles acute triangle ABC such that $AB^2+AC^2 = 2BC^2$. Let H and O be the orthocenter and the circumcenter of triangle ABC, respectively. Let M be the midpoint of BC and let D be the intersection of MH with the circumcircle of triangle ABC such that H lies between M and D. Prove that AD, BC, and the Euler line of triangle ABC are concurrent.

Proposed by Daniel Campos Salas, Costa Rica

First solution by Andrei Iliasenco, Chisinau, Moldova

It is well known that the intersection N of MH with the circumcircle is the same as the intersection of AO with the circumcircle. Hence AN is a diameter of the circumcircle and $AD \perp MH$. Let PQ be the intersection of AD with BC. We have that $AH \perp MQ$ and $MH \perp AQ$. Therefore H is the orthocenter of triangle AMQ, so $QH \perp AM$. If we prove that $OH \perp AM$ we are done. Let OA be $\vec{a}$, OB be $\vec{b}$, OC be $\vec{c}$. It is clear that $\vec{a}^2 = \vec{b}^2 = \vec{c}^2 = r^2$. Then:

$$6\vec{OH}*\vec{AM} = 6\tfrac{\vec{a}+\vec{b}+\vec{c}}{3}*(a-\tfrac{\vec{b}+\vec{c}}{2}) == 2\vec{a}^2-\vec{b}^2-\vec{c}^2+\vec{a}\vec{c}+\vec{a}\vec{b}-2\vec{b}\vec{c} = = \vec{a}\vec{c}+\vec{a}\vec{b}-2\vec{b}\vec{c} = \vec{b}^2+\vec{c}^2-2\vec{a}^2+\vec{a}\vec{c}+\vec{a}\vec{b}-2\vec{b}\vec{c} = 2(\vec{b}-\vec{c})^2-(\vec{a}-\vec{c})^2-(\vec{a}-\vec{b})^2 = 2BC^2-AC^2-AB^2 = 0$$

Second solution by Vinoth Nandakumar, Sydney University, Australia

First we prove the following Lemma:

Lemma: If ABC is a triangle such that $AB^2 + AC^2 = 2BC^2$, then the Euler Line is perpendicular to the median through A.

Proof: Let G be the centroid of triangle ABC, and let M be the midpoint of BC. We seek to prove that OG is perpendicular to AM. Since G divides the line AM in the ratio $2:1$, we have $AG = \frac{2}{3}AM$, and $GM = \frac{1}{3}AM$. Using Apollonius Theorem, we compute:

$$\begin{aligned} AG^2 - GM^2 &= \left(\frac{2}{3}AM\right)^2 - \left(\frac{1}{3}AM\right)^2 \\ &= \frac{AM^2}{3} \\ &= \frac{2(b^2+c^2)-a^2}{12} \\ &= \frac{a^2}{4} \end{aligned}$$

Furthermore, using Pythagoras' Theorem in triangle BOM,

$$\begin{aligned} AO^2 - OM^2 &= BO^2 - OM^2 \\ &= BM^2 \\ &= \frac{a^2}{4} \end{aligned}$$

Thus it follows $AO^2 - OM^2 = AG^2 - GM^2$. Suppose the foot of the perpendicular from O to AM is G'. Then $AG^2 - GM^2 = AO^2 - OM^2 = (AG'^2 + OG'^2) - (MG'^2 + OG'^2) = AG'^2 - MG'^2$. However, as the point G' moves along the line AM from A to M, the quantity $AG'^2 - MG'^2$ decreases, and since G, G' both lie on AM, it follows that G and G' coincide, so OG is perpendicular to AM, as required.

Now consider the point H', the reflection of H in the midpoint M of BC. Clearly $BHCH'$ is a rhombus, so $\angle BH'C = \angle BHC = 180 - A$, so $\angle BH'C + \angle BAC = 180$, so H' lies on the cirumcircle of ABC. But, $\angle ACH' = \angle H'CB + \angle ACB = C + \angle HBC = 90$, so AH' is a diameter of the circumcircle. Thus $\angle ADM = \angle ADH' = 90$.
Consider triangle AHM. From results above, the three altitudes of this triangle are HG, AD, and BC. Since the three altitudes of any triangle are concurrent, it follows that AD, BC, and the Euler line of ABC, HG, meet at a point.

Also solved by Andrei Frimu, Chisinau, Moldova; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Pak Hin Lee, Hong Kong.

O75. Let a, b, c, d be positive real numbers such that $a^2 + b^2 + c^2 + d^2 = 1$. Prove that

$$\sqrt{1-a} + \sqrt{1-b} + \sqrt{1-c} + \sqrt{1-d} \geq \sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d}.$$

Proposed by Vasile Cartoaje, Ploiesti, Romania

First solution by Athanasios Magkos, Kozani, Greece

The inequality takes the form $\sum(\sqrt{1-a} - \sqrt{a}) \geq 0$. We have

$$\sqrt{1-a} - \sqrt{a} = \frac{1-a-a}{\sqrt{1-a}+\sqrt{a}} = \frac{1-2a}{\sqrt{1-a}+\sqrt{a}} \geq \frac{1}{\sqrt{2}} \sum(1-2a),$$

where in the last step we used the estimate $(x+y)^2 \leq 2(x^2+y^2)$ applied to the numbers $\sqrt{1-a}, \sqrt{a}$. Hence, it suffices to prove that

$$\sum(1-2a) \geq 0 \Leftrightarrow 4-2(a+b+c+d) \geq 0 \Leftrightarrow 4(a^2+b^2+c^2+d^2) \geq (a+b+c+d)^2,$$

which is clearly true (for instance using Power Mean.)

Second solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

If one of the coordinates is equal to 1, the inequality is clearly true being $\sqrt{1-a} + \sqrt{1-b} + \sqrt{1-c} + \sqrt{1-d} - \sqrt{a} - \sqrt{b} - \sqrt{c} - \sqrt{d} = 2$ by $a^2 + b^2 + c^2 + d^2 = 1$. By continuity (1) is true for any quadruple (a, b, c, d) where one of the coordinates is sufficiently close to the value 1.

If one of the coordinates, say a, is equal to zero the inequality is true as well. Indeed we have

$$1 + \sqrt{1-b} + \sqrt{1-c} + \sqrt{1-d} \geq \sqrt{b} + \sqrt{c} + \sqrt{d} \tag{2}$$

i.e.

$$1 + \frac{1-2b}{\sqrt{b}+\sqrt{1-b}} + \frac{1-2c}{\sqrt{c}+\sqrt{1-c}} + \frac{1-2d}{\sqrt{d}+\sqrt{1-d}} \geq 0.$$

The concavity of $\sqrt{x}$ yields

$$1 + \sqrt{2}(1-2b) + \sqrt{2}(1-2c) + \sqrt{2}(1-2d) \geq 0 \quad \Longrightarrow \quad b+c+d \leq \frac{1+3\sqrt{2}}{2\sqrt{2}}.$$

By power–maeans–inequality and $b^2 + c^2 + d^2 = 1$ we have

$$\frac{b+c+d}{3} \leq \Big(\frac{b^2+c^2+d^2}{3}\Big)^{1/3} = 1/\sqrt{3} \implies b+c+d \leq \sqrt{3},$$

and $\sqrt{3} < (1+3\sqrt{2})/2\sqrt{2}$ (the inequality is strict). It follows that if one of the coordinates close enough to zero, (1) is true.

Motivated by these observations we study (1) as a constrained minimum problem introducing the Lagrange multipliers. Let

$$f(a,b,c,d) = \sqrt{1-a} + \sqrt{1-b} + \sqrt{1-c} + \sqrt{1-d} - \sqrt{a} - \sqrt{b} - \sqrt{c} - \sqrt{d}$$

defined on the set $D \doteq \{\delta \leq a,b,c,d \leq \delta'\}$ where δ and δ' are so close respectively to 0 and 1 that (1) holds. The lack of continuity of the derivative of $\sqrt{x}$ at $x=0$ and $\sqrt{1-x}$ at $x=1$ forces us to introduce δ and δ'. Let's define

$$F(a,b,c,d,\lambda) = f(a,b,c,d) - \lambda(a^2+b^2+c^2+d^2-1).$$

We have to solve the system $S \doteq \{F_a = F_b = F_c = F_d = 0, F_\lambda = 0\}$ and

$$F_a = -\frac{1}{2\sqrt{1-a}} - \frac{1}{2\sqrt{a}} - 2\lambda a.$$

The equation $F_a - F_b = 0$ is equivalent to

$$H(a,b) \doteq \frac{F_a}{a} - \frac{F_b}{b} = \frac{1}{2b\sqrt{b}} - \frac{1}{2a\sqrt{a}} + \frac{1}{2b\sqrt{1-b}} - \frac{1}{2a\sqrt{1-a}} = 0.$$

The graph of the function $\frac{1}{2x\sqrt{x}} + \frac{1}{2x\sqrt{1-x}}$ has the two asymptotes at $x=0$ and $x=1$ and only one minimum at $x = x_m \neq 1/2$ with $0 < x_m < 1$. It follows that the equation $H(a,b) = 0$ is solved by $a = b = a_0$ or $a = a_0$ and $b = b_0 \neq a_0$ with $\delta \leq a_0 < x_m$ and $x_m < b_0 \leq \delta'$. The other five combinations: $\frac{F_a}{a} - \frac{F_b}{c} = 0$, $\frac{F_a}{a} - \frac{F_b}{d} = 0$, $\frac{F_a}{b} - \frac{F_b}{c} = 0$, $\frac{F_a}{b} - \frac{F_b}{d} = 0$, $\frac{F_a}{c} - \frac{F_b}{d} = 0$, have the same structure hence the possible solutions of S are:

$$\begin{aligned}
&\{a=b=c=d=1/2\}, \qquad \{a=b=a_0,\ c=d=b_0\},\\
&\{a=b=c=a_0, d=b_0\},\\
&\{a=a_0, b=c=d=b_0\}
\end{aligned}$$

and the other eleven combinations. In five of them two of the coordinates are equal to a_0 and the other two are equal to b_0. In six of them three coordinates are equal to a_0 or b_0 and the fourth to b_0 or a_0. We prove that

$$1)\{a=b=a_0,\ c=d=b_0\} \text{ or } 2)\{a=b=c=a_0, d=b_0\} \text{ or } 3)\{a=a_0, b=c=d=b_0\}$$

are all incompatible with $a^2+b^2+c^2+d^2=1$.

1) Being $a_0 < b_0$, $\frac{F_a}{a} - \frac{F_c}{c} = 0$ implies $\frac{1}{a_0\sqrt{1-a_0}} > \frac{1}{b_0\sqrt{1-b_0}}$ hence $(a_0-b_0)(b_0 + a_0 - a_0^2 - a_0b_0 - b_0^2) > 0$ or $b_0 + a_0 - a_0b_0 - a_0^2 - b_0^2 = b_0 + a_0 - a_0b_0 - \frac{1}{2} < 0$. $a_0^2 + b_0^2 = \frac{1}{2}$ has been used. Moreover we have $(a_0+b_0)^2 = \frac{1}{2} + 2a_0b_0$ hence

$(\frac{1}{2} + 2a_0b_0)^{1/2} < \frac{1}{2} + a_0b_0$ or $4x^2 - 4x - 1 > 0$ where $x = a_0b_0$. We would end with $x < (1-\sqrt{2})/2$ or $x > (1+\sqrt{2})/2$ and both of them are impossible.

2) $a_0 < b_0$. $\frac{F_a}{a} - \frac{F_d}{d} = 0$ implies $\frac{1}{a_0\sqrt{1-a_0}} > \frac{1}{b_0\sqrt{1-b_0}}$ namely $a_0 + b_0 - a_0b_0 - a_0^2 - b_0^2 < 0$ as before. Employing $3a_0^2 + b_0^2 = 1$ we get $(a_0+b_0) - (a_0+b_0)^2 + a_0b_0 < 0$. We note that $a_0 + b_0 < 1$ and then the inequality is false. That $a_0 + b_0 < 1$ follows by $3a_0^2 + b_0^2 = 1$.

3) $a_0 < b_0$. That $\frac{F_a}{a} - \frac{F_d}{d} = 0$ is incompatible with $3a_0^2 + b_0 = 1$ follows exactly as in 2) due to the symmetry of the inequality $a_0 + b_0 - a_0b_0 - a_0^2 - b_0^2 < 0$

The conclusion of our argument is that the only critical point of the gradient of the function $F(a,b,c,d,\lambda)$ is the point $(1/2, 1/2, 1/2, 1/2, -\sqrt{2})$ and $f(1/2, 1/2, 1/2, 1/2) = 0$.

On the boundary of D we have already shown the validity of (1).

The conclusion is that on the boundary of the domain we have $f(a,b,c,d) > 0$. The Weierstrass theorem on the continuous functions on compact sets assures the existence of maximum and minimum of f and then the point $(1/2, 1/2, 1/2, 1/2)$ must be a minimum.

Third solution by Oleh Faynshteyn, Leipzig, Germany

From the condition it follows that $(a,b,c,d) \in (0,1)$ and

$$1 = a^2 + b^2 + c^2 + d^2 \geq \frac{1}{4}(a+b+c+d)^2, \ a+b+c+d \leq 2.$$

We have

$$\sqrt{a} = \frac{\sqrt{a(1-a)}}{\sqrt{1-a}} \leq \frac{1}{2\sqrt{1-a}}.$$

Then

$$\sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d} \leq \frac{1}{2}\left(\frac{1}{\sqrt{1-a}} + \frac{1}{\sqrt{1-b}} + \frac{1}{\sqrt{1-c}} + \frac{1}{\sqrt{1-d}}\right).$$

It suffices to prove that

$$2(\sqrt{1-a} + \sqrt{1-b} + \sqrt{1-c} + \sqrt{1-d}) \geq \frac{1}{\sqrt{1-a}} + \frac{1}{\sqrt{1-b}} + \frac{1}{\sqrt{1-c}} + \frac{1}{\sqrt{1-d}}$$

or

$$\sum_{cyc}\left(2\sqrt{1-a} - \frac{1}{\sqrt{1-a}}\right) \geq 0$$

which is equivalent to

$$\sum_{cyc} \frac{1-2a}{\sqrt{1-a}} \geq 0.$$

By applying the Cauchy-Schawrz inequality, we have

$$\sum_{cyc} \frac{1-2a}{\sqrt{1-a}} \geq \frac{4(4-2(a+b+c+d))}{\sqrt{1-a}+\sqrt{1-b}+\sqrt{1-c}+\sqrt{1-d}} \geq 0.$$

Equality holds if $a=b=c=d=\frac{1}{2}$.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; G.R.A.20 Math Problems Group, Roma, Italy; Vinoth Nandakumar, Sydney University, Australia; Roberto Bosch Cabrera, Cuba.

O76. A triple of different subsets S_i, S_j, S_k of a set with n elements is called a "triangle". Define its perimeter by

$$|(S_i \cap S_j) \cup (S_j \cap S_k) \cup (S_k \cap S_i)|\,.$$

Prove that the number of triangles with perimeter n is $\frac{1}{3}(2^{n-1}-1)(2^n-1)$.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Let us call s_m, $m = 1, 2, ..., n$ the elements of the n-element set S out of which subsets S_1, S_2 and S_3 are chosen. Each triangle may be represented by a $n \times 3$ matrix, where element in position (m, l) equals 1 if s_m is in S_l, 0 otherwise, for $l = 1, 2, 3$. The conditions of the problem require that 1) no two columns are equal, otherwise the corresponding subsets would be equal, and 2) each row contains at least 2 1's, otherwise an element of S would not be in any of the intersections, and the cardinal of the union would be less than n. Provided that these two conditions are met, it is obvious that the sets are distinct, and that the cardinal of the union equals the cardinal n of S.

Therefore, each row can take 4 different values, $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ and $(1, 1, 1)$, for a total of 4^n possible combinations. However, some are not allowed, since they would result in two columns being equal. In fact, columns 1 and 2 are equal if and only if the only values taken by the rows are $(1, 1, 0)$ and $(1, 1, 1)$, out of which there are a total of 2^n possible combinations. We may calculate in the same way the number of permutations that we need to discard in order to avoid the combinations such that columns 1 and 3 are equal, and columns 2 and 3 are equal. Note however that, out of the 2^n combinations such that two columns are equal, one corresponds to the entire matrix being 1's, and will appear thus in the three calculations, or the total number of combinations that are not allowed out of the 4^n is $3(2^n - 1) + 1 = 3 \cdot 2^n - 2$, for a total of $4^n - 3 \cdot 2^n + 2$ allowed combinations. Finally, note that in counting these combinations, we have counted each triangle six times, since a permutation of the three columns leaves the triangle unchanged but produces a different matrix. The total number of triangles with perimeter n is then

$$\frac{4^n - 3 \cdot 2^n + 2}{6} = \frac{2\left(2^{n-1}\right)^2 - 3 \cdot 2^{n-1} + 1}{3} = \frac{\left(2 \cdot 2^{n-1} - 1\right)\left(2^{n-1} - 1\right)}{3}.$$

Second solution by G.R.A.20 Math Problems Group, Roma, Italy

Let X_n be the set of n elements. We say that a triple of subsets of X_n is a *degenerate triangle* if the subsets are not all different. If $S_i = S_j$ then

$$(S_i \cap S_j) \cup (S_j \cap S_k) \cup (S_k \cap S_i) = S_i \cup (S_i \cap S_k) = S_i$$

and it follows that a degenerate triangle has perimeter n if and only if at least two of its subsets are equal to X_n. Therefore the number of degenerate triangles of X_n with perimeter n is the number of subsets of X_n, that is 2^n. Let a_n be the number of non-degenerate triangles of X_n with perimeter n, then we would like to prove that

$$a_n = \frac{1}{3}(2^{n-1} - 1)(2^n - 1).$$

For $n = 1$ there are no non-degenerate triangles and $a_1 = 0$. We can construct any non-degenerate triangle of X_{n+1} from the triangles of X_n. If the triangle of X_n is non-degenerate then the new element have be added to at least two subsets: there are $3 + 1 = 4$ different ways to do that. If the triangle of X_n is degenerate and different from $\{X_n, X_n, X_n\}$ then the new element has to be added to one of the two sets which are equal to X_n and to third one: only 1 way. Hence we have the linear recurrence:

$$a_1 = 0, \quad a_{n+1} = 4 \cdot a_n + 1 \cdot (2^n - 1) = 4a_n + 2^n - 1$$

which can be easily solved

$$a_n = \frac{1}{6}4^n - \frac{1}{2}2^n + \frac{1}{3} = \frac{1}{3}(2^{n-1} - 1)(2^n - 1).$$

Third solution by John Mangual, New York, USA

Let us consider the disjoint union of, $T_1 = (S_i \cap S_j) \backslash S_k$, $T_2 = (S_j \cap S_k) \backslash S_i$, $T_3 = (S_k \cap S_i) \backslash S_j$ and $T_4 = S_i \cap S_j \cap S_k$. The union of these four sets had better contain all n elements. Therefore let us try to place each of the numbers $\{1, 2, \ldots, n\}$ into T_1, T_2, T_3 or T_4 so that S_i, S_j, S_k are all distinct. By inclusion-exclusion, we can initially place elements into the T_i arbitrarily and remove those sets which do not fit our criterion. There are 4^n ways of doing this first step. However, we have to remove the case where exactly two of S_i, S_j, S_k are distinct. If $S_i = S_j$, then $S_i = S_i \cap S_j = S_j$, so we are placing elements into T_1 or T_4. There are 2^n ways of doing this. However, we could have set $S_i = S_k$ or $S_j = S_k$ instead, so there are 3 ways of picking the "different" set totaling $3 \cdot 2^n$ exceptions. This removes the case when $S_i = S_j = S_k$ three times, so we must add back 2. Now we've counted the number of ways of placing $\{1, 2, \ldots, n\}$ into distinct labeled sets S_i, S_j, S_k, but we've overcounted the triangles by a factor of 6 so:

$$\#\{\text{Triangles}\} = \frac{4^n - 3 \cdot 2^n + 2}{6} = \frac{1}{3}(2^n - 1)(2^{n-1} - 1).$$

Fourth solution by Vinoth Nandakumar, Sydney University, Australia

Name the elements of the subset $1, 2, ...n$. First we count the number of subsets S_i, S_j, S_k, which have perimeter n, such that S_i and S_j are identical, but not the same as S_k. Call such a subset a bad triangle. Clearly the repeated subset must be the complete subset $(1, 2...n)$ - since if an element, say i, is missing, then i will not appear in any of the sets $S_i \cap S_j, S_i \cap S_k, S_j \cap S_k$. Once S_i and S_j are both the complete set, $(1, 2, 3...n)$, then S_k can be chosen arbitrarily, and will satisfy the condition. Thus the number of bad triangles is $2^n - 1$ (since S_k cannot be the same as S_i and S_j).

Now we prove by induction that the number of triangles with perimeter n is $\frac{1}{3}(2^{n-1} - 1)(2^n - 1)$. For $n = 1$, the claim is true, since there are no triangles with perimeter 1, and $\frac{1}{3}(2^0 - 1)(2^1 - 1) = 0$. For $n = 2$, the claim is true, since there is one triangle with perimeter 1: $(1), (2), (1, 2)$. Suppose the claim is true for $n = m$. We seek to prove the claim for $n = m + 1$.

First, list the $2^m - 1$ bad triangles, which are triples of subsets of $(1, 2, 3...m)$. For each bad triangle (S_i, S_j, S_k), where $S_i = S_j$, consider adding the new element, $m + 1$, to sets S_i, S_k - now these 3 subsets form a satisfactory triangle of perimeter $m + 1$, with respect to the set $(1, 2, 3...m + 1)$ (since S_i and S_k are different, so the subsets are different). Also, we can easily check that if we are only allowed to add the element $m + 1$ to any set (S_i, S_j, S_k), or to keep it the same, there is no other way of obtaining a triangle with perimeter $m+1$ from a bad triangle with perimeter m. Thus, from each bad triangle, by adding $m+1$, we obtain exactly one triangle of perimeter $m+1$ - so in this fashion, we obtain $2^m - 1$ triangles of perimeter $m + 1$ (with respect to set $(1, 2, 3...m + 1)$).

Now consider any triangle with respect to set $(1, 2...m)$. For each triangle, (S_i, S_j, S_k), from it we can obtain 4 triangles with respect to the set $(1, 2...m + 1)$, by: adding the element $m + 1$ to sets S_i, S_j; or sets S_i, S_k; or sets S_j, S_k; or sets S_i, S_j and S_k. Thus for each of the $\frac{1}{3}(2^{m-1} - 1)(2^m - 1)$ triangles with perimeter m, we can obtain 4 triangles with perimeter $m+1$ - so in this fashion, we obtain $\frac{4}{3}(2^{m-1} - 1)(2^m - 1)$ triangles.

Together, the number of triangles with perimeter $m + 1$ that we have constructed, is:

$$\begin{aligned}
2^m - 1 + (\frac{4}{3}(2^{m-1} - 1)(2^m - 1)) &= (2^m - 1)(1 + \frac{4}{3}(2^{m-1} - 1)) \\
&= \frac{1}{3}(2^m - 1)(3 + 4 * 2^{m-1} - 4) \\
&= \frac{1}{3}(2^m - 1)(2^{m+1} - 1)
\end{aligned}$$

Now, it remains to see that we have not missed any triangles with perimeter $m + 1$ in our construction. But this is simple: For any triangle of perimeter $m + 1$ with respect to set $(1, 2, 3...m + 1)$, the element $m + 1$ must occur in at least two of the three subsets S_i, S_j, S_k - and if we remove the element $m+1$ in whichever subsets it occurs in, we will have, (with respect to the set $(1, 2...m)$), either a bad triangle, or a triangle with perimeter m. In either case, we have accounted for the triangle of perimeter $m+1$ in our construction; this completes the induction.

O77. Consider the polinomials $f, g \in \mathbb{R}[X]$. Prove that there is a nonzero polynomial $P \in \mathbb{R}[X, Y]$ such that $P(f, g) = 0$.

Proposed by Iurie Boreico, Harvard University, USA

Solution by Vinoth Nandakumar, Sydney University, Australia

Define polynomial $P_{a,b}(x) = f(x)^a g(x)^b$, for a, b non-negative integers. First suppose 2 polynomials, $P_{a,b}(x)$ and $P_{c,d}(x)$ are the same, for two distinct pairs of integers a, b and c, d. Then we have $f(x)^a g(x)^b - f(x)^c g(x)^d = 0$, so we can choose $P(x, y) = x^a y^b - x^c y^d$, and $P(f, g) = 0$, as required. Now suppose all polynomial $P_{a,b}(x)$ are distinct. Let $f(x)$ have degree k and $g(x)$ have degree l. Consider all polynomials of the form $P_{a,b}(x)$, which have degree, at most $4kl$. We can count, that there must be at least $8kl$ such polynomials: the polynomial $P_{a,b}(x)$ has degree $ka + lb$, so the number of such polynomials is equal to the number of pairs (a, b) of integers such that $ka+lb \leq 4kl$. Consider the quadrant of Cartesian plane, with both x and y co-ordinate being non-negative. For each lattice point (a, b), with a, b integers, assign it the value $ka + lb$. The number of pairs (a, b) with $ka + lb \leq 4kl$ is equal to the number of lattice points lying underneath, or on, the line $ka + lb = 4kl$, which passes through $(0, 4k)$ and $(4l, 0)$. The number of such points, is at least half the number of points lying in a rectangle of dimensions $4k$ and $4l$, which is $8kl$. Thus we have at least $8kl$ polynomials $P_{a,b}(x)$, with degree at most $4kl$. Consider the vector space of polynomials, $P(\mathbb{R}, 4kl)$, which have degree at most $4kl$. This vector space has a basis of $(1, x, x^2, ...x^{4kl})$, and thus has dimension $4kl + 1$. It is a well-known result in linear algebra, that in any finite dimensional vector space, the length of any linearly independent list is at most the length of a basis - so thus, any $4kl + 2$ polynomials in the vector space $P(R, 4kl)$, cannot be linearly independent. Consequently, our list of $8kl$ distinct polynomials of form $P_{a,b}(x)$, cannot be linearly independent. Thus we can find suitable constants $\lambda_{i,j}$, not all of which are zero, where $0 \leq ki + lj \leq 4kl$, such that $\sum_{i,j} \lambda_{i,j} * P_{i,j}(x) = 0$. Now choose polynomial $P(x, y) = \sum_{i,j} \lambda_{i,j} * x^i * y^j$. It follows that $P(f, g) = 0$, and since P is a non-zero polynomial, we are done.

O78. Let ABC be a triangle and let M, N, P be the midpoints of sides BC, CA, AB, respectively. Denote by X, Y, Z the midpoints of the altitudes emerging from vertices A, B, C, respectively. Prove that the radical center of the circles AMX, BNY, CPZ is the center of the nine-point circle of triangle ABC.

Proposed by Cosmin Pohoata, Bucharest, Romania

First solution by Andrei Frimu, Chisinau, Moldova

Let ω be the center of the nine-point circle. Our aim is to show that the power of ω with respect to each of the circumscribed circles of the triangles AMX, BNY and CPZ is constant. Let $\Gamma(T, x)$ be the circumcircle of $\triangle AMX$. Let us calculate the power of ω wrt Γ.
The power is $\rho(\omega, \Gamma) = TM^2 - T\omega^2$. Let us compute $T\omega^2$. We use the well-known fact that $OG : G\omega = 2 : 1$ and apply Stewart Theorem for triangle $T\omega O$ and cevian TG. We get

$$3TG^2 = 2T\omega^2 + TO^2 - \frac{2}{3}O\omega^2.$$

Our aim is to prove that $TM^2 - T\omega^2$ is constant, or, ignoring the $\frac{2}{3}O\omega^2$ (since it does not depend on our choice of AMX), we must prove that

$$E_A = 2TM^2 - 3TG^2 + TO^2$$

does not depend on the initial choice of AMX. Note that $\rho(O, \Gamma) = TM^2 - TO^2$. Let R be a point in the plane such that quadrilateral $RAXM$ is an isosceles trapezoid. Then $R \in \Gamma$ and $RM \| AX \perp BC$, hence $O \in RM$. Assume the parallel through A to BC intersects OR in S. Then $\triangle ARS \equiv \triangle HMA'$, where A' is the foot of the altitude from A. Hence $SR = XA_1 = \dfrac{h_a}{2}$, and $OS = h - OM$. Let $t = OM$. Then $TM^2 - TO^2 = \rho(O, \Gamma) = OM \cdot OR = t\left(\dfrac{3h}{2} - t\right)$.
Then

$$TO^2 = TM^2 - t\left(\frac{3h}{2} - t\right),$$

and consequently

$$E_A = 2TM^2 - 3TG^2 + TM^2 - t\left(\frac{3h}{2} - t\right) = 3TM^2 - 3TG^2 - t\left(\frac{3h}{2} - t\right).$$

By Leibniz, $3TG^2 = TA^2 + TB^2 + TC^2 - \frac{1}{3}(a^2 + b^2 + c^2)$, hence it is enough to show that

$$3TM^2 - TA^2 - TB^2 - TC^2 - t\left(\frac{3h}{2} - t\right)$$

is constant. Note that $TM = TA$ and $2TM^2 = TB^2 + TC^2 - \frac{1}{2}BC^2$, thus we must show that

$$P_A = \frac{1}{2}BC^2 + t\left(\frac{3h}{2} - t\right)$$

is constant.
Now that we got rid of T we are left with an identity in the triangle, namely, we are going to prove that

$$P_A = \frac{1}{2}BC^2 + t\left(\frac{3h}{2} - t\right) = \frac{3(a^2+b^2+c^2)}{8} - R^2,$$

hence it is a constant value and the proof ends.
To see this, note that $t = R\cos A$, $t^2 = R^2 - \dfrac{a^2}{4}$ and $h = \dfrac{2S}{a} = \dfrac{bc\sin A}{a} = \dfrac{bc}{2R}$.
Hence,

$$P_A = \frac{1}{2}BC^2 + t\left(\frac{3h}{2} - t\right) = \frac{a^2}{2} + \frac{3}{4}bc\cos A - \left(R^2 - \frac{a^2}{4}\right)$$

Since $bc\cos A = \frac{1}{2}(b^2 + c^2 - a^2)$, we finally get

$$P_A = \frac{3(a^2+b^2+c^2)}{8} - R^2$$

Second solution by Donlapark Pornnopparat, Thailand

Let ω_1, ω_2 and ω_3 be circumcircles of triangles AMX, BNY and CPZ respectively. Let H be the orthocenter and R be the center of the nine-point circle ω of triangle ABC. Let RM intersect ω_1 at point E and RN intersect ω_2 at point F. Denote by H_1 and H_2 the midpoints of AH and BH, respectively. Because EM passes through the center R of ω and it is well-known that H_1 is on ω and H_1M is a diameter of ω, therefore E, H_1, R, M lie on a line. In the same way, we have F, H_2, R, N lie on a line. Denote be U and V the feet of altitudes from A and B of triangle ABC, respectively. It is obvious that quadrilateral $ABUV$ is cyclic. By the Power of a Point Theorem, we have,

$$\begin{aligned}
HA \cdot HU &= HB \cdot HV \\
\left(\frac{HA}{2}\right)\left(\frac{AU}{2} - \frac{HA}{2}\right) &= \left(\frac{HB}{2}\right)\left(\frac{BV}{2} - \frac{HB}{2}\right) \\
AH_1(AX - AH_1) &= BH_2(BY - BH_2) \\
AH_1 \cdot H_1X &= BH_2 \cdot H_2Y \\
EH_1 \cdot H_1M &= FH_2 \cdot H_2N.
\end{aligned}$$

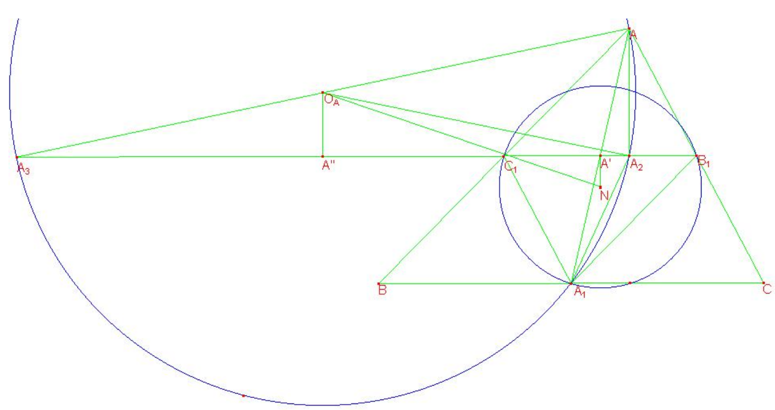
But $H_1M = H_2N$ are diameters of ω, therefore we have $EH_1 = FH_2$. Then we have,

$$\begin{aligned} RM \cdot RE &= RM(RH_1 + EH_1) \\ &= RN(RH_2 + FH_2) \\ &= RN \cdot RF. \end{aligned}$$

Thus the Powers of Point of R with respect to ω_1 and ω_2 are equal. In the same way, we get that the Powers of Point R with respect to ω_1, ω_2 and ω_3 are equal. Thus we conclude that R is the radical center of ω_1, ω_2 and ω_3.

Third solution by Vinoth Nandakumar, Sydney University, Australia

For convenience, we first relabel the vertices:
Problem: Let the midpoints of sides BC, AC, AB of scalene triangle ABC be A_1, B_1, C_1, and the midpoints of the altitudes from A, B, C, be A_2, B_2, C_2. Prove that the radical centre of the circumcircles of $AA_1A_2, BB_1B_2, CC_1C_2$ is the nine-point centre N of ABC.
Clearly, A_2 lies on B_1C_1 (the dilation $(A, \frac{1}{2})$ maps the foot of the altitude from A, which lies on BC to A_2, so A_2 lies on the image of line BC, which is C_1B_1). A', the midpoint of B_1C_1, is the foot of perpendicular from N to B_1C_1. Let the circle Γ_A through A, A_1, A_2 meet B_1C_1 again at A_3, and let its centre be O_A. Let the foot of the perpendicular from O_A to A_2A_3, i.e., the midpoint of A_2A_3, be A''. Let the sidelengths of triangle $A_1B_1C_1$ be a, b, c respectively. We compute that the power of N with respect to Γ_A is symmetric with respect to the three sides (from here the conclusion follows, since the power of N with respect to the three circles will be same).



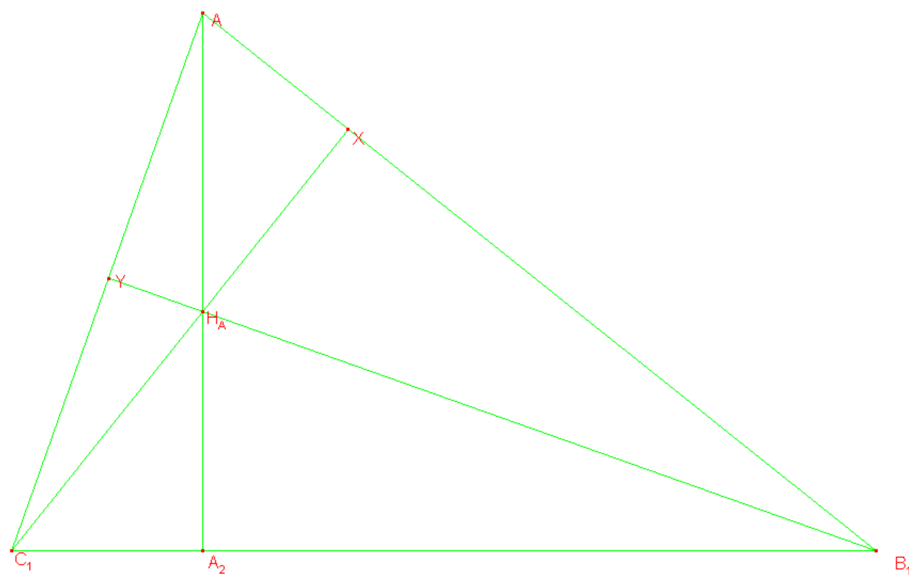
Since A_2 lies on Γ_A, the power of N with respect to Γ_A is $O_AN^2 - O_AA_2^2$:

$$\begin{aligned}
O_AN^2 - O_AA_2^2 &= ((O_AA'' + A'N)^2 + A'A''^2) - (O_AA''^2 + A_2A''^2) \\
&= ((O_AA'' + A'N)^2 - O_AA''^2) - (A''A_2^2 - A'A''^2) \\
&= A'N(2O_AA'' + A'N) - (A''A_2 - A'A'')(A''A_2 + A'A'') \\
&= A'N(AA_2 + A'N) - A'A_2(A''A' + A''A_3) \\
&= A'N(AA_2 + A'N) - A'A_2 * A'A_3
\end{aligned}$$

Here $A'A_2 * A'A_3 = AA'^2 = \frac{2b^2+2c^2-a^2}{4}$ using Power of a Point theorem in $AA_1A_2A_3$ and Apollonius' Theorem. If H_A is orthocentre of AB_1C_1, since $\triangle AB_1C_1 \equiv \triangle A_1B_1C_1$, $AA_2 = AH_A + H_AA_2$. Also, $A'N = \frac{1}{2}AH_A$ (if N', G are the circumcentre and centroid of AB_1C_1 then $A'N \parallel AH_A$ and $\triangle AH_AG \sim \triangle A'N'G$, so the result follows since $AG = 2GA'$). Thus:

$$\begin{aligned}
O_AN^2 - O_AA_2^2 &= A'N(AA_2 + A'N) - AA_2 * A'A_3 \\
&= \frac{1}{2}AH_A(\frac{3}{2}AH_A + H_AA_2) - \frac{2b^2+2c^2-a^2}{4} \\
&= \frac{3}{4}AH_A^2 + \frac{1}{2}AH_A * H_AA_2 - \frac{a^2+b^2+c^2}{2} + \frac{3a^2}{4} \\
&= \frac{3}{4}(AH_A^2 + a^2) + \frac{1}{2}AH_A * H_AA_2 - \frac{a^2+b^2+c^2}{2}
\end{aligned}$$

Now, it is sufficient to prove that both expressions $AH_A^2 + a^2$ and $AH_A * H_AA_2$ are symmetric in the variables a, b and c. Indeed, if the feet of the perpendicular from C_1 to AB_1 is X, then firstly we must prove that $AH_A^2 + B_1C_1^2 = C_1H_A^2 + AB_1^2$. This follows from the fact that the quadrilateral $AH_AB_1C_1$ has perpendicular diagonals, and the sum of the squares of opposite sides of any such quadrilateral are equal. Secondly, we must prove that $AH_A * H_AA_2 = C_1H_A * H_AX$. This is a consequence of Power of a Point theorem in quadrilateral AXC_1A_2, which is concyclic since $\angle AXC_1 = \angle AA_2C_1 = 90$.



Fourth solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

If ABC is equilateral, circles AMX, BNY, CPZ degenerate to lines AM, BN, CP, which meet at the center (and also nine-point center) of ABC; the result trivially follows in this particular case.

If ABC is not equilateral but isosceles, wlog at A, then circle AMX degenerates to line AM, which is a symmetry axis of ABC. Therefore, circles BNY, CPZ are symmetric around AM, and must hence meet at two points on AM. The radical center cannot then be defined; however, the nine-point center lies somewhere between the two points where circles BNY, CPZ meet, on line AM.

We will thus assume henceforth that ABC is not isosceles. Choose a Cartesian coordinate system such that the x-axis coincides with line BC, and A is on the positive half of the y-axis. Then, the origin $(0,0)$ is the foot of the altitude from A, and $A \equiv (0,h)$, $X \equiv \left(0, \frac{h}{2}\right)$, where h is the length of the altitude from A. Furthermore, we may take wlog $B \equiv (-b\cos C, 0)$ and $C \equiv (c\cos B, 0)$, and since M is the midpoint of BC, using the theorem of the sine and calling R the circumradius of ABC, then $M \equiv (R\sin(C-B), 0)$. The circumcenter $O_A \equiv (x_A, y_A)$ and circumradius R_A of circle AMX satisfy then $y_A = \frac{3h}{4}$ (since O_A is on the perpendicular bisector $y = \frac{3h}{4}$ of AX), and

$$x_A^2 + \frac{h^2}{16} = R_A^2 = (x_A - R\sin(C-B))^2 + \frac{9h^2}{16};$$

$$O_A \equiv \left(\frac{R\sin(C-B)}{2} + \frac{h^2}{4R\sin(C-B)}, \frac{3h}{4} \right),$$

$$R_A^2 = \frac{h^4}{16R^2\sin^2(C-B)} + \frac{5h^2}{16} + \frac{R^2\sin^2(C-B)}{4}.$$

It is also well known that the nine-point circle of ABC passes through M and through the foot of the altitude from A, or its center F has x-coordinate $x_F = \frac{R\sin(C-B)}{2}$, since it is on the perpendicular bisector $x = \frac{R\sin(C-B)}{2}$ of AM. Since the nine-point circle has radius $\frac{R}{2}$, then the y-coordinate of the nine-point center must be $y_F = \frac{R\cos(C-B)}{2}$, which is consistent with the well-known triangle center function $\cos(B-C)$ for F. The power of the nine-point center F with respect to the circumcircle of AMX is then $R_A^2 - FO_A^2$, where

$$FO_A^2 = \left(\frac{h^4}{16R^2\sin^2(C-B)} \right) + \left(\frac{R^2\cos^2(C-B)}{4} - \frac{3Rh\cos(C-B)}{4} + \frac{9h^2}{16} \right).$$

Now, $h = c\sin B = b\sin C = 2R\sin B\sin C = R\cos(C-B) + R\cos A$, and

$$FO_A^2 = \frac{h^4}{16R^2\sin^2(C-B)} + R^2\frac{\cos^2(C-B) + 6\cos(C-B)\cos A + 9\cos^2 A}{16},$$

$$R_A^2 = \frac{h^4}{16R^2\sin^2(C-B)} + R^2\frac{4 + \cos^2(C-B) + 10\cos(C-B)\cos A + 5\cos^2 A}{16},$$

$$R_A^2 - FO_A^2 = \frac{R^2}{4} + \frac{R^2}{4}\cos A\left(\cos(C-B) - \cos(A)\right) = R^2\frac{1 + 2\cos A\cos B\cos C}{4}.$$

This last expression is invariant under permutation of A, B, C, or it will also be the power of F with respect to circles BNY and CPZ. The result follows.