SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We belatedly acknowledge a correct solution to #3340 by "Solver X", dedicated to the memory of Jim Totten, which we had previously classified as incorrect. Our apologies.

3376. [2008 : 430, 432] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

The vertices of quadrilateral $ABCD$ lie on a circle. Let K, L, M, and N be the incentres of $\triangle ABC$, $\triangle BCD$, $\triangle CDA$, and $\triangle DAB$, respectively. Show that quadrilateral $KLMN$ is a rectangle.

Comment by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.

This problem has already appeared in Crux. The proof of the result and the proof of its converse was published together with comments and references in [1980 : 226-230].

Solutions, comments, and other references were sent by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Geupel provides the link `www.mathlinks.ro/viewtopic.php?t=137589`. *Janous provides the link* `http://www.gogeometry.com/problem/p035_incenter_cyclic_quadrilateral`. *Konečný and Schlosberg found a solution at* `www.cut-the-knot.org/Curriculum/Geometry/CyclicQuadrilateral.shtml` *Schlosberg also gives two other links,* `forumgeom.fau.edu/FG2002volume2/FG200223.ps` *and* `mathworld.wolfram.com/CyclicQuadrilateral.html`.

3377. [2008 : 430, 432] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let ABC be a triangle with $\angle B = 2\angle C$. The interior bisector of $\angle BAC$ meets BC at D. Let M and N be the midpoints of AC and BD, respectively. Suppose that A, M, D, and N are concyclic. Prove that $\angle BAC = 72^\circ$.

Solution by Michel Bataille, Rouen, France.

We will use the familiar notations for the elements of the triangle ABC. First, we note that from $\angle B = 2\angle C$, we have

$$b^2 \;=\; c(c+a) \qquad (1)$$

(this has been proven in the April 2006 issue of this journal, [2006 : 159]).

It then suffices to prove that $a = b$, for then we will have $B = A$, and then

$$A \;=\; \frac{2}{5}\left(A + A + \frac{A}{2}\right) \;=\; \frac{2}{5}\,(A + B + C) \;=\; 72^\circ\,.$$

From $\dfrac{DB}{c} = \dfrac{DC}{b} = \dfrac{a}{b+c}$, we obtain $DB = \dfrac{ac}{b+c}$ and $DC = \dfrac{ab}{b+c}$, so that

$$CN \;=\; \frac{ab}{b+c} + \frac{ac}{2(b+c)} \;=\; \frac{a(2b+c)}{2(b+c)}\,.$$

Since A, M, D, and N are concyclic, we have $CM \cdot CA = CD \cdot CN$, or, after a simple calculation,

$$c^2 b \;=\; c\left(a^2 - 2b^2\right) + b\left(2a^2 - b^2\right)\,. \tag{2}$$

The relation (1), rewritten as $c^2 b = b^3 - abc$, together with (2) yields

$$c(a-b)(2b+a) \;=\; 2b(b-a)(b+a)\,.$$

Now, if $a \neq b$, then $c < 0$, which is impossible. Therefore, $a = b$, which completes the proof.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; SOUTHEAST MISSOURI STATE UNIVERSITY MATH CLUB; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; ALBERT STADLER, Herrliberg, Switzerland; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Amengual Covas noted that the relation (1) *had already appeared in several other issues of **CRUX**, namely,* [1976 : 74], [1984 : 278], *and* [1996 : 265-267].

3378. [2008 : 430, 432] *Proposed by Mihály Bencze, Brasov, Romania.*

Let x, y, and z be positive real numbers. Prove that

$$\sum_{\text{cyclic}} \frac{xy}{xy + x + y} \;\leq\; \sum_{\text{cyclic}} \frac{x}{2x + z}\,.$$

Counterexample by George Apostolopoulos, Messolonghi, Greece.

The inequality is false in general. For example, if $x = 2$ and $y = z = 1$, then the inequality becomes $\frac{2}{5} + \frac{1}{3} + \frac{2}{5} \leq \frac{2}{5} + \frac{1}{4} + \frac{1}{3}$, or $\frac{2}{5} < \frac{1}{4}$, which is clearly false.

Also disproved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State

University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; STAN WAGON, Macalester College, St. Paul, MN, USA; and TITU ZVONARU, Comănești, Romania.

Let $f(x,y,z) = \sum_{\text{cyclic}} \frac{x}{2x+z} - \sum_{\text{cyclic}} \frac{xy}{xy+x+y}$. *Then Curtis showed that* $f(x,x,x) < 0$ *for all* $x > 1$. *Perfetti showed that, in general,* $f(x,y,z) < 0$ *if* x, y, z *all lie in* $(1,\infty)$, *while* $f(x,y,z) \geq 0$ *if* x, y, z *all lie in* $(0,1]$.

Janous reports that a similar inequality, $\sum_{\text{cyclic}} \frac{xy}{xy+x^2+y^2} \leq \sum_{\text{cyclic}} \frac{x}{2x+z}$, *where* x, y, z *are positive, was problem 4 of the 2009 Mediterranean Mathematics Competition.*

3379. [2008 : 430, 433] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a_1, a_2, $\dots$, a_n be positive real numbers. Prove that

$$\sum_{i=1}^{n} \frac{a_i}{a_i + (n-1)a_{i+1}} \geq 1,$$

where the subscripts are taken modulo n.

Solution by Michel Bataille, Rouen, France.

Set $x_i = \dfrac{(n-1)a_{i+1}}{a_i}$ for each i; the problem is then to prove that

$$\sum_{i=1}^{n} \frac{1}{1+x_i} \geq 1 \qquad (1)$$

subject to the constraint $x_1 x_2 \cdots x_n = (n-1)^n$.

Let $y_i = \dfrac{1}{1+x_i}$ for each i, so that $x_i = \dfrac{1-y_i}{y_i}$.

Suppose on the contrary that $\sum_{i=1}^{n} y_i < 1$. Then $1 - \sum_{i=1}^{n} y_i > 0$, and hence for each i we have by the AM–GM Inequality that

$$1 - y_i > \sum_{j \neq i} y_j \geq (n-1)\left(\prod_{j \neq i} y_j\right)^{\frac{1}{n-1}}. \qquad (2)$$

Multiplying across the inequalities in (2) over all i, we then obtain

$$\prod_{i=1}^{n}(1-y_i) > (n-1)^n \prod_{i=1}^{n} y_i,$$

or $\prod_{i=1}^{n}\left(\frac{1-y_i}{y_i}\right) > (n-1)^n$, which violates the constraint on x_1, x_2, $\dots$, x_n.

This contradiction establishes (1), and we are done.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; CRISTINEL MORTICI, Valahia University of Târgoviște, Romania; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

For positive α Janous proved the generalization

$$\sum_{i=1}^{n} \frac{a_i}{a_i + \alpha a_{i+1}} \ \geq \ \min\left\{\frac{n}{1+\alpha}, 1\right\} .$$

3380. [2008 : 430, 433] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, c, x, y, and z be real numbers. Show that

$$\frac{(a^2+x^2)(b^2+y^2)}{(c^2+z^2)(a-b)^2} + \frac{(b^2+y^2)(c^2+z^2)}{(a^2+x^2)(b-c)^2} + \frac{(c^2+z^2)(a^2+x^2)}{(b^2+y^2)(c-a)^2}$$
$$\geq \ \frac{a^2+x^2}{|(a-b)(a-c)|} + \frac{b^2+y^2}{|(b-a)(b-c)|} + \frac{c^2+z^2}{|(c-a)(c-b)|} .$$

Similar solutions by Arkady Alt, San Jose, CA, USA; and Michel Bataille, Rouen, France.

Let

$$u \ = \ \sqrt{\frac{(a^2+x^2)(b^2+y^2)}{(c^2+z^2)(a-b)^2}} ; \qquad v \ = \ \sqrt{\frac{(b^2+y^2)(c^2+z^2)}{(a^2+x^2)(b-c)^2}} ;$$
$$w \ = \ \sqrt{\frac{(c^2+z^2)(a^2+x^2)}{(b^2+y^2)(c-a)^2}} .$$

Then

$$uv \ = \ \sqrt{\frac{(b^2+y^2)^2}{(a-b)^2(b-c)^2}} \ = \ \frac{b^2+y^2}{|(a-b)(b-c)|} ,$$

and similarly

$$uw \ = \ \frac{a^2+x^2}{|(a-b)(c-a)|} ; \qquad vw \ = \ \frac{c^2+z^2}{|(b-c)(c-a)|} .$$

The original inequality now follows from the well-known inequality

$$u^2+v^2+w^2 \ \geq \ uv+uw+vw .$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, *Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; TITU ZVONARU, Comănești, Romania; and the proposer.*

3381★. [2008 : 431, 433] *Proposed by Shi Changwei, Xi`an City, Shaan Xi Province, China.*

Let n be a positive integer. Prove that

(a) $\left(1-\frac{1}{6}\right)\left(1-\frac{1}{6^2}\right)\cdots\left(1-\frac{1}{6^n}\right) > \frac{4}{5}$;

(b) Let $a_n = aq^n$, where $0 < a < 1$ and $0 < q < 1$. Evaluate

$$\lim_{n\to\infty}\prod_{i=1}^{n}(1-a_i)\,.$$

Solution to part (a) *by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

We have $(1-aq)\left(1-aq^2\right) > 1-a\left(q+q^2\right)$. In general, if we know that $\prod\limits_{k=1}^{n}\left(1-aq^k\right) > 1-\sum\limits_{k=1}^{n} aq^k$ holds for some $n \geq 2$, then

$$\begin{aligned}\prod_{k=1}^{n+1}\left(1-aq^k\right) &> \left(1-aq^{n+1}\right)\left(1-\sum_{k=1}^{n} aq^k\right)\\ &= 1-\sum_{k=1}^{n+1} aq^k + aq^{n+1}\left(\sum_{k=1}^{n} aq^k\right)\\ &> 1-\sum_{k=1}^{n+1} aq^k\,.\end{aligned}$$

By induction, the inequality holds for each $n \geq 2$, and furthermore we have $\prod\limits_{k=1}^{n}\left(1-aq^k\right) > 1-\sum\limits_{k=1}^{n} aq^k > 1-\dfrac{aq}{(1-q)}$. Taking $a=1$ and $q=\frac{1}{6}$ yields $\left(1-\frac{1}{6}\right)\left(1-\frac{1}{6^2}\right)\cdots\left(1-\frac{1}{6^n}\right) > 1-\frac{1}{5} = \frac{4}{5}$.

Also solved by ARKADY ALT, San Jose, CA, USA (*part* (a))*; GEORGE APOSTOLOPOULOS, Messolonghi, Greece* (*part* (a))*; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA* (*part* (a))*; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ALBERT STADLER, Herrliberg, Switzerland; and STAN WAGON, Macalester College, St. Paul, MN, USA. There was one incorrect solution to part* (b) *submitted.*

Although it is a fundamental problem to evaluate the limit in part (b)*, there appears to be no simple way to tame this infinite product.*

Bracken notes that the product is related to Ramanujan's q-extension of the Gamma function, the customary notations being $(a;q)_k = (1-a)(1-aq)\cdots\left(1-aq^{k-1}\right)$ and $(a;q)_\infty = \prod_{k=0}^{\infty}\left(1-aq^k\right)$. The q-binomial theorem implies that for $|x|<1$, $|q|<1$ we have $\frac{1}{(a;q)_\infty} = \sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n}$, which leads to an expression for the required product.

Geupel gives the expansion

$$\prod_{k=1}^{\infty}\left(1-q^k\right) = \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt[24]{q}}\cdot\vartheta_2\left(\frac{\pi}{6},\sqrt[6]{q}\right),$$

where ϑ_2 is a Jacobi theta function. For further information he refers to the online survey `http://mathworld.wolfram.com/q-PochhammerSymbol.html`.

Stadler provided six different expansions related to the required product, one involving the Dedekind eta function and the others due to Euler and Jacobi. He refers the interested reader to Marvin I. Knopp, Modular Functions in Analytic Number Theory (2nd ed.), *Chelsea, New York (2003).*

Wagon also refers to the Wolfram website, as well as B. Gordon and R.J. McIntosh, "Some Eighth Order Mock Theta Functions", J. London Math. Soc. **62**, *pp. 321-335 (2000). There is given the asymptotic estimate $(q;q)_\infty = \sqrt{\frac{\pi}{t}}\exp\left(\frac{t}{12}-\frac{\pi^2}{12t}\right)+o(1)$; $t = -\frac{1}{2}\ln q$, which yields a value for $(1/6;1/6)_\infty$ that is within $2\cdot 10^{-10}$ of the true value.*

3382. [2008 : 431, 433] *Proposed by José Luis Díaz-Barrero and Josep Rubió-Massegú, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let n be a positive integer. Prove that

$$\sin\left(\frac{P_{n+2}}{4P_nP_{n+1}}\right) + \cos\left(\frac{P_{n+2}}{4P_nP_{n+1}}\right) < \frac{3}{2}\sec\left(\frac{3P_n+2P_{n-1}}{4P_nP_{n+1}}\right),$$

where P_n is the n^{th} Pell number, which is defined by $P_0 = 0$, $P_1 = 1$, and $P_n = 2P_{n-1}+P_{n-2}$ for $n \ge 2$.

Solution by Oliver Geupel, Brühl, NRW, Germany.

For all real numbers x and y with $0 < y < \frac{\pi}{2}$, we have

$$\sin x + \cos x = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right) \le \sqrt{2} < \frac{3}{2} \le \frac{3}{2}\sec y\,.$$

Setting $x = \dfrac{P_{n+2}}{4P_nP_{n+1}}$ and $y = \dfrac{3P_n+2P_{n-1}}{4P_nP_{n+1}}$ gives the desired result, if we can show that $0 < y < \frac{\pi}{2}$.

Because the Pell numbers form a strictly increasing sequence, we obtain for positive n that

$$0 < \frac{3P_n+2P_{n-1}}{4P_nP_{n+1}} < \frac{5P_n}{4P_n^2} \le \frac{5}{4} < \frac{\pi}{2},$$

which completes the proof.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposers.

It is clear from the featured solution that on the right side of the inequality $\frac{3}{2}$ *can be replaced by the smaller coefficient* $\sqrt{2}$.

3383. [2008 : 431, 433] *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle with $\angle BAC \neq 90^\circ$, let O be its circumcentre and let M be the midpoint of BC. Let P be a point on the ray MA such that $\angle BPC = 180^\circ - \angle BAC$. Let BP meet AC at E and let CP meet AB at F. If D is the projection of the midpoint of EF onto BC, show that

(a) AD is a symmedian of $\triangle ABC$;

(b) O, P, and the orthocentre of $\triangle EDF$ are collinear.

A composite of solutions by Madhav R. Modak, formerly of Sir Parashurambhau College, Pune, India and the proposer.

We first show that there is a unique point P on the ray MA such that $\angle BPC = 180^\circ - \angle BAC$, and P can be constructed by extending the median AM to where it meets the circumcircle Γ of $\triangle ABC$ at P'. *Claim*: The line through B parallel to $P'C$ meets AM at P and AC at E. To see this, note that $\triangle PBM \cong \triangle P'CM$ ($BM = MC$ and corresponding angles are equal), whence $PBP'C$ is a parallelogram. It follows that $\angle BPC = \angle CP'B$. But P', A lie on opposite arcs BC of Γ, so $\angle BPC = \angle CP'B = 180^\circ - \angle BAC$, whence P is the unique point on the ray MA that forms the required angle. Moreover, since $\angle BAC \neq 90^\circ$, P is different from A. The homothety with centre A that takes P' to P will take C to E, B to F, and Γ to the circumcircle Γ' of the quadrilateral $AFPE$. Note that EF is parallel to BC; its midpoint, call it N, lies on AP.

We turn now to part (a). Since AP and FE intersect at N, by the classical construction N is the pole of the line BC with respect to Γ' (because B and C are the diagonal points other than N of quadrilateral $AFPE$). The polar of D, therefore, passes through N and, since DN passes through the centre of Γ' and is perpendicular to EF, this polar is actually EF. As a result, DE and DF are tangent to Γ' at E and F. It follows that AD is the symmedian from A in $\triangle EAF$. (See, for example, Roger A. Johnson, *Advanced Euclidean Geometry*, (Dover, 2007), page 215, no. 347, or Nathan Altshiller Court, *College Geometry*, (Dover, 2007), page 248, Theorem 560.) The result in (a) follows since $\triangle BAC$ and $\triangle FAE$ are homothetic. For part (b) note that the homothety that takes the circle BAC (namely Γ) to FAE (namely Γ') will take OB to the radius of Γ' through F; since DF is the tangent at F, its preimage is tangent to Γ at B, and is therefore perpendicular to OB. We conclude that $OB \perp DF$. Similarly, $OC \perp DE$. Let H' denote the orthocentre of $\triangle EDF$. The homothety with centre P that takes B to E will take C to F. Since $OB \| H'E$ and $OC \| H'F$, this homothety must take O to H', whence H', P, and O are collinear, which completes the proof of part (b).

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

All the submitted solutions except Bataille's were based on the property that AD is a symmedian of $\triangle ABC$ if and only if D divides the side BC in the ratio $c^2 : b^2$ (Court, page 248, Theorem 561).

3384. [2008 : 431, 434] *Proposed by Michel Bataille, Rouen, France.*

Show that, for any real number x,

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n-1} k \cdot \left\lfloor x + \frac{n-k-1}{n} \right\rfloor = \frac{\lfloor x \rfloor + \{x\}^2}{2},$$

where $\lfloor a \rfloor$ is the integer part of the real number a and $\{a\} = a - \lfloor a \rfloor$.

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

Let $T_n = \sum\limits_{k=1}^{n-1} k \cdot \left\lfloor x + \frac{n-k-1}{n} \right\rfloor$. Given an integer $n \geq 2$, let m be the unique integer such that $\frac{m}{n} \leq \{x\} < \frac{m+1}{n}$. Note that $m = m(n)$ is a function of n, and that $\frac{m}{n} \to \{x\}$ as $n \to \infty$ since $\left|\{x\} - \frac{m}{n}\right| < \frac{1}{n}$. Now, the first $m-1$ terms of T_n are $\big(\lfloor x \rfloor + 1\big)\big(1+2+\cdots+(m-1)\big)$, and the final $n-m$ terms of T_n are $\lfloor x \rfloor \big(m + (m+1) + \cdots + (n-1)\big)$. Collect all terms involving $\lfloor x \rfloor$ and close the sums to obtain $\frac{1}{n^2} T_n = \frac{1}{2} \lfloor x \rfloor \left(\frac{n-1}{n}\right) + \frac{1}{2}\left(\frac{m}{n}\right)\left(\frac{m-1}{n}\right)$; it then follows that $\frac{1}{n^2} T_n \to \frac{1}{2}\left(\lfloor x \rfloor + \{x\}^2\right)$ as $n \to \infty$.

Also solved by ARKADY ALT, San Jose, CA, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

The Missouri State University Problem Solving Group reduced the problem to calculating $\int_0^1 y \lfloor x + 1 - y \rfloor \, dy$ by noting that the required limit is the limit of a Riemann sum for the integrand on $[0, 1]$ plus a term that vanishes as $n \to \infty$.

3385. [2008 : 431, 434] *Proposed by Michel Bataille, Rouen, France.*

Let p_1, p_2, $\dots$, p_6 be primes with $p_{k+1} = 2p_k + 1$ for $k = 1, 2, \dots, 5$. Show that

$$\sum_{1 \le i < j \le 6} p_i p_j$$

is divisible by 15.

A composite of similar solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Cristinel Mortici, Valahia University of Târgovişte, Romania; and Titu Zvonaru, Comăneşti, Romania.

If $p_1 = 3$, then $p_2 = 7$ and $p_3 = 15$, whence p_3 is not prime and, therefore, our sequence cannot begin with the prime $p_1 = 3$. Nor can it begin with $p_1 \equiv 1 \pmod 3$, otherwise we would have $p_2 \equiv 0 \pmod 3$, so that p_2 could not be prime. We conclude that necessarily, $p_1 \equiv -1 \pmod 3$, and hence that $p_i \equiv -1 \pmod 3$ for each p_i. The resulting sum satisfies

$$\sum_{1 \le i < j \le 6} p_i p_j \equiv \sum_{1 \le i < j \le 6} (-1)^2 \equiv \binom{6}{2} \equiv 0 \pmod 3 .$$

Similarly, working modulo 5, we see that

(a) If $p_1 = 5$, then $p_5 \equiv 0 \pmod 5$ and is not prime.
(b) If $p_1 \equiv 1 \pmod 5$, then $p_4 \equiv 0 \pmod 5$ and is not prime.
(c) If $p_1 \equiv 2 \pmod 5$, then $p_6 \equiv 0 \pmod 5$ and is not prime.
(d) If $p_1 \equiv 3 \pmod 5$, then $p_3 \equiv 0 \pmod 5$ and is not prime.

Once again, the only possible value for p_i is $-1 \pmod 5$, whence

$$\sum_{1 \le i < j \le 6} p_i p_j \equiv \sum_{1 \le i < j \le 6} (-1)^2 \equiv \binom{6}{2} \equiv 0 \pmod 5 .$$

We have seen that the sum is divisible by 3 and by 5, and thus by 15 as claimed. Finally, we note that the result is not vacuously true: 89, 179, 359, 719, 1439, 2879 is an example of such a sequence (and is easily seen to be the smallest example).

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ROBERT P. SEALY, Mount Allison University, Sackville, NB; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Manes points out that a prime p is called a Sophie Germain prime *if $2p+1$ is also a prime; moreover, a sequence of $n-1$ Sophie Germain primes, p, $2p+1$, $2(2p+1)+1$, $\dots$, that cannot be extended in either direction (that is, the first prime is not of the form $2q+1$ for q a prime, while the final prime of the sequence is not a Sophie Germain prime) is called a* Cunningham chain of the first kind of length n. *Aside from the Cunningham chain of length 5 that begins with 2 (namely, 2, 5, 11, 23, 47), the final digit of any prime in a Cunningham chain of length four or greater must be a 9 (because the final digit cycles 1, 3, 7, 5, $\dots$). According* *to the Cunningham Chain Records web page, the longest known Cunningham chain of the first kind has length* 17.

On a related note, compare Problem 10 on the Mathematical Competition Baltic Way 2004 [2008 : 212; 2009 : 153] *where one is asked to prove a result implying that a Cunningham chain can never be infinite.*

3386. [2008 : 432, 434] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Evaluate the integral

$$\int_0^\infty e^{-x}\left(\int_0^x \frac{e^{-t}-1}{t}\,dt\right)\ln x\,dx\,.$$

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria and Gerhard Kirchner, University of Innsbruck, Innsbruck, Austria.

We will use the fact that $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n} = -\ln 2$ and $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$. Also, from the product representation of the Gamma function, we have

$$\begin{aligned}
\frac{1}{\Gamma(x)} &= xe^{\gamma x}\prod_{k=1}^{\infty}\left(1+\frac{x}{k}\right)e^{-x/k} \\
\implies\quad -\frac{\Gamma'(x)}{\Gamma(x)} &= \frac{1}{x}+\gamma+\sum_{k=1}^{\infty}\left(\frac{1}{x+k}-\frac{1}{k}\right) \\
\implies\quad -\frac{\Gamma'(n+1)}{\Gamma(n+1)} &= \frac{1}{n+1}+\gamma+\sum_{k=1}^{\infty}\left(\frac{1}{n+1+k}-\frac{1}{k}\right) \\
&= \gamma-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\,,
\end{aligned}$$

where γ is Euler's constant. [*Ed. For properties of the Gamma function see* `http://en.wikipedia.org/wiki/Gamma_function`.]

We now compute

$$\begin{aligned}
&\int_0^\infty e^{-x}\left(\int_0^x \frac{e^{-t}-1}{t}\,dt\right)\ln x\,dx \\
=\ &\int_0^\infty e^{-x}\left(\int_0^x \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}t^{n-1}\,dt\right)\ln x\,dx \\
=\ &\int_0^\infty e^{-x}\ln x\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{x^n}{n}\,dx \;=\; \sum_{n=1}^{\infty}\frac{(-1)^n}{n\cdot n!}\int_0^\infty e^{-x}x^n\ln x\,dx \\
=\ &\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cdot\frac{\Gamma'(n+1)}{\Gamma(n+1)} \;=\; \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(-\gamma+\sum_{k=1}^{n}\frac{1}{k}\right) \\
=\ &\gamma\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sum_{k=1}^{n}\frac{1}{k}
\end{aligned}$$

$$
\begin{aligned}
&= \gamma\ln 2 + \int_0^1\int_0^1 \sum_{n=1}^{\infty}(-1)^n x^{n-1}\sum_{k=1}^{n} y^{k-1}\,dxdy \\
&= \gamma\ln 2 + \int_0^1\int_0^1 \sum_{n=1}^{\infty}(-1)^n x^{n-1}\frac{y^n-1}{y-1}\,dxdy \\
&= \gamma\ln 2 + \int_0^1\int_0^1 \frac{\dfrac{-y}{1+xy}+\dfrac{1}{1+x}}{y-1}\,dxdy \\
&= \gamma\ln 2 - \int_0^1\biggl(\int_0^1 \frac{1}{(1+xy)(1+x)}\,dy\biggr)dx \;=\; \gamma\ln 2 - \int_0^1 \frac{\ln(1+x)}{x(1+x)}\,dx \\
&= \gamma\ln 2 - \int_0^1 \ln(1+x)\left(\frac{1}{x}-\frac{1}{1+x}\right)dx \\
&= \gamma\ln 2 - \int_0^1 \frac{\ln(1+x)}{x}\,dx + \frac{1}{2}\ln^2 2 \\
&= \gamma\ln 2 + \frac{1}{2}\ln^2 2 + \sum_{n=1}^{\infty}\int_0^1 \frac{(-1)^n}{n}x^{n-1}\,dx \\
&= \gamma\ln 2 + \frac{1}{2}\ln^2 2 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \;=\; \gamma\ln 2 + \frac{1}{2}\ln^2 2 - \frac{\pi^2}{12}\,.
\end{aligned}
$$

Also solved by KEE-WAI LAU, Hong Kong, China; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; MOUBINOOL OMARJEE, Paris, France; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; NGUYEN VAN VINH, Belarusian State University, Minsk, Belarus; and the proposer. There were two incomplete solutions submitted.

3387. [2008 : 432, 434] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $k > l \geq 0$ be fixed integers. Find

$$\lim_{x\to\infty} 2^x\left(\zeta(x+k)^{\zeta(x+k)} - \zeta(x+l)^{\zeta(x+l)}\right),$$

where ζ is the Riemann zeta function.

Similar solutions by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy and Albert Stadler, Herrliberg, Switzerland.

For real-valued functions $f(x)$ and $g(x)$ defined on the interval $(1,\infty)$ the notation $f(x) = O\big(g(x)\big)$ will mean that there exists an $x_0 > 1$ and a positive constant C such that $|f(x)| \leq C|g(x)|$ whenever $x \geq x_0$.

We have $\zeta(x) = 1 + 2^{-x} + 3^{-x} + \sum\limits_{k=4}^{\infty} k^{-x}$ and

$$\frac{1}{(x-1)4^{x-1}} = \int_4^{\infty} \frac{ds}{s^x} \le \sum_{k=4}^{\infty} k^{-x} \le \int_3^{\infty} \frac{ds}{s^x} = \frac{1}{(x-1)3^{x-1}},$$

hence $\sum\limits_{k=4}^{\infty} k^{-x} = O\big(3^{-x}\big)$ and $\zeta(x) = 1 + 2^{-x} + O\big(3^{-x}\big)$. Moreover,

$$\begin{aligned} \zeta(x)^{\zeta(x)} &= \exp\Big(\Big(1 + 2^{-x} + O\big(3^{-x}\big)\Big) \cdot \ln\Big(1 + 2^{-x} + O\big(3^{-x}\big)\Big)\Big) \\ &= \exp\Big(\Big(1 + 2^{-x} + O\big(3^{-x}\big)\Big) \cdot \Big(2^{-x} + O\big(3^{-x}\big)\Big)\Big) \\ &= \exp\Big(2^{-x} + O\big(3^{-x}\big)\Big) = 1 + 2^{-x} + O\big(3^{-x}\big). \end{aligned}$$

Finally, we have

$$\begin{aligned} &\lim_{x\to\infty} 2^x \left(\zeta(x+k)^{\zeta(x+k)} - \zeta(x+l)^{\zeta(x+l)}\right) \\ = &\lim_{x\to\infty} 2^x\Big(2^{-x-k} - 2^{-x-l} + O\big(3^{-x}\big)\Big) = 2^{-k} - 2^{-l}. \end{aligned}$$

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

3388. [2008 : 432, 434] *Proposed by Paul Bracken, University of Texas, Edinburg, TX, USA, in memory of Murray S. Klamkin.*

For all real $x \ge 1$, show that

$$\frac{1}{2}\sqrt{x-1} + \frac{(x-1)^2}{\sqrt{x-1} + \sqrt{x+1}} < \frac{x^2}{\sqrt{x} + \sqrt{x+2}}.$$

Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy, modified by the editor.

The inequality holds for $x = 1$, so let $x > 1$. Since $\sqrt{x+1} > \sqrt{x-1}$ and $\sqrt{x} + \sqrt{x+2} < 2\sqrt{x+1}$ by the concavity of $\sqrt{x}$, it suffices to prove

$$\frac{1}{2}\sqrt{x-1} + \frac{1}{2}(x-1)^{3/2} < \frac{x^2}{2\sqrt{x+1}}, \tag{1}$$

as the right side of (1) is less than the right side of the desired inequality and the left side of (1) is greater than the left side of the desired inequality. Inequality (1) is successively equivalent to

$$\begin{aligned} \sqrt{x^2-1} + \sqrt{x^2-1}(x-1) &< x^2, \\ x\sqrt{x^2-1} &< x^2, \\ \sqrt{x^2-1} &< x, \end{aligned}$$

and the last inequality is true. The proof is complete.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy (two solutions); ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. One incorrect solution was submitted.

STAN WAGON, Macalester College, St. Paul, MN, USA verified the inequality using a computer algorithm.

In this small space that remains, we again put out the call for more problem proposals from our readers in the areas of Geometry, Algebra, Logic, and Combinatorics. We have a vast store of interesting inequalities and we will continue to process them and accept new ones, but the other areas are wanting!

Regarding articles, we now have a small backlog. This is due to the diligence of the Articles Editor, James Currie, but also due to the fact that space for articles in ***Crux*** is extremely limited. For this reason we ask that, if possible, authors with articles appearing in ***Crux*** wait about 18–20 months before submitting another manuscript to ***Crux***. In the meantime, we will gear up and attempt to clear the backlog in 2010 by running the occasional 96 page issue.

Václav Linek