PROBLEM DEPARTMENT

ASHLEY AHLIN AND HAROLD REITER*

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to `hbreiter@uncc.edu`*. Electronic submissions using LaTeX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by October 1, 2010. Solutions identified as by students are given preference.*

Problems for Solution.

1214. *William Gosnell, Amherst MA, Herb Bailey, Rose-Hulman Inst. Tech.*

Tatyana loves track and trig. She runs one lap around a circular track with center O and radius r. Let T be her position on the track at any time and S her starting point. Let $\theta = \angle SOT$, $0 \leq \theta \leq 2\pi$ and d the distance she has run along the track from S to T. For fixed r, how many values of θ are there such that $d = \sec^2 \theta$?

1215. *Tuan Le, Fairmont High School, Anaheim, CA.*

Let a, b, c be non-negative real numbers no two of which are zero. Prove that

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} + \frac{3\sqrt[3]{abc}}{2(a+b+c)} \geq 2.$$

1216. *Proposed by Cecil Rousseau, University of Memphis, Memphis, TN.*

Evaluate the sum

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \binom{2n}{2k}^{-1}.$$

1217. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Calculate

$$\int_0^1 \left\{ (-1)^{\lfloor \frac{1}{x} \rfloor} \cdot \frac{1}{x} \right\} dx,$$

where $\lfloor a \rfloor$ denotes the **floor** of a and $\{a\} = a - \lfloor a \rfloor$ denotes the **fractional part** of a.

*University of North Carolina Charlotte

1218. *Proposed by Mohammad K. Azarian, University of Evansville, Indiana.*

Find the following infinite sum

$$S = \frac{1}{1\cdot 2} - \frac{1}{1\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} - \frac{1}{5\cdot 7} + \frac{1}{7\cdot 8} + \frac{1}{9\cdot 10} - \frac{1}{9\cdot 11} + \frac{1}{11\cdot 12}$$
$$+ \frac{1}{13\cdot 14} - \frac{1}{13\cdot 15} + \frac{1}{15\cdot 16} + \frac{1}{17\cdot 18} - \frac{1}{17\cdot 19} + \frac{1}{19\cdot 20} + \frac{1}{21\cdot 22} - \frac{1}{21\cdot 23} + \ldots.$$

1219. *Proposed by Sam Vandervelde, St. Lawrence University, Canton, NY*

A set of discs with radii 1, 3, 5, $\ldots$, 2009 are stacked on a peg from smallest on top to largest at bottom. Another set of discs having radii 2, 4, 6, $\ldots$, 2010 are similarly stacked on a second peg. A third peg is available but is initially empty. Let N be the fewest number of moves needed to transfer all 2010 discs to the empty peg, if each move consists of moving a single disc from the top of one stack onto another peg which is either empty or has only larger discs. (Following the familiar "Tower of Hanoi" rules.) Find, with proof, the number of 1's in the binary representation of N.

1220. *Proposed by Robert Gebhardt, Hopatcong, NJ*

Find the area of the region bounded by the curve $x^{2n+1} + y^{2n+1} = (xy)^n$, $n = 1, 2, 3 \ldots$.

1221. *Proposed by Stas Molchanov, University of North Carolina Charlotte*

Let p and q be different prime numbers greater than 2. Prove that $x^p + y^p = z^q$ has infinitely many solutions over the nonnegative integers.

1222. *Proposed by Arthur Holshouser, Charlotte, NC and Ben Klein, Davidson, NC.*

A positive integer n bigger than 1 can be *split* into two positive integer summands, called its *offspring*, a and b, usually in several ways. Notice that 1 cannot be split. Starting with an integer n greater than 1, we can successively split n and its offspring to eventually arrive at all 1s. For a fixed real number θ, define the *value* of the split to be $V(a,b) = (\sin a\theta)(\sin b\theta)(\cos(a+b-1)\theta)$. Let n be a positive integer.

1. Prove that the number of splits must be exactly $n-1$.
2. Prove that the sum of the values of the $n-1$ splits does not depend on the sequence of splits and determine its constant value.

1223. *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abington, PA.*

Show that the sum of any $2(2n+1)$ consecutive terms of the Fibonacci sequence is divisible by the $(2n+1)^{\text{st}}$ Lucas number.

1224. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let a, b and c be the sides of an acute-angled triangle ABC. Let H be the orthocenter, and let d_a, d_b, and d_c be the distances from H to the sides BC, CA, and AB, respectively. Prove or disprove that

$$\sqrt[6]{\frac{a^3b^3c^3}{(-a+b+c)(a-b+c)(a+b-c)}} \geq \frac{2\sqrt{3}}{3}(d_a + d_b + d_c).$$

Solutions. The editors regret that we did not give credit to Kazuo Tezuka, Ichikawa City, Chiba, Japan, for his solution to problem 1196 in the spring issue of 2009.

1204. *Proposed by Sam Vandervelde, St. Lawrence University, Canton, NY*

Prove that there exists a real number α satisfying $\lfloor \alpha^n \rfloor \equiv n \pmod 5$ for all $n \in \mathbb{N}$.

Solution by **Eugen J. Ionascu**, *Columbus State University, Columbus, GA*

We will show that such a real number exists and it can be defined as a limit

$$\alpha = \lim_{n\to\infty} x_n^{\frac{1}{n}}, \tag{1}$$

where x_n is a sequence of positive integers defined (not uniquely) by the following conditions:

$$\begin{cases} x_1 = 6, \\ x_n \equiv \ n \ (mod\ 5), \ n \in \mathbb{N}, \\ x_{n+1} \in (x_n^{\frac{n+1}{n}}, (x_n+1)^{\frac{n+1}{n}} - 1), \ n \in \mathbb{N} \\ x_n^{\frac{1}{n}} < 7, \ n \in \mathbb{N}. \end{cases} \tag{2}$$

Let us observe first that if a sequence of integers x_n satisfying (2) exists, then $a_n = x_n^{\frac{1}{n}}$ being strictly increasing and bounded by 7 it is convergent, so α is well defined by (1). We will show that $(x_n+1)^{\frac{1}{n}}$ is strictly decreasing and for every $n > k$ we have $x_n^{\frac{1}{n}} \in (x_k^{\frac{1}{k}}, (x_k+1)^{\frac{1}{k}})$. This will give that $\alpha^k \in (x_k, x_k+1)$ and so $\lfloor \alpha^k \rfloor = x_k \equiv k \pmod 5$ for all $k \in \mathbb{N}$ by (2).

For $n = 1$ we need to find x_2, a positive integer such that $x_2 \equiv 2 \pmod 5$ such that $x_2 \in (36, 48)$. Clearly we can take x_2 to be 37, 42 or 47. For the sake of exemplification, let us say we chose $x_2 = 42$. Then x_3 needs to satisfy $x_3 \equiv 3 \pmod 5$ and $x_3 \in (42^{\frac{3}{2}}, 43^{\frac{3}{2}} - 1) \approx (272.2, 280)$. This gives two choices: $x_3 = 273$ or $x_3 = 278$. We took $x_3 = 273$ and continued this way for a few more steps and obtained a number $\alpha \approx 6.490161791$.

By induction on n, let us observe that, if $x_1, x_2, x_3, ..., x_n$ are defined and satisfy the appropriate relations in (2). We observe that, we must have

$$6 = x_1 < x_2^{\frac{1}{2}} < ... < x_n^{\frac{1}{n}}.$$

In order to be able to define x_{n+1} we need to have enough space in the interval $(x_n^{\frac{n+1}{n}}, (x_n+1)^{\frac{n+1}{n}} - 1)$ in order to pick an integer congruent to $n+1$ modulo 5. One can see that this can be accomplished if we can show that $(x_n+1)^{\frac{n+1}{n}} - 1 - x_n^{\frac{n+1}{n}} > 5$. For this purpose let us consider the function $g(x) = x^{\frac{n+1}{n}}$ defined for positive values of x. Since $g'(x) = \frac{n+1}{n} x^{\frac{1}{n}}$, by Mean Value Theorem, we have

$$g(x_n+1) - g(x_n) = \frac{n+1}{n} c^{\frac{1}{n}}, c \in (x_n, x_n+1).$$

Therefore

$$(x_n+1)^{\frac{n+1}{n}} - x_n^{\frac{n+1}{n}} = \frac{n+1}{n}c^{\frac{1}{n}} > x_n^{\frac{1}{n}} > 6.$$

The next thing is to show that $(x_n+1)^{\frac{1}{n}}$ is strictly decreasing. Because $x_{n+1} < (x_n+1)^{\frac{n+1}{n}} - 1$, we see that the inequality $(x_{n+1}+1)^{\frac{1}{n+1}} < (x_n+1)^{\frac{1}{n}}$ follows. Finally if $n > k$ we have $x_n^{\frac{1}{n}} \in (x_k^{\frac{1}{k}}, (x_k+1)^{\frac{1}{k}})$ because

$$6 = x_1 < x_2^{\frac{1}{2}} < ... < x_n^{\frac{1}{n}} < (x_n+1)^{\frac{1}{n}} < ... < (x_2+1)^{\frac{1}{2}} < x_1 + 1 = 7.$$

It seems like there is quite a bit of freedom to construct such numbers. Even more so if one starts with $x_1 \in \{11, 16, ...\}$ for instance.

Note from the solver. This problem is just a different version of Problem A-5 from The William Putnam Mathematical Competition of 1983 (solutions in AMM 1984, page 493) which is essentially our solution (except theirs contains the modulo 2 analysis instead of 5). A little note appears in AMM 1987 (Even Older Than We Thought" by the editors of Problems and Solutions) where more precise examples are given in that case. An editorial note including a little history of various approaches and some even older references appears with the note. It is interesting to see if 2 those ideas work out or not for different values of the modulo. Some other natural questions may be studied, like how may solutions for the problem can be found or what is the smallest positive solution for a certain modulo and base.

Also solved by **Paul S. Bruckman**, Sointula, BC; **Arthur Holshouser**, Charlotte, NC; **Northwestern University Math Problem Solving Group**, Evanston, IL; **Pedro Henrique O. Pantoja, student**, UFRN, Natal, Brazil; and the **Proposer**.

1205. *Proposed by Peter A. Lindstrom, Batavia, NY*

The concatenation of positive integers p and q, denoted $p \mid\mid q$, is defined as

$$p \mid\mid q = p \cdot \left(10^{\lfloor \log_{10} q \rfloor + 1}\right) + q,$$

where $\lfloor x \rfloor$ is the floor function. For example $37 \mid\mid 21 = 37(10^{\lfloor \log_{10} 21 \rfloor + 1} + 21 = 37(10^{1+1}) + 21 = 3721$.

Find infinitely many triplets of positive integers a, b, c such that

1. $(a \mid\mid b) \mid\mid c$ is not a palindrome, and
2. $(a^2 \mid\mid b^2) \mid\mid c^2$ is a palindrome.

Solution by **Kathy Lewis**, *SUNY Oswego, Oswego, NY.*

The triplet $(21, 2, 2)$ meets the condition, since $21||2||2 = 2122$ is not a palindrome, but $21^2||2^2||2^2 = 44144$ is. From this one triplet, we can create infinitely many such triplets by an appropriate insertion of zeroes: $(20010, 20, 2), (20000100, 200, 2)$, etc. In each case, the initial string of zeroes is twice the length of the other two strings. Since there is always just one 2 before the 1 and two 2's after it, the string $a||b||c$ is never a palindrome, but the strings $a^2||b^2||c^2$ result in $2002001002002, 200002000010000200002 \ldots.$

Solution by **Alan Upton and Alin A. Stancu**, *Columbus State University, Columbus, GA.*

Note that, for each nonnegative integer n, the terms of the sequence $b_n = 1\underbrace{00...0}_{n}1$ are palindromic. Moreover, it is easy to see that b_n^2 is palindromic for each n. Indeed, we have $b_1^2 = 121$, $b_2^2 = 10201$, $b_3^2 = 1002001$, $b_4^2 = 100020001$, etc.

Take now $a = c = 26$ and $b = b_n$ for some n. Since $a^2 = c^2 = 676$ we obtain that the number $(a^2 \parallel b^2) \parallel c^2$ is palindromic. Also, the number $(a \parallel b) \parallel c$ is not palindromic since its first digit is 2 and its last digit is 6. Because there are infinitely many positive integers b_n we produced infinitely many triplets (a, b, c) with the required properties.

Also solved by **Dionne T. Bailey, Elsie M. Campbell, Charles Diminnie**, Angelo State University, San Angelo, TX; **David Burwell, student**, Alma College, Alma, MI; **Paul S. Bruckman**, Sointula, BC; **Mark Evans**, Louisville, KY; **Joe Pleso**, Southern Illinois University Carbondale, Carbondale, IL; **James A. Sellers**, Pennsylvania State University, University Park, PA; and the **Proposer**.

1206. *Proposed by Stas Molchanov, University of North Carolina Charlotte*

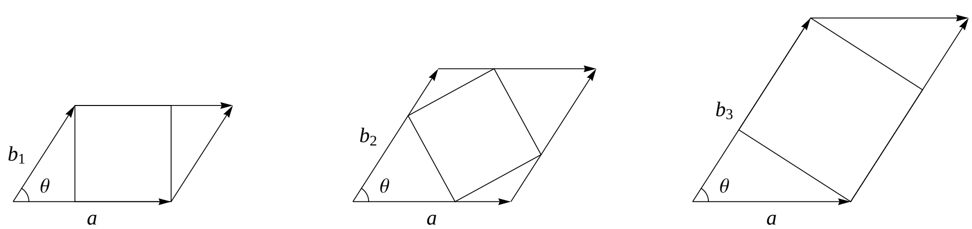
A square is said to be inscribed in a quadrilateral if each vertex of the square belongs to a different edge of the quadrilateral. Find necessary and sufficient condition on a parallelogram in order to have an inscribed square.

Solution by **Nathan Caudill, student and Tim McDevitt**, *Elizabethtown College, Elizabethtown, PA.*

For a given parallelogram, let vectors a and b coincide with two of the sides of the parallelogram so that the (tail-to-tail) angle between a and b, $\theta \in (0, \pi/2]$. The following figure shows three parallelograms with a common vector a and angle θ with vectors b of different lengths. The shortest b can be is shown in the first figure, where one edge of the inscribed square is coincident with a. In this case, elementary trigonometry shows that $\|b_1\| = \|a\|/(\cos\theta + \sin\theta)$. Similarly, the longest possible b is shown in the third figure where one edge of the inscribed square coincides with b_3 and $\|b_3\| = \|a\|(\cos\theta + \sin\theta)$. Therefore, a square can be inscribed in only those parallelograms for which

$$\frac{1}{\cos\theta + \sin\theta} \leq \frac{\|b\|}{\|a\|} \leq \cos\theta + \sin\theta.$$

If we interpret the problem as requiring that a vertex of an inscribed square cannot share a vertex with the parallelogram, then the inequalities should be strict.



Also solved by the **Proposer**.

1207. *Proposed by R. M. Welukar, K. S. Bhanu, and M. N. Deshpande, Open University and Institute of Science, India.*

Let $F_k, F_{k+1}, \ldots F_{k+4n-1}$ be arranged in a $2 \times 2n$ matrix as shown below.

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & 2n \end{array}$$
$$\begin{pmatrix} F_k & F_{k+3} & F_{k+4} & F_{k+7} & \cdots & F_{k+4n-1} \\ F_{k+1} & F_{k+2} & F_{k+5} & F_{k+6} & \cdots & F_{k+4n-2} \end{pmatrix}$$

Show that the sum of the elements of the first and second rows denoted by R_1 and R_2 respectively can be expressed as

$$R_1 = 2F_{2n}F_{2n+k}$$
$$R_2 = F_{2n}F_{2n+k+1}$$

where $\{F_n, n \geq 1\}$ denotes the Fibonacci sequence.

Solution by **Hongwei Chen**, *Christopher Newport University, Newport News, VA.*

We use the following three identities:

$$\sum_{i=0}^{N} F_{ij+k} = \frac{F_{(N+1)j+k} - (-1)^j F_{Nj+k} - F_k + (-1)^j F_{k-j}}{L_j - (-1)^j - 1}, \tag{1}$$

$$F_m L_{m+n} = F_{2m+n} - (-1)^m F_n, \text{ and} \tag{2}$$

$$5F_n = L_{n+1} + L_{n-1}, \tag{3}$$

where $\{L_n, n \geq 1\}$ denotes the Lucas sequence.

Appealing to Binet's formula $F_n = F_{n-1} + F_{n-2}$, we have

$$\begin{aligned} R_1 + R_2 &= \sum_{i=0}^{4n-1} F_{i+k} = F_{4n+k} + F_{4n+k-1} - F_k - F_{k-1} \qquad \text{(using (1) with } j = 1) \\ &= F_{4n+k+1} - F_{k+1} = F_{2n}L_{2n+k+1} \qquad \text{(using (2) with } m = 2n) \\ &= F_{2n}(2F_{2n+k} + F_{2n+k+1}) \qquad \text{(using } L_n = F_{n+1} + F_{n-1}). \end{aligned}$$

On the other hand, regrouping and using Binet's formula, we have

$$R_1 - R_2 = F_k + F_{k+4} + \cdots + F_{k+4(n-1)} = \sum_{i=0}^{n-1} F_{4i+k}.$$

Thus,

$$\begin{aligned} R_1 - R_2 &= \frac{1}{5}\,(F_{4n+k} - F_{4n+k-4} - F_k + F_{k-4})) \qquad \text{(using (1) with } j = 4) \\ &= \frac{1}{5}\,(F_{2n}\,L_{2n+k} - F_{2n}\,L_{2n+k-4}) \qquad \text{(using (2) with } m = 2n) \\ &= \frac{1}{5}\,F_{2n}\,[(L_{2n+k} + L_{2n+k-2}) - (L_{2n+k-2} + L_{2n+k-4})] \\ &= F_{2n}\,(F_{2n+k-1} - F_{2n+k-3}) \qquad \text{(using (3) twice)} \\ &= F_{2n}(2F_{2n+k} - F_{2n+k+1}). \end{aligned}$$

Now, solving for R_1 and R_2 yields the required expressions.

Also solved by **Paul S. Bruckman**, Sointula, BC; **G. C. Greubel**, Newport News, VA; **Parviz Khalili**, Christopher Newport University, Newport News, VA; **Yoshinobu Murayoshi**, Naha City, Okinawa, Japan; **Angel Plaza and Sergio Falcon**, University of Las Palmas de Gran Canaria, Spain; **John Postl, student**, St. Bonaventure University; St. Bonaventure NY; and the **Proposer**.

1208. *Proposed by Matthew McMullen, Otterbein College, Westerville, OH.*

Let k be a positive integer with $k \equiv 1 \pmod 4$. Define x_k to be the solution to $\tan x = x$ on the interval $\left[\frac{(k-1)\pi}{2}, \frac{k\pi}{2}\right)$. Let $A = A(k) > 0$ be chosen so that the equation $\sin x = Ax$ has exactly k solutions. Show that A is unique and that $x_k < 1/A < k\pi/2$.

Solution by **the Proposer.**

If $k = 1$, then $A = 1$ is the only possible value for A; so suppose $k \geq 5$. Because of the symmetries inherent in the graphs of $y = \sin x$ and $y = Ax$, we must have exactly $(k-3)/2$ solutions on the interval $(2\pi, \infty)$. Also, for $k > 5$, each interval of the form $(2i\pi, (2i+1)\pi)$, for $i = 1, 2, \ldots, (k-5)/4$, contributes two more solutions. Since we have one more root to account for, A is the unique positive number that ensures that $y = Ax$ is tangent to $y = \sin x$ on the interval $[(k-1)\pi/2, k\pi/2)$.

Next, notice that the line tangent to $y = \sin x$ at the point $(x_k, \sin x_k)$ is given by the equation

$$\begin{aligned} y &= (\cos x_k)(x - x_k) + \sin x_k \\ &= (\cos x_k)x - (\cos x_k)x_k + \sin x_k \\ &= (\cos x_k)x. \end{aligned}$$

Therefore, $A = \cos x_k$.

We are left to show

$$x_k < \frac{1}{\cos x_k} < \frac{k\pi}{2}. \tag{1}$$

Since $x_k \cos x_k = \sin x_k < 1$, the left-hand inequality of (1) holds. Also, since the function $y = x/\sin x$ has a local minimum at x_k on the interval $((k-1)\pi/2, (k+1)\pi/2)$, we see that

$$\frac{1}{\cos x_k} = \frac{x_k}{\sin x_k} < \frac{k\pi/2}{\sin(k\pi/2)} = k\pi/2.$$

Therefore, (1) holds.

Also solved by **Paul S. Bruckman**, Sointula, BC; and **Yoshinobu Murayoshi**, Naha City, Okinawa, Japan.

1209. *Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain.*

Let R and r denote the radii of the circumcircle and the incircle of a triangle ABC with sides a, b, c and semi-perimeter s. Prove that

$$\frac{(s-a)^4}{c(s-b)} + \frac{(s-b)^4}{a(s-c)} + \frac{(s-c)^4}{b(s-a)} \geq \frac{3r}{4}\sqrt[3]{\frac{Rs^2}{2}}$$

Solution by **Francisco Javier García Capitán***, I. E. S. Álvarez Cubero, Priego de Córdoba Spain and Ercole Suppa, Liceo Scientifico "A. Einstein", Teramo, Italy*

By using the well known inequalities

$$a^2 + b^2 + c^2 \geq ab + bc + ca, \qquad 3(a^2 + b^2 + c^2) \geq (a + b + c)^2$$

as well as the Cauchy-Schwartz inequality in Engel form, we have

$$\textbf{LHS} \geq \frac{\left[(s-a)^2+(s-b)^2+(s-c)^2\right]^2}{c(s-b)+a(s-c)+b(s-a)} = \frac{\left[3s^2-2s(a+b+c)+a^2+b^2+c^2\right]^2}{(a+b+c)s-ab-bc-ca} =$$

$$= \frac{\left(a^2+b^2+c^2-s^2\right)^2}{2s^2-ab-bc-ca} = \frac{\left(3a^2+3b^2+3c^2-2ab-2ac-2bc\right)^2}{8(a^2+b^2+c^2)} \geq$$

$$\geq \frac{\left(a^2+b^2+c^2\right)^2}{8(a^2+b^2+c^2)} = \frac{a^2+b^2+c^2}{8} \geq \frac{(a+b+c)^2}{24} = \frac{s^2}{6} \geq \frac{3r}{4}\cdot\sqrt[3]{4Rs^2}.$$

To justify the last step, from $\frac{1}{3}(a+b+c) \geq \sqrt[3]{abc}$, we get $8s^3 \geq 27\cdot 4R\Delta$, or $2s^2 \geq 27Rr$. Now, since $R \geq 2r$, we have

$$4s^4 \geq 729\cdot R^2r^2 \geq 729\cdot 2Rr^3 \quad \Rightarrow$$
$$8s^6 \geq 729r^3\cdot 4Rs^2 \quad \Rightarrow \quad 2s^2 \geq 9r\cdot\sqrt[3]{4Rs^2}$$

whence, dividing by 12, we have the last step above, that is a refinement of the proposed inequality and where equality holds if and only if the triangle is equilateral.

Also solved by **Yoshinobu Murayoshi**, Naha City, Okinawa, Japan; and the **Proposer**.

1210. *Proposed by Robert Gebhardt, Hopatcong, NJ.*

Find an equation for the plane tangent to the ellipsoid $a^2x^2+b^2y^2+c^2z^2=d^2$ in the first octant such that the volume in the first octant bounded by this plane and the coordinate planes is minimum.

Solution by **Ronald L. Persky**, *Department of Mathematics, Christopher Newport University, Newport News, VA.*

A triangular section of the plane is shown. The point (x_0, y_0, z_0) is the point of tangency. The volume describe in this problem is a pyramid. So, $V = \frac{1}{3}$(Base Area) (Height) = $\frac{1}{3}$ ($\frac{1}{2}$ (x-intercept$\cdot y$-intercept)$\cdot z$-intercept.

We write the equation of the ellipsoid as

$$f(x,y) = \sqrt{\frac{d^2}{c^2}-\frac{a^2}{c^2}x^2-\frac{b^2}{c^2}y^2} = \frac{1}{c}\sqrt{d^2-a^2x^2-b^2y^2} \tag{1}$$

Treating (x_0, y_0, z_0) as fixed but variable, the equation of the tangent plane at (x_0, y_0, z_0) can be written as

$$z = \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) + \frac{\partial f\,(x_0,y_0)}{\partial y}\,(y-y_0)+z_0. \tag{2}$$

From (1),

$$\frac{\partial f}{\partial x} = \frac{1}{c}\frac{-a^2x}{\sqrt{d^2-a^2x^2-b^2y^2}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{c}\frac{-b^2y}{\sqrt{d^2-a^2x^2-b^2y^2}}\ .$$

Substituting into (2), the equation of the tangent plane is

$$z = \frac{1}{c}\frac{-a^2x_0}{\sqrt{d^2 - a^2x_0^2 - b^2y_0^2}}(x - x_0) + \frac{1}{c}\frac{-b^2y_0}{\sqrt{d^2 - a^2x_0^2 - b^2y_0^2}}(y - y_0)$$
$$+\frac{1}{c}\frac{d^2 - a^2x_0^2 - b^2y_0^2}{\sqrt{d^2 - a^2x_0^2 - b^2y_0^2}} \ .$$

Combine the three terms on the right to get a simplified version for the equation of the tangent plane.

$$z = \frac{-a^2x_0x - b^2y_0y + d^2}{c\sqrt{d^2 - a^2x_0^2 - b^2y_0^2}} \tag{3}$$

For the tangent plane, we want the x, y and z intercepts.

To get the plane's x intercept, set $z = 0$ and $y = 0$ in (3):

$$0 = \frac{-a^2x_0x + d^2}{c\sqrt{d^2 - a^2x_0^2 - b^2y_0^2}}$$

or,

$$0 = -a^2x_0x + d^2$$

or

$$x \text{ intercept } = \frac{d^2}{a^2x_0} \ .$$

For the plane's y intercept, set $z = 0$ and $x = 0$ in (3):

$$0 = \frac{-b^2y_0y + d^2}{c\sqrt{d^2 - a^2x_0^2 - b^2y_0^2}}$$

or

$$0 = -b^2y_0y + d^2$$

or

$$y \text{ intercept } = \frac{d^2}{b^2y_0} \ .$$

For the plane's z intercept, set $x = 0$ and $y = 0$ in (3):

$$z \text{ intercept } = \frac{d^2}{c\sqrt{d^2 - a^2x_0^2 - b^2y_0^2}} \ .$$

The figure at the start of the proof now becomes:

The volume of the problem is:

$$V = \frac{1}{6}\frac{d^2}{a^2x_0}\frac{d^2}{b^2y_0}\frac{d^2}{c\sqrt{d^2 - a^2x_0^2 - b^2y_0^2}} \ .$$

As stated earlier, $(x_0, y_{0,} z_0)$ is treated as fixed but variable. Rewrite V without the naught subscripts:

$$V = \frac{d^6}{6a^2b^2cxy\sqrt{d^2 - a^2x^2 - b^2y^2}} \ .$$

The geometry of the problem makes it obvious that there is no maximum volume. A minimum volume will be found by solving simultaneously $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$. Here we enlist the aid of Maple.

From Maple:

$$v := (x, y) \to \frac{1}{6}\frac{d^6}{a^2b^2cxy\sqrt{d^2 - a^2x^2 - b^2y^2}}$$

$$vx := diff(v(x, y), x);$$
$$-\frac{1}{6}\frac{d^6}{a^2b^2cx^2y\sqrt{d^2 - a^2x^2 - b^2y^2}} + \frac{1}{6}\frac{d^6}{b^2cy\left(d^2 - a^2x^2 - b^2y^2\right)^{\frac{3}{2}}}$$

$$v := diff(v(x, y), y);$$
$$-\frac{1}{6}\frac{d^6}{a^2b^2cxy^2\sqrt{d^2 - a^2x^2 - b^2y^2}} + \frac{1}{6}\frac{d^6}{a^2cx\left(d^2 - a^2x^2 - b^2y^2\right)^{\frac{3}{2}}}$$

$$solve(\{vx = 0, vy = 0\}, \{x, y\});$$
$$\{x = \frac{0.5773502693d}{a}, y = \frac{0.5773502693d}{b}\} \ .$$

The decimal, $0.5773502693\ldots$, is $\frac{1}{\sqrt{3}}$.

The problem asks for an equation for the tangent plane. From Maple, the minimum volume occurs at $x_0 = \frac{d}{\sqrt{3}a}$, $y_0\frac{d}{\sqrt{3}b}$. Using these values for x_0 and y_0 in equation (3), we have an equation for the tangent plane yielding minimum volume:

$$z = \frac{-a^2\frac{d}{\sqrt{3}a}x - b^2\frac{d}{\sqrt{3}b}y + d^2}{c\sqrt{d^2 - a^2\frac{d^2}{3a^2} - b^2\frac{d^2}{3b^2}}}$$

or

$$z = \frac{\frac{-adx}{\sqrt{3}} - \frac{bdy}{\sqrt{3}} + d^2}{c\sqrt{\frac{d^2}{3}}}$$

or

$$z = \frac{-ax - by + d\sqrt{3}}{c} .$$

This plane has its x intercept at $\left(\frac{d\sqrt{3}}{a}, 0, 0\right)$, its y intercept at $\left(0, \frac{d\sqrt{3}}{b}, 0\right)$, and its z intercept at $\left(0, 0, \frac{d\sqrt{3}}{c}\right)$.

In summary, for the minimum volume, the point of tangency on the ellipsoid is $\left(\frac{d}{a\sqrt{3}}, \frac{d}{b\sqrt{3}}, \frac{d}{c\sqrt{3}}\right)$. Putting all of this together presents a nice picture.

Also solved by **Paul S. Bruckman**, Sointula, BC; **Rod Hardy, student**, Columbus State University, Columbus, GA; and the **Proposer**.

1211. *Proposed by Greg Oman, Otterbein College, Westerville, OH.*

Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with positive terms. Prove that $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges. *Note from the poser:* Since this problem is well-known, the challenge is to provide an elementary solution, one that does not invoke the Cauchy-Schwartz Inequality.

Solution by **Paolo Perfetti**, *Dipartimento di matematica, Università degli Studi di Roma "Tor Vergata", Rome, Italy.*

By $(a-b)^2 \geq 0$ it follows $a^2 + b^2 \geq 2ab$ hence

$$0 \leq \sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} \leq \frac{1}{2} \sum_{n=1}^{\infty} \left(a_n + \frac{1}{n^2}\right) = \frac{1}{2} \sum_{n=1}^{\infty} a_n + \frac{\pi^2}{12} < \infty$$

Also solved by **Paul S. Bruckman**, Sointula, BC; **Hongwei Chen**, Christopher Newport University, Newport News, VA; **Mark Evans**, Louisville, KY; **Andrs Fielbaum**, Universidad de Chile; **Rod Hardy, student**, Columbus State University, Columbus, GA; **Sean Householder, student**, Northwest Missouri State University, Maryville, MO; **Peter Lindstrom**, Batavia, NY; **Yoshinobu Murayoshi**, Naha City, Okinawa, Japan; **Pedro Henrique O. Pantoja, student**, UFRN, Natal, Brazil;

José Hernández Santiago, student, UTM, Oaxaca, México; **Maxwell Simpson**, Northeastern University, Boston, MA; and the **Proposer**.

1212. *Proposed by Arthur Holshouser, Charlotte, NC.*

Is it possible to paint the 750 faces of 125 unit cubes using five colors, say red, white, blue, green and yellow, in such a way that for each color c, a $5 \times 5 \times 5$ cube can be built whose entire surface is color c.

Solution by **Dale K. Hathaway**, *Olivet Nazarene University, Bourbonnais, IL*

It is possible to paint the cubes in the manner described below. Let r-red, w-white, b-blue, g-green, and y-yellow. Each of the five colors must be represented as a corner on 8 cubes, an edge on 36 cubes, and a center on 54 cubes. In the following table one coloring is given, others are possible. A color listed once in the table is a center color in the $5 \times 5 \times 5$ representation (one side only of the unit cube has the color), a color listed twice for the same cube represents an edge for that color (two adjacent sides of the unit cube are colored the same color), and a color listed three times for the same cube represents a corner for that color (three sides that meet in a corner).

Number of cubes	Color of sides	Number of cubes	Color of sides
7	rrrwwb	5	rrwbgy
7	wwwbbg	5	wwrbgy
7	bbbggy	5	bbwrgy
7	gggyyr	5	ggwrby
7	yyyrrw	5	yywrbg
12	rrwwgy	1	rrrwbg
12	wwbbry	1	wwwbgy
12	bbggwr	1	bbbgyr
12	ggyywb	1	gggyrw
12	yyrrbg	1	yyyrwb
	125		

It is also possible to color the unit cubes of a $2 \times 2 \times 2$ cube with 2 colors, a $3 \times 3 \times 3$ cube with 3 colors, and a $4 \times 4 \times 4$ cube with 4 colors so that each can be reformed into a cube with each color as the entire surface. A $2 \times 2 \times 2$ coloring is trivial and the following charts show one coloring for the unit cubes for $3 \times 3 \times 3$ and $4 \times 4 \times 4$ with 3 and 4 colors respectively.

$3 \times 3 \times 3$ Number of cubes	Color of sides
6	rrrwwb
6	wwwbbr
6	bbbrrw
1	rrrwww
1	rrrbbb
1	wwwbbb
6	wwbbrr
27	

$4 \times 4 \times 4$ Number of cubes	Color of sides
8	rrrwwb
8	wwwbbg
8	bbbggr
8	gggrrw
8	wwbbrg
8	bbrrgw
8	rrggwb
8	ggwwbr
64	

The faces of a set of n^3 cubes can be painted with n colors in such a way that the cubes can be used to construct a $n \times n \times n$ cube n different ways with the entire exterior surface showing a different one of the n colors each time.

The result seems at least plausible because there are n^3 cubes with a total of $6n^3$ faces. Each color requires $6n^2$ faces so there are $6n^2 \cdot n$ faces needed to color using n colors. The result will be proved using mathematical induction.

The basis step is shown above for $n = 5$

Inductive step: Assuming that n^3 cubes can be painted appropriately with n colors, we need to show that $(n+1)^3$ cubes can be successfully painted with $n+1$ colors. There are $(n+1)^3 - n^3 = 3n^2 + 3n + 1$ new cubes with no color on any face. Label the original n colors as $c_1, c_2, \ldots, c_n$ and the new color as c_{n+1}. Color c_{n+1} needs distinct cubes with 8 corners, $12(n-1)$ edges, and $6(n-1)^2$ centers. While each of the n old colors need an additional 12 edges and $12n - 18$ centers, or a total of $12n$ edges and $12n^2 - 18n$ centers colored with old colors . To achieve this result color the new cubes as follows:

Number	c_{n+1}	Edges: old colors	Centers: old colors
8	8 corners	8	8
$12n-12$	$12n-12$ edges	$12n-12$	$2(12n-12)$
4	4 centers	4	
$3n^2-9n+1$	$3n^2-9n+1$ centers		$5(3n^2-9n+1)$
$3n^2+3n+1$	$3n^2+3n+1$	$12n$	$15n^2-21n+1$

This results in all of the appropriately colored corners and edges for the new color and $3n^2-9n+5$ of the needed center faces with color c_{n+1}. This leaves $3n^2-3n+1$ center faces needed for the new color among the previous n^3 cubes. The additional cubes also provide all $12n$ additional edges for colors $c_1, \ldots, c_n$, and $15n^2-21n+1$ of the needed centers, which is larger than the number needed. But since $3n^2-3n+1$ centers faces among the old cubes will be swapped out for the new color, this can be subtracted from the available centers, $15n^2-21n+1-(15n^2-21n+1) = 12n^2-18n$. Since we used $n=5$ as our basis step we have at least 6 colors so that even a cube that needs to be colored with all centers can be colored with distinct colors.

Also solved by **David Burwell, student**, Alma College, Alma, MI; **Nathan Caudill, student**, Elizabethtown College, Elizabethtown, PA; **Pete Kosek, student**, SUNY, The College at Brockport; **Alan Upton, Student**, Columbus State University, Columbus GA; and the **Proposer**.

1213. *Proposed by Scott D. Kominers, student, Harvard University, Cambridge, MA. and Paul M. Kominers, student, MIT, Cambridge, MA.*

In Problem 1176 of this *Journal*, readers were challenged to determine for which nonzero ρ there exists a real (or complex) sequence $\{a_1, a_2, a_3, \ldots\}$ such that

$$\rho = \frac{a_1 + \cdots + a_n}{a_{n+1} + \cdots + a_{2n}} \tag{1}$$

for all $n > 0$.

Here, we ask a similar question: For which nonzero ρ does there exist an *integer* sequence $\{a_1, a_2, a_3, \ldots\}$ such that (1) holds for all $n > 0$?

Solution by **Northwestern University Math Problem Solving Group**, *Evanston, IL.*

The answer is all rational numbers of the form $\rho = \frac{1}{b}$, $b \in \mathbb{Z} \setminus \{0, -1\}$.

First, we prove that ρ must have that form.

Obviously ρ must be a rational number, so it can be written $\rho = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$, $\gcd(a, b) = 1$. We can assume $a \in \mathbb{Z}^+$.

Let $S_n = a_1 + \cdots + a_n$. Then $\rho = \frac{a}{b} = \frac{S_n}{S_{2n} - S_n}$, which implies $aS_{2n} = (a+b)S_n$. From here we get $S_{2^k} = (a+b)^k a^{-k} a_1$ for $k \geq 0$. Since that expression must be an integer for all $k \geq 0$, a must divide $a+b$, but $\gcd(a, a+b) = 1$, hence $a = 1$.

The case $b = -1$ must be ruled out because $a_1/a_2 = -1$ implies $a_1 = -a_2$, and $(a_1 + a_2)/(a_3 + a_4) = -1$ implies $a_3 + a_4 = -a_1 - a_2 = 0$, which is impossible.

Hence ρ is of the form asserted.

It remains to prove that all ρ of that form have the desired property. In fact, let a_n be defined $a_1 = 1$, $a_{2^k} = b(1+b)^{k-1}$ for $k \geq 1$, and $a_n = 0$ if n is not a power of 2. Then, we have $a_1/a_2 = 1/b = \rho$, and for $n \geq 2$:

$$\frac{a_1 + \cdots + a_n}{a_{n+1} + \cdots + a_{2n}} = \frac{1 + \sum_{i=0}^{k-1} b(1+b)^i}{b(1+b)^k} = \frac{1 + b\frac{(1+b)^k - 1}{b}}{b(1+b)^k} = \frac{1}{b} = \rho\,.$$

where k is the greatest integer such that $2^k \leq n$.

This completes the proof.

Also solved by **Paul S. Bruckman**, Sointula, BC; and the **Proposers**.