

SOLUTIONS

Aucun problème n'est immuable. L'éditeur est toujours heureux d'envisager la publication de nouvelles solutions ou de nouvelles perspectives portant sur des problèmes antérieurs.

3363. [2008 : 362, 364] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let ABC be a triangle with $\angle ACB = 90^\circ + \frac{1}{2}\angle ABC$. Let M be the midpoint of BC . Prove that $\angle AMC < 60^\circ$.

Composite of similar solutions by Ricardo Barroso Campos, University of Seville, Seville, Spain; Richard I. Hess, Rancho Palos Verdes, CA, USA; Matti Lehtinen, National Defence College, Helsinki, Finland; Madhav R. Modak, formerly of Sir Parashurambhau College, Pune, India; and Peter Y. Woo, Biola University, La Mirada, CA, USA.

In $\triangle ABC$ let $4\beta = \angle B$ and let W be the foot of the bisector of $\angle A$. We are given that $\angle C = 90^\circ + 2\beta$, whence $\angle A = 90^\circ - 6\beta$. We deduce that $6\beta < 90^\circ$, so that $\beta < 15^\circ$. Moreover, because $\angle AWC$ is an external angle of $\triangle ABW$, it satisfies

$$\angle AWC = 4\beta + (45^\circ - 3\beta) = 45^\circ + \beta < 60^\circ.$$

It remains to show that $\angle AMC < \angle AWC$. Because we are given that $\angle C$ is obtuse, we have $AB > AC$, whence the foot W of the angle bisector satisfies $BW > WC$. That is, W lies between M and C ; thus $\angle AWC$ is an external angle of $\triangle AMW$ and therefore satisfies $\angle AMC < \angle AWC < 60^\circ$, as desired.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; GERALD EDGECOMB and JULIE STEELE, students, California State University, Fresno, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; GEORGE TSAPAKIDIS, Agrinio, Greece; TITU ZVONARU, Comănești, Romania; and the proposer.

3364. [2008 : 362, 364] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let ABC be a triangle with $\angle BAC = 120^\circ$ and $AB > AC$. Let M be the midpoint of BC . Prove that $\angle MAC > 2\angle ACB$.

I. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let $\beta = \angle CBA$ and $\gamma = \angle BCA$; then (because $\angle BAC = 120^\circ$) $\beta + \gamma = 60^\circ$. Let D be the point of segment BC for which $\angle BAD = 2\beta$; then $\angle DAC = 2\gamma$. The Law of Sines tells us that

$$BD = \frac{AD \sin 2\beta}{\sin \beta} = 2AD \cos \beta;$$

also,

$$CD = \frac{AD \sin 2(60^\circ - \beta)}{\sin(60^\circ - \beta)} = 2AD \cos(60^\circ - \beta).$$

But $\beta + \gamma = 60^\circ$ and $\gamma > \beta$; hence, $60^\circ - \beta > \beta$ so that $\cos(60^\circ - \beta) < \cos \beta$. Consequently, $CD < BD$, whence M lies between B and D . It follows that $\angle MAC > \angle DAC = 2\angle ACB$, as desired.

II. *Solution by Matti Lehtinen, National Defence College, Helsinki, Finland.*

—Let O be the centre of the circumcircle Γ of $\triangle ABC$, and D be the midpoint of the arc CAB of Γ . Note that A is on the smaller arc DC . Since $120^\circ = \angle BAC = \angle BDC$, $DB = DC = DO$. Draw the circle Γ' with centre D and radius DB (passing through B, O , and C). Because M is the common midpoint of BC and DO , circles Γ and Γ' are symmetric with respect to M . Let AM meet Γ' at E (so that M is also the midpoint of AE); note that $\angle EDC = \angle AOB = 2\angle ACB$. Extend AE to meet Γ again at F and extend DE to meet Γ again at G . Because the line FA separates D from G and C , G is on the arc FC opposite D and A ; consequently, $\angle FAC > \angle GDC$. But, $\angle FAC = \angle MAC$, and $\angle GDC = \angle EDC = \angle AOB = 2\angle ACB$, so we are done.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; GEORGE TSAPAKIDIS, Agrinio, Greece; TITU ZVONARU, Comănești, Romania; and the proposer.

3365. [2008 : 362, 364] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

A square $ABCD$ is inscribed in a circle Γ . Let P be a point on the minor arc AD of Γ , and let E and F be the intersections of AD with PB and PC , respectively. Prove that

$$AE \cdot DF = 2([PAE] + [PDF]),$$

where $[KLM]$ denotes the area of triangle KLM .

Solution by John G. Heuver, Grande Prairie, AB.

The altitudes from point P to segments AB and CD lead us to conclude that

$$[PAB] + [PDC] = \frac{1}{2}AB^2,$$

Furthermore,

$$\frac{1}{2}AB^2 = [PAE] + [AEB] + [PDF] + [DFC],$$

Since

$$[AEB] + [DFC] = \frac{1}{2}AB(AE + DF),$$

we have

$$\begin{aligned} [PAE] + [PDF] &= \frac{1}{2}AB^2 - ([AEB] + [DFC]) \\ &= \frac{1}{2}AB^2 - \frac{1}{2}AB(AE + DF) \\ &= \frac{1}{2}AB(AB - AE - DF) = \frac{1}{2}AB \cdot EF. \end{aligned}$$

Now, $\angle BPC = \angle CPD = \angle 45^\circ$ as angles subtended by arcs equal to a quarter of the circle. Hence PF is the bisector of $\angle EPD$, and therefore,

$$\frac{EF}{FD} = \frac{PE}{PD}.$$

Also, $\angle ABP = \angle PDA$ as angles subtended by the arc AP , so that triangles PDE and ABE are similar, and therefore,

$$\frac{PE}{PD} = \frac{AE}{AB}.$$

Consequently,

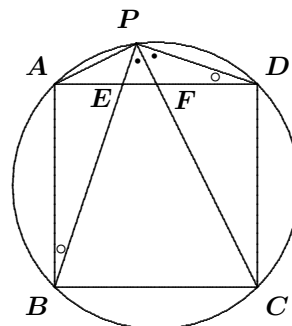
$$\frac{EF}{FD} = \frac{AE}{AB},$$

or $EF \cdot AB = AE \cdot DF$, and then

$$[PAE] + [PDF] = \frac{1}{2}AB \cdot EF = \frac{1}{2}AE \cdot DF,$$

which completes the proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; SALEM



MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer. There was one incorrect comment submitted.

Konečný and the proposer each began by letting Z be the foot of the perpendicular from P to AD , then deducing that $\triangle DPZ \sim \triangle BEA$ and $\triangle APZ \sim \triangle CFD$, which then enables the calculation of $AE \cdot DF$.

3366. [2008 : 362, 365] Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.

Let $\{x\}$ denote the fractional part of the real number x ; that is, $\{x\} = x - [x]$, where $[x]$ is the greatest integer not exceeding x . Evaluate

$$\int_0^1 \left\{ \frac{1}{x} \right\}^4 dx.$$

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

Let I be the integral to be evaluated. Then

$$\begin{aligned} I &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \int_{\frac{1}{n+1}}^{\frac{1}{n}} \left\{ \frac{1}{x} \right\}^4 dx = \lim_{N \rightarrow \infty} \sum_{n=1}^N \int_n^{n+1} \frac{\{y\}^4}{y^2} dy \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \int_n^{n+1} \frac{(y-n)^4}{y^2} dy = \lim_{N \rightarrow \infty} \sum_{n=1}^N I_n, \end{aligned}$$

where

$$I_n = \int_n^{n+1} \frac{(y-n)^4}{y^2} dy.$$

We have

$$\begin{aligned} I_n &= \int_n^{n+1} \left(y^2 - 4ny + 6n^2 - \frac{4n^3}{y} + \frac{n^4}{y^2} \right) dy \\ &= \left(\frac{y^3}{3} - 2ny^2 + 6n^2y - 4n^3 \ln y - \frac{n^4}{y} \right) \Big|_n^{n+1} \\ &= 3n^2 - n + \frac{1}{3} - \frac{n^4}{n+1} + n^3 - 4n^3 \ln \left(1 + \frac{1}{n} \right) \\ &= 3n^2 - n + \frac{1}{3} + \left(n^2 - n - \frac{1}{n+1} + 1 \right) - 4n^3 \ln \left(1 + \frac{1}{n} \right) \\ &= 4n^2 - 2n + \frac{4}{3} - \frac{1}{n+1} - 4n^3 \ln \left(1 + \frac{1}{n} \right). \end{aligned}$$

Substituting the series expansions

$$\frac{x}{x+1} = \sum_{k=0}^{\infty} (-1)^k x^{k+1} \quad \text{and} \quad \ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k},$$

with $x = \frac{1}{n}$ ($n \geq 2$) into the last expression for I_n yields

$$\begin{aligned} I_n &= 4n^2 - 2n + \frac{4}{3} - \sum_{k=0}^{\infty} \frac{(-1)^k}{n^{k+1}} - 4n^3 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{kn^k} \\ &= 4n^2 - 2n + \frac{4}{3} - 4n^3 \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} \right) \\ &\quad - \sum_{k=0}^{\infty} \frac{(-1)^k}{n^{k+1}} - 4n^3 \sum_{k=4}^{\infty} \frac{(-1)^{k+1}}{kn^k} \\ &= - \sum_{k=0}^{\infty} \frac{(-1)^k}{n^{k+1}} - 4n^3 \sum_{k=4}^{\infty} \frac{(-1)^{k+1}}{kn^k} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k}{n^k} - 4 \sum_{k=4}^{\infty} \frac{(-1)^{k+1}}{kn^{k-3}} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k}{n^k} - 4 \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+3)n^k} \\ &= \sum_{k=2}^{\infty} \left(-\frac{1}{n} \right)^k - 4 \sum_{k=2}^{\infty} \frac{(-1)^k}{(k+3)n^k} \\ &= \frac{1}{n(n+1)} - 4 \sum_{k=2}^{\infty} \frac{(-1)^k}{(k+3)n^k}, \end{aligned}$$

and the last formula is valid for I_1 as well. Thus,

$$\begin{aligned} I &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} - 4 \sum_{n=1}^{\infty} \sum_{k=2}^{\infty} \frac{1}{k+3} \left(-\frac{1}{n} \right)^k \\ &= 1 - 4 \sum_{k=2}^{\infty} \frac{(-1)^k}{k+3} \sum_{n=1}^{\infty} \frac{1}{n^k} \\ &= 1 - 4 \sum_{k=2}^{\infty} \frac{(-1)^k}{k+3} \zeta(k). \end{aligned}$$

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer. There were two incorrect solutions submitted.

The proposer obtained the answer $-\frac{1}{3} - \gamma + 2 \ln 2\pi - 12 \ln A + 12 \ln B$, where γ is Euler's constant and where

$$A = \lim_{n \rightarrow \infty} \frac{1^1 2^2 \dots n^n}{n^{\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}} e^{-\frac{n^2}{4}}}; \quad B = \lim_{n \rightarrow \infty} \frac{1^{1^2} 2^{2^2} \dots n^{n^2}}{n^{\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}} e^{-\frac{n^3}{9} + \frac{n}{12}}}$$

are the Glaisher–Kinkelin constants of order 1 and order 2, respectively.

Geupel obtained $I = 24 \sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k(k+1)(k+2)(k+3)} = 0.14553289 \dots$, which he

remarked converges more rapidly than the series $1 - \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k)}{k+3}$ and is thus better suited for numerical approximation.

Janous reports the generalization

$$\int_0^1 \left\{ \frac{1}{x} \right\}^N dx = \left(\sum_{j=0}^{N-2} (-1)^j \binom{N}{j} \frac{1}{N-j-1} (2^{N-j-1} - 1) \right) - (-1)^N \left(N \ln 2 - \frac{1}{2} \right) + \sum_{p=2}^{\infty} \frac{(-1)^p (p-1)}{p+N-1} (\zeta(p) - 1).$$

Keith Ekblaw, Walla Walla, WA, USA, obtained the estimate $I \approx 0.146$ by a Monte Carlo approach, which is correct to 3 decimal places after rounding.

3367. [2008 : 362, 365] Proposed by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x$ be a polynomial with integer coefficients, where $a_n > 0$ and $\sum_{k=1}^n a_k = 1$. Prove or disprove that there are infinitely many pairs of positive integers (k, ℓ) such that $p(k+1) - p(k)$ and $p(\ell+1) - p(\ell)$ are relatively prime.

Solution by Cristinel Mortici, Valahia University of Targoviste, Romania, modified slightly by the editor.

Let $Q(x) = p(x+1) - p(x)$. Then we have

$$Q(0) = p(1) - p(0) = \sum_{k=1}^n a_k - 0 = 1.$$

Hence, $Q(x) = xq(x) + 1$ for some polynomial $q(x)$ of degree $n-1$.

We need to find infinitely many positive integers k and l such that $\gcd(Q(k), Q(l)) = 1$.

Since the leading term in $Q(x)$ is $na_n x^{n-1}$ we have $Q(k) > 0$ for sufficiently large positive integers k . For such k let $l = Q(k) = kq(k) + 1$, so that l is a positive integer. If, on the contrary, $\gcd(Q(k), Q(l)) \neq 1$, then there is a prime number p such that $p \mid Q(k)$ and $p \mid Q(l)$. Then, since $Q(l) = lq(l) + 1 = Q(k)q(l) + 1$, we conclude that $p \mid 1$, a contradiction.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

Mortici remarked that the conclusion in fact holds for any polynomial $Q(k)$ whose constant term is either 1 or -1 . This is clear from the proof featured above. Barbara proved the stronger results that (1) if $a_n \neq 0$ and $\sum_{k=1}^n a_k = 1$, then there are infinitely many positive

integers k such that $p(k+1) - p(k)$ are all pairwise relatively prime, and (2) if $f(x)$ is a polynomial with integer coefficients and positive degree such that $f(0) = 1$, then there are infinitely many integers $k_1 < k_2 < k_3 < \dots$ such that the $f(k_i)$'s are all pairwise relatively prime.

3368. [2008 : 363, 365] *Proposed by Neven Jurič, Zagreb, Croatia.*

Let m be an integer, $m \geq 2$, and let $A = [A_{ij}]$ be a block matrix of dimension $2^m \times 2^m$ with $A_{ij} \in M_{4,4}(\mathbb{N})$ for $1 \leq i, j \leq 2^{m-2}$, defined by $A_{ij} = 2^m B_{ij} + C_{ij}$, where

$$B_{ij} = \begin{bmatrix} 2^m - 4i + 4 & 4i - 4 & 4i - 4 & 2^m - 4i + 4 \\ 4i - 3 & 2^m - 4i + 3 & 2^m - 4i + 3 & 4i - 3 \\ 4i - 2 & 2^m - 4i + 2 & 2^m - 4i + 2 & 4i - 2 \\ 2^m - 4i + 1 & 4i - 1 & 4i - 1 & 2^m - 4i + 1 \end{bmatrix},$$

$$\text{and } C_{ij} = \begin{bmatrix} 4 - 4j & 4j - 2 & 4j - 1 & 1 - 4j \\ 4j - 3 & 3 - 4j & 2 - 4j & 4j \\ 4j - 3 & 3 - 4j & 2 - 4j & 4j \\ 4 - 4j & 4j - 2 & 4j - 1 & 1 - 4j \end{bmatrix}.$$

Show that matrix A is a magic square of order 2^m .

Solution by Oliver Geupel, Brühl, NRW, Germany.

For convenience, write $B_{ij} = B^{(i)} = (b_{k\ell}^{(i)})$ and $C_{ij} = C^{(j)} = (c_{k\ell}^{(j)})$, and note that these 4×4 matrices have integer entries.

Claim 1 For each entry a of A , $1 \leq a \leq 2^{2m}$.

Proof: We have $a = 2^m b_{k\ell}^{(i)} + c_{k\ell}^{(j)}$ for some i, j, k , and ℓ . If we also have $(k, \ell) \in \{(1, 2), (1, 3)\}$, then $b_{k\ell}^{(i)} \geq 0$ and $c_{k\ell}^{(j)} \geq 1$; hence $a \geq 1$. Otherwise, $b_{k\ell}^{(i)} \geq 1$ and $c_{k\ell}^{(j)} \geq 1 - 2^m$; hence $a \geq 2^m + 1 - 2^m = 1$. This proves the lower bound. For the upper bound, we observe that for $(k, \ell) \in \{(1, 1), (1, 4)\}$ we have $b_{k\ell}^{(i)} \leq 2^m$ and $c_{k\ell}^{(j)} \leq 0$; hence $a \leq 2^m \cdot 2^m = 2^{2m}$. In any other case $b_{k\ell}^{(i)} \leq 2^m - 1$ and $c_{k\ell}^{(j)} \leq 2^m$; so again $a \leq 2^m(2^m - 1) + 2^m = 2^{2m}$. ■

Claim 2 The entries of A are distinct.

Proof: If two entries of A are equal then $2^m b_{k\ell}^{(i)} + c_{k\ell}^{(j)} = 2^m b_{k'\ell'}^{(i')} + c_{k'\ell'}^{(j')}$ for some j, j', k, k', ℓ , and ℓ' . Let $b = b_{k\ell}^{(i)}$, $b' = b_{k'\ell'}^{(i')}$, $c = c_{k\ell}^{(j)}$, and $c' = c_{k'\ell'}^{(j')}$. We then have $1 \leq |c - c'| \leq 4 \cdot 2^{m-2} - (1 - 4 \cdot 2^{m-2}) = 2^{m+1} - 1 < 2^{m+1}$. Without loss of generality, suppose that $c < c'$. Then $c' - c = 2^m$ and $b = b' + 1$, hence $c \equiv c' \pmod{4}$ and $b \equiv b' + 1 \pmod{4}$. There are four cases for c and c' :

Case 1 $(k, \ell) \in \{(1, 1), (4, 1)\}$ and $(k', \ell') \in \{(2, 4), (3, 4)\}$;

Case 2 $(k, \ell) \in \{(1, 4), (4, 4)\}$ and $(k', \ell') \in \{(2, 1), (3, 1)\}$;

Case 3 $(k, \ell) \in \{(2, 3), (3, 3)\}$ and $(k', \ell') \in \{(1, 2), (4, 2)\}$;

Case 4 $(k, \ell) \in \{(2, 2), (3, 2)\}$ and $(k', \ell') \in \{(1, 3), (4, 3)\}$;

each of them incompatible with the condition $b \equiv b' + 1 \pmod{4}$. ■

Claim 3 The sum of the entries along any horizontal, vertical, or main diagonal line of A is $2^{3m-1} + 2^{m-1}$.

Proof: The sum along any given horizontal or vertical line of $B^{(i)}$ and $C^{(j)}$ is $S_B = 2^{m+1}$ and $S_C = 2$, respectively. Therefore, the sum of the entries in each line of A is $2^{m-2}(2^m S_B + S_C) = 2^{m-2}(2^m \cdot 2^{m+1} + 2) = 2^{3m-1} + 2^{m-1}$.

The main diagonal sums of $B^{(i)}$ and $C^{(j)}$ are $b^{(i)} = 2^{m+2} - 16i + 10$ and $c^{(j)} = 10 - 16j$. The entries in each main diagonal of A then also have the

$$\text{sum } \sum_{i=1}^{2^{m-2}} 2^m b^{(i)} + \sum_{j=1}^{2^{m-2}} c^{(j)} = 2^{3m-1} + 2^{m-1}. \quad \blacksquare$$

Also solved by the proposer. One incomplete solution was received that verified the row, column, and diagonal sums, but did not show that the entries of the magic square consisted of $1, 2, \dots, 2^{2m}$.

3369. [2008 : 363, 365] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $A_1 A_2 A_3 A_4$ be a tetrahedron which contains the centre O of its circumsphere as an interior point. Let ρ_i be the distance from O to the face opposite vertex A_i . If R is the radius of the circumsphere, prove that

$$\frac{4}{3}R \geq \sum_{i=1}^4 \rho_i.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

The claim is false; the following counterexample is adapted from [1].

First, consider the degenerate tetrahedron $A'_1 A'_2 A'_3 A'_4$, where

$$\begin{aligned} A'_1 &= A'_2 = (-1, 1, 0), \\ A'_3 &= (1, 1, 0), \\ A'_4 &= (-1, -1, 0), \end{aligned}$$

$O = (0, 0, 0)$, and $R = \sqrt{2}$. We have $\rho_1 = \rho_2 = 0$, and $\rho_3 = \rho_4 = 1$, so that

$$\frac{4R}{\sum_{i=1}^4 \rho_i} = \frac{4\sqrt{2}}{2} = 2\sqrt{2} < 3.$$

Now, let

$$\begin{aligned} A_1 &= (-1 + \varepsilon, 1 - \varepsilon, \sqrt{2\varepsilon(2 - \varepsilon)}) , \\ A_2 &= (-1 + \varepsilon, 1 - \varepsilon, -\sqrt{2\varepsilon(2 - \varepsilon)}) , \\ A_3 &= (\sqrt{2} \cos(\frac{\pi}{4} - \varepsilon), \sqrt{2} \sin(\frac{\pi}{4} - \varepsilon), 0) , \\ A_4 &= (\sqrt{2} \cos(-\frac{3\pi}{4} + \varepsilon), \sqrt{2} \sin(-\frac{3\pi}{4} + \varepsilon), 0) . \end{aligned}$$

It is easy to verify that point O is in the interior of the tetrahedron $A_1A_2A_3A_4$, and that $A_i \rightarrow A'_i$ for each i as $\varepsilon \rightarrow 0$, as well as

$$\frac{4R}{\sum_{i=1}^4 \rho_i} \rightarrow 2\sqrt{2} .$$

The bound $2\sqrt{2}$ is the best possible. This is a corollary from the following:

Theorem [1]. Let $A_1A_2A_3A_4$ be a nondegenerate tetrahedron whose circumcentre O is not an exterior point. Let P be a point not exterior to $A_1A_2A_3A_4$. Let the distances from P to the vertices and to the faces of $A_1A_2A_3A_4$ be denoted by R_i and ρ_i , respectively. Then

$$\frac{\sum_{i=1}^4 R_i}{\sum_{i=1}^4 \rho_i} > 2\sqrt{2} ,$$

and $2\sqrt{2}$ is the greatest lower bound.

References

- [1] Nicholas D. Kazarinoff, "D.K. Kazarinoff's inequality for tetrahedra", *Michigan Mathematical Journal*, Vol. 4, No. 2 (1957), pp. 99-104.

Counterexample also given by Peter Y. Woo, Biola University, La Mirada, CA, USA.

George Apostolopoulos, Messolonghi, Greece, cited the book by Kazarinoff, Geometric Inequalities, Yale University Press, 1961, p. 116.

No complete solutions or other comments were submitted.

3370. [2008 : 363, 365] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let a_i and b_i be positive real numbers for $1 \leq i \leq k$, and let n be a positive integer. Prove that

$$\left(\sum_{i=1}^k a_i^{\frac{1}{n}} \right)^n \leq \left(\sum_{i=1}^k \frac{a_i}{b_i} \right) \left(\sum_{i=1}^k b_i^{\frac{1}{n-1}} \right)^{n-1} .$$

Composite of similar or identical solutions submitted by all the solvers whose names appear below (except the two solvers identified by a "*" before their names).

Let $p = n$ and $q = \frac{n}{n-1}$. Then $p > 1$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

[Ed: clearly, $n > 1$ for the given inequality to make sense.]

Let $x_i = \left(\frac{a_i}{b_i}\right)^{\frac{1}{n}}$ and $y_i = b_i^{\frac{1}{n}}$. Then x_i and y_i are positive for each i .

By Hölder's Inequality, we have $\sum_{i=1}^k x_i y_i \leq \left(\sum_{i=1}^k x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^k y_i^q\right)^{\frac{1}{q}}$ which becomes $\sum_{i=1}^k a_i^{\frac{1}{n}} \leq \left(\sum_{i=1}^k \frac{a_i}{b_i}\right)^{\frac{1}{n}} \left(\sum_{i=1}^k b_i^{n-1}\right)^{\frac{n-1}{n}}$.

The result follows by raising both sides to the n^{th} power.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; *CRISTINEL MORTICI, Valahia University of Targoviste, Romania; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JOEL SCHLOSBERG, Bayside, NY, USA; *PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3371. [2008 : 363, 368] Proposed by George Tsintsifas, Thessaloniki, Greece.

Let ABC be a triangle with a , b , and c the lengths of the sides opposite the vertices A , B , and C , respectively, and let M be an interior point of $\triangle ABC$. The lines AM , BM , and CM intersect the opposite sides at the points A_1 , B_1 , and C_1 , respectively. Lines through M perpendicular to the sides of $\triangle ABC$ intersect BC , CA , and AB at A_2 , B_2 , and C_2 , respectively. Let p_1 , p_2 , and p_3 be the distances from M to the sides BC , CA , and AB , respectively. Prove that

$$\frac{[A_1B_1C_1]}{[A_2B_2C_2]} = \frac{(ap_1 + bp_2)(bp_2 + cp_3)(cp_3 + ap_1)}{8a^2b^2c^2} \left(\frac{a}{p_1} + \frac{b}{p_2} + \frac{c}{p_3} \right),$$

where $[KLM]$ denotes the area of triangle KLM .

Solution by Joel Schlosberg, Bayside, NY, USA.

By the basic formula for the area of a triangle, $[ACM] = \frac{1}{2}bp_2$ and $[CMB] = \frac{1}{2}ap_1$. It is well known that if segments YZ and $Y'Z'$ lie on the

same line and X is any point, then $[XYZ] : [XY'Z'] = YZ : Y'Z'$, so that

$$\begin{aligned}\frac{[ACC_1]}{[ACM]} &= \frac{[CC_1B]}{[CMB]} = \frac{CC_1}{CM}; \\ \frac{[ACC_1]}{[CC_1B]} &= \frac{[ACM]}{[CMB]} = \frac{AC_1}{C_1B} = \frac{bp_2}{ap_1}.\end{aligned}$$

By similar reasoning, $\frac{BA_1}{A_1C} = \frac{cp_3}{bp_2}$ and $\frac{CB_1}{B_1A} = \frac{ap_1}{cp_3}$. A known formula (see Eric W. Weisstein, "Routh's Theorem," at <http://mathworld.wolfram.com/RouthsTheorem.html>) states that if A' , B' , and C' are points on the sides BC , CA , and AB of $\triangle ABC$, respectively, then

$$[A'B'C'] = \frac{\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} + 1}{\left(\frac{AC'}{C'B} + 1\right) \left(\frac{BA'}{A'C} + 1\right) \left(\frac{CB'}{B'A} + 1\right)} [ABC].$$

Therefore,

$$\begin{aligned}[A_1B_1C_1] &= \frac{\frac{bp_2}{ap_1} \cdot \frac{cp_3}{bp_2} \cdot \frac{ap_1}{cp_3} + 1}{\left(\frac{bp_2}{ap_1} + 1\right) \left(\frac{cp_3}{bp_2} + 1\right) \left(\frac{ap_1}{cp_3} + 1\right)} [ABC] \\ &= \frac{2abcp_1p_2p_3}{(ap_1 + bp_2)(bp_2 + cp_3)(cp_3 + ap_1)} [ABC].\end{aligned}$$

Since $\angle MA_2C$ and $\angle MB_2C$ are right angles,

$$\begin{aligned}\angle A_2MB_2 &= 360^\circ - \angle MA_2C - \angle MB_2C - \angle A_2CB_2 \\ &= 180^\circ - \angle ACB,\end{aligned}$$

so $\sin \angle A_2MB_2 = \sin C$, and since all four angles of quadrilateral A_2MB_2C are less than 180° , A_2MB_2C is convex. The well-known area formula for a triangle, $[XYZ] = \frac{1}{2}XY \cdot XZ \sin \angle YXZ$, yields

$$\begin{aligned}[MA_2B_2] &= \frac{1}{2}MA_2 \cdot MB_2 \sin \angle A_2MB_2 \\ &= \frac{p_1p_2}{ab} \left(\frac{1}{2}ab \sin C\right) = \frac{p_1p_2}{ab} [ABC].\end{aligned}$$

By similar reasoning, $[MB_2C_2] = \frac{p_2p_3}{bc} [ABC]$ and $[MC_2A_2] = \frac{p_3p_1}{ca} [ABC]$, and B_2MC_2A and C_2MA_2B are convex. Since A_2MB_2C , B_2MC_2A , and C_2MA_2B are convex, M is outside of $\triangle A_2B_2C$, $\triangle B_2C_2A$, and $\triangle C_2A_2B$, and since M is in the interior of $\triangle ABC$, M must be in the interior of $\triangle A_2B_2C_2$. Therefore,

$$[A_2B_2C_2] = [MA_2B_2] + [MB_2C_2] + [MC_2A_2],$$

and so

$$\begin{aligned} [A_2B_2C_2] &= \left(\frac{p_1p_2}{ab} + \frac{p_2p_3}{bc} + \frac{p_3p_1}{ca} \right) [ABC] \\ &= \frac{p_1p_2p_3}{abc} \left(\frac{a}{p_1} + \frac{b}{p_2} + \frac{c}{p_3} \right) [ABC]. \end{aligned}$$

Finally,

$$\begin{aligned} \frac{[A_2B_2C_2]}{[A_1B_1C_1]} &= \frac{\frac{p_1p_2p_3}{abc} \left(\frac{a}{p_1} + \frac{b}{p_2} + \frac{c}{p_3} \right) [ABC]}{\frac{2abc p_1 p_2 p_3}{(ap_1 + bp_2)(bp_2 + cp_3)(cp_3 + ap_1)} [ABC]} \\ &= \frac{(ap_1 + bp_2)(bp_2 + cp_3)(cp_3 + ap_1)}{2a^2b^2c^2} \left(\frac{a}{p_1} + \frac{b}{p_2} + \frac{c}{p_3} \right). \end{aligned}$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and TITU ZVONARU, Comănești, Romania. There was one incorrect solution submitted.

Both Bataille and Geupel calculated with barycentric coordinates. Geupel also used the formula for the area of the pedal triangle of M , to wit $[A_2B_2C_2] = |R^2 - OM^2|/4R^2$, where O and R are the circumcentre and circumradius of triangle ABC , respectively.

3372. [2008 : 364, 366] Proposed by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.

If $x, y, z \geq 0$ and $xy + yz + zx = 1$, prove that

$$\begin{aligned} \text{(a)} \quad & \frac{1}{\sqrt{2x^2 + 3yz}} + \frac{1}{\sqrt{2y^2 + 3zx}} + \frac{1}{\sqrt{2z^2 + 3xy}} \geq \frac{2\sqrt{6}}{3}; \\ \text{(b)} \star & \frac{1}{\sqrt{x^2 + yz}} + \frac{1}{\sqrt{y^2 + zx}} + \frac{1}{\sqrt{z^2 + xy}} \geq 2\sqrt{2}. \end{aligned}$$

Solution by George Apostolopoulos, Messolonghi, Greece.

(a) Since $2x^2 \leq 3x^2$ we have $2x^2 + 3yz \leq 3x^2 + 3yz$, thus

$$\frac{1}{\sqrt{2x^2 + 3yz}} \geq \frac{1}{\sqrt{3x^2 + 3yz}} = \frac{1}{\sqrt{3}\sqrt{x^2 + yz}}.$$

Similar inequalities hold for the other two terms on the left side of the desired inequality, and we now have

$$\begin{aligned} & \frac{1}{\sqrt{2x^2 + 3yz}} + \frac{1}{\sqrt{2y^2 + 3zx}} + \frac{1}{\sqrt{2z^2 + 3xy}} \\ & \geq \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{x^2 + yz}} + \frac{1}{\sqrt{y^2 + zx}} + \frac{1}{\sqrt{z^2 + xy}} \right). \end{aligned}$$

The desired inequality follows now from (b), which is proven below.

(b)★ More generally, we will prove that if $x, y, z \geq 0$ and $xy + yz + zx > 0$, then

$$\frac{1}{\sqrt{x^2 + yz}} + \frac{1}{\sqrt{y^2 + zx}} + \frac{1}{\sqrt{z^2 + xy}} \geq \frac{2\sqrt{2}}{\sqrt{xy + yz + zx}}. \quad (1)$$

Since this is symmetric in x, y, z , we may assume that $x \geq y \geq z$. Notice that

$$\frac{1}{\sqrt{y^2 + zx}} + \frac{1}{\sqrt{z^2 + xy}} \geq \frac{2\sqrt{2}}{\sqrt{y^2 + z^2 + xy + zx}}.$$

[Ed: we have $\frac{1}{\sqrt{y^2 + zx}} + \frac{1}{\sqrt{z^2 + xy}} \geq 2\sqrt{\frac{1}{\sqrt{y^2 + zx}} \frac{1}{\sqrt{z^2 + xy}}}$ from the AM–GM Inequality, and also $\frac{1}{\sqrt{y^2 + zx}\sqrt{z^2 + xy}} \geq \frac{2}{y^2 + zx + z^2 + xy}$; the inequality above now follows from these two.]

So it suffices to prove that

$$\frac{1}{\sqrt{x^2 + yz}} + \frac{2\sqrt{2}}{\sqrt{y^2 + z^2 + xy + zx}} \geq \frac{2\sqrt{2}}{\sqrt{xy + yz + zx}}.$$

Let $K = xy + yz + zx$ and $L = y^2 + z^2 + xy + zx$. Then

$$\frac{2\sqrt{2}}{\sqrt{K}} - \frac{2\sqrt{2}}{\sqrt{L}} = \frac{2\sqrt{2}(\sqrt{L} - \sqrt{K})}{\sqrt{KL}} = \frac{2\sqrt{2}(y^2 - yz + z^2)}{\sqrt{KL}(\sqrt{L} + \sqrt{K})}.$$

It is clear that $L \geq K$, $L \geq 2(y^2 - yz + z^2)$, and $K \geq y\sqrt{x^2 + yz}$.

[Ed: Since $x \geq y \geq z$, we have $L = y^2 + z^2 + xy + zx \geq y^2 + z^2 + y^2 + z^2 \geq 2(y^2 - yz + z^2)$ and $K = xy + yz + zx \geq xy + yz + yz = y\sqrt{x^2 + 4xz + 4z^2} \geq y\sqrt{x^2 + 4yz + 4z^2} \geq y\sqrt{x^2 + yz}$.]

Thus,

$$\begin{aligned} \frac{2\sqrt{2}(y^2 - yz + z^2)}{\sqrt{KL}(\sqrt{L} + \sqrt{K})} &\leq \frac{2\sqrt{2}(y^2 - yz + z^2)}{\sqrt{KL}(2\sqrt{K})} = \frac{\sqrt{2}(y^2 - yz + z^2)}{K\sqrt{L}} \\ &\leq \frac{\sqrt{2}(y^2 - yz + z^2)}{y\sqrt{x^2 + yz}\sqrt{2(y^2 - yz + z^2)}} \\ &= \frac{\sqrt{y^2 - yz + z^2}}{y\sqrt{x^2 + yz}} \leq \frac{\sqrt{y^2}}{y\sqrt{x^2 + yz}} \\ &= \frac{1}{\sqrt{x^2 + yz}}, \end{aligned}$$

and we have proved (1).

Since $xy + yz + zx = 1$, inequality (b) now follows from (1).

Part (a) also solved by Oliver Geupel, Brühl, NRW, Germany; and the proposer.

Geupel notes that the problem is Problem 1.89 on page 74.f with solution in the e-paper toanhocmuonmaumain.pdf written by the proposer in Vietnamese, available on the MathLinks forum website <http://www.mathlinks.ro/viewtopic.php?t=197674>.

3373. Proposed by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.

Let x , y , z , and t be positive real numbers. Prove that

$$(x + y)(x + z)(x + t)(y + z)(y + t)(z + t) \geq 4xyz t(x + y + z + t)^2.$$

Solution by Cristinel Mortici, Valahia University of Targoviste, Romania.

Dividing by $xyz t(x + z)(y + t)(x + t)(y + z)$ the inequality becomes

$$\frac{x + y}{xy} \cdot \frac{z + t}{zt} \geq \frac{2(x + y + z + t)}{(x + z)(y + t)} \cdot \frac{2(x + y + z + t)}{(x + t)(y + z)}.$$

Thus we need to show:

$$\left(\frac{2}{x + z} + \frac{2}{y + t}\right) \left(\frac{2}{x + t} + \frac{2}{y + z}\right) \leq \left(\frac{1}{x} + \frac{1}{y}\right) \left(\frac{1}{z} + \frac{1}{t}\right).$$

By the AM–GM Inequality we have

$$\left(\frac{2}{x + z} + \frac{2}{y + t}\right) \left(\frac{2}{x + t} + \frac{2}{y + z}\right) \leq \left(\frac{1}{\sqrt{xz}} + \frac{1}{\sqrt{yt}}\right) \left(\frac{1}{\sqrt{xt}} + \frac{1}{\sqrt{yz}}\right),$$

while by the Cauchy–Schwartz Inequality we have

$$\frac{1}{\sqrt{xz}} + \frac{1}{\sqrt{yt}} \leq \sqrt{\left(\frac{1}{x} + \frac{1}{y}\right) \left(\frac{1}{z} + \frac{1}{t}\right)}$$

and

$$\frac{1}{\sqrt{xt}} + \frac{1}{\sqrt{yz}} \leq \sqrt{\left(\frac{1}{x} + \frac{1}{y}\right) \left(\frac{1}{z} + \frac{1}{t}\right)}.$$

Combining the last three inequalities we obtain the desired inequality.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College.

Sarajevo, Bosnia and Herzegovina; KHANH BAO NGUYEN, High School for Gifted Students, Hanoi University of Education, Hanoi, Vietnam; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Howard notes that this problem is similar to CRUX 2393(a) [1999 : 447]. In fact, Malikić observes that by using 2393(a) for $(a, b, c, d) = (x, y, z, t)$, $(a, b, c, d) = (x, z, t, y)$, and $(a, b, c, d) = (x, t, y, z)$ (that is, all the cyclic permutations of y, z, t) and multiplying the resulting inequalities together we get exactly the desired inequality.

Geupel notes that the problem is Problem 1.73 on page 64.f with solution in the e-paper toanhocmuonmaumain.pdf written by the proposer in Vietnamese, available on the MathLinks forum website <http://www.mathlinks.ro/viewtopic.php?t=197674>.

3374. [2008 : 364, 366] Proposed by Pham Huu Duc, Ballajura, Australia.

Let a, b , and c be positive real numbers. Prove that

$$\frac{a^2}{a^2 + bc} + \frac{b^2}{b^2 + ca} + \frac{c^2}{c^2 + ab} \leq \frac{a + b + c}{2\sqrt[3]{abc}}.$$

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

By homogeneity, we may take $abc = 1$ and $a \leq b \leq c$. Also, if $a = 1$, then $a = b = c = 1$ and there is nothing to prove, so we take $a < 1$.

For convenience, write $x = a^3$, $y = b^3$, and $z = c^3$; let $f(x) = \frac{x}{x+1}$, and let the left and right sides of the inequality be $L = f(x) + f(y) + f(z)$ and $R = \frac{1}{2}(x^{1/3} + y^{1/3} + z^{1/3})$, respectively.

Setting $m = \sqrt{yz}$ and $r = \sqrt{\frac{z}{y}}$, we have that $m > 1$ (because $a < 1$ implies that $yz > 1$), $y = \frac{m}{r}$, and $z = mr$. We then find that

$$\begin{aligned} 2f(m) - f(y) - f(z) &= \frac{2m}{m+1} - \frac{m}{m+r} - \frac{mr}{mr+1} \\ &= \frac{m(m-1)(r-1)^2}{(m+1)(m+r)(mr+1)} \geq 0, \end{aligned}$$

hence L cannot decrease if each of y and z are replaced by their geometric mean. On the other hand, from $\sqrt{bc} \leq \frac{b+c}{2}$, we see that R cannot increase if y and z are each replaced by their geometric mean.

Therefore, it suffices to prove the inequality under the additional assumption that $b = c$ and $a = \frac{1}{b^2}$. This new relation yields

$$R - L = \frac{(1+2y)(y^3 + y^2 + y + 1) - 4b^{11} - 6b^5 - 2b^2}{2b^2(y^3 + y^2 + y + 1)},$$

where the denominator is positive and (after some computation) the numerator becomes

$$(b-1)^2 \left((b-1)(2b^9 + 2b^8 - b^6 - b^5) + 5b^4 + 3b^3 + b^2 + 2b + 1 \right),$$

which is also a positive quantity since $b > 1$.

Thus, $R - L > 0$ in this last case, and the proof is complete.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA (2 solutions); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; and the proposer. There were two incorrect solutions submitted.

3375. Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.

Let p be a non-negative integer and x any real number. Find the sum

$$\sum_{n=1}^{\infty} (-1)^n \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^{n+p}}{(n+p)!} \right).$$

Solution by Cristinel Mortici, Valahia University of Targoviste, Romania.

If $r > 0$ and f is a function with

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad x \in (-r, r),$$

then we will show that

$$\begin{aligned} & \sum_{n=1}^{\infty} (-1)^n (f(x) - a_0 - a_1 x - \cdots - a_{n+p} x^{n+p}) \\ &= \begin{cases} \sum_{i=0}^{p/2} a_{2i} x^{2i} - \frac{f(x) + f(-x)}{2}, & p \text{ even,} \\ \sum_{i=0}^{(p-1)/2} a_{2i+1} x^{2i+1} - \frac{f(x) - f(-x)}{2}, & p \text{ odd.} \end{cases} \end{aligned} \quad (1)$$

Taking $f(x) = e^x$, we obtain the answer

$$\begin{aligned} & \sum_{n=1}^{\infty} (-1)^n \left(f(x) - 1 - \frac{x}{1!} - \cdots - \frac{x^{n+p}}{(n+p)!} \right) \\ &= \begin{cases} \sum_{i=0}^{p/2} \frac{x^{2i}}{(2i)!} - \frac{e^x + e^{-x}}{2}, & p \text{ even,} \\ \sum_{i=0}^{(p-1)/2} \frac{x^{2i+1}}{(2i+1)!} - \frac{e^x - e^{-x}}{2}, & p \text{ odd.} \end{cases} \end{aligned}$$

To prove (1), note that the general term

$$a_n = (-1)^n (f(x) - a_0 - a_1 x - \cdots - a_{n+p} x^{n+p})$$

converges to zero, so it suffices to find the limit of the sequence $\{s_{2n}\}$, where

$$s_n = \sum_{k=1}^n a_k,$$

because $s_{2n+1} = s_{2n} + a_{2n+1}$ and $a_{2n+1} \rightarrow 0$. We have

$$s_{2n} = \sum_{k=1}^n (a_{2k-1} + a_{2k}) = \sum_{k=1}^n (-a_{2k+p} x^{2k+p}).$$

If p is even, then

$$s_{2n} = \sum_{i=0}^{p/2} a_{2i} x^{2i} - \sum_{i=0}^{n+p/2} a_{2i} x^{2i};$$

while if p is odd, then

$$s_{2n} = \sum_{i=0}^{(p-1)/2} a_{2i+1} x^{2i+1} - \sum_{i=0}^{n+(p-1)/2} a_{2i+1} x^{2i+1}.$$

Now the relation (1) follows from the equations

$$\sum_{i=0}^{\infty} a_{2i} x^{2i} = \frac{f(x) + f(-x)}{2}; \quad \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1} = \frac{f(x) - f(-x)}{2}.$$

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNÁNDEZ, and LUZ RONCAL, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer. There were three incorrect solutions submitted.

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