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Dynamical Entropy Through Quantum Markov Chains

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Abstract. Classical dynamical entropy is an important tool to analyze communication processes. For instance, it may represent a transmission capacity for one letter. In this paper, we formulate the notion of dynamical entropy through a quantum Markov chain and calculate it for some simple models.

1. Introduction

Classical dynamical (or Kolmogorov-Sinai) entropy was introduced in [10, 11, 16], and relates to classical coding theorems of Shannon [5, 9, 13]. Quantum dynamical (QD) entropy has been studied by Emch [8], Connes, Størmer [7], Connes, Narnhofer, Thirring [6] and many others. Recently, the quantum dynamical entropy and the quantum dynamical mutual entropy were defined by Ohya in terms of complexity [12, 14, 15]. Classical Markov chain is a fundamental concept in stochastic processes. The notion of quantum Markov chain (QMC) was formulated by means of the transition expectation introduced by Accardi [1, 2].

In Section 1, we review the notion of dynamical entropy through a classical Markov chain. In Section 2, we define dynamical entropy through a quantum Markov chain, and, in Section 3, we calculate it for some simple models.

2. Formulation of Dynamical Entropy in Classical Markov Chain

Let $(\Omega, \mathcal{F}, \mu)$ be a probability measure space, T be a measure preserving (i.e., $\mu \circ T = \mu$) automorphism on Ω and $\mathcal{C} \equiv \{C_k\}$ be a finite partition of Ω. Let $L^\infty(\Omega)$ be the set of all functions f on Ω satisfying $\|f\|_\infty \equiv \inf\{\alpha; |f| \le \alpha, \mu-\text{a.e.}\} < +\infty$. We denote the set of all $n \times n$ diagonal matrices by D_n. Then there exists a one to one correspondence between $L^\infty(\{1,\ldots,n\})$ and D_n, that is, a characteristic function $\chi_{\{k\}} \in L^\infty(\{1,\ldots,n\})$ relates to a diagonal matrix $e_{kk} \in D_n$, where e_{ij} is the matrix unit (i.e., (i,i)-element $= 1$ and other elements $= 0$). The transition expectation $\mathcal{E}_\mathcal{C}$ from $L^\infty(\{1,\ldots,n\}) \otimes L^\infty(\Omega)$ to $L^\infty(\Omega)$ is defined by

$$\mathcal{E}_\mathcal{C}(\chi_{\{k\}} \otimes f) \;\equiv\; \chi_{C_k} \cdot f \tag{2.1}$$

for any $\chi_{\{k\}} \otimes f \in L^\infty(\{1,\ldots,n\}) \otimes L^\infty(\Omega)$ and $k \in \{1,2,\ldots,n\}$. Let θ be a $*$-automorphism on $L^\infty(\Omega)$ defined by

$$(\theta f)(x) \;\equiv\; f(Tx) \tag{2.2}$$

for any $f \in L^\infty(\Omega)$ and any $x \in \Omega$. A classical transition expectation with respect to θ from $L^\infty(\{1,\ldots,n\}) \otimes L^\infty(\Omega)$ to $L^\infty(\Omega)$ is

$$\mathcal{E}_{\mathcal{C},\theta} \;\equiv\; \theta \circ \mathcal{E}_\mathcal{C}\,. \tag{2.3}$$

The classical Markov chain on $\otimes_N L^\infty(\{1,\ldots,n\})$ is given by a pair $\psi \equiv \{\mu, \mathcal{E}_\mathcal{C}\}$. The Markov chain $\psi = \{\mu, \mathcal{E}_\mathcal{C}\}$ is stationary and its joint correlation is characterized by the following property:

$$\begin{aligned}\psi(C_{k_1} \cap TC_{k_2} \cap \ldots \cap T^{n-1}C_{k_n})\\ = \mu(\mathcal{E}_{\mathcal{C},\theta}(\chi_{\{k_1\}} \otimes \mathcal{E}_{\mathcal{C},\theta}(\chi_{\{k_2\}} \otimes \mathcal{E}_{\mathcal{C},\theta}(\ldots \mathcal{E}_{\mathcal{C},\theta}(\chi_{\{k_n\}} \otimes I)\ldots))))\end{aligned} \tag{2.4}$$

for any $n \in \mathbb{N}$ and $k_1,\ldots,k_n \in \{1,2,\ldots,n\}$.

The entropy for the stationary Markov chain $\psi = \{\mu, \mathcal{E}_\mathcal{C}\}$ is given by

$$\begin{aligned}\tilde{\mathrm{S}}(\mathcal{C},\theta) \;\equiv\; \lim_{n\to\infty} \frac{-1}{n} \sum_{k_1,k_2,\ldots,k_n \in \{1,2,\ldots,n\}} \psi(C_{k_1} \cap TC_{k_2} \cap \ldots \cap T^{n-1}C_{k_n})\\ \times\; \log \psi(C_{k_1} \cap TC_{k_2} \cap \ldots \cap T^{n-1}C_{k_n})\,.\end{aligned} \tag{2.5}$$

DEFINITION 2.1. The dynamical entropy of the system $(\Omega, \mathcal{F}, \mu, \theta)$ is defined by

$$\tilde{\mathrm{S}}(\theta) \;\equiv\; \sup_\mathcal{C} \tilde{\mathrm{S}}(\mathcal{C},\theta)\,, \tag{2.6}$$

where the supremum is taken over all finite partitions $\mathcal{C}$ of Ω.

3. Construction of Dynamical Entropy Through Quantum Markov Chain

Let $(\mathcal{A}, \Sigma(\mathcal{A}))$ be a von Neumann algebraic system, that is, $\mathcal{A}$ is a von Neumann algebra with an identity operator I acting on a Hilbert space $\mathcal{H}$ and $\Sigma(\mathcal{A})$ is the set of all normal states on $\mathcal{A}$. We denote a finite partition of $I \in \mathcal{A}$ by $\gamma \equiv \{\gamma_j\}$; $\sum_j \gamma_j = I$ $\gamma_i\gamma_j = \gamma_j\delta_{ij}$. Let M_d be the set of all $d \times d$ matrices. For a finite partition γ, a transition expectation $\mathcal{E}_\gamma$ from $M_d \otimes \mathcal{A}$ to $\mathcal{A}$ introduced in [1, 2] is given by

$$\mathcal{E}_\gamma(\tilde{A}) \equiv E_e(p^*_{\gamma,e}\tilde{A}p_{\gamma,e}), \qquad \tilde{A} \in M_d \otimes \mathcal{A}, \tag{3.1}$$

where $p_{\gamma,e} \equiv \sum_j e_{jj} \otimes \gamma_j$ with the matrix unit $e_{jj} \in M_d$, and E_e is a transition expectation from $M_d \otimes \mathcal{A}$ to $\mathcal{A}$ defined by

$$E_e\Big(\sum_{i,j} e_{ij} \otimes A_{ij}\Big) = \sum_i A_{ii}.$$

Let θ be a $*$-automorphism on $\mathcal{A}$ and φ be a state on $\mathcal{A}$. The transition expectation $\mathcal{E}_{\gamma,\theta}$ with respect to θ is given by

$$\mathcal{E}_{\gamma,\theta} \equiv \theta \circ \mathcal{E}_\gamma. \tag{3.2}$$

A quantum Markov chain on $\otimes^{\mathbb{N}} M_d$ is defined by $\psi \equiv \{\varphi, \mathcal{E}_{\gamma,\theta}\} \in \Sigma(\otimes^{\mathbb{N}} M_d)$, where φ is called the initial distribution of ψ. The quantum Markov chain $\psi = \{\varphi, \mathcal{E}_{\gamma,\theta}\}$ is characterized by the following joint correlation

$$\begin{aligned}\psi(j_1(a_1)j_2(a_2)\ldots j_n(a_n)) \\ = \varphi(\mathcal{E}_{\gamma,\theta}(a_1 \otimes \mathcal{E}_{\gamma,\theta}(a_2 \otimes \ldots \otimes \mathcal{E}_{\gamma,\theta}(a_n \otimes I)\ldots)))\end{aligned} \tag{3.3}$$

for each $n \in \mathbb{N}$ and each $a_1, \ldots, a_n \in M_d$, where j_k is an embedding map from M_d into the k-th factor of the tensor product $\otimes^{\mathbb{N}} M_d$ such that

$$j_k(a) \equiv I \otimes \ldots \otimes I \otimes a \otimes I \otimes \ldots$$

Let $P_{\gamma,\theta}$ be a forward Markovian operator from $\mathcal{A}$ to $\mathcal{A}$ given by

$$P_{\gamma,\theta}(A) \equiv \mathcal{E}_{\gamma,\theta}(I \otimes A) = \theta \circ \sum_j \gamma_j A \gamma_j \tag{3.4}$$

for any $A \in \mathcal{A}$. When φ is a stationary state on $\mathcal{A}$, $\varphi \circ \theta = \varphi$, we have $\varphi(P_{\gamma,\theta}A) = \sum_j \varphi(\gamma_j A \gamma_j)$. Only when γ_j is an element of the centralizer $\mathcal{A}_\varphi$ of φ, $\varphi(P_{\gamma,\theta}A) = \varphi(A)$ holds. Suppose that for φ with stationarity there exists unique density operator ρ such that $\varphi(A) = \operatorname{tr}\rho A$ for any $A \in \mathcal{A}$, For any $a_1 \otimes I \in M_d \otimes \mathcal{A}$, we have

$$\begin{aligned}\psi(j_1(a_1)) &= \varphi(\mathcal{E}_{\gamma,\theta}(a_1 \otimes I)) \\ &= \operatorname{tr}_{\mathcal{A}}\rho\mathcal{E}_{\gamma,\theta}(a_1 \otimes I) \\ &= \operatorname{tr}_{\mathcal{A}}\rho\mathcal{E}_\gamma(a_1 \otimes I) \\ &= \operatorname{tr}_{\mathcal{A}}\rho E_e(p^*_{\gamma,e}(a_1 \otimes I)p_{\gamma,e})\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{tr}_{\mathcal{A}}\, \rho E_e\Big((\sum_i e_{ii} \otimes \gamma_i)(a_1 \otimes I)(\sum_k e_{kk} \otimes \gamma_k) \Big) \\
&= \mathrm{tr}_{M_d \otimes \mathcal{A}} (\sum_{i,k} e_{kk} e_{ii} \otimes \gamma_i \rho \gamma_k)(a_1 \otimes I) \\
&= \mathrm{tr}_{M_d \otimes \mathcal{A}} (\sum_k e_{kk} \otimes \gamma_k \rho \gamma_k)(a_1 \otimes I) \\
&= \psi_{[0,1]}(a_1 \otimes I) \\
&= \mathrm{tr}_{M_d} (\sum_k (\mathrm{tr}_{\mathcal{A}}\, \rho \gamma_k) e_{kk}) a_1 \\
&= \psi_1(a_1)\,,
\end{aligned}
$$

where $\mathrm{tr}_{M_d \otimes \mathcal{A}}$ is the trace on $M_d \otimes \mathcal{A}$. The state $\psi_{[0,1]}$ on $M_d \otimes \mathcal{A}$ is constructed by a lifting $\mathcal{E}^*_{\gamma,\theta}$ from $\Sigma(\mathcal{A})$ to $\Sigma(M_d \otimes \mathcal{A})$ in the sense of [3], and the density operator $\rho_{[0,1]}$ of $\psi_{[0,1]}$ is obtained by

$$
\rho_{[0,1]} \;=\; \sum_i e_{ii} \otimes \gamma_i \theta^*(\rho) \gamma_i \;=\; \sum_i e_{ii} \otimes \theta(\gamma_i) \rho \theta(\gamma_i)\,.
$$

Hence the density operator ρ_1 of $\psi_{[0,1]}|M_d$ is given by

$$
\rho_1 \;=\; \mathrm{tr}_{\mathcal{A}}\, \rho_{[0,1]} \;=\; \sum_k (\mathrm{tr}_{\mathcal{A}}\, \rho \theta(\gamma_k)) e_{kk}\,. \tag{3.5}
$$

Similarly we have

$$
\begin{aligned}
&\psi(j_1(a_1) j_2(a_2) \ldots j_n(a_n)) \\
&\quad= \varphi(\mathcal{E}_{\gamma,\theta}(a_1 \otimes \mathcal{E}_{\gamma,\theta}(a_2 \otimes \ldots \mathcal{E}_{\gamma,\theta}(a_n \otimes I) \ldots))) \\
&\quad= \mathrm{tr}_{M_d \otimes \mathcal{A}}\, \mathcal{E}^*_{\gamma}(\rho)(a_1 \otimes \mathcal{E}_{\gamma,\theta}(a_2 \otimes \ldots \mathcal{E}_{\gamma,\theta}(a_n \otimes I) \ldots)) \\
&\quad= \mathrm{tr}_{M_d \otimes \mathcal{A}} (\sum_{i_1} e_{i_1 i_1} \otimes \gamma_{i_1} \rho \gamma_{i_1})(a_1 \otimes \mathcal{E}_{\gamma,\theta}(a_2 \otimes \ldots \theta \circ \mathcal{E}_{\gamma,\theta}(a_n \otimes I) \ldots)) \\
&\quad= \mathrm{tr}_{M_d \otimes \mathcal{A}} \sum_{i_1} e_{i_1 i_1} a_1 \otimes \gamma_{i_1} \rho \gamma_{i_1} \mathcal{E}_{\gamma,\theta}(a_2 \otimes \ldots \theta \circ \mathcal{E}_{\gamma,\theta}(a_n \otimes I) \ldots) \\
&\quad= \mathrm{tr}_{M_d \otimes M_d \otimes \mathcal{A}} \sum_{i_1} \sum_{i_2} e_{i_1 i_1} a_1 \\
&\qquad\qquad \otimes\, (e_{i_2 i_2} \otimes \gamma_{i_2} \theta^*(\gamma_{i_1} \rho \gamma_{i_1}) \gamma_{i_2})\, (a_2 \otimes \mathcal{E}_{\gamma,\theta}(a_3 \ldots \theta \circ \mathcal{E}_{\gamma,\theta}(a_n \otimes I) \ldots)) \\
&\quad= \mathrm{tr}_{M_d \otimes M_d \otimes \mathcal{A}} \sum_{i_1} \sum_{i_2} e_{i_1 i_1} a_1 \otimes e_{i_2 i_2} a_2
\end{aligned}
$$

$$
\begin{aligned}
&\otimes\ (\gamma_{i_2}\theta^*(\gamma_{i_1}\rho\gamma_{i_1})\gamma_{i_2})\,\mathcal{E}_{\gamma,\theta}\,(a_3\ldots\theta\circ\mathcal{E}_{\gamma,\theta}(a_n\otimes I)\ldots) \\
=\ &\mathrm{tr}_{(\overset{n}{\underset{1}{\otimes}} M_d)\otimes\mathcal{A}}\sum_{i_1}\cdots\sum_{i_{n-1}}\sum_{i_n} e_{i_1i_1}a_1\otimes\ldots\otimes e_{i_{n-1}i_{n-1}}a_{n-1}\otimes e_{i_ni_n}a_n \\
&\otimes\gamma_{i_n}\theta^*(\gamma_{i_{n-1}}\ldots\theta^*(\gamma_{i_1}\rho\gamma_{i_1})\ldots\gamma_{i_{n-1}})\gamma_{i_n} \\
=\ &\mathrm{tr}_{(\overset{n}{\underset{1}{\otimes}} M_d)\otimes\mathcal{A}}\sum_{i_1}\cdots\sum_{i_{n-1}}\sum_{i_n} e_{i_1i_1}\otimes\ldots\otimes e_{i_{n-1}i_{n-1}}\otimes e_{i_ni_n} \\
&\otimes\gamma_{i_n}\theta^*(\gamma_{i_{n-1}}\ldots\theta^*(\gamma_{i_1}\rho\gamma_{i_1})\ldots\gamma_{i_{n-1}})\gamma_{i_n}(a_1\otimes\ldots a_{n-1}\otimes a_n\otimes I) \\
=\ &\mathrm{tr}_{(\overset{n}{\underset{1}{\otimes}} M_d)\otimes\mathcal{A}}\rho_{[0,n]}(a_1\otimes\ldots a_{n-1}\otimes a_n\otimes I) \\
=\ &\mathrm{tr}_{(\overset{n}{\underset{1}{\otimes}} M_d)}\sum_{i_1}\cdots\sum_{i_{n-1}}\sum_{i_n}(\mathrm{tr}_{\mathcal{A}}\,\gamma_{i_n}\theta^*(\gamma_{i_{n-1}}\ldots\theta^*(\gamma_{i_1}\rho\gamma_{i_1})\ldots\gamma_{i_{n-1}})\gamma_{i_n}) \\
&\times e_{i_1i_1}\otimes\ldots\otimes e_{i_{n-1}i_{n-1}}\otimes e_{i_ni_n}(a_1\otimes\ldots a_{n-1}\otimes a_n) \\
=\ &\mathrm{tr}_{(\overset{n}{\underset{1}{\otimes}} M_d)}\rho_n(a_1\otimes\ldots a_{n-1}\otimes a_n)\,.
\end{aligned}
$$

Thus, we obtain the density operator $\rho_{[0,n]}$ of $\psi_{[0,n]}$ on $\left(\overset{n}{\underset{1}{\bigotimes}} M_d\right)$ as

$$
\begin{aligned}
\rho_{[0,n]}\ &=\ \sum_{i_1}\cdots\sum_{i_{n-1}}\sum_{i_n} e_{i_1i_1}\otimes\ldots\otimes e_{i_{n-1}i_{n-1}}\otimes e_{i_ni_n} \\
&\qquad\otimes\gamma_{i_n}\theta^*(\gamma_{i_{n-1}}\ldots\theta^*(\gamma_{i_1}\rho\gamma_{i_1})\ldots\gamma_{i_{n-1}})\gamma_{i_n} \\
&=\ \sum_{i_1}\cdots\sum_{i_{n-1}}\sum_{i_n} e_{i_1i_1}\otimes\ldots\otimes e_{i_{n-1}i_{n-1}}\otimes e_{i_ni_n} \\
&\qquad\otimes\theta^{n-1}(\gamma_{i_n})\theta^{n-2}(\gamma_{i_{n-1}})\ldots\gamma_{i_1}\rho\gamma_{i_1}\ldots\theta^{n-2}(\gamma_{i_{n-1}})\theta^{n-1}(\gamma_{i_n})\,.
\end{aligned}
$$

Put $,_{i_n\ldots i_1}=\theta^{n-1}(\gamma_{i_n})\ldots\theta(\gamma_{i_2})\gamma_{i_1}$. Then

$$
\begin{aligned}
\rho_{[0,n]}\ &=\ \sum_{i_1}\cdots\sum_{i_{n-1}}\sum_{i_n} e_{i_1i_1}\otimes\ldots\otimes e_{i_{n-1}i_{n-1}}\otimes e_{i_ni_n}\otimes\,,_{i_ni_{n-1}\ldots i_1}\rho\,,^*_{i_ni_{n-1}\ldots i_1}\,, \\
\rho_n\ &=\ \mathrm{tr}_{\mathcal{A}}\,\rho_{[0,n]}\ =\ \sum_{i_1}\cdots\sum_{i_n}\mathrm{tr}_{\mathcal{A}}\,,_{i_n\ldots i_1}\rho\,,^*_{i_n\ldots i_1}e_{i_1i_1}\otimes\ldots\otimes e_{i_ni_n} \\
&=\ \sum_{i_1}\cdots\sum_{i_n}P_{i_n\ldots i_1}e_{i_1i_1}\otimes\ldots\otimes e_{i_ni_n}\,,
\end{aligned}
\tag{3.6}
$$

where $P_{i_n\ldots i_1}=\mathrm{tr}_{\mathcal{A}}|,_{i_n\ldots i_1}|^2\rho$.

Under the above settings, we define the entropy with respect to γ,θ and n as

$$
\mathrm{S}_n(\gamma,\theta)\ \equiv\ -\mathrm{tr}\,\rho_n\log\rho_n\ =\ -\sum_{i_1,\ldots,i_n}P_{i_n\ldots i_1}\log P_{i_n\ldots i_1}\,, \tag{3.7}
$$

and, subsequantly, the dynamical entropy through a quantum Markov chain with respect to γ and θ is given by

$$\begin{aligned}\tilde{\mathrm{S}}(\gamma;\theta) &\equiv \limsup_{n\to\infty} \frac{1}{n}\mathrm{S}_n(\gamma,\theta) \\ &= \limsup_{n\to\infty} \frac{1}{n}\big(-\sum_{i_1,\ldots,i_n} P_{i_n\ldots i_1}\log P_{i_n\ldots i_1}\big)\,. \end{aligned}\tag{3.8}$$

If the joint probability $P_{i_n\ldots i_1}$ satisfies the Markov property, then the above equality is written as

$$\tilde{\mathrm{S}}(\gamma;\theta) = -\sum_{i_1,i_2} P_{i_1}P(i_2|i_1)\log P(i_2|i_1)\,, \tag{3.9}$$

where $P(i_2|i_1)$ is the conditional probability from i_1 to i_2. $\tilde{\mathrm{S}}(\gamma;\theta)$ has the additivity property in the following sense.

PROPOSITION 3.1. *For two pairs* $(\gamma^{(1)},\theta_1)$ *and* $(\gamma^{(2)},\theta_2)$, *we have*

$$\tilde{\mathrm{S}}(\gamma^{(1)}\otimes\gamma^{(2)};\theta_1\otimes\theta_2) = \tilde{\mathrm{S}}(\gamma^{(1)};\theta_1)+\tilde{\mathrm{S}}(\gamma^{(2)};\theta_2)\,.$$

Proof. Since

$$\theta_1^{n-1}\otimes\theta_2^{n-1}\left(\gamma_{i_n}^{(1)}\otimes\gamma_{k_n}^{(2)}\right) = \theta_1^{n-1}\left(\gamma_{i_n}^{(1)}\right)\otimes\theta_2^{n-1}\left(\gamma_{k_n}^{(2)}\right),$$

$$\begin{aligned}, _{(i_n,k_n)\ldots(i_1,k_1)} &\equiv \theta_1^{n-1}\otimes\theta_2^{n-1}(\gamma_{i_n}^{(1)}\otimes\gamma_{k_n}^{(2)})\ldots\theta_1\otimes\theta_2(\gamma_{i_2}^{(1)}\otimes\gamma_{k_2}^{(2)})\gamma_{i_1}^{(1)}\otimes\gamma_{k_1}^{(2)} \\ &= ,\,^{(1)}_{i_n\ldots i_1}\otimes ,\,^{(2)}_{k_n\ldots k_1}\,,\end{aligned}$$

we have

$$\begin{aligned}P_{(i_n,k_n)\ldots(i_1,k_1)} &\equiv \mathrm{tr}_{\mathcal{A}_1\otimes\mathcal{A}_2}\, ,_{(i_n,k_n)\ldots(i_1,k_1)}\rho_1\otimes\rho_2\, ,^{*}_{(i_n,k_n)\ldots(i_1,k_1)} \\ &= \mathrm{tr}_{\mathcal{A}_1}|,\,^{(1)}_{i_n\ldots i_1}|^2\rho_1\cdot\mathrm{tr}_{\mathcal{A}_2}|,\,^{(2)}_{j_n\ldots j_1}|^2\rho_2\,.\end{aligned}$$

The additivity of $\tilde{\mathrm{S}}(\gamma;\theta)$ follows. □

DEFINITION 3.1. The dynamical entropy through a quantum Markov chain with respect to θ and a subalgebra $\mathcal{B}$ of $\mathcal{A}$ is

$$\tilde{\mathrm{S}}_{\mathcal{B}}(\theta) \equiv \sup\{\tilde{\mathrm{S}}(\gamma;\theta);\ \gamma\subset\mathcal{B}\}\,,$$

where the supremum is taken over all finite partitions of identity $I\in\mathcal{B}$. When $\mathcal{B}=\mathcal{A}$, we simply write $\tilde{\mathrm{S}}_{\mathcal{A}}(\theta)=\tilde{\mathrm{S}}(\theta)$, which is called the dynamical entropy through a quantum Markov chain with respect to θ.

When we take the transition expectation from $M_d \otimes \mathcal{A}$ to $\mathcal{A}$ such that

$$E_e(\sum_{i,j} e_{ij} \otimes A_{ij}) = \frac{1}{d} \sum_{i,j} A_{ij} ,$$

ρ_n is given by

$$\rho_n \equiv \mathrm{tr}_{\mathcal{A}}\, \rho_{[0,n]} = \sum_{i_1,k_1} \dots \sum_{i_n,k_n} (\mathrm{tr}_{\mathcal{A}}\, ,\ {}_{i_n \dots i_1}\rho, \overset{*}{{}_{k_n \dots k_1}})\, e_{i_1 k_1} \otimes \dots \otimes e_{i_n k_n} .$$

In this case, although the joint distribution $P_{i_n \dots i_1}$ is not directly induced, the dynamical entropy through a quantum Markov chain with respect to θ and a subalgebra $\mathcal{A}_1$ of $\mathcal{A}$ can be defined in the same way as above. We will discuss this general case elsewhere.

Our setting for the dynamical entropy through a quantum Markov chain can be further generalized as follows.

Let $\mathcal{A}$ be a $*$-algebra, φ be a state on $\mathcal{A}$ and θ be an endomorphism of $\mathcal{A}$. The triple $(\mathcal{A}, \theta, \varphi)$ is called a $*$-dynamical system with a stationary state φ if $\varphi \circ \theta = \varphi$ holds.

Two such systems $(\mathcal{A}_1, \theta_1, \varphi_1)$ and $(\mathcal{A}_2, \theta_2, \varphi_2)$ are called isomorphic if there exists an isomorphism $v\colon \mathcal{A}_1 \to \mathcal{A}_2$ such that

$$\begin{aligned} \varphi_2 \circ v &= \varphi_1 , \\ v \circ \theta_1 &= \theta_2 \circ v . \end{aligned}$$

Let $(\mathcal{A}, \theta, \varphi)$ be a discrete C^*-dynamical system. For each $d \in \mathbb{N}$ and $(n_1, \dots, n_d) \in \mathbb{N}^d$, we define the map

$$w_{(n_1,\dots,n_d)} : \mathcal{A}^d \times \mathcal{A}^d \to \mathbb{C}$$

by

$$\begin{aligned} w_{(n_1,\dots,n_d)}(A_1, \dots, A_d; B_1, \dots, B_d) &= \\ &\varphi_1(\theta_1^{n_1}(A_1)^* \dots \theta_1^{n_d}(A_d)^* \theta_1^{n_d}(B_d) \cdot \theta_1^{n_1}(B_1)) \end{aligned} \tag{3.10}$$

for any $(A_1, \dots, A_d)$, $(B_1, \dots, B_d) \in \mathcal{A}^d$. It is clear that the family of the maps (3.10) is a projective family of correlation kernels in the sense of [2], hence, by the reconstruction theorem, there exists a stochastic process $\{\otimes^{\mathbb{N}} \mathcal{A}, (j_n)_{n \in \mathbb{N}}\}$ over $\mathcal{A}$ indexed by $\mathbb{N}$ whose family of correlation kernels is given by (3.10). This process is unique up to stochastic equivalence. When φ is stationary, the process is stationary, that is, there exists an endomorphism $u \in \mathrm{End}\,(\mathcal{A})$ such that

$$u \circ j_n = j_{n+1} .$$

In the commutative case, this construction gives the usual stationary process associated to a dynamical system [2].

DEFINITION 3.2. Let $\mathcal{A}_1, \mathcal{A}_2$ be two C^*-algebras and let $w^{(i)}$ be a projective family of correlation kernels over $\mathcal{A}_i$ $(i = 1, 2)$ indexed by $\mathbb{N}$. The two families of correlation kernels $w^{(1)}, w^{(2)}$ are called equivalent if there exists an isomorphism

$$v : \mathcal{A}_1 \to \mathcal{A}_2$$

which intertwines them, that is, for each $d \in \mathbb{N}$, $(n_1, \ldots, n_d) \in \mathbb{N}^d$ and $A_1, \ldots A_d$, $B_1, \ldots, B_d \in \mathcal{A}_1$ one has

$$w^{(1)}_{(n_1,\ldots,n_d)}(A_1, \ldots, A_d; B_1, \ldots, B_d) = w^{(2)}_{(n_1,\ldots,n_d)}(v(A_1), \ldots, v(A_d); v(B_1), \ldots, v(B_d)) .$$

We shall use the notion of equivalence also for families $w_{(n_1,\ldots,n_d)}$ indexed by a proper subset of $\mathbb{N}^d$. Now we introduce the time ordered correlation kernels. They are the kernels $w_{\langle n\rangle}$ with $\langle n\rangle$ of the form

$$\langle n\rangle = (1, 2, \ldots, n)$$

for some $n \in \mathbb{N}$ and we shall use the notation

$$w_{(1,2,\ldots,n)} = w_{\langle n\rangle} .$$

Thus, by definition

$$w_{\langle n\rangle}(A_1, \ldots, A_n; B_1, \ldots, B_n) \;=\; \varphi_1(A_1^*\theta_1(A_2^* \ldots \theta_1(A_{n-1}^*\theta_1(A_n^* B_n)B_{n-1}) \ldots B_2)B_1) .$$

If the state φ_1 is regular enough (e.g., faithful) then the time ordered correlation kernels, even if it is not enough to specify uniquely up to stochastic equivalence the stochastic process associated to the dynamical system, are sufficient to determine the isomorphism class of the dynamical system.

PROPOSITION 3.2. *Two dynamical systems $(\mathcal{A}_i, \theta_i, \varphi_i)$ with faithful state φ_i $(i = 1, 2)$ are isomorphic if and only if the associated time-ordered correlation kernels are equivalent.*

Proof. The necessity is obvious. Assume that the two given processes are isomorphic and let $v: \mathcal{A}_1 \to \mathcal{A}_2$ be an isomorphism such that

$$\varphi_2 \circ v \;=\; \varphi_1 , \qquad v^{-1} \circ \theta_2 \circ v \;=\; \theta_1 .$$

Then, one has $v \circ \theta_1 = \theta_2 \circ v$

$$\begin{aligned} &\varphi_1(A_1^*\theta_1(A_2^* \ldots \theta_1(A_{n-1}^*\theta_1(A_n^* B_n)B_{n-1}) \ldots B_2)B_1) \\ &\quad= \varphi_2(v(A_1)^*\theta_2(v(A_2)^* \ldots \theta_1(A_{n-1}^*\theta_1(A_n^* B_n)B_{n-1}) \ldots))v(B_1)) \\ &\qquad\qquad \vdots \\ &\quad= \varphi_2(v(A_1)^*\theta_2(v(A_2)^* \ldots \theta_2(v(A_{n-1})^*\theta_2(v(A_n)^* v(B_n))v(B_{n-1}) \ldots v(B_2))v(B_1)) , \end{aligned}$$

hence the correlation kernels are equivalent. Conversely, if there exists an isomorphism $v: \mathcal{A}_1 \to \mathcal{A}_2$ which intertwines the time-ordered correlation kernels, then, in particular, for every $A_1, B_1 \in \mathcal{A}_1$ one has

$$\begin{aligned} w^{(1)}_{\langle 2\rangle}(A_1, A_1; B_1, B_1) &= \varphi_1(A_1^* \theta_1(A_1^* B_1) B_1) \\ &= w^{(2)}_{\langle 2\rangle}(v(A_1), v(A_1); v(B_1), v(B_1)) \\ &= \varphi_2(v(A_1)^* \theta_2(v(A_1)^* v(B_1)) v(B_1))) \,. \end{aligned} \tag{3.11}$$

Letting $A_1 = B_1 = B$ in (3.11) we deduce

$$\varphi_1(B) = \varphi_2(v(B)) \,, \qquad B \in \mathcal{A}_1 \,,$$

so that $\varphi_1 = \varphi_2 \circ v$. Using this identity, we can write (3.11) as

$$\varphi_1(A_1^* \theta_1(A_1^* B_1) B_1) = \varphi_1(A_1^* \cdot v^{-1} \cdot \theta_2 \cdot v(A_1^* B_1) B_1)$$

and since A_1, B_1 are arbitrary in $\mathcal{A}_1$, this implies $\theta_1 = v^{-1} \circ \theta_2 \circ v$ which shows the isomorphism of the two dynamical systems because of the faithfulness of φ_1, φ_2. □

The relevance of the above proposition is that, as long as we are interested only in the isomorphism class of the dynamical system $(\mathcal{A}, \theta, \varphi)$, we need only to consider its time ordered correlation kernels.

To every family of time ordered correlation kernels, one can naturally associate an entropy.

Let $\gamma \equiv \{\gamma_j\}_j \in I(\gamma)$, where $I(\gamma)$ is a finite or countable set of discrete partitions of the identity with projections in $\mathcal{A}$. We shall denote

$$P_{i_n,\dots,i_1} = w_{\langle n\rangle}(\gamma_{i_1}, \dots, \gamma_{i_n}; \gamma_{i_1}, \dots, \gamma_{i_n}) \,.$$

The entropy of the probability measure $P_{i_n,\dots,i_1}$ on the space $I(\gamma)^n$ is defined in the usual way

$$\mathrm{S}_n(\gamma; w_{\langle n\rangle}) = - \sum_{i_1,\dots,i_n} P_{i_n,\dots,i_1} \log P_{i_n,\dots,i_1} \,.$$

Because of the projective property of the correlation kernels $w_{\langle n\rangle}$, it follows that the family of probability measures is projective in the sense that

$$P_{i_n,\dots,i_1} = P(i_n | i_{n-1}) P_{i_{n-1},\dots,i_1} \,,$$

hence it defines a unique probability measure P on the space of sequences $I(\gamma)$. Since the family of correlation kernels is stationary, it follows that the probability measure P will also be stationary. Therefore the limit

$$\tilde{\mathrm{S}}(\gamma; w) = \lim_{n \to \infty} \frac{1}{n} \mathrm{S}_n(\gamma; w_{\langle n\rangle})$$

exists. Let $\mathcal{P}(\mathcal{B})$ be a family of partitions of the identity in a subalgebra $\mathcal{B}$ of $\mathcal{A}$.

DEFINITION 3.3. The dynamical scattering entropy of the correlation kernel $w = \{w_{\langle n\rangle} : n \in \mathbb{N}\}$ is

$$\tilde{\mathrm{S}}_{\mathcal{B}}(w) \;=\; \sup\{\tilde{\mathrm{S}}(\gamma; w);\ \gamma \in \mathcal{P}(\mathcal{B})\}\,,$$

where the supremum is taken over all finite or countable partitions of the identity in $\mathcal{P}(\mathcal{B})$ with projections in $\mathcal{B}$. When $\mathcal{B} = \mathcal{A}$, we simply write $\tilde{\mathrm{S}}_{\mathcal{A}}(w) = \tilde{\mathrm{S}}(w)$, which is called the dynamical scattering entropy of the correlation kernel $w = \{w_{\langle n\rangle} : n \in \mathbb{N}\}$.

When $\mathcal{A}$ is a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and φ is a faithful normal state on $\mathcal{A}$ with an automorphism θ such that $\varphi \circ \theta = \varphi$, the dynamical scattering entropy is exactly the same as the dynamical entropy through a quantum Markov chain discussed before. That is, in this case, the correlation kernel becomes

$$w_{\langle n\rangle}(\gamma_{i_1}, \ldots, \gamma_{i_n}; \gamma_{i_1}, \ldots, \gamma_{i_n}) \equiv \varphi(\gamma_{i_1}^* \theta(\gamma_{i_2})^* \ldots \theta^{n-1}(\gamma_{i_n})^* \theta^{n-1}(\gamma_{i_n}) \ldots \gamma_{i_1}) = P_{i_n \ldots i_1}.$$

The example 2 suggests the term dynamical scattering entropy.

4. Calculation of Dynamical Entropy Through QMC For Some Simple Models

In this section, we compute the dynamical entropy through a QMC for several simple models.

4.1. Model 1

Let M_d be a matrix algebra induced by the set of all $d \times d$ matrices acting on d-dimensional Hilbert space $\mathcal{H}_0$, and $\mathcal{A}$ (resp. $\mathcal{H}$) be the infinite tensor product space of M_d (resp. $\mathcal{H}_0$) expressed by

$$\mathcal{A} \;\equiv\; \otimes^{\mathbb{Z}} M_d\,,$$
$$\mathcal{H} \;\equiv\; \otimes^{\mathbb{Z}} \mathcal{H}_0\,.$$

We denote a finite partition of identity $I \in M_d$ by $\gamma_0 \equiv \{\gamma_j^{(0)} = |z_i^{(0)}\rangle\langle z_i^{(0)}|\}$, where $\{z_i^{(0)}\}$ is a CONS (complete orthonormal system) of $\mathcal{H}_0$. Let τ_k be an embedding map from M_d into the k-th factor of the tensor product $\otimes^{\mathbb{Z}} M_d = \mathcal{A}$. For any finite partitions of $\otimes^{\mathbb{Z}} I$ given by $\gamma \equiv \{\gamma_i = \tau_0(\gamma_i^{(0)})\}$, let θ be a Berunoulli shift on $\mathcal{A}$ defined by

$$\theta(\gamma_i) \;\equiv\; \tau_1(\gamma_i)\,.$$

By iteration, θ^k is a map given by

$$\theta^k(\gamma_i) \;=\; \tau_k(\gamma_i)\,.$$

Let ρ_0 be an arbitrary state on $\mathcal{H}_0$ and ρ be $\otimes^{\mathbb{Z}}\rho_0 \in \Sigma(\mathcal{H})$, the set of all density operators on $\mathcal{H}$. Then $,\, _{i_n \dots i_1}$ is obtained by

$$,\, _{i_n \dots i_1} \;=\; \theta^{n-1}(\gamma_{i_n}) \dots \theta(\gamma_{i_2})\gamma_{i_1} \,.$$

For any $\rho = \otimes^{\mathbb{Z}}\rho_0 \in \Sigma(\mathcal{H})$, we have

$$\begin{aligned} \rho_{[0,n]} &= \sum_{i_1,\dots,i_n} e_{i_1 i_1} \otimes \dots \otimes e_{i_n i_n} \otimes ,\, _{i_n \dots i_1}\rho ,\, ^{*}_{i_n \dots i_1} \,, \\ \rho_n &= \operatorname{tr}_{\mathcal{A}} \rho_{[0,n]} \,. \end{aligned}$$

The entropy with respect to γ, θ and n is

$$\mathrm{S}_n(\gamma,\theta) \;=\; -\operatorname{tr}\rho_n \log \rho_n \,.$$

Therefore the dynamical entropy through a quantum Markov chain with respect to γ and θ becomes

$$\begin{aligned} \tilde{\mathrm{S}}(\gamma,\theta) &\equiv \lim_{n\to\infty} \frac{1}{n}\mathrm{S}_n(\gamma,\theta) \\ &= -\sum_i \langle z_i, \rho_0 z_i\rangle \log\langle z_i, \rho_0 z_i\rangle \\ &= \mathrm{S}(\rho_0) \,, \end{aligned}$$

which is exactly the von Neumann entropy of ρ_0.

4.2. Model 2

Let $\mathcal{A}$ be a matrix algebra M_d acting on a Hibert space $\mathcal{H}_0$. For unitary operator U, θ is given by $\theta(A) \equiv UAU^*$ for any $A \in \mathcal{A}$. Let $\{z_j\}$ be a CONS in $\mathcal{H}_0$ and γ_j be $|z_j\rangle\langle z_j|$. Since the following equations

$$\begin{aligned} \theta^{k-1}\gamma_{j_k} &= |U^{k-1}z_{j_k}\rangle\langle U^{k-1}z_{j_k}| \\ ,\, _{j_n \dots j_1} &= \theta^{n-1}\gamma_{j_k} \dots \theta\gamma_{j_2}\gamma_{j_1} \\ &= \prod_{k=1}^{n-1} \langle U z_{j_{k+1}}, z_{j_k}\rangle |U^{n-1} z_{j_n}\rangle\langle z_{j_1}| \end{aligned}$$

hold for any $\rho \in \Sigma(\mathcal{H}_0)$, the set of all density operators on $\mathcal{H}_0$, we have

$$\begin{aligned} \rho_{[0,n]} &= \sum_{i_1,\dots,i_n} e_{i_n i_n} \otimes \dots \otimes e_{i_1 i_1} \\ &\qquad \otimes \prod_{k=1}^{n-1} |\langle U z_{i_{k+1}}, z_{i_k}\rangle|^2 \langle z_{i_1}, \rho z_{i_1}\rangle |U^{n-1} z_{i_n}\rangle\langle U^{n-1} z_{i_n}| \\ \rho_n &\equiv \operatorname{tr}_{\mathcal{A}} \rho_{[0,n]} \end{aligned}$$

$$= \sum_{i_1,\dots,i_n} \sum_{j_1,\dots,j_n} \prod_{k=1}^{n-1} |\langle U z_{j_{k+1}}, z_{j_k}\rangle|^2 \langle z_{j_1}, \rho z_{j_1}\rangle \times e_{i_n j_n} \otimes \ldots \otimes e_{i_1 j_1} ,$$

$P_{i_n \dots i_1}$, $P(i_{k+1}|i_k)$ and P_{i_1} are $\prod_{k=1}^{n-1} |\langle U z_{i_{k+1}}, z_{i_k}\rangle|^2 \langle z_{i_1}, \rho z_{i_1}\rangle$, $|\langle U z_{i_{k+1}}, z_{i_k}\rangle|^2$ ($k = 1, \dots, n$) and $\langle z_{i_1}, \rho z_{i_1}\rangle$. Since the joint probability $P_{i_n \dots i_1}$ satisfies the Markov property, the dynamical entropy $\tilde{\mathrm{S}}_\rho(\gamma;\theta)$ through a QMC with respect to γ and θ is given by

$$\begin{aligned} \tilde{\mathrm{S}}_\rho(\gamma;\theta) &= -\sum_{i_1,i_2} P_{i_1} P(i_2|i_1) \log P(i_2|i_1) \\ &= -\sum_{i_1,i_2} \langle z_{i_1}, \rho z_{i_1}\rangle |\langle U z_{i_2}, z_{i_1}\rangle|^2 \log |\langle U z_{i_2}, z_{i_1}\rangle|^2 . \end{aligned}$$

We have the following result.

PROPOSITION 4.1.

(1) For any $\rho \in \Sigma(\mathcal{H}_0)$ and any $\gamma = \{\gamma_j\}$,

$$0 \le \tilde{\mathrm{S}}_\rho(\gamma;\theta) \le \log d .$$

(2) There exists $\rho^{(u)} \in \Sigma(\mathcal{H}_0)$ and $\gamma^{(u)} = \{\gamma_j^{(u)}\}$ such that

$$\tilde{\mathrm{S}}_{\rho^{(u)}}(\gamma^{(u)};\theta) = \log d .$$

(3) There exists $\rho^{(l)} \in \Sigma(\mathcal{H}_0)$ and $\gamma^{(l)} = \{\gamma_j^{(l)}\}$ such that

$$\tilde{\mathrm{S}}_{\rho^{(l)}}(\gamma^{(l)};\theta) = 0 .$$

Moreover, all intermediate values between 0 and $\log d$ are assumed for some choice of U.

Proof.

(1) Since $-\log P(i_2|i_1) \ge 0$ and $P(i_2|i_1)P_{i_1} \ge 0$ hold for any $i_1, i_2 = 1, \dots, d$,

$$\tilde{\mathrm{S}}_{\rho^{(u)}}(\gamma,\theta) \ge 0 .$$

Moreover the following inequality

$$-P_{i_1} \sum_{i_2} P(i_2|i_1) \log P(i_2|i_1) \le -P_{i_1} \sum_{i_2} \frac{1}{d} \log \frac{1}{d}$$

holds for any $P_{i_i} \in [0,1]$, hence we have

$$-\sum_{i_1} \sum_{i_2} P_{i_1} P(i_2|i_1) \log P(i_2|i_1) \le \log d .$$

(2) When $\rho^{(u)} = \frac{I}{d}$, $\tilde{\mathrm{S}}_{\rho^{(u)}}(\gamma, \theta) = \log d$.

(3) When $\rho^{(u)} = |z_j\rangle\langle z_j|$, $\tilde{\mathrm{S}}_{\rho^{(u)}}(\gamma, \theta) = 0$. □

By taking the eigenvectors of U as z_j, one finds the deterministic chain with minimum entropy. This rules out the use of the dynamical scattering entropy as a dynamical invariant for finite dimensional deterministic systems (they have all the same dynamical scattering entropy).

4.3. Model 3

Let $\mathcal{A}$ be $\otimes^{\mathbb{N}} M_d = B(\otimes^{\mathbb{N}}\mathcal{H}_0)$ and θ be a cyclic shift; that is, (1) $\theta \circ \tilde{\jmath}_k \equiv \tilde{\jmath}_{k+1}$ for $k \in \{1, 2, \ldots, N-1\}$ and (2) $\theta \circ \tilde{\jmath}_N \equiv \tilde{\jmath}_1$. Let γ_{j_1} be $|z_{j_1}\rangle\langle z_{j_1}|$, where $z_{j_1} = \sum_{\tilde{i}_1} \lambda_{\tilde{i}_1}^{(j_1)} |x_{i_1(1)} \otimes \ldots \otimes x_{i_1(N)}\rangle$, $\tilde{i}_k \equiv (i_k(1), \ldots, i_k(N))$, and $\{x_{i_1(k)}\}$ be a CONS of $\mathcal{H}_0$. Since the following equations

$$
\begin{aligned}
\theta^{k-1}\gamma_{j_k} &= |z_{j_k}^{(k-1)}\rangle\langle z_{j_k}^{(k-1)}| \,, \\
z_{j_k}^{(k-1)} &= \sum_{\tilde{i}_k} \lambda_{\tilde{i}_k}^{(j_k)} |x_{i_k(k \bmod N)} \otimes \ldots \otimes x_{i_k(N-k+1 \bmod N)}\rangle \,, \\
{}_{,\, j_n \ldots j_1} &= \theta^{n-1}\gamma_{j_k} \ldots \theta\gamma_{j_2}\gamma_{j_1} \\
&= \prod_{k=1}^{n-1} \langle z_{j_{k+1}}^{(k)}, z_{j_k}^{(k-1)}\rangle |z_{j_n}^{(n-1)}\rangle\langle z_{j_1}| \,,
\end{aligned}
$$

hold for any $\rho = \otimes^{\mathbb{N}}\rho_0 \in \otimes^{\mathbb{N}}\Sigma(\mathcal{H}_0)$, we have

$$
\begin{aligned}
\rho_{[0,n]} &= \sum_{j_1,\ldots,j_n} e_{j_n j_n} \otimes \ldots \otimes e_{j_1 j_1} \\
&\quad \otimes \prod_{k=1}^{n-1} |\langle z_{j_{k+1}}^{(k)}, z_{j_k}^{(k-1)}\rangle|^2 \langle z_{j_1}, \rho z_{j_1}\rangle |z_{j_n}^{n-1}\rangle\langle z_{j_n}^{n-1}| \\
\rho_n &\equiv \mathrm{tr}_{\mathcal{A}}\, \rho_{[0,n]} \\
&= \sum_{j_1,\ldots,j_n} e_{j_n j_n} \otimes \ldots \otimes e_{j_1 j_1} \\
&\quad \times \prod_{k=1}^{n-1} |\langle z_{j_{k+1}}^{(k)}, z_{j_k}^{(k-1)}\rangle|^2 \langle z_{j_1}, \rho z_{j_1}\rangle \,,
\end{aligned}
$$

$P_{j_n \ldots j_1}$, $P(j_{k+1}|j_k)$ and P_{j_1} are $\prod_{k=1}^{n-1} |\langle z_{j_{k+1}}^{(k)}, z_{j_k}^{(k-1)}\rangle|^2 \langle z_{j_1}, \rho z_{j_1}\rangle$, $|\langle z_{j_{k+1}}^{(k)}, z_{j_k}^{(k-1)}\rangle|^2$ $(k = 1, \ldots, n-1)$ and $\langle z_{j_1}, \rho z_{j_1}\rangle$, respectively. Since the joint probability $P_{j_n \ldots j_1}$ satisfies the Markov property, the dynamical entropy $\tilde{\mathrm{S}}_\rho(\gamma;\theta)$ through a QMC with respect to γ and θ is given by

$$
\tilde{\mathrm{S}}_\rho(\gamma;\theta)^{(N)} = -\frac{1}{N} \sum_{j_1, j_2} P_{j_1} P(j_2|j_1) \log P(j_2|j_1)
$$

$$= -\frac{1}{N}\sum_{j_1,j_2}\Big|\sum_{\tilde{i}''_1}\langle z_{j_1},\rho z_{j_1}\rangle\bar{\lambda}^{(i_2)}_{i''_1(2)\dots i''_1(N)i''_1(1)}\lambda^{(i_1)}_{\tilde{i}''_1}\Big|^2$$
$$\times \log\Big|\sum_{\tilde{i}''_1}\bar{\lambda}^{(i_2)}_{i''_1(2)\dots i''_1(N)i''_1(1)}\lambda^{(i_1)}_{\tilde{i}''_1}\Big|^2 .$$

The above coefficients $\lambda_{\tilde{i}}$ satisfy the following conditions:

$$\sum_{j_1}\gamma_{j_1} = I \;\Rightarrow\; \sum_{j_1}\lambda^{(j_1)}_{\tilde{i}_1}\bar{\lambda}^{(j_1)}_{\tilde{i}'_1} = \prod_{k=1}^{N}\delta_{i_1(k)i'{}_1(k)} ,$$
$$\gamma_{j_1}\gamma_{j_2} = \delta_{j_1j_2}\gamma_{j_1} \;\Rightarrow\; \sum_{\tilde{i}_1}\lambda^{(j_1)}_{\tilde{i}_1}\bar{\lambda}^{(j_2)}_{\tilde{i}_1} = \delta_{j_1j_2} ,$$
$$\gamma^*_{j_1} = \gamma_{j_1} \;\Rightarrow\; \lambda^{(j_1)}_{\tilde{i}_1}\bar{\lambda}^{(j_1)}_{\tilde{i}'_1} = \bar{\lambda}^{(j_1)}_{\tilde{i}_1}\lambda^{(j_1)}_{\tilde{i}'_1} ,$$

from the properties of the partition $\gamma = \{\gamma_j\}$. We have the same result of the model 2. Its proof is essentially the same, so that we omit it here.

PROPOSITION 4.2. *(1) For any $\rho \in \otimes^{\mathbb{N}}\Sigma(\mathcal{H}_0)$ and any $\gamma = \{\gamma_j\}$,*

$$0 \;\le\; \tilde{\mathrm{S}}_\rho(\gamma;\theta) \;\le\; \log d$$

holds.

(2) There exists $\rho^{(u)} \in \otimes^{\mathbb{N}}\Sigma(\mathcal{H}_0)$ and $\gamma^{(u)} = \{\gamma^{(u)}_j\}$ such that

$$\tilde{\mathrm{S}}_{\rho^{(u)}}(\gamma^{(u)};\theta) \;=\; \log d .$$

(3) There exists $\rho^{(l)} \in \otimes^{\mathbb{N}}\Sigma(\mathcal{H}_0)$ and $\gamma^{(l)} = \{\gamma^{(l)}_j\}$ such that

$$\tilde{\mathrm{S}}_{\rho^{(l)}}(\gamma^{(l)};\theta) \;=\; 0 .$$

4.4. Model 4

Let $\mathcal{A}$ be $\otimes^{\mathbb{Z}}(M_d\otimes M_d)$ and θ be a shift defined by $\theta(A_1\otimes A_2) \equiv I\otimes A_1\otimes A_2$ for any $A_i \in M_d$ $(i = 1,2)$ and $I \in M_d$. Let γ_{j_1} be $|z_{j_1}\rangle\langle z_{j_1}|$, where $z_{j_1} = \sum_{i_1,k_1}\lambda^{(j_1)}_{i_1k_1}x_{i_1}\otimes x_{k_1}$ and $\{x_{i_1}\}$ be a CONS in $\mathcal{H}_0$. Since the following equations

$$\theta^{k-1}\gamma_{j_k} \;=\; I\otimes\ldots\otimes I\otimes\gamma_{j_k}$$
$$,\; {}_{j_n\dots j_1} \;=\; \theta^{n-1}\gamma_{j_k}\ldots\theta\gamma_{j_2}\gamma_{j_1}$$
$$= \sum\lambda^{(j_1)}_{i_1k_1}\bar{\lambda}^{(j_1)}_{i'_1k'_1}\Big(\prod_{\ell=2}^{n-1}\lambda^{(j_\ell)}_{i_\ell k_\ell}\bar{\lambda}^{(j_\ell)}_{k_{\ell-1}k'_\ell}\Big)\lambda^{(j_n)}_{i_nk_n}\bar{\lambda}^{(j_n)}_{i'_nk'_n}$$

$$\times\ |x_{i_1}\rangle\langle x_{i_1'}|\otimes\Big(\bigotimes_{r=1}^{n-2}|x_{i_{r+1}}\rangle\langle x_{k_r'}|\Big)\otimes|x_{k_n}\rangle\langle x_{k_n'}|$$

hold for any $\rho=\bigotimes_{-\infty}^{\infty}\rho_0\in\bigotimes_{-\infty}^{\infty}\Sigma(\mathcal{H}_0\otimes\mathcal{H}_0)$, we have

$$\begin{aligned}\rho_{[0,n]} &= \sum_{j_1,\dots,j_n} e_{j_nj_n}\otimes\ldots\otimes e_{j_1j_1}\otimes\Big|\sum\big(\prod_{\ell=2}^{n-1}\lambda_{i_\ell k_\ell}^{(j_\ell)}\bar{\lambda}_{k_{\ell-1}k_\ell'}^{(j_\ell)}\big)\Big|^2\\ &\quad\times\prod_{r=1}^{n-2}\langle x_{k_r'},\rho_0x_{x_r'}\rangle\langle z_{j_1},\rho z_{j-1}\rangle\Big(\bigotimes_{t=1}^{n-2}|x_{i_{t+1}}\rangle\langle x_{k_{t+1}}|\Big)\,,\\ \rho_n &\equiv \operatorname{tr}_{\mathcal{A}}\rho_{[0,n]}\\ &= \sum_{j_1,\dots,j_n} e_{j_nj_n}\otimes\ldots\otimes e_{j_1j_1}\otimes\Big|\sum\big(\prod_{\ell=2}^{n-1}\lambda_{i_\ell k_\ell}^{(j_\ell)}\bar{\lambda}_{k_{\ell-1}k_\ell'}^{(j_\ell)}\big)\Big|^2\\ &\quad\times\prod_{r=1}^{n-2}\langle x_{k_r'},\rho_0x_{k_r''}\rangle\langle z_{j_1},\rho z_{j-1}\rangle\,,\end{aligned}$$

$P_{j_n\dots j_1}$, $P(j_2|j_1)$ and P_{j_1} are $\Big|\sum\big(\prod_{\ell=2}^{n-1}\lambda_{i_\ell k_\ell}^{(j_\ell)}\bar{\lambda}_{k_{\ell-1}k_\ell'}^{(j_\ell)}\big)\Big|^2$, $\sum\lambda_{i_1k_1}^{(j_1)}\bar{\lambda}_{i_1k_1'''}^{(j_1)}\bar{\lambda}_{k_1k_2'}^{(j_2)}\lambda_{k_1'''k_2''}^{(j_2)}\ \langle x_{k_2'},\rho_0x_{k_2''}\rangle$ and $\langle z_{j_1},\rho z_{j_1}\rangle$. Since the joint probability $P_{j_n\dots j_1}$ satisfies the Markov property, the dynamical entropy $\tilde{\mathrm{S}}_\rho(\gamma;\theta)$ through a QMC with respect to γ and θ becomes

$$\begin{aligned}\tilde{\mathrm{S}}_\rho(\gamma;\theta)^{(2)} &= -\frac{1}{2}\sum_{j_1,j_2}P_{j_1}P(j_2|j_1)\log P(j_2|j_1)\\ &= -\frac{1}{2}\sum_{j_1,j_2}\sum\lambda_{i_1k_1}^{(j_1)}\langle z_{j_1},\rho z_{j_1}\rangle\bar{\lambda}_{i_1k_1'''}^{(j_1)}\bar{\lambda}_{k_1k_2'}^{(j_2)}\lambda_{k_1'''k_2''}^{(j_2)}\langle x_{k_2'},\rho_o x_{k_2''}\rangle\\ &\quad\times\ \log\sum\lambda_{i_1k_1}^{(j_1)}\bar{\lambda}_{i_1k_1'''}^{(j_1)}\bar{\lambda}_{k_1k_2'}^{(j_2)}\lambda_{k_1'''k_2''}^{(j_2)}\langle x_{k_2'},\rho_0x_{k_2''}\rangle\,.\end{aligned}$$

4.5. Model 5

Let $\mathcal{A}_1,\mathcal{A}_2$ be two von Neumann algebras acting on Hilbert spaces $\mathcal{H}_1,\mathcal{H}_2$, respectively. Let U_k be a partial isometry operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ ($k=1,\dots,d$). We define a transition expectation $\mathcal{E}$ from $\mathcal{A}_2\otimes\mathcal{A}_1$ to $\mathcal{A}_1$ by

$$\mathcal{E}(B\otimes A)\ =\ \sum_{k=1}^{d}U_k^*BU_k\varphi_0(\xi_k^{1/2}A\xi_k^{1/2})\,,$$

where φ_0 is a stationary state on $\mathcal{A}_1$, and $\xi_k\in\mathcal{A}_1$ satisfies: (1) $\xi_k\geq 0$; (2) $\sum_k\xi_k=I$. Put $p_k=\varphi_0(\xi_k)$. Then

$$\mathcal{E}(B\otimes 1)\ =\ \sum_k U_k^*BU_kp_k\mathcal{E}(\gamma_{j_{n-1}}\otimes\mathcal{E}(\gamma_{j_n}\otimes I))\ =\ \sum_{k_n}\mathcal{E}(\gamma_{j_{n-1}}\otimes U_{k_n}^*\gamma_{j_n}U_{k_n})p_{k_n}\,,$$

$$
\begin{aligned}
P_{j_n \ldots j_1} &= \varphi_0(\mathcal{E}(\gamma_{j_1} \otimes \mathcal{E}(\gamma_{j_2} \otimes \ldots \otimes \mathcal{E}(\gamma_{j_n} \otimes I) \ldots))) \\
&= \sum_{k_n} \varphi_0(\mathcal{E}(\gamma_{j_1} \otimes \mathcal{E}(\gamma_{j_2} \otimes \ldots \mathcal{E}(\gamma_{j_{n-1}} \otimes U^*_{x_n} \gamma_{j_n} U_{k_n}) \ldots))) p_{k_n} \\
&= \sum_{k_{n-1}, k_n} \varphi_0(\mathcal{E}(\gamma_{j_1} \otimes \mathcal{E}(\gamma_{j_2} \otimes \ldots \mathcal{E}(\gamma_{j_{n-2}} \otimes U^*_{k_{n-1}} \gamma_{j_{n-1}} U_{k_{n-1}}) \ldots))) \\
&\qquad \times \varphi_0(\xi_{k_{n-1}} U^*_{k_n} \gamma_{j_n} U_{k_n}) p_{k_n} \\
&= \sum_{k_1, \ldots, k_n} \varphi_0(U^*_{k_1} \gamma_{j_1} U_{k_1}) \varphi_0(\xi_{k_1} U^*_{k_2} \gamma_{j_2} U_{k_2}) \ldots \varphi_0(\xi_{k_{n-2}} U^*_{k_{n-1}} \gamma_{j_{n-1}} U_{k_{n-1}}) \\
&\qquad \times \varphi_0(\xi_{k_{n-1}} U^*_{k_n} \gamma_{j_n} U_{k_n}) p_{k_n} \, . \qquad (4.1)
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\tilde{\mathrm{S}}_\rho(\gamma; U) &= -\lim_{n \to \infty} \Big(\frac{1}{n} \sum_{j_1, \ldots, j_n} P_{j_n, \ldots, j_1} \log P_{j_n, \ldots, j_1} \Big) \\
= -\lim_{n \to \infty} \Big[\frac{1}{n} & \sum_{j_1, \ldots, j_n} \Big(\sum_{k_1, \ldots, k_n} \varphi_0(U^*_{k_1} \gamma_{j_1} U_{k_1}) \\
&\times \varphi_0(\xi_{k_1} U^*_{k_2} \gamma_{j_2} U_{k_2}) \ldots \varphi_0(\xi_{k_{n-2}} U^*_{k_{n-1}} \gamma_{j_{n-1}} U_{k_{n-1}}) \varphi_0(\xi_{k_{n-1}} U^*_{k_n} \gamma_{j_n} U_{k_n}) p_{k_n} \Big) \\
&\times \log \Big(\sum_{k_1, \ldots, k_n} \varphi_0(U^*_{k_1} \gamma_{j_1} U_{k_1}) \qquad (4.2) \\
&\times \varphi_0(\xi_{k_1} U^*_{k_2} \gamma_{j_2} U_{k_2}) \ldots \varphi_0(\xi_{k_{n-2}} U^*_{k_{n-1}} \gamma_{j_{n-1}} U_{k_{n-1}}) \varphi_0(\xi_{k_{n-1}} U^*_{k_n} \gamma_{j_n} U_{k_n}) p_{k_n} \Big) \Big] \, .
\end{aligned}
$$

The relation between the dynamical entropies by complexity and by QMC is discussed in [4].

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