Junior problems

J61. Find all pairs (m, n) of positive integers such that

$$m^2 + n^2 = 13(m + n).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Magkos Athanasios, Kozani Greece

Because our equation is symmetric without loss of generality we can assume that $m \geq n$. Rewrite the equation as $n^2 - 13n + m^2 - 13m = 0$. This is a quadratic equation with respect to n. Its discriminant $D = -m^2 + 52m - 169$ must be non-negative and m is a positive integer. Thus $0 < m \leq \frac{13}{2}(1 + \sqrt{2})$, hence $0 < m \leq 15$.

Now, writing the equation in the form $m(m - 13) + n(n - 13) = 0$ we see that it is impossible for m, n to be both greater that 13 or less that 13. Bceause we assumed that $m \geq n$, we get $m \in \{13, 14, 15\}$.

If $m = 13$, we get the solution $(m, n) = (13, 13)$.

If $m = 14$, our equation becomes $n^2 - 13n + 14 = 0$, which has no integral solutions.

If $m = 15$, the equation becomes $n^2 - 13n + 30 = 0$ and its integral solutions are 3 and 10.

We conclude the solutions are $(13, 13), (3, 15), (15, 3), (10, 15), (15, 10)$.

Second solution by Andrea Munaro, Italy

We have $(m-6)^2+(n-6)^2-(m+n) = 72$ and $(m-7)^2+(n-7)^2+(m+n) = 98$. Adding these two equations we get

$$(m - 6)^2 + (m - 7)^2 + (n - 6)^2 + (n - 7)^2 = 170.$$

Clearly $m, n \leq 15$. Because the equation is symmetric in m, n and it is quadratic, we have to check solutions for $7 \leq m \leq 15$. Remembering that a positive integer is a sum of two squares if and only iff each prime factor of the form $4k+3$ appears an even number of times in its factorization, the cases when $m = 14, 11, 8$ give no solutions. For $m = 15$, we get $n = 3$ and $n = 10$. For $m = 13$, we get $n = 13$. For $m = 12$ there are no solutions. For $m = 10$ we get symmetric solution $n = 15$. Finally, for $m = 9$ and for $m = 7$ there are no solutions.

Thus the solutions are: $(15, 3), (15, 10), (10, 15), (3, 15), (13, 13)$.

Third solution by Vishal Lama, Southern Utah University, Utah, USA

We have

$$m^2 + n^2 = 13(m + n) \qquad (1),$$

where $m, n \in \mathbb{N}$.

1^{st} case: $m = n$. Then the only solution we obtain is $m = n = 13$.

Note that if (m, n) is a solution to the given equation, then so is (n, m). Thus it is enough to consider one more case.

2^{nd} case: $m < n$. From the Cauchy-Schwarz inequality we have

$$(m + n)^2 < 2(m^2 + n^2).$$

Therefore, $(m + n)^2 < 2 \cdot 13(m + n)$, which implies $m + n < 26$. Thus, we have $m + n \leq 25$. Because $m < n$ we get $1 \leq m \leq 12$.

Now, consider $n^2 - 13n + m^2 - 13m = 0$ as a quadratic equation in n. Since n is a positive integer, the discriminant $\Delta = 169 - 4(m^2 - 13m)$ must be a perfect odd square. Let $169 - 4(m^2 - 13m) = (2k + 1)^2$ for some $k \in \mathbb{N}$. Simplifying we get

$$k(k + 1) = 42 + m(13 - m).$$

The only possible values of the product $k(k + 1)$ are

$$k(k + 1) = 54, 64, 72, 78, 82, 84.$$

Out of the six values for $k(k + 1)$ above, only $72 = 8 \cdot 9$ is expressible as the product of two consecutive positive integers. This corresponds to $m = 3, 10$.

Plugging $m = 3$ into (1) yields $(n - 15)(n + 2) = 0$. Hence $(3, 15)$ is a solution.

Plugging $m = 10$ into (1) again yields $(n - 15)(n + 2) = 0$. So $(10, 15)$ is another solution. These two pairs are the only possible solutions for $m < n$.

Hence, the only ordered pairs (m, n) satisfying (1) are

$$(3, 15), (10, 15), (13, 13), (15, 3), (15, 10).$$

Also solved by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Spain; Shukurjon Shokirov, Samarqand, Uzbekistan Brian Bradie, Christopher Newport University, USA; Salem Malikic Sarajevo, Bosnia and Herzegovina; Daniel Campos Salas, Costa Rica; Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Son Hong Ta, High School for Gifted Students at Ha Noi University of Education, Vietnam; G.R.A.20 Math Problems Group, Roma, Italy; Courtis G. Chryssostomos, Larissa, Greece

J62. Consider a right-angled triangle ABC with $\angle A = 90^\circ$. Let $E \in AC$ and $F \in AB$ such that $\angle AEF = \angle ABC$ and $\angle AFE = \angle ACB$. Denote by E' and F' the projections of E and F onto BC, respectively. Prove that

$$E'E + EF + FF' \leq BC$$

and determine when equality holds.

Proposed by Alex Anderson, New Trier High School, Winnetka, USA

First solution by Daniel Campos Salas, Costa Rica

Note that $\triangle ABC \sim \triangle AEF \sim \triangle E'EC \sim \triangle F'BF$. Let a, b, c be the lengths of BC, CA, AB, respectively.

From $\triangle ABC \sim \triangle AEF$ we have that $AE = cx$, $AF = bx$, $EF = ax$, for some $x < \min\{\frac{b}{c}, \frac{c}{b}\}$. In addition, from $\triangle ABC \sim \triangle E'EC$ we have

$$E'E = \frac{AB \cdot EC}{BC} = \frac{c(b - cx)}{a},$$

and analogously, $FF' = \dfrac{b(c - bx)}{a}$. Then,

$$\begin{aligned} E'E + EF + FF' &= \frac{c(b-cx)}{a} + ax + \frac{b(c-bx)}{a} = \frac{x(a^2 - b^2 - c^2) + 2bc}{a} \\ &= \frac{2bc}{a} \leq \frac{b^2 + c^2}{a} = BC, \end{aligned}$$

as we wanted to prove. Equality holds if and only if $AB = AC$.

Second solution by Son Hong Ta, High School for Gifted Students at Ha Noi University of Education, Vietnam

Denote by H, K the reflections of F' and E' across AB, AC, respectively. Let G be the orthogonal projection of C on BH. We have $\angle KEC = \angle CEE' = \angle B = \angle AEF$, which implies that E, F and K are collinear. Similarly, H also lies on EF. Hence we have

$$E'E + EF + FF' = KE + EF + FH = KH = CG \leq BC.$$

Equality holds when $CG = CB$, i.e. $G \equiv B$. Thus $BH \perp BC$, yielding $EF \parallel BC$. Hence $\angle B = \angle C$, or triangle ABC is an isosceles right-angled triangle.

Also solved by Miguel Amengual Covas, Mallorca, Spain; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Ricardo Barroso Campos Universidad de Sevilla, Spain; Courtis G. Chryssostomos, Larissa, Greece; Vicente Vicario Garcia, Huelva, Spain; Vishal Lama, Southern Utah University, Utah, USA

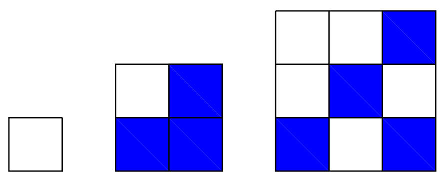
J63. Find the least n such that no matter how we color an $n \times n$ lattice point grid in two colors we can always find a parallelogram with all vertices to be monochromatic.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

First solution by Arnau Messegue Buisan, Barcelona, Spain

Let us change the $n \times n$ lattice point grid for a $n \times n$ cell grid, having a parallelogram all whose vertices are monochromatic when the centers of the colored cells form a parallelogram. Also for simplicity, we will say that the cells are or colored, with the same color, or uncolored.

First, we see that there exist colorings for each $n < 4$ that do not contain any parallelogram:



n=1 n=2 n=3

As a result of this, we have $n \geq 4$. We prove that $n = 4$. For convenience we will call the different values of uncolored cells separating two colored cells in the same row, cell-distances in a row. For instance, in our 4×4 grid the different possible values of cell-distances in a row are 2, 1, or 0. We prove that no matter how you color $\frac{16}{2} = 8$ cells in a 4×4 grid, you always have in two different rows two cell-distances in each one which are the same, proving then that these cells form a parallelogram all whose vertices are monochromatic.

We distinguish three cases:

(i). We have one colored row: then the other four cells are in three different rows, so at least there are two in the same row. Since the colored row has all the possible cell-distances values, it is clear that the cell-distance in the two-colored-cells row will coincide with one of them.

(ii). We have three colored cells in the same row, so the cell-distances in this row can take at least two different values: then there are five colored left cells that can be put in these ways:

- We have four of them in the same row: we are again in the case (i), which has already been proven.

- We have three colored cells in the same row: if we have two rows that have three colored cells, in each row there are at least two possible values of cell-distances. So, they necessary have on cell-distance which is the same.

- We have two colored cells in two different row: then, if the cell-distance of these two rows is the same, the four colored cells form a parallelogram. Otherwise, there is at least one cell-distance value which coincides with one of the cell-distances in the three-colored-cells row.

(iii). We color two cells in each row: Then in each row there is only one cell-distance value. Since there exists only three possible values, at least two of the four rows have the same cell-distance, and we are done.

Second solution by Andrea Munaro, Italy

For $n = 3$, we color the points $(1, 1), (1, 2), (2, 3), (3, 2)$ in the color X and the rest in the color Y. Clearly, there is no monochromatic parallelogram.

For $n = 4$, we make the following observation: if there are two rows with three points of the same color, then there exist a monochromatic parallelogram. Consider three cases:

(i) There is a row a with four points of the same color X. Thus there is no row from the remainings with at least two X colored points. Hence every other row contains at least three points of color Y, and we are done.

(ii) There is a row a with exactly three X colored points. There exist at least two rows b and c from the remainings with at least two points of color X. The distance between points of the same color in each row can be $1, 2$ or 3. In the row a three points generate at least two different distances. Rows b and c generate other two distances. By the Pigeonhole Principle the exist a pair of points that have the same distance and they lie in the different rows. They are parallel and hence form a parallelogram.

(iii) All rows have two X colored points. Recall that the distance between points of the same color in each row can be $1, 2$ or 3. We have four rows, thus by the Pigeonhole Principle there will exist two pairs that are parallel and with the same distance. They form a parallelogram and we are done.

Third solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

The minimum number is $n = 4$. For $n = 3$, color the top row and one of the diagonals of the grid in one color, the other four lattice points in another. This coloring yields no parallelogram with monochromatic vertices, so $n \geq 4$.

Assume that the colors are red and blue in a 4×4 lattice with no parallelogram with monochromatic vertices. The lengths of the segments defined by three points in a given row may take values $(1, 1, 2)$ or $(1, 2, 3)$. So, if two rows contain three red points each, two parallel segments of length 2 are defined by red vertices, and a parallelogram with monochromatic vertices exists. Furthermore, if one of the rows in a 4×4 lattice has its 4 grid points red, then either there is a row that contains two red points, or there exist three rows that have three blue points, forming in each case a parallelogram with monochromatic vertices. Because the distance of any two points in the same row is either 1, 2 or 3, if there are at least two red points in each row, then two rows will contain red points at the same distance, defining a parallelogram with monochromatic vertices. So, since without loss of generality at least 8 of the 16 lattice points are red, then there must be one row with 3 red points, two rows with 2 red points, and one row with 1 red point. The distances between points in the row with 3 red points must be $(1, 1, 2)$, since otherwise a parallelogram with monochromatic vertices would be defined by two of the points in this row, and the two points in one of the rows with 2 red points. Thus, both rows with 2 red points are such that the distance between red points is 3. But then these 4 points define a rectangle with monochromatic vertices, a contradiction. Therefore for any $n \geq 4$ we can always find a 4×4 sublattice of the $n \times n$ lattice that always contains a parallelogram with monochromatic vertices.

Also solved by Jose H. Nieto S., Universidad del Zulia, Venezuela; Vishal Lama, Southern Utah University, Utah, USA; G.R.A.20 Math Problems Group, Roma, Italy

J64. Let a, b, c be positive real numbers. Prove that

$$\frac{b+c}{a+\sqrt[3]{4(b^3+c^3)}}+\frac{c+a}{b+\sqrt[3]{4(a^3+c^3)}}+\frac{a+b}{c+\sqrt[3]{4(a^3+b^3)}} \leq 2.$$

Proposed by José Luis Díaz-Barrero, Barcelona, Spain

First solution by Oleh Faynshteyn, Leipzig, Germany

Observe that $b^3+c^3=(b+c)^3-3bc(b+c) \geq (b+c)^3-\frac{3}{4}(b+c)^2=\frac{1}{4}(b+c)^3$. Using this fact we get

$$a+\sqrt[3]{4(b^3+c^3)} \geq a+b+c.$$

Analogously, we write for $b+\sqrt[3]{4(a^3+c^3)}$ and $c+\sqrt[3]{4(a^3+b^3)}$. We have

$$\frac{b+c}{a+\sqrt[3]{4(b^3+c^3)}}+\frac{c+a}{b+\sqrt[3]{4(a^3+c^3)}}+\frac{a+b}{c+\sqrt[3]{4(a^3+b^3)}} \leq$$

$$\leq \frac{b+c}{a+b+c}+\frac{c+a}{a+b+c}+\frac{a+b}{a+b+c}=2,$$

and we are done.

Second solution by Magkos Athanasios, Kozani Greece

For positive reals x, y we have $4(x^3+y^3) \geq (x+y)^3$, since

$$4(x^3+y^3)-(x+y)^3=3(x+y)(x-y)^2 \geq 0.$$

Therefore, the LHS of the inequality is less than or equal to

$$\frac{b+c}{a+b+c}+\frac{c+a}{a+b+c}+\frac{a+b}{a+b+c}=2.$$

Remark: For positive reals $x_j, j=1,2,...,n$, we have

$$n^2(x_1^3+x_2^3+...+x_n^3) \geq (x_1+x_2+...+x_n)^3.$$

This follows, for instance, from Power Mean Inequality or the Chebychev Inequality. Thus we get the following generalization.

$$\sum \frac{x_2+x_3+...+x_n}{x_1+\sqrt[3]{(n-1)^2(x_2^3+x_3^3+...+x_n^3)}} \leq n-1.$$

Also solved by Nguyen Manh Dung, Hanoi University of Science, Vietnam; Perfetti Paolo, Dipartimento di matematica Universita degli studi di Tor Vergata, Italy; Daniel Lasaosa, Universidad Publica de Navarra, Spain; O.O.Ibrogimov Samarqand State University, Uzbekistan; Arkady Alt, San Jose, California, USA; Salem Malikic Sarajevo, Bosnia and Herzegovina; Samin Riasat, Notre Dame College, Bangladesh; Arnau Messegue Buisan, Barcelona, Spain; Son Hong Ta, High School for Gifted Students at Ha Noi University of Education, Viet Nam, Andrea Munaro, Italy

J65. Prove that the interval $(2^n+1, 2^{n+1}-1), n \geq 2$ contains an integer that can be represented as a sum of n prime numbers.

Proposed by Radu Sorici, University of Texas at Dallas, USA

First solution by Arnau Messegue Buisan, Barcelona, Spain

Recall Bertrand's Postulate, which was proved by P. Chebyshev, that states that for each positive integer $n > 1$ there is a prime number p such that $n < p < 2n$. Using this postulate there exist prime numbers $p_1, p_2, \ldots, p_n$, for $n \geq 2$ such that $p_1 = 3$ and

$$\begin{aligned} 2 < & \; p_2 < 2^2 \\ 2^2 < & \; p_3 < 2^3 \\ & \; \ldots \\ 2^{n-1} < & \; p_n < 2^n. \end{aligned}$$

Adding these up we get the desired result. Namely,

$$2+(1+2+2^2+\ldots+2^{n-1}) = 2^n+1 < p_1+p_2+\ldots+p_n < 1+2+2^2+\ldots+2^n = 2^{n+1}-1.$$

Second solution by G.R.A.20 Math Problems Group, Roma, Italy

For $n = 1$ the interval $(2^n+1, 2^{n+1}-1)$ is empty. For $n = 2, 3$ we have that $3+3 = 6 \in (5, 7)$ and $2+3+5 = 10 \in (9, 15)$.

Now we prove that there is a prime p such that $np \in (2^n+1, 2^{n+1}-1)$ for any $n > 3$. This means that

$$\frac{2^n+1}{n} < p < \frac{2^{n+1}-1}{n}.$$

By Bertrand's postulate, for any integer $a > 1$ there is a prime p such that $a < p < 2a$ that is $a+1 \leq p \leq 2a-1$. Therefore it suffices to prove that there is an integer $a > 1$ such that

$$\frac{2^n+1}{n} < a+1 \quad \text{and} \quad 2a-1 < \frac{2^{n+1}-1}{n}.$$

Hence

$$\frac{2^n+1}{n} - 1 < a < \frac{2^{n+1}-1}{2n} + \frac{1}{2}$$

and the existence of a is proved as soon as the length of this open interval is greater than 1

$$1 < \left(\frac{2^{n+1}-1}{2n} + \frac{1}{2}\right) - \left(\frac{2^n+1}{n} - 1\right) = \frac{3}{2}\left(1 - \frac{1}{n}\right)$$

that is true for $n > 3$.

Also solved by Vishal Lama, Southern Utah University, Utah, USA; Vicente Vicario Garcia, Huelva, Spain; Salem Malikic Sarajevo, Bosnia and Herzegovina.

J66. Let $a_0 = a_1 = 1$ and $a_{n+1} = 2a_n - a_{n-1} + 2$ for $n \geq 1$. Prove that $a_{n^2+1} = a_{n+1}a_n$ for all $n \geq 0$.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

First solution by O.O.Ibrogimov, Samarqand State University, Uzbekistan

Because $a_{n+1} + a_{n-1} = 2a_n + 2$, we have

$$a_2 + a_0 = 2a_1 + 2$$
$$a_3 + a_1 = 2a_2 + 2$$
$$\dots$$
$$a_{m-1} + a_{m+1} = 2a_m + 2$$

Summing up we get

$$a_0 + a_1 + 2(a_2 + \ldots + a_{m-1}) + a_m + a_{m+1} = 2(a_1 + a_2 + \ldots + a_m) + 2m,$$

yielding

$$a_{m+1} = a_m + m.$$

From here it is not difficult to find that $a_m = m^2 - m + 1$. Then

$$a_{n^2+1} = (n^2+1)^2 - (n^2+1) + 1 = ((n+1)^2 - (n+1) + 1)(n^2 - n + 1) = a_{n+1}a_n.$$

Second solution by Arkady Alt, San Jose, California, USA

Observe that $a_{n+1} - a_n - 2n = a_n - a_{n-1} - 2(n-1)$, for $n \geq 1$. Therefore, $a_{n+1} - a_n - 2n = c$, where c is some constant. Because $a_{n+1} - a_n - 2n = c$ we can conclude $a_n = (n-1)\,n + cn + b$, for $n \geq 0$. Initial conditions $a_0 = a_1 = 1$ give us $c = 0$ and $b = 1$, i.e. $a_n = n^2 - n + 1$, for $n \geq 0$. Hence

$$a_{n^2+1} = \left(n^2+1\right)^2 - \left(n^2+1\right) + 1 = \left(n^2 - n + 1\right)\left(n^2 + n + 1\right) = a_n a_{n+1}.$$

Third solution by Brian Bradie, Christopher Newport University, USA

The characteristic equation associated with the difference equation $a_{n+1} = 2a_n - a_{n-1}$ has a double root of 1; therefore, the complementary solution associated with the difference equation $a_{n+1} = 2a_n - a_{n-1} + 2$ is

$$c_1 + c_2 n$$

for some constants c_1 and c_2. A particular solution for the nonhomogeneous difference equation takes the form λn^2 for some constant λ. Substituting this form into the nonhomogeneous difference equation yields

$$\lambda(n+1)^2 = 2\lambda n^2 - \lambda(n-1)^2 + 2 \quad \text{or} \quad \lambda = 1.$$

The general solution of $a_{n+1} = 2a_n - a_{n-1} + 2$ is then

$$a_n = c_1 + c_2 n + n^2.$$

To satisfy the initial conditions $a_0 = a_1 = 1$, we find $c_1 = 1$ and $c_2 = -1$. Thus,

$$a_n = 1 - n + n^2.$$

Now,

$$\begin{aligned} a_{n^2+1} &= 1 - (n^2+1) + (n^2+1)^2 \\ &= n^4 + n^2 + 1 \\ &= (n^2 - n + 1)(n^2 + n + 1) \\ &= (n^2 - n + 1)\left[(n+1)^2 - (n+1) + 1\right] \\ &= a_n a_{n+1}. \end{aligned}$$

Also solved by Salem Malikic Sarajevo, Bosnia and Herzegovina; Arnau Messegue Buisan, Barcelona, Spain; Andrea Munaro, Italy; Daniel Campos Salas, Costa Rica; Jose Hernandez Santiago, UTM, Oaxaca, Mexico; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Jose H. Nieto S., Universidad del Zulia, Venezuela; Son Hong Ta, Ha Noi University, Vietnam; Vicente Vicario Garcia, Huelva, Spain; Vishal Lama, Southern Utah University, Utah, USA

Senior problems

S61. Let ABC be a triangle. Prove that

$$\frac{1}{\sin\frac{A}{2}}+\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\geq 4\sqrt{\frac{R}{r}},$$

where R and r are its circumradius and inradius, respectively.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

* There was a mistake in the initially published inequality. A lot of readers pointed out this mistake. Originally proposed inequality was

$$\frac{1}{\sin\frac{A}{2}}+\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\geq 6\sqrt[3]{\frac{R}{2r}}.$$

But due to the square root that we had, we changed it to the following

$$\frac{1}{\sin\frac{A}{2}}+\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\geq 4\sqrt{\frac{R}{r}}.$$

Solution by Daniel Campos Salas, Costa Rica

Recall the identity $\frac{r}{R}=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$. Our inequality is equivalent to

$$\sqrt{\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}}+\sqrt{\frac{\sin\frac{A}{2}\sin\frac{C}{2}}{\sin\frac{B}{2}}}+\sqrt{\frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\sin\frac{C}{2}}}\geq 2.$$

Let $(x,y,z)=(s-a,s-b,s-c)$, where s is the semiperimeter and a,b,c are the sidelengths of triangle ABC. Using the fact that $\sin\frac{A}{2}=\sqrt{\frac{yz}{(x+y)(x+z)}}$, our inequality transforms to

$$\sqrt{\frac{x}{y+z}}+\sqrt{\frac{y}{x+z}}+\sqrt{\frac{z}{x+y}}\geq 2.$$

Note that $x+y+z\geq 2\sqrt{x(y+z)}$. Thus

$$\sqrt{\frac{x}{y+z}}+\sqrt{\frac{y}{x+z}}+\sqrt{\frac{z}{x+y}}\geq\frac{2x}{x+y+z}+\frac{2y}{x+y+z}+\frac{2z}{x+y+z}=2,$$

and we are done.

Also solved by Salem Malikic Sarajevo, Bosnia and Herzegovina; Son Hong Ta, Ha Noi University, Vietnam; Oleh Faynshteyn, Leipzig, Germany; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Vicente Vicario Garcia, Huelva, Spain

S62. Let $ABCD$ be a parallelogram and let $X \in AB, Y \in BC, Z \in CD$, and $K \in AD$ such that $XZ \parallel BC \parallel AD$ and $YK \parallel AB \parallel CD$. Let $P = XZ \cap YK$ and $Q = BZ \cap DY$. Prove that A, P, Q are collinear.

Proposed by Proposed by Juan Bosco Marquez and Francisco Javier Garcia Capitan, Spain

First solution by Andrea Munaro, Italy

Applying Menelaus's Theorem to the triangle CZB with transversal DQY we get

$$\frac{CY}{YB} \cdot \frac{BQ}{QZ} \cdot \frac{ZD}{DC} = 1.$$

Because $CYPZ$ and $YPXB$ are parallelograms we have $\frac{CY}{YB} = \frac{ZP}{PX}$. Similarly for $AXZD$ and $XBCZ$ we get $\frac{ZD}{DC} = \frac{XA}{AB}$. It follows that

$$\frac{ZP}{PX} \cdot \frac{XA}{AB} \cdot \frac{BQ}{QZ} = 1,$$

and by the converse of Menelaus's Theorem applied to the triangle BZX we conclude that A, P, Q are collinear.

Second solution by Son Hong Ta, Ha Noi University, Vietnam

Denote by M the intersection of AP with BC and N the intersection of XZ with DY. By the Thales theorem, we have

$$\frac{MY}{YB} = \frac{MY}{AK} = \frac{PY}{PK} = \frac{PY}{ZD} = \frac{PN}{NZ}.$$

On the other hand, $MB \parallel ZP$, hence we conclude that P, Q, and M are collinear. Therefore A, P and Q are collinear as desired.

Third solution by Arnau Messegue Buisan, Barcelona, Spain

Let $R = DX \cap BZ$, $S = AR \cap DB$ and $S' = CQ \cap DB$. Applying Ceva's theorem to the triangles ADB and BCD we get

$$\frac{DS \cdot BS'}{SB \cdot S'D} = \frac{AX \cdot DK \cdot CZ \cdot BY}{XB \cdot KA \cdot ZD \cdot YC} = 1$$

Thus $S' = S$. Now, consider $M = KX \cap AR$, $N = ZY \cap CQ$, $T = KX \cap DB$ and $T' = ZY \cap DB$. We will use the following Lemma which we won't prove, because it has appeared in the article of Cosmin Pohoata, published in Mathematical Reflections in the 2007 volume.

*Lemma. I*n a triangle ABC consider three points X Y, Z on the side BC, CA, AB, respectively. If X' is the point of intersection of YZ with the extended side BC, the the four-point $(BXCX')$ forms and harmonic division if and only if the cevians AX, BY and CZ are concurrent.

Applying the lemma to triangles ADB with cevians DX, BK and AS and DBC with the cevians DY, ZB and CS we get that the four point $(BSDT)$ and $(BSDT')$ are harmonic. Thus, we have that $T = T'$, that is, lines XK, BD and YZ are concurrent.

Finally, we recall Desargue's theorem. It states that if two triangles are in perspective with respect to a center they are in perspective with respect to an axis. Since triangles XBY and KDZ are in perspective with respect to T, they are in perspective with respect to an axis. This means that points $A = XB \cap KD$, $Q = BZ \cap DY$ and $P = ZX \cap YK$ are collinear which is what we wanted to prove.

Also solved by Courtis G. Chryssostomos, Larissa, Greece; Ricardo Barroso Campos Universidad de Sevilla, Spain; Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Publica de Navarra, Spain; O.O.Ibrogimov student of Samarqand State University, Uzbekistan; Salem Malikic Sarajevo, Bosnia and Herzegovina; Andrei Iliasenco, Chisinau, Moldova; Vicente Vicario Garcia, Huelva, Spain; Vishal Lama, Southern Utah University, Utah, USA

S63. Let a, b, c be positive real numbers such that $ab + bc + ca \geq 3$. Prove that

$$\frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} \geq \frac{3}{\sqrt{2}}.$$

Proposed by Pham Huu Duc, Ballajura, Australia

First solution by Son Hong Ta, Ha Noi University, Vietnam

By the Holders Inequality, we get

$$\left(\sum \frac{a}{\sqrt{a+b}}\right)\left(\sum \frac{a}{\sqrt{a+b}}\right)\left(\sum a(a+b)\right) \geq (a+b+c)^3.$$

Therefore, it remains to prove that

$$2(a+b+c)^3 \geq 9(a^2+b^2+c^2+ab+bc+ca.$$

Let $x = a+b+c$, $y = ab+bc+ca$, we have $y \geq 3$ and the inequality above transforms to

$$2x^3 \geq 9(x^2 - y).$$

It is enough to prove $2x^3 + 27 \geq 9x^2$, that immediately follows from the AM-GM inequality.

Second solution by Perfetti Paolo, Dipartimento di matematica Universita degli studi di Tor Vergata, Italy

By the Cauchy-Schwarz inequality we have

$$\Big(\sum_{\text{cyc}} a\sqrt{a+b}\Big)\Big(\sum_{\text{cyc}} \frac{a}{\sqrt{a+b}}\Big) \geq (a+b+c)^2$$

Again using the Cauchy-Schwarz inequality we get

$$\sqrt{a+b+c}\sqrt{a(a+b)+b(b+c)+c(c+a)} \geq a\sqrt{a+b} + b\sqrt{b+c} + c\sqrt{a+c}.$$

Then it is enough to prove that

$$\frac{(a+b+c)^{3/2}}{\sqrt{a^2+b^2+c^2+ab+bc+ca}} \geq \frac{3}{\sqrt{2}},$$

or

$$2(a+b+c)^3 \geq 9(a^2+b^2+c^2+ab+bc+ca).$$

After elementary algebra we have

$$2(a+b+c)^3 - 9(a+b+c)^2 + 9(ab+bc+ca) \geq 0.$$

Using the fact $ab+bc+ca \geq 3$, it is enough to prove that

$$2(a+b+c)^3 - 9(a+b+c)^2 + 27 \geq 0.$$

Denote $x = a+b+c$, then $2x^3 - 9x^2 + 27 = (x-3)^2(x+\frac{3}{2}) \geq 0$, and we are done.

Third solution by Daniel Campos Salas, Costa Rica

Let $f(x) = \frac{1}{\sqrt{x}}$ for all positive reals x. Note that $f''(x) = \frac{3}{4\sqrt{x^5}} > 0$, then f is convex. From Jensen's weighted inequality we have

$$\sum_{cyc} af(a+b) \geq (a+b+c)f\left(\frac{\displaystyle\sum_{cyc} a(a+b)}{a+b+c}\right).$$

This implies that

$$\sum_{cyc} \frac{a}{\sqrt{a+b}} \geq \sqrt{\frac{(a+b+c)^3}{a^2+b^2+c^2+ab+bc+ca}}.$$

Then, it is enough to prove that

$$2(a+b+c)^3 \geq 9(a^2+b^2+c^2+ab+bc+ca),$$

or equivalently,

$$4(a+b+c)^6 \geq 81(a^2+b^2+c^2+ab+bc+ca)^2.$$

We will prove that

$$4(a+b+c)^6 \geq 27(ab+bc+ca)(a^2+b^2+c^2+ab+bc+ca)^2,$$

which combined with the hypothesis gives the desired result. Let
$x = a^2+b^2+c^2$ and $y = ab+bc+ca$. Note that

$$\begin{aligned} & 4(a+b+c)^6 - 27(ab+bc+ca)(a^2+b^2+c^2+ab+bc+ca)^2 \\ = & \ 4(x+2y)^3 - 27y(x+y)^2 \\ = & \ (x-y)^2(4x+5y), \end{aligned}$$

which is clearly nonnegative and this completes the proof.

Fourth solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Denote by $x = \sqrt{b+c}$, $y = \sqrt{c+a}$, $z = \sqrt{a+b}$. We have to prove

$$\frac{y^2+z^2-x^2}{2z} + \frac{z^2+x^2-y^2}{2x} + \frac{x^2+y^2-z^2}{2y} \geq \frac{3}{\sqrt{2}}.$$

Assume that $x+y+z$ is known. Then, using Lagrange's multipliers method in order to find the minimum of the LHS, we find

$$\begin{aligned} xyz(\lambda - 1) &= \frac{y^3 z - z^3 y + x^3(z-y)}{x} \\ &= \frac{z^3 x - x^3 z + y^3(x-z)}{y} \\ &= \frac{x^3 y - y^3 x + z^3(y-x)}{z}, \end{aligned}$$

or $xyz(x+y+z)(\lambda-1) = 0$. Since x, y, z are positive, $\lambda = 1$, and

$$\begin{aligned} (y-z)\left(y^2 z + z^2 y - x^3\right) &= (z-x)\left(z^2 x + x^2 z - y^3\right) \\ &= (x-y)\left(x^2 y + y^2 x - z^3\right) = 0. \end{aligned}$$

Assume without loss of generality that $x \leq y, z$. Then, $y = z$, and the LHS is $y + \frac{x}{2} = \frac{x+y+z}{2}$. Therefore it is sufficient to prove that $x+y+z \geq 3\sqrt{2}$.

Let us consider triangle ΔXYZ such that the lengths of the sides opposite angles X, Y, Z are respectively x, y, z. Then,

$$0 < \cos X = \frac{y^2+z^2-x^2}{2yz} = \frac{a}{y\sqrt{a+b}} = \frac{a}{\sqrt{c+a}\sqrt{a+b}} < 1,$$

and similarly its cyclic permutations. Our triangle is well defined and has acute angles. The condition $ab+bc+ca \geq 3$ can be written as

$$\begin{aligned} 3 &\leq xyz\left(z\cos X\cos Y + x\cos Y\cos Z + y\cos Z\cos X\right) \\ &= 2Rxyz\sin X\sin Y\sin Z = \frac{x^2y^2z^2}{4R^2} = 4S^2, \end{aligned}$$

where R and S are the circumradius and area of ΔXYZ. But it is well known that, of all triangles that have a certain area, the perimeter is minimum when the triangle is equilateral. Since $S \geq \frac{\sqrt{3}}{2}$, the perimeter $x+y+z$ of ΔXYZ is larger than or equal to the perimeter of an equilateral triangle with sides $\sqrt{2}$, and we are done.

Also solved by Salem Malikic Sarajevo, Bosnia and Herzegovina; Oleh Faynshteyn, Leipzig, Germany; Son Hong Ta, Ha Noi University, Vietnam; Vishal Lama, Southern Utah University, Utah, USA

S64. Let ABC be a triangle with centroid G and let g be a line through G. Line g intersects BC at a point X. The parallels to lines BG and CG through A intersect line g at points X_b and X_c, respectively. Prove that

$$\frac{1}{\overrightarrow{GX}}+\frac{1}{\overrightarrow{GX_b}}+\frac{1}{\overrightarrow{GX_c}}=0.$$

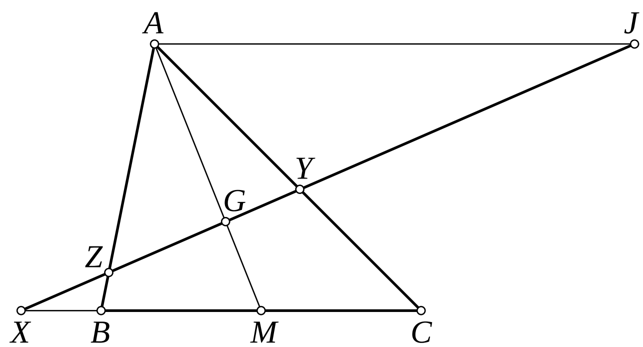
Proposed by Darij Grinberg, Germany

First solution by Francisco Javier Garcia Capitan, Spain

Let us prove the following lemma:

Lemma. If a line g through G intersects BC, CA and AB at X, Y and Z, respectively, then

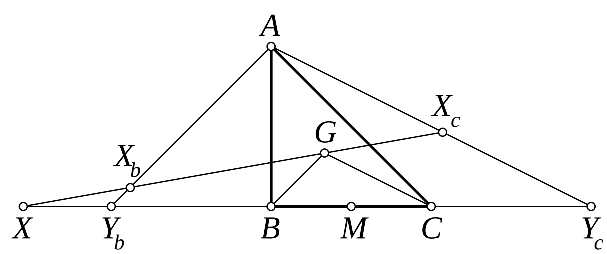
$$\frac{1}{\overrightarrow{GX}}+\frac{1}{\overrightarrow{GY}}+\frac{1}{\overrightarrow{GZ}}=0. \qquad (1)$$



Proof. We have $(BCM\infty)=-1$, where M is the midpoint of BC and ∞ denotes the infinite point of BC. After a projection of the line BC onto g with respect to the point A we get $(ZYGJ)=-1$, where J is the intersection of g with the parallel to BC through A, i.e. J is the harmonic conjugate of G with respect to the pair B, C. Thus we have

$$\frac{2}{\overrightarrow{GJ}}=\frac{1}{\overrightarrow{GY}}+\frac{1}{\overrightarrow{GZ}}. \qquad (2)$$

Now, the parallelism of AJ and BC and the relation $AG:GM=2:1$ gives $\overrightarrow{GJ}=-2\overrightarrow{GX}$, and this with (2) proves (1).



Let the parallel lines through A to the BG and CG intersect BC at points Y_b and Y_c, respectively. It is not difficult to see that triangle AY_bY_c is obtained by a homotethy with center M and ratio 3. In other words Y_b and Y_c are reflections of C and B with respect to B, C. We can write $\overrightarrow{Y}_b = 2\overrightarrow{B} - \overrightarrow{C}$ and $\overrightarrow{Y}_c = 2\overrightarrow{C} - \overrightarrow{B}$, and therefore we have $\overrightarrow{A} + \overrightarrow{Y}_b + \overrightarrow{Y}_c = \overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C} = 3\overrightarrow{G}$, hence G is also the centroid of triangle AY_bY_c. Applying our lemma to this triangle we get the desired result.

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

In homogeneous barycentric coordinates, we may express any point X on the line BC (whose equation is $w_B + w_C = 1$), as $(w_A, w_B, w_C) \equiv (0, \rho, 1-\rho)$.

If $w_B = w_C = \frac{1}{2}$, then X is the midpoint of BC, and g passes through A, or $X_b = X_c = A$, and $\overrightarrow{GX_b} = \overrightarrow{GX_b} = -2\overrightarrow{GX}$, clearly satisfying the given relation.

If $w_B = 0$ or $w_B = 1$, then $X = C$ and $X = B$, respectively, resulting in g being coincident with GC and GB, respectively, or $\overrightarrow{GX_b} = -\overrightarrow{GX}$ and $\overrightarrow{GX_c} = -\overrightarrow{GX}$, with X_c and X_b being at infinity, respectively, which again satisfy the given relation. In any other case, $\rho \neq 0, \frac{1}{2}, 1$.

Line g passes through $G \equiv \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and $X \equiv (0, \rho, 1-\rho)$. The equation of g is $(1-2\rho)\, w_A + (\rho - 1)\, w_B + \rho w_C = 0$. The respective parallels to GB and GC through A have respective equations $w_B + 2w_C = 0$ and $2w_B + 2w_C = 0$, or united to the normalization condition $w_A + w_B + w_C = 1$, we may find

$$X_b \equiv \left(\frac{2-\rho}{3-3\rho}, \frac{2-\rho}{2-4\rho}, \frac{2\rho-1}{3-3\rho}\right),$$

$$X_c \equiv \left(\frac{1+\rho}{3\rho}, \frac{1-2\rho}{3\rho}, \frac{4\rho-2}{3\rho}\right).$$

Now, since X, X_b, X_c are collinear with G, then denoting by $[XYZ]$ the area of triangle XYZ,

$$\frac{\overrightarrow{XX_b}}{\overrightarrow{XG}} = \frac{[BX_bC]}{[BGC]} = \frac{\frac{2-\rho}{3-3\rho}}{\frac{1}{3}} = \frac{2-\rho}{1-\rho},$$

$$\overrightarrow{GX_b} = \left(1 - \frac{2-\rho}{1-\rho}\right)\overrightarrow{GX} = \frac{\overrightarrow{GX}}{\rho - 1}.$$

Similarly,

$$\frac{\overrightarrow{XX_c}}{\overrightarrow{XG}} = \frac{[BX_cC]}{[BGC]} = \frac{\frac{1+\rho}{3\rho}}{\frac{1}{3}} = \frac{1+\rho}{\rho},$$

$$\overrightarrow{GX_c} = \left(1 - \frac{1+\rho}{\rho}\right)\overrightarrow{GX} = -\frac{\overrightarrow{GX}}{\rho}.$$

Finally,

$$\frac{1}{\overrightarrow{GX}} + \frac{1}{\overrightarrow{GX_b}} + \frac{1}{\overrightarrow{GX_c}} = \frac{1 + (\rho - 1) - \rho}{\overrightarrow{GX}} = 0,$$

and we are done.

Third solution by Son Hong Ta, Ha Noi University, Vietnam

Denote by M, N the intersections of the lines AX_b, AX_c with the sideline BC, respectively. We can easily prove that $\overrightarrow{MB} = \overrightarrow{BC} = \overrightarrow{CN}$. Now, we have

$$\begin{aligned}
\overrightarrow{GX}\left(\frac{1}{\overrightarrow{GX}} + \frac{1}{\overrightarrow{GX_b}} + \frac{1}{\overrightarrow{GX_c}}\right) &= 1 + \frac{\overrightarrow{GX}}{\overrightarrow{GX_b} + \overrightarrow{GX}}\overrightarrow{GX_c} \\
&= 1 + \frac{\overrightarrow{BX}}{\overrightarrow{BM}} + \frac{\overrightarrow{CX}}{\overrightarrow{CN}} \\
&= 1 + \frac{\overrightarrow{BX}}{\overrightarrow{CB}} - \frac{\overrightarrow{CX}}{\overrightarrow{CB}} \\
&= 1 + \frac{\overrightarrow{BC}}{\overrightarrow{CB}} \\
&= 0.
\end{aligned}$$

Hence $\frac{1}{\overrightarrow{GX}} + \frac{1}{\overrightarrow{GX_b}} + \frac{1}{\overrightarrow{GX_c}} = 0$, and we are done.

Also solved by Andrei Iliasenco, Chisinau, Moldova

S65. Let n be an integer greater than 1 and let X be a set with $n+1$ elements. Let $A_1, A_2, \ldots, A_{2n+1}$ be subsets of X such that the union of any n has at least n elements. Prove that among these $2n+1$ subsets there exist three such that any two of them have a common element.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

First solution by Jose H. Nieto S., Universidad del Zulia, Venezuela

We will prove that among the $2n+1$ subsets there are three whose intersection is non-empty. Let us call *classes* the given subsets A_i. If some $x \in X$ belongs to three classes,we are done. Otherwise, each $x \in X$ belongs at most to two classes, hence the number p of ordered pairs (x, A_i) with $x \in A_i$ is at most $2(n+1)$. But clearly $p = \sum_{i=1}^{2n+1} |A_i|$, where $|A_i|$ is the number of elements in A_i. Therefore there are at most $n+1$ classes with 2 or more elements, and at least n classes with 0 or 1 elements. If some class A_i is empty, the union of A_i and $n-1$ other classes with 0 or 1 elements would contain less than n elements, a contradiction. Hence no class is empty. If k is the number of classes with 2 or more elements, then the other $2n+1-k$ classes contain exactly one element each. Hence $2k + (2n+1-k) \leq p \leq 2(n+1)$, i.e., $k \leq 1$. This means that at least $2n$ classes contain exactly one element. But there are only $n+1$ elements, and $n+1 < 2n$ (since $n > 1$), hence there are two identical classes $A_i = A_j$ with one element. If we take other $n-2$ one-element classes, we have n classes whose union contain at most $n-1$ elements, a contradiction.

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Let us assume that $2n+1$ sets may be found such that no three of them have pairwise non-empty intersection, and let us work our way to a contradiction. Each element of X appears at most in two sets A_i. Thus the sum of the cardinals of the A_i cannot be larger than $2n+2$.

Let order the sets $A_1, A_2, \ldots, A_{2n+1}$ such that set A_{i+1} has at least as many elements as set A_i for $i = 1, 2, \ldots, 2n$. It is clear that the sum of the cardinals of $A_1, A_2, \ldots, A_n$ is at least n, since their union contains at least n elements. Similarly, the sum of the cardinals of $A_{n+1}, A_{n+2}, \ldots, A_{2n}$ is at least n. Let us consider two different cases:

a) Assume that A_{2n+1} contains one element. Then no A_i contains more than one element, or each A_i must contain exactly one element for the sums of cardinals considered above to be at least n. But all elements should be different, a contradiction.

b) Assume that A_{2n+1} contains at least two elements. Then the sum of the cardinals of $A_1, A_2, \ldots, A_{2n}$ is at most $(2n+2)-2 = 2n$, and it is at least $2n$, or it is exactly $2n$, and the sum of the cardinals of $A_1, A_2, \ldots, A_n$ and the sum of the cardinals of $A_{n+1}, A_{n+2}, \ldots, A_{2n}$ are both exactly n. Since A_{n+j} has at least as many elements as A_{n+1-j} ($j = 1, 2, \ldots, n$), and the sum of the cardinals of the first set is equal to the sum of the cardinals of the second set, then A_1 and A_{2n}, and hence $A_2, A_3, \ldots, A_{2n-1}$, contain all exactly the same number of elements, which can only be 1.

Therefore each A_i contains exactly one element, except for A_{2n+1}, which may contain either one or two. Furthermore, no two A_i ($i = 1, 2, \ldots, 2n$) may contain the same element, since otherwise, taking these two sets, and any other $n-2$ different sets out of $A_1, A_2, \ldots, A_{2n}$, their union would contain at most $n-1$ different elements, a contradiction. But then there would be at least $2n$ different elements in X, a contradiction. The conclusion follows.

S66. Consider a triangle ABC and let D and E be the reflections of vertices B and C into AC and AB, respectively. Let $F = BE \cap CD$ and let H_a be the projection of the altitude from A onto BC. Denote by F_a, F_b, F_c the projections of F onto BC, CA, AB, respectively. Prove that F_a, F_b, F_c, H_a are concyclic.

Proposed by Mihai Miculita, Oradea, Romania

First solution by Ricardo Barroso Campos Universidad de Sevilla, Spain

Note that F_a lies on the circle with diamter FB. Thus $\angle F_cF_aF = \angle F_cBF = \angle B = 180 - \angle F_aBF_c = \angle F_cFF_a$, hence triangle F_cF_aF is isosceles. Because F_a lies on the circle with diamter FC, we have that triangle F_bF_aF is also isosceles, so line F_cF_b is perpendicular to FF_a and is parallel to BC.

Now, let J be the second intersection point of the circumcircle of triangle $F_cF_aF_b$ with the line BC. Clearly, $F_aJF_bF_c$ is a regular trapezium. We have

$$\angle JF_bF_c = \angle F_aF_cF_b = \frac{1}{2}\angle F_aF_cF = \frac{1}{2}\angle F_aBF = \frac{1}{2}(180^\circ - 2\angle B) = 90^\circ - \angle B.$$

Hence F_bJ is perpendicular to AF_c. Similarly, F_cJ is perpendicular to AF_b and J is the orthocenter of the triangle AF_bF_c. Thus AJ is perpendicular to BC and $H_a = J$ and therefore points F_a, F_b, F_c, H_a are concyclic.

Second solution by Andrei Iliasenco, Chisinau, Moldova

First we prove that $BC \parallel F_bF_c$, or $CF_b/AC = BF_c/AB$. Using definitions of the points F, F_a, F_b we can easily express the angles of $\triangle CFF_b, \triangle BCF$ through the angles of $\triangle ABC$. Using the Law of Sines all these triangles we get

$$\frac{CF_b}{AC} = \frac{CF\cos C}{AC} = \frac{BC\dfrac{\sin(180-2B)}{\sin(180-2A)}\cos C}{AC} = \frac{BC}{AC}\frac{\sin 2B}{\sin 2A}\cos C =$$

$$= \frac{2\sin A\sin B\cos B}{2\sin B\sin A\cos A}\cos C = \frac{\cos B\cos C}{\cos A} \qquad (1)$$

Analogously to (1) we have

$$\frac{BF_c}{AB} = \frac{\cos B\cos C}{\cos A} = \frac{CF_b}{AC} \qquad (2)$$

Hence $BC \parallel F_bF_c$. Let H be the projection of the altitude from A onto F_bF_c. Let us prove, that $\triangle ACH_a$ is similar to $\triangle F_cH_aH$, or $AH_a/CH_a = F_cH/HH_a$. By the Law of Sines for $\triangle ACH_A$ we have

$$\frac{AH_a}{CH_A} = \frac{AC\sin C}{AC\cos C} = \frac{\sin C}{\cos C} \qquad (3)$$

Using $BC \parallel F_cF_b$, similarity of $\triangle ABH_a$ and $\triangle AF_cH$ and (2), we get

$$\frac{H_aH}{AH_a} = \frac{BF_c}{AB} = \frac{\cos B \cos C}{\cos A} \tag{4}$$

$$\frac{F_cH}{BH_a} = \frac{AF_c}{AB} = \frac{BF_c}{AB} + 1 = \frac{\cos B \cos C}{\cos A} + 1 = \frac{\cos B \cos C + \cos A}{\cos A} = \frac{\sin B \sin C}{\cos A} \tag{5}$$

Using (3), (4) and (5) we have

$$\frac{F_cH}{HH_a} = \frac{BH_a \dfrac{\sin B \sin C}{\cos A}}{AH_a \dfrac{\cos B \cos C}{\cos A}} = \frac{AB \cos B \sin B \sin C}{AB \sin B \cos B \cos C} = \frac{\sin C}{\cos C} = \frac{AH_a}{CH_A}.$$

Hence $\triangle ACH_a$ is similar to $\triangle F_cH_aH$. Analogously, $\triangle ABH_a$ is similar to $\triangle F_bH_aH$.

Thus $\angle F_bH_aF_c = \angle F_bH_AH + \angle F_cH_aH = \angle ACH_a + \angle ABH_a = \angle B + \angle C$. But the quadrilaterals FF_aCF_b and FF_aBF_c are cyclic, because $\angle FF_aC = \angle FF_bC = \pi/2$ and $\angle FF_aB = \angle FF_cB = \pi/2$. So $\angle F_cF_aF_b = \angle F_cF_aF + \angle FF_aF_b = \angle F_cBF + \angle FCF_b = \angle B + \angle C = \angle F_bH_aF_c$. It follows that quadrilateral $F_aF_bF_cH_a$ is cyclic.

Also solved by Salem Malikic Sarajevo, Bosnia and Herzegovina; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Son Hong Ta, High School for Gifted Students at Ha Noi University of Education, Viet Nam; Andrea Munaro, Italy

Undergraduate problems

U61. Find the sum of the series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{i!j!}{(i+j+1)!}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Brian Bradie, Christopher Newport University, USA

For each i,

$$\begin{aligned}\frac{j!}{(i+j+1)!} &= \frac{1}{(j+1)(j+2)\cdots(j+i+1)}\\ &= \frac{1}{i}\left(\frac{1}{(j+1)(j+2)\cdots(j+i)}-\frac{1}{(j+2)(j+3)\cdots(j+i+1)}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{i!j!}{(i+j+1)!} &= \sum_{i=1}^{\infty}\left(i!\cdot\sum_{j=1}^{\infty}\frac{j!}{(i+j+1)!}\right)\\ &= \sum_{i=1}^{\infty}\left(i!\cdot\frac{1}{i(i+1)!}\right)\\ &= \sum_{i=1}^{\infty}\frac{1}{i(i+1)}\\ &= \sum_{i=1}^{\infty}\left(\frac{1}{i}-\frac{1}{i+1}\right)=1.\end{aligned}$$

Second solution by G.R.A.20 Math Problems Group, Roma, Italy

Letting $n^{\underline{k}}=n(n-1)\cdots(n-k+1)$, we note that

$$\begin{aligned}\Delta_j\left(\frac{1}{(i+j)^{\underline{i}}}\right) &= \frac{1}{(i+j+1)^{\underline{i}}}-\frac{1}{(i+j)^{\underline{i}}}=\frac{j+1-(i+j+1)}{(i+j+1)(i+j)\cdots(j+1)}\\ &= \frac{-i}{(i+j+1)^{\underline{i}}(j+1)},\end{aligned}$$

Therefore

$$\frac{i!j!}{(i+j+1)!} = \frac{i!}{(i+j+1)^{\underline{i}}(j+1)} = -(i-1)!\Delta_j\left(\frac{1}{(i+j)^{\underline{i}}}\right),$$

and it follows that

$$\begin{aligned}
\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{i!j!}{(i+j+1)!} &= \sum_{i=1}^{\infty} -(i-1)!\sum_{j=1}^{\infty}\Delta_j\left(\frac{1}{(i+j)^{\underline{i}}}\right) \\
&= \sum_{i=1}^{\infty} -(i-1)!\left(-\frac{1}{(i+1)^{\underline{i}}}\right) \\
&= \sum_{i=1}^{\infty}\frac{1}{i(i+1)} = 1
\end{aligned}$$

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; O.O.Ibrogimov student of Samarqand State University, Uzbekistan; Vishal Lama, Southern Utah University, Utah, USA

U62. Let $x_1, x_2, \ldots, x_n > 0$ such that $x_1 + x_2 + \ldots + x_n = n$ and let $y_k = n - x_k$, $k = 1, 2, \ldots, n$. Prove that

$$x_1^{x_1} \cdot x_2^{x_2} \cdots x_n^{x_n} \geq \left(\frac{y_1}{n-1}\right)^{y_1} \cdot \left(\frac{y_2}{n-1}\right)^{y_2} \cdots \left(\frac{y_n}{n-1}\right)^{y_n}.$$

Proposed by Cezar Lupu, University of Bucharest, Romania

First solution by Perfetti Paolo, Dipartimento di matematica Universita degli studi di Tor Vergata, Italy

Applying the logarithm at both sides of the inequality we get

$$\sum_{k=1}^{n} x_k \ln x_k \geq \sum_{k=1}^{n} (n - x_k) \ln\left(\frac{n - x_k}{n-1}\right) = (n-1) \sum_{k=1}^{n} \frac{n - x_k}{n-1} \ln\left(\frac{n - x_k}{n-1}\right)$$

Recall the general inequality for convex functions

$$\sum_{k=1}^{n} F(x_k) + n(n-2)F\left(\frac{\sum_{k=1}^{n} x_k}{n}\right) \geq (n-1) \sum_{k=1}^{n} F\left(\frac{S - x_k}{n-1}\right),$$

where $S = \sum_{k=1}^{n} x_k$. Because the function $x \ln x$ is convex and $S = n$ in our case, we conclude the proof.

Second solution by Vishal Lama, Southern Utah University, Utah, USA

We use the Generalized Cauchy Inequality:

$$\frac{p_1 a_1 + p_2 a_2 + \ldots + p_n a_n}{p_1 + p_2 + \ldots + p_n} \geq (a_1^{p_1} a_2^{p_2} \ldots a_n^{p_n})^{\frac{1}{p_1 + p_2 + \ldots + p_n}} \geq \frac{p_1 + p_2 + \ldots + p_n}{\frac{p_1}{a_1} + \frac{p_2}{a_2} + \ldots + \frac{p_n}{a_n}},$$

where $a_i > 0, p_i > 0, \forall\ 1 \leq i \leq n$.

Let $L = x_1^{x_1} \cdot x_2^{x_2} \cdots x_n^{x_n}$ and $R = \left(\frac{y_1}{n-1}\right)^{y_1} \cdot \left(\frac{y_2}{n-1}\right)^{y_2} \cdots \left(\frac{y_n}{n-1}\right)^{y_n}$.

We have to prove that $L \geq R$. Setting $a_i = x_i$ and $p_i = x_i, \forall\ 1 \leq i \leq n$, and using the right hand inequality of the GCI above, we get

$$L^{\frac{1}{x_1 + x_2 + \ldots + x_n}} \geq \frac{x_1 + x_2 + \ldots + x_n}{\frac{x_1}{x_1} + \frac{x_2}{x_2} + \ldots + \frac{x_n}{x_n}} = 1.$$

Now, setting $a_i = \dfrac{y_i}{n-1}$ and $p_i = y_i, \forall\ 1 \leq i \leq n$, and using the left inequality of the GCI above, we get

$$\frac{\frac{y_1^2}{n-1} + \frac{y_2^2}{n-1} + \ldots + \frac{y_n^2}{n-1}}{y_1 + y_2 + \ldots + y_n} \geq R^{\frac{1}{y_1 + y_2 + \ldots + y_n}},$$

or

$$\frac{(n-x_1)^2+(n-x_2)^2+\ldots+(n-x_n)^2}{n(n-1)^2} \geq R^{\frac{1}{n(n-1)}},$$

because $y_1+y_2+\ldots+y_n=(n-x_1)+(n-x_2)+\ldots+(n-x_n)=n(n-1)$.

Consider $f(x)=(n-x)^2$, where $x\in(0,n)$. We have $f''(x)=-2$, which implies that $f(x)$ is concave down. Using Jensen's inequality we have

$$f\left(\frac{x_1+x_2+\ldots+x_n}{n}\right) \geq \frac{f(x_1)+f(x_2)+\ldots+f(x_n)}{n},$$

hence

$$f(1) \geq \frac{1}{n}\left((n-x_1)^2+(n-x_2)^2+\ldots+(n-x_n)^2\right),$$

or

$$1 \geq \frac{(n-x_1)^2+(n-x_2)^2+\ldots+(n-x_n)^2}{n(n-1)^2}$$

Combining obtained inequalities we get $L\geq 1\geq R$, and we are done.

Third solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

The natural logarithm is a strictly increasing function, thus we have to prove

$$\sum_{k=1}^{n}(y_k\ln y_k - x_k\ln x_k) \leq \ln(n-1)\sum_{k=1}^{n}y_k = \left(n^2-n\right)\ln(n-1).$$

Let us assume that we know the values of $x_3,x_4,\ldots,x_n$ such that the LHS is maximum, and let us call $n-k=x_3+x_4+...+x_n<n$. Then, we need to find the maximum of

$$y_2\ln y_2 - x_2\ln x_2 + y_1\ln y_1 - x_1\ln x_1 = (n-k+x_1)\ln(n-k+x_1)$$

$$-(k-x_1)\ln(k-x_1)+(n-x_1)\ln(n-x_1)-x_1\ln x_1,$$

where $0<x_1<k<n$. Taking the first derivative of this last expression with respect to x_1, we find

$$\ln\left(\frac{(n-k+x_1)(k-x_1)}{x_1(n-x_1)}\right).$$

This first derivative is continuous in the $(0,k)$ interval, reaching a value of 0 when $k(n-k)=2(n-k)x_1$, i.e., then $x_1=x_2=\frac{k}{2}$. The second derivative, evaluated for this value of x_1, is $-\frac{8(n-k)}{k(2n-k)}$, which is negative, and the maximum of the expression in the given interval is achieved when $x_1=x_2$. Due to symmetry between variables, we may perform this same optimization procedure for all x_i, x_j for $i \neq j$, which proves that the maximum of the LHS of the proposed expression is obtained when all x_i are equal to 1, yielding

$$\sum_{k=1}^{n} (y_k \ln y_k - x_k \ln x_k) \leq n\left((n-1)\ln n - 1\ln 1\right) = \left(n^2 - n\right)\ln(n-1), \text{ q.e.d.}$$

U63. Let f and g be polynomials with complex coefficients and let a be a nonzero complex number. Prove that if

$$(f(x))^3 = (g(x))^2 + a$$

for all $x \in C$, then the polynomials f and g are constant.

Proposed by Magkos Athanasios, Kozani, Greece

First solution by G.R.A.20 Math Problems Group, Roma, Italy

Consider polynomial $(f(x))^3 - (g(x))^2$ which is identically equal to the constant a. Therefore the degree of f is $2n$ and the degree of g is $3n$ for some non-negative integer n. Assume to the contrary that f and g are not constant, that is $n > 0$. Since $a \neq 0$, if $f(x_0) = 0$, then $g(x_0) \neq 0$. Taking the derivative we get

$$3(f(x))^2 f'(x) = 2g(x)g'(x).$$

Hence if $f(x_0) = 0$, then $g'(x_0) = 0$. This means that every zero of the polynomial $(f(x))^2$ is a zero of the polynomial $g'(x)$ with at least same multiplicity of f^2. Hence 2 times the degree of f, $4n$, is less or equal than the degree of g', $3n - 1$, a contradiction. Thus the polynomials f and g are constant.

Second solution by Suchandal Pal, Florida State University

We prove the following lemma:

Lemma. There do not exist non-constant polynomials $f, g \in \mathbb{C}[x]$ and a constant $\alpha \in \mathbb{C}$ such that $f^2(x) = \alpha + g^2(x)$.

Proof. $f^2(x) - g^2(x) = (f(x) - g(x))\,(f(x) + g(x)) = \alpha$. Thus $f - g$ and $f + g$ are constant, so their linear combinations, specifically f and g are constant.

Returning to the problem we have $g^2(x) = (f(x) - a^{1/3})(f^2(x) + a^{1/3} f(x) + a^{2/3})$. Note that if $f(x) - a^{1/3}$ and $f^2(x) + a^{1/3} f(x) + a^{2/3}$ share a common root x_0, then $f(x_0) = a^{1/3}$ and $f^2(x_0) + a^{1/3} f(x_0) + a^{2/3} = 3a^{1/3} = 0, \Leftrightarrow a = 0$, a contradiction.

Therefore each of them must be the square of a polynomial in $\mathbb{C}[x]$. For simplification we write $a^{1/3} = \beta$. Hence $f(x) = m_1^2(x) + \beta$, thus

$$\begin{aligned}(m_1^2(x) + \beta)^2 + \beta(m_1^2(x) + \beta) + \beta &= m_1^4(x) + 3m_1^2(x)\beta + 3\beta^2 \\ &= (m_1^2(x) + \gamma_1)(m_1^2(x) + \gamma_2),\end{aligned}$$

where $\gamma_1 = \frac{-3\beta + \beta\sqrt{-3}}{2}, \gamma_2 = \frac{-3\beta - \beta\sqrt{-3}}{2}$.

By the same argument these two polynomials $(m_1^2(x)+\gamma_1), (m_1^2(x)+\gamma_2)$ share no roots. Thus each of them must be the square of a polynomial. Thus, $m_1^2(x)+\gamma_1 = m_2^2(x)$ and $m_1^2(x)+\gamma_2 = m_3^2(x)$. Therefore $m_2^2(x)-\gamma_1+\gamma_2 = m_3^2(x)$ and using our lemma we get the desired contradiction.

Third solution by Vishal Lama, Southern Utah University, Utah, USA

We use the following theorem in our solution below.

Mason-Stothers Theorem: If $f, g, h \in C[t]$ are relatively prime polynomials, not all constant, and if $f = g + h$, then

$$\max\{\deg(f), \deg(g), \deg(h)\} \le N_0(fgh) - 1,$$

where $\deg(f)$ denotes the degree of f, and $N_0(fgh)$ denotes the number of distinct zeroes of the polynomial fgh.

In our problem, we have for all $x \in C$, $(f(x))^3 = (g(x))^2 + a$, where $f, g \in C[t]$ and a is a non-zero complex number.

If f is a constant, then so is g, and vice versa, and we are done. Suppose f and g are not constant functions. We prove that this leads to a contradiction.

We first note that f and g are relatively prime polynomials, for otherwise, if f and g have a common factor $x - c$, say, then, we have $(f(c))^3 = (g(c))^2 + a \Rightarrow 0 = 0 + a \Rightarrow a = 0$, a contradiction. Hence, f, g and a are relatively prime, which implies f^3, g^2 and a are relatively prime as well. Again, we note that if $\deg(f) = n$ and $\deg(g) = m$, then we must have $3n = 2m$, which implies $2 \mid n$. Therefore, $n = 2k$ for some $k \in \mathbb{N}$. Hence, $\deg(f) = 2k$ and $\deg(g) = 3k$ for some $k \in \mathbb{N}$.

Thus $\max\{\deg(f^3), \deg(g^2), \deg(a)\} = \max\{6k, 6k, 0\} = 6k$.

Also $N_0(f^3g^2a) - 1 = N_0(fg) - 1 \le \deg(f) + \deg(g) - 1 = 2k + 3k - 1 = 5k - 1$.

Hence using the Mason-Stothers theorem, we conclude $6k \le 5k - 1$, which implies $k \le -1$, a contradiction! The problem is solved.

Remark. The Mason-Stother's theorem is considered the polynomial version of the yet unproved ABC conjecture in modern Number Theory.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain

U64. Let x be a real number. Define the sequence $(x_n)_{n\geq 1}$ recursively by $x_1 = 1$ and $x_{n+1} = x^n + nx_n$, $n \geq 1$. Prove that

$$\prod_{n=1}^{\infty}\left(1 - \frac{x^n}{x_{n+1}}\right) = e^{-x}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Arkady Alt, San Jose, California, USA

From $x_{n+1} = x^n + nx_n$ we get $\dfrac{x_{n+1}}{n!} = \dfrac{x^n}{n!} + \dfrac{x_n}{(n-1)!}$, for $n \geq 1$. Then

$$\sum_{k=1}^{n}\frac{x^k}{k!} = \sum_{k=1}^{n}\left(\frac{x_{k+1}}{k!} - \frac{x_k}{(k-1)!}\right) = \frac{x_{n+1}}{n!} - \frac{x_1}{0!} = \frac{x_{n+1}}{n!} - 1,$$

yielding

$$\frac{x_{n+1}}{n!} = \sum_{k=0}^{n}\frac{x^k}{k!}.$$

It follows that

$$\prod_{k=1}^{n}\left(1 - \frac{x^k}{x_{k+1}}\right) = \prod_{k=1}^{n}\left(\frac{x_{k+1} - x^k}{x_{k+1}}\right) = \prod_{k=1}^{n}\frac{kx_k}{x_{k+1}} = n!\prod_{k=1}^{n}\frac{x_k}{x_{k+1}} = \frac{n!}{x_{n+1}}.$$

Thus

$$\prod_{n=1}^{\infty}\left(1 - \frac{x^n}{x_{n+1}}\right) = \lim_{n\to\infty}\frac{n!}{x_{n+1}} = \frac{1}{\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\dfrac{x^k}{k!}} = \frac{1}{e^x} = e^{-x}.$$

Second solution by Brian Bradie, Christopher Newport University, USA Using the recurrence relation, we find

$$\begin{aligned}\prod_{n=1}^{N}\left(1 - \frac{x^n}{x_{n+1}}\right) &= \prod_{n=1}^{N}\frac{x_{n+1} - x^n}{x_{n+1}} \\ &= \prod_{n=1}^{N}\frac{nx_n}{x_{n+1}} = \frac{x_1}{x_2}\cdot\frac{2x_2}{x_3}\cdot\frac{3x_3}{x_4}\cdot\dots\cdot\frac{Nx_N}{x_{N+1}} \\ &= \frac{N!}{x_{N+1}}. \qquad (1)\end{aligned}$$

Next, we establish that

$$x_n = (n-1)!\sum_{k=0}^{n-1}\frac{x^k}{k!}. \qquad (2)$$

For $n = 1$,

$$x_1 = 0! \sum_{k=0}^{0} \frac{x^k}{k!} = 1,$$

as required. Moreover, if equation (2) holds, then

$$x_{n+1} = x^n + nx_n = x^n + n! \sum_{k=0}^{n-1} \frac{x^k}{k!} = n! \sum_{k=0}^{n} \frac{x^k}{k!}.$$

Thus, equation (2) holds for all n by induction. Finally, combining equations (1) and (2), we find

$$\begin{aligned} \prod_{n=1}^{\infty} \left(1 - \frac{x^n}{x_{n+1}}\right) &= \lim_{N\to\infty} \frac{N!}{x_{N+1}} \\ &= \lim_{N\to\infty} \frac{1}{\sum_{k=0}^{N} \frac{x^k}{k!}} \\ &= \frac{1}{e^x} = e^{-x}. \end{aligned}$$

Third solution by Jose Hernandez Santiago, UTM, Oaxaca, Mexico

Our proof leans heavily on the following straightforward property of the sequence $\{x_n\}_{n=1}^{\infty}$: for all $n \in \mathbb{N}$

$$x_{n+1} = x^n + nx^{n-1} + n(n-1)x^{n-2} + \ldots + n!x + n! \qquad (1)$$

Let us denote with $\mathcal{P}_k$ the k-th partial product of $\prod_{n=0}^{\infty}\left(1 - \frac{x^n}{x_{n+1}}\right)$. Using mathematical induction we prove that the equality

$$\begin{aligned} \mathcal{P}_k &= \left(1 - \frac{x}{x_2}\right)\left(1 - \frac{x^2}{x_3}\right) \cdot \ldots \cdot \left(1 - \frac{x^k}{x_{k+1}}\right) & (2) \\ &= \frac{1}{\frac{x^k}{k!} + \frac{x^{k-1}}{(k-1)!} + \ldots + \frac{x}{1!} + 1} & (3) \end{aligned}$$

holds true for every $k \in \mathbb{N}$. This is clearly the case for $k = 1$. Let us suppose our identity remains valid for k. Since

$$\begin{aligned} \mathcal{P}_{k+1} &= \mathcal{P}_k \left(1 - \frac{x^{k+1}}{x_{k+2}}\right) \\ &= \left(\frac{1}{\frac{x^k}{k!} + \frac{x^{k-1}}{(k-1)!} + \ldots + \frac{x}{1!} + 1}\right)\left(1 - \frac{x^{k+1}}{x_{k+2}}\right) \\ &= \frac{x_{k+2} - x^{k+1}}{\left(\frac{x^k}{k!} + \frac{x^{k-1}}{(k-1)!} + \ldots + \frac{x}{1!} + 1\right) x_{k+2}}, \end{aligned}$$

and (1) allows us to conclude that

$$\begin{aligned}
\mathcal{P}_{k+1} &= \frac{(k+1)x^k + (k+1)kx^{k-1} + \ldots + (k+1)!x + (k+1)!}{\left(\frac{x^k}{k!} + \frac{x^{k-1}}{(k-1)!} + \ldots + \frac{x}{1!} + 1\right) x_{k+2}} \\
&= \frac{(k+1)!}{x_{k+2}} \\
&= \frac{(k+1)!}{x^{k+1} + (k+1)x^k + \ldots + (k+1)!x + (k+1)!} \\
&= \frac{1}{\frac{x^{k+1}}{(k+1)!} + \frac{x^k}{k!} + \ldots + \frac{x}{1!} + 1}.
\end{aligned}$$

From (2) and (3) it follows that

$$\begin{aligned}
\prod_{n=0}^{\infty}\left(1 - \frac{x^n}{x_{n+1}}\right) &= \prod_{n=1}^{\infty}\left(1 - \frac{x^n}{x_{n+1}}\right) \\
&= \lim_{k\to\infty} \mathcal{P}_k \\
&= \lim_{k\to\infty} \frac{1}{\frac{x^k}{k!} + \frac{x^{k-1}}{(k-1)!} + \ldots + \frac{x}{1!} + 1} \\
&= \frac{1}{\lim_{k\to\infty}\left(\frac{x^k}{k!} + \frac{x^{k-1}}{(k-1)!} + \ldots + \frac{x}{1!} + 1\right)} \\
&= \frac{1}{\sum_{k=0}^{\infty} \frac{x^k}{k!}} \\
&= \frac{1}{e^x} \\
&= e^{-x},
\end{aligned}$$

and we are done.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; O.O.Ibrogimov student of Samarqand State University, Uzbekistan; G.R.A.20 Math Problems Group, Roma, Italy; Vishal Lama, Southern Utah University, Utah, USA

U65. Let A, B, C be 3×3 invertible matrices such that their elements are in the interval $[0, 1]$ and entries in each row sum up to 1. Prove that $AC^{-1}BA^{-1}CB^{-1}$ and $CA^{-1}BC^{-1}AB^{-1}$ have the same trace.

Proposed by Jean-Charles Mathieux, Dakar University, Sénégal

Solution by G.R.A.20 Math Problems Group, Roma, Italy

Let $X = AC^{-1}BA^{-1}CB^{-1}$, then $\det(X) = 1$. Moreover, since each row of A, B and C sum up to 1, then $Av = Bv = Cv = v$ with $v = (1, 1, 1)'$. Hence $A^{-1}v = B^{-1}v = C^{-1}v = v$ and $Xv = X^{-1}v = v$. This means that 1 is one of the eigenvalues of X. Let λ and μ be the other eigenvalues of X. Therefore

$$\operatorname{tr}(X^{-1}) = 1 + \lambda^{-1} + \mu^{-1} = \frac{1 + \lambda + \mu}{\lambda\mu} = \frac{\operatorname{tr}(X)}{\det(X)} = \operatorname{tr}(X).$$

Finally,

$$\begin{aligned} \operatorname{tr}(AC^{-1}BA^{-1}CB^{-1}) &= \operatorname{tr}(X) = \operatorname{tr}(X^{-1}) \\ &= \operatorname{tr}(BC^{-1}AB^{-1}CA^{-1}) = \operatorname{tr}(CA^{-1}BC^{-1}AB^{-1}), \end{aligned}$$

because the trace of a matrix product is invariant by cyclic permutation.

Another solution by G.R.A.20 Math Problems Group, Roma, Italy

Let $M = AC^{-1}BA^{-1}CB^{-1}$ and $N = CA^{-1}BC^{-1}AB^{-1}$.

Since $N, M \in \mathbb{R}^{3\times 3}$, then

$$P_N(x) = \det(xI - N) = x^3 - \operatorname{tr}(N)x^2 + a\,x - \det(N)$$

and

$$P_M(x) = \det(xI - M) = x^3 - \operatorname{tr}(M)x^2 + b\,x - \det(M).$$

Moreover, $N^{-1} = BA^{-1}CB^{-1}AC^{-1}$ and

$$P_{N^{-1}}(x) = \det(xI - N^{-1}) = x^3 - a\ \det(N)^{-1}x^2 + \operatorname{tr}(N)\det(N)^{-1}x - \det(N)^{-1}.$$

Because the characteristic polynomial is invariant with respect to cyclic permutations then $P_M(x) = P_{N^{-1}}(x)$, which implies that

$$\begin{cases} a = \operatorname{tr}(M)\ \det(N) \\ b = \operatorname{tr}(N)\ \det(M) \end{cases}.$$

Since $\lambda = 1$ is an eigenvalue of A, B, C with respect to the same eigenvector $[1\,1\,1]^t$, then the same holds for M and N and therefore

$$P_N(1) = 1 - \operatorname{tr}(N) + \operatorname{tr}(M)\ \det(N) - \det(N) = 0$$

Finally, we note that

$$\det(N) = \det(C)\ \det(A)^{-1}\det(B)\ \det(C)^{-1}\det(A)\ \det(B)^{-1} = 1.$$

Thus $\operatorname{tr}(N) = \operatorname{tr}(M)$ and we are done.

U66. Let $V = \{v_1, v_2, \ldots, v_k, \ldots\}$ be a set of vectors in R^n containing n linearly independent vectors. A finite subset $S \subset V$ is called "crucial" if the set $V \setminus S$ contains no n independent vectors, but every set $V \setminus T$ where $T \subset S$ does. Prove that there are finitely many "crucial" subsets.

Proposed by Iurie Boreico, Harvard University, USA

Solution by Iurie Boreico, Harvard University, USA

Let S be a crucial set. Let V_S be the vector space spanned by $V \setminus S$, then adding any vector from S to V_S we will be able to span R^n, which implies that V_S is a subspace of dimension $n-1$ and all but the vectors in S are in V_S. Now let $W = \bigcap V_S$, where the intersection is taken over all crucial subsets. It is clear that W is a intersection of some finite collection of vector spaces V_S, say $W = V_{S_1} \bigcap V_{S_2} \ldots \bigcap V_{S_m}$ (we can also prove $m \leq n$). As all but finitely many vectors belong to V_{S_i}, we conclude that all but finitely many vectors belong to W. But we know that the vectors from S do not belong to V_S, so not to W, for any crucial subset S. It remains to see that we have finitely many vectors not belonging to W, so finitely many ways to compose S from them.

Olympiad problems

O61. Let a, b, c be positive numbers such that $4abc = a + b + c + 1$. Prove that

$$\frac{b^2+c^2}{a} + \frac{c^2+a^2}{b} + \frac{a^2+b^2}{c} \geq 2(ab+bc+ca).$$

Proposed by Ciupan Andrei, Bucharest, Romania

First solution by Magkos Athanasios, Kozani Greece

From the condition given and AM-GM Inequality we have

$$4abc = a + b + c + 1 \geq 4\sqrt[4]{abc},$$

hence $abc \geq 1$ and $a + b + c = 4abc - 1 \geq 3abc$. Clearly $x^2 + y^2 \geq 2xy$, then LHS is greater than or equal to

$$\frac{2bc}{a} + \frac{2ca}{b} + \frac{2ab}{c} = \frac{2}{abc}\left((ab)^2 + (bc)^2 + (ca)^2\right) \geq \frac{2}{3abc}(ab+bc+ca)^2.$$

The last inequality follows from the inequality $3(x^2 + y^2 + z^2) \geq (x + y + z)^2$.

Therefore, it suffices to prove that $ab + bc + ca \geq 3abc$. Recall well known inequality $ab + bc + ca \geq \sqrt{3abc(a + b + c)}$. Hence it suffices to prove that $a + b + c \geq 3abc$, and we are done.

Second solution by Daniel Campos Salas, Costa Rica

Observe that it is enough to prove that

$$\frac{2bc}{a} + \frac{2ca}{b} + \frac{ab}{c} \geq 2(ab+bc+ca),$$

or equivalently

$$\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \geq ab+bc+ca.$$

From the Cauchy-Schwarz inequality we have

$$\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \geq \frac{(ab+bc+ca)^2}{abc+abc+abc} = \frac{(ab+bc+ca)^2}{3abc}.$$

Then, it is enough to prove that $ab+bc+ca \geq 3abc$, or equivalently, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 3$. This inequality is the left hand side inequality of problem J41, and this completes the proof.

Third solution by Vishal Lama, Southern Utah University, Utah, USA

Applying the AM-GM inequality we get

$$abc = \frac{a+b+c+1}{4} \geq \sqrt[4]{a \cdot b \cdot c \cdot 1}, \quad (1)$$

yielding $abc \geq 1$. Without any loss of generality assume that $a \geq b \geq c$.

We have $\frac{1}{a} \leq \frac{1}{b} \leq \frac{1}{c}$ and $b^2+c^2 \leq c^2+a^2 \leq a^2+b^2$ Applying Chebyshev's inequality to the sequences above we obtain

$$\frac{1}{3}\left(\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}+\frac{a^2+b^2}{c}\right) \geq \frac{b^2+c^2+c^2+a^2+a^2+b^2}{3} \cdot \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3},$$

or

$$\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}+\frac{a^2+b^2}{c} \geq \frac{2(a^2+b^2+c^2)(ab+bc+ca)}{3abc}, \quad (2)$$

Using Cauchy-Schwarz inequality we get

$$(a^2+b^2+c^2+1^2)(1^2+1^2+1^2+1^2) \geq (a \cdot 1+b \cdot 1+c \cdot 1+1 \cdot 1)^2 = (4abc)^2,$$

or

$$a^2+b^2+c^2 \geq 4(abc)^2-1.$$

Using the above result in (2), we get

$$\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}+\frac{a^2+b^2}{c} \geq \frac{2(4(abc)^2-1)(ab+bc+ca)}{3abc} \geq 2(ab+bc+ca).$$

The last inequality follows from the fact that $4(abc)^2-3abc-1 \geq 0$, which is equivalent to $(abc-1)(4abc+1) \geq 0$, and we are done.

Also solved by Kee-Wai Lau, Hong Kong, China; Daniel Lasaosa, Universidad Publica de Navarra, Spain; G.R.A.20 Math Problems Group, Roma, Italy; Salem Malikic Sarajevo, Bosnia and Herzegovina; Perfetti Paolo, Dipartimento di matematica Universita degli studi di Tor Vergata, Italy; Arkady Alt, San Jose, California, USA

O62. Consider the Cartesian plane. Let us call a point X rational if both its coordinates are rational numbers. Prove that if a circle passes through three rational points, then it passes through infinitely many of them.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

Solution by G.R.A.20 Math Problems Group, Roma, Italy

Knowing the coordinates of these three points $P_i = (x_i, y_i)$ for $i = 1, 2, 3$ we can find the center $O = (x_0, y_0)$ by intersecting the axes of the segments P_1P_2 and P_1P_3:

$$\begin{cases} (y_2 - y_1)(x_0 - (x_1 + x_2)/2) - (x_2 - x_1)(y_0 - (y_1 + y_2)/2) = 0 \\ (y_3 - y_1)(x_0 - (x_1 + x_3)/2) - (x_3 - x_1)(y_0 - (y_1 + y_3)/2) = 0 \end{cases}$$

Since the coefficients of this linear system are rational also the solution (x_0, y_0) is rational. Now let $P_n = (x_n, y_n)$ for $n \geq 4$ where

$$\begin{cases} x_n = x_0 + p_n(x_3 - x_0) - q_n(y_3 - y_0) \\ y_n = y_0 + q_n(x_3 - x_0) + p_n(y_3 - y_0) \end{cases} \quad \text{with } p_n = \frac{n^2 - 1}{n^2 + 1} \text{ and } q_n = \frac{2n}{n^2 + 1}$$

then

$$||P_n - P_0||^2 = (x_n - x_0)^2 + (y_n - y_0)^2 = (x_3 - x_0)^2 + (y_3 - y_0)^2 = ||P_3 - P_0||^2$$

that is $\{P_n\}_{n \geq 4}$ is a sequence of infinite points on the circle.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Vicente Vicario Garcia, Huelva, Spain; Vishal Lama, Southern Utah University, Utah, USA

O63. Let M and N be two points inside the circle $C(O)$ such that O is the midpoint of MN and let S be an arbitrary point on this circle. Let E and F be the second intersections of the lines SM and SN with the circle. Tangents at E and F to $C(O)$ intersect each other at I. Prove that the perpendicular bisector of the segment MN passes through the midpoint of SI.

Proposed by Son Hong Ta, Ha Noi University, Vietnam

First solution by Arnau Messegue Buisan, Barcelona, Spain

Consider the complex plane. We denote by z de complex number which represents the point Z. Without loss of generality we may assume that $C(O)$ is the unit circle, and m and n are parallel to the real axis. Taking this fact into account, we have that $m = -n$ and $\bar{n} = n$. Let s be the complex which represents S, then the points E and F are represented by the complex numbers e and f which can be easily found:

$$\frac{s-n}{\bar{s}-\bar{n}} = \frac{e-s}{\bar{e}-\bar{s}} \Rightarrow e = \frac{s-n}{ns-1},$$

$$\frac{s+n}{\bar{s}+\bar{n}} = \frac{f-s}{\bar{f}-\bar{s}} \Rightarrow f = -\frac{s+n}{ns+1}.$$

From this one can find the complex number which represents the point I. We will denote it by j:

$$j = \frac{2ef}{e+f} \Rightarrow j = \frac{(s-n)(s+n)}{s(n^2-1)}.$$

The midpoint of the segment SI, represented by the complex number k will be

$$k = \frac{n^2(s^2-1)}{2s(n^2-1)}.$$

As we can see, $\bar{k} = -\dfrac{n^2(s^2-1)}{2s(n^2-1)} = -k$. Thus k belongs to the imaginary axis, that is to the perpendicular bisector of MN.

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Consider a system of coordinates such that the center of the circle is the origin $O \equiv (0,0)$, the radius of the circle is 1, and the horizontal axis X coincides with line MN. Then, $0 < \rho < 1$ exists such that $M \equiv (\rho, 0)$, $N \equiv (-\rho, 0)$. Furthermore, angle $0 \leq \alpha < 2\pi$ exists such that $S \equiv (x_S, y_S) = (\cos\alpha, \sin\alpha)$. Point E has coordinates that are solutions of the system formed by the circle equation $x^2+y^2=1$ and the equation of line SM, $\frac{y}{x-\rho}=\frac{\sin\alpha}{(\cos\alpha-\rho)}$. Expressing y as a function of x in the second equation, and substituting in the first, we obtain

$$\frac{\sin^2\alpha\,(x-\rho)^2}{(\cos\alpha-\rho)^2}+x^2=1;$$

$$(x-\cos\alpha)\left(\left(1+\rho^2-2\rho\cos\alpha\right)x-\left(2\rho-\left(1+\rho^2\right)\cos\alpha\right)\right)=0.$$

The first solution corresponds to point S, and $E\equiv(x_E,y_E)$, where

$$x_E=\frac{2\rho-\left(1+\rho^2\right)\cos\alpha}{1+\rho^2-2\rho\cos\alpha},$$

and substitution in the equation for line SM gives

$$y_E=-\frac{\left(1-\rho^2\right)\sin\alpha}{1+\rho^2-2\rho\cos\alpha}.$$

In an entirely analogous way, we may find that $F\equiv(x_F,y_F)$, where

$$x_F=-\frac{2\rho+\left(1+\rho^2\right)\cos\alpha}{1+\rho^2+2\rho\cos\alpha},\,y_F=-\frac{\left(1-\rho^2\right)\sin\alpha}{1+\rho^2+2\rho\cos\alpha}.$$

Now, tangents to circle $x^2+y^2=1$ at E and F obey, respectively, equations $y=\frac{1-x_Ex}{y_E}$ and $y=\frac{1-x_Fx}{y_F}$. Their point of intersection $I\equiv(x_I,y_I)$ satisfies both, and

$$x_I=\frac{y_E-y_F}{x_Fy_E-x_Ey_F}=-\frac{4\rho\left(1-\rho^2\right)\sin\alpha\cos\alpha}{4\rho\,(1-\rho^2)\sin\alpha}=-\cos\alpha=-x_S.$$

Now, the x coordinate of the midpoint of SI is $\frac{x_I+x_S}{2}=0$, or it is on the line with equation $x=0$, which is the perpendicular bisector of MN, and we are done.

O64. Let F_n be the n-th Fibonacci number. Prove that for all $n \geq 4$, $F_n + 1$ is not a prime.

Proposed by Dorin Andrica, Cluj-Napoca, Romania

First solution by Vicente Vicario Garcia, Huelva, Spain

Let us recall two famous identities.

Cassini's identity: If F_k is the k-th Fibonacci number, then

$$F_k^2 = F_{k-1}F_{k+1} + (-1)^{k+1}.$$

Gelin-Cesaro identity: If F_k is the k-th Fibonacci number, then

$$F_k^4 = F_{k-2}F_{k-1}F_{k+1}F_{k+2} + 1.$$

Using the Gelin-Cesaro identity we obtain the following factorization

$$(F_k^2 + 1)(F_k - 1)(F_k + 1) = F_{k-2}F_{k-1}F_{k+1}F_{k+2} + 1.$$

Then if $F_k + 1 = p$, where p is prime, we get $p \mid F_{k+2}$. This goes from the fact $F_{k-2}, F_{k-1} < p, F_{k+1} < 2F_k = 2p$ and $\gcd(F_i, F_{i+1}) = 1$. Observe that $2F_k < F_{k+2} < 3F_k$, thus $2p < F_{k+2} < 3p$, for $k \geq 4$, a contradiction. The conclusion follows.

Second solution by Nguyen Trong Tung, Hanoi University of Technology, Hanoi, Vietnam

We need the following lemma:

Lemma. $A_n = F_{n+2}^4 - F_nF_{n+2}F_{n+3}F_{n+4} = 1$ for all $n \geq 0$.

Proof. Indeed,for all $n \geq 1$ we have

$$\begin{aligned}
A_n &= (F_{n+1} + F_n)^4 - F_nF_{n+1}(2F_{n+1} + F_n)(2F_{n+1} + 3F_n + F_{n-1}) \\
&= F_{n+1}^4 + F_n^4 - 2F_{n+1}^2F_nF_{n-1} - (F_{n-1} + F_n)F_n^2(F_n + 3F_{n-1}) \\
&= F_{n+1}^4 - F_nF_{n-1}(F_n^2 + 3F_nF_{n+1} + 2F_{n+1}^2) \\
&= F_{n+1}^4 - F_nF_{n-1}(F_n + F_{n+1})(F_n + 2F_{n+1}) \\
&= F_{n+1}^4 - F_nF_{n-1}F_{n+2}F_{n+3} \\
&= A_{n-1}
\end{aligned}$$

Thus we get $A_n = A_0 = 1$ for all $n \geq 0$. The lemma is proved.

Returning back the original problem, using our lemma we have

$$(F_n - 1)(F_n + 1)(F_n^2 + 1) = F_{n-2}F_{n-1}F_{n+1}F_{n+2}.$$

Suppose that $F_n + 1 = p$, for some prime p. Hence we must have one of two situations

(i) $p \mid F_{n+1}$, thus $p \mid F_{n-1} - 1$, false!

(ii) $p \mid F_{n+2}$, hence $p \mid F_{n+1} - 1$ and $p \mid F_{n-1} - 2$, also false for $n > 4$. The problem is solved.

Third solution by G.R.A.20 Math Problems Group, Roma, Italy

By Cesaro identity, for $n \geq 2$, we have

$$F_{n-2}F_{n-1}F_{n+1}F_{n+2} = F_n^4 - 1 = (F_n^2 + 1)(F_n + 1)(F_n - 1).$$

Assume that $p = F_n + 1$ is a prime and $n \geq 5$ ($F_4 + 1 = 4$ is not prime), then

$$F_{n-2}F_{n-1}F_{n+1}F_{n+2} = (p^2 - 2p + 2)p(p - 2).$$

and therefore p divides one of the factors F_{n-2}, F_{n-1}, F_{n+1}, F_{n+2}.

Because $0 < 2 = F_3 \leq F_{n-2} < F_{n-1} < F_n = p - 1 < p$, p cannot divide F_{n-2} and F_{n-1}.

Because $p < p + 2 = F_4 + F_n \leq F_{n+1} = F_{n-1} + F_n < 2F_n = 2p - 2 < 2p$, thus p cannot divide F_{n+1}.

Because $2p < 2p+1 = (p-1)+(p+2) \leq F_{n+2} = F_n+F_{n+1} < 3F_n = 3p-3 < 3p$, p cannot divide F_{n+2}.

Therefore we get a contradiction and $F_n + 1$ is not a prime for $n \geq 4$.

Also solved by Kee-Wai Lau, Hong Kong, China; Daniel Lasaosa, Universidad Publica de Navarra, Spain

O65. Let ABC be a triangle and let D, E, and F be the tangency points of its incircle $\gamma(I)$ with BC, CA, and AB, respectively. Let X_1 and X_2 be the intersections of line EF with the circumcircle $\rho(O)$ of triangle ABC. Similarly, define Y_1, Y_2, and Z_1, Z_2. Prove that the radical center of the circles DX_1X_2, EY_1Y_2, and FZ_1Z_2 lies on line OI.

Proposed by Cosmin Pohoata, Romania and Darij Grinberg, Germany

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Denote by A', B', C' the second points where AI, BI, CI intersect the circumcircle of ABC, respectively. Denote by O_D, O_E, O_F the midpoints of $B'C'$, $C'A'$, $A'B'$, respectively.

Lemma 1. O_D, O_E, O_F are the centers of the circles DX_1X_2, EY_1Y_2, FZ_1Z_2.

Proof. Recalling the length of the median we get

$$O_DA^2 = \frac{AB'^2 + AC'^2}{2} - \frac{B'C'^2}{4},\ O_DB^2 = \frac{BB'^2 + BC'^2}{2} - \frac{B'C'^2}{4},$$

$$\text{and } O_DC^2 = \frac{CB'^2 + CC'^2}{2} - \frac{B'C'^2}{4},$$

and similarly for O_E and O_F. Using Stewart's theorem we have

$$O_DD^2 = \frac{CD \cdot O_DB^2 + BD \cdot O_DC^2}{BC} - BD \cdot CD$$

and

$$O_DE^2 = \frac{AE \cdot O_DC^2 + CE \cdot O_DA^2}{CA} - CE \cdot AE.$$

Finally, $B'C' \perp IA$, as $\angle B'A'A = \frac{\angle B}{2}$, $\angle A'B'C' = \angle A'B'B + \angle C'B'B = \frac{\angle A + \angle C}{2}$. Therefore $A'B' \parallel X_1X_2$ because $EF \perp IA$. Since $A'B', X_1X_2$ are parallel chords of a given circle, the midpoint O_E of $A'B'$ is at the same distance to X_1 and X_2. Using Stewart's theorem,

$$O_DE^2 = \frac{EX_2 \cdot O_DX_1^2 + EX_1 \cdot O_DX_2^2}{X_1X_2} - EX_1 \cdot EX_2 = O_DX_1^2 - AE \cdot CE,$$

where we have also used that $EX_1 \cdot EX_2$ is the power of E with respect to the circumcircle of ABC. Thus we have to prove $O_DX_1 = O_DD$, using the previous expressions we find the equivalent condition

$$\frac{AE \cdot O_DC^2 + CE \cdot O_DA^2}{CA} = \frac{CD \cdot O_DB^2 + BD \cdot O_DC^2}{BC} - BD \cdot CD.$$

It is known that $AE = AF = \frac{b+c-a}{2}$, $BF = BD = \frac{c+a-b}{2}$ and $CD = CE = \frac{a+b-c}{2}$. Using the Law of Sines, it easy to prove that $CA' = BA' = 2R\sin\frac{A}{2}$, $AB' = CB' = 2R\sin\frac{B}{2}$ and $BC' = AC' = 2R\sin\frac{C}{2}$, and also that

$$AA' = 2R\cos\frac{2B+A}{2} = 2R\cos\frac{B-C}{2},\ BB' = 2R\cos\frac{C-A}{2} \text{ and } CC' = 2R\cos\frac{A-B}{2}.$$

Plugging these results, our expression is equivalent to

$$(b-a)\left(CC'^2 - BC'^2\right) - b\left(BB'^2 - CB'^2\right) + ab\left(c+a-b\right) = 0.$$

Now,

$$\frac{CC'^2 - BC'^2}{4R^2} = \cos^2\frac{A-B}{2} - \cos^2\frac{A+B}{2} = \sin A \sin B = \frac{ab}{4R^2},$$

$$\frac{BB'^2 - CB'^2}{4R^2} = \cos^2\frac{C-A}{2} - \cos^2\frac{C+A}{2} = \sin C \sin A = \frac{ca}{4R^2},$$

and the conclusion follows.

Lemma 2. The radical center R of the circles DX_1X_2, EY_1Y_2 and FZ_1Z_2 is the orthocenter of DEF.

Proof. The power of the point D with respect to the circles EY_1Y_2 and FZ_1Z_2 are equal, since $DY_1 \cdot DY_2$ and $DZ_1 \cdot DZ_2$ are equal. Therefore, D is in the radical line of EY_1Y_2 and FZ_1Z_2. Furthermore, this radical line is perpendicular to the line O_EO_F that joins the centers of both circles, and since O_E, O_F are the respective midpoints of $C'A', A'B'$, then $O_EO_F \parallel B'C' \parallel EF$, and the radical line is the altitude from D onto EF. Analogously, we prove that the radical lines of circles FZ_1Z_2 and DX_1X_2, and of circles DX_1X_2 and EY_1Y_2, are respectively the altitudes from E and F onto FD and DE. These radical lines meet at the radical center, which is the orthocenter R of DEF.

Lemma 3. The Euler line of $O_DO_EO_F$ is line OI.

Proof. Because $AI \perp B'C'$, $BI \perp C'A'$ and $CI \perp A'B'$, I is the orthocenter of $A'B'C'$, and since O is also the circumcenter of $A'B'C'$, the midpoint N of OI is the nine-point center of $A'B'C'$, i.e., the circumcenter of $O_DO_EO_F$. Furthermore, OO_D, OO_E and OO_F are the perpendicular bisectors of $B'C'$, $C'A'$ and $A'B'$, or they are the corresponding altitudes from O_D, O_E and O_F onto O_EO_F, O_FO_D and O_DO_E, and O is the orthocenter of $O_DO_EO_F$. Therefore, ON, and hence OI, is the Euler line of $O_DO_EO_F$.

We are now ready to prove that the orthocenter of DEF, and hence the radical center of circles DX_1X_2, EY_1Y_2 and FZ_1Z_2, lies on the line OI. Since DEF and $O_DO_EO_F$ are homothetic, there exist a point P where DO_D, EO_E and FO_F concur. The line joining the respective circumcenters I and N of DEF and $O_DO_EO_F$, also passes through P. Therefore, P is on OI. But the line joining the respective orthocenters R and O of DEF and $O_DO_EO_F$, passes also through P. Since P is on OI, then R is also on OI, and we are done.

O66. Let m a fixed positive integer. Prove that there is a constant $c(m)$ such that for each integer $n > 0$, there is a prime number $p < c(m)n$ with the following property: the equation $k^{2^m} \equiv n (\text{mod } p)$ has integer solutions while the equation $k^{2^m} \equiv -n (\text{mod } p)$ does not have integer solutions.

Proposed by Adrian Zahariuc, Princeton University, USA

Solution by Adrian Zahariuc, Princeton University, USA

The main fact is the following classical result

Lemma. If $a, b, m \in \mathbb{N}$ and p is an odd prime such that $p \mid a^{2^m} + b^{2^m}$, then either $p \mid a$ and $p \mid b$ or $p \equiv 1 (\text{mod } 2^{m+1})$.

Proof. If $p \mid a$, then $p \mid b$, so take $(p, a) = (p, b) = 1$. We have $a^{2^m} \equiv -b^{2^m} (\text{mod } p)$, so

$$\left(ab^{-1}\right)^{2^m} \equiv -1 (\text{mod } p), \left(ab^{-1}\right)^{2^{m+1}} \equiv 1 (\text{mod } p) \Rightarrow \left(ab^{-1}\right)^{(2^{m+1}, p-1)} \equiv 1 (\text{mod } p)$$

If 2^{m+1} does not divide $p-1$, then $(p-1, 2^{m+1}) \mid 2^m$ hence

$$\left(ab^{-1}\right)^{2^m} \equiv 1 (\text{mod } p) \Rightarrow -1 \equiv 1 (\text{mod } p),$$

which is clearly false, so $2^{m+1} \mid p - 1$.

Let us solve the original problem. Define $N_j = 2^{2^m} n - \left(2^{2^m} j^{2^m} + 1\right)^{2^m}$ and set j to be the largest such positive integer for which $N_j > 0$. Clearly, $N_j \equiv -1 (\text{mod } 2^{m+1})$, hence it has a prime factor p which is not congruent to 1 modulo 2^{m+1} because otherwise (taking $N_j > 0$ into account) $1 \equiv N_j \equiv -1 (\text{mod } p)$, a contradiction. We claim that this p has the desired properties. By the maximality of j, it follows by straightforward calculations that $j = O(n^{\frac{1}{4^m}})$ and $p < N_j = O(n^{1-\frac{1}{4^m}})$. It is clear that $k^{2^m} \equiv n (\text{mod } p)$ has integer solutions because

$$2^{2^m} n \equiv \left(2^{2^m} j^{2^m} + 1\right)^{2^m} (\text{mod } p)$$

and 2 is invertible mod p. So it remains to prove that $k^{2^m} \equiv -n (\text{mod } p)$ does not have integer solutions. Suppose by the way of contradiction that it does, so there is some $k \in \mathbb{Z}$ such that

$$p \mid k^{2^m} + n \Rightarrow p \mid (2k)^{2^m} + 2^{2^m} n = (2k)^{2^m} + (2^{2^m} j^{2^m} + 1)^{2^m} + N_j \Rightarrow$$

$$\Rightarrow p \mid (2k)^{2^m} + (2^{2^m} j^{2^m} + 1)^{2^m}.$$

Due to the lemma, $p \mid 2k$ and $p \mid (2j)^{2^m} + 1$ since $p \neq 1 (\text{mod } 2^{m+1})$, but this implies $p \mid 2j$ and $p \mid 1$, which is false.