# SOLUTIONS

*Aucun problème n'est immuable. L'éditeur est toujours heureux d'envisager la publication de nouvelles solutions ou de nouvelles perspectives portant sur des problèmes antérieurs.*

We have received a late batch of correct solutions to problems 3478, 3479, 3480, 3481, 3482, 3483, and 3486 from Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

**3488**. [2009 : 515, 517] *Proposed by Pham Huu Duc, Ballajura, Australia.*

Let $a$, $b$, and $c$ be positive real numbers. Prove that

$$\frac{a}{2a^2+bc}+\frac{b}{2b^2+ca}+\frac{c}{2c^2+ab}\leq\sqrt{\frac{a^{-1}+b^{-1}+c^{-1}}{a+b+c}}\,.$$

*Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.*

Let $t=(t_1,t_2,\ldots,t_n)$ and $s=(s_1,s_2,\ldots,s_n)$ be arbitrary $n$-tuples of nonnegative real numbers. We will write $t\succ s$ if

(i) $t_1\geq\cdots\geq t_n$ and $s_1\geq\cdots\geq s_n$,

(ii) $\sum\limits_{i=1}^{k}t_i\geq\sum\limits_{i=1}^{k}s_i$ for all $k=1,2,\ldots,n$, with equality when $k=n$.

Let $\mathbb{R}_+$ denote the set of positive real numbers, let $P$ be the set of all permutations of $\{1,2,\ldots,n\}$, and define $[t]:\mathbb{R}_+^n\to\mathbb{R}$ by

$$[t](x)\;=\;\sum_{\sigma\in P}x_{\sigma(1)}^{t_1}x_{\sigma(2)}^{t_2}\cdots x_{\sigma(n)}^{t_n}\qquad\text{for all }\;x=(x_1,x_2,\ldots,x_n)\,.$$

Muirhead's inequality states that if $t\succ s$, then $[t]\geq[s]$. Here, as usual, $[t]\geq[s]$ means that $[t](x)\geq[s](x)$ for all $x\in\mathbb{R}_+^n$. Now, by squaring and simplifying, the given inequality is equivalent to $A\geq B$, where

$$\begin{aligned}A\;&=\;12[8,5,1]+23[7,4,3]+16[6,6,2]+12[8,4,2]+4[7,6,1]\\&\qquad\qquad+4[9,3,2]+8[7,7,0]\,,\\B\;&=\;12[7,5,2]+22[6,5,3]+26[6,4,4]+\frac{5}{2}[8,3,3]+\frac{33}{2}[5,5,4]\,.\end{aligned}$$

But this last inequality holds by these applications of Muirhead's inequality:

$$\begin{aligned}[8,5,1]\;&\geq\;[7,5,2]\,,\\ [7,4,3]\;&\geq\;[6,5,3]\quad\text{and}\quad[7,4,3]\geq[6,4,4]\,,\\ [8,4,2]\;&\geq\;[8,3,3]\quad\text{and}\quad[8,4,2]\geq[6,4,4]\,,\\ [6,6,2]\;&\geq\;[6,4,4]\quad\text{and}\quad[6,6,2]\geq[5,5,4]\,,\end{aligned}$$

$$
\begin{aligned}
[7,6,1] &\geq [5,5,4]\,, \\
[9,3,2] &\geq [5,5,4]\,, \\
[7,7,0] &\geq [5,5,4]\,.
\end{aligned}
$$

*Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. One incomplete solution was submitted.*

**3489**. [2009 : 515, 517] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $n$ be a nonnegative integer. Prove that

$$\frac{1}{2^{n-1}}\sum_{k=0}^{n}\sqrt{k}\binom{2n}{k} \;\leq\; \sqrt{n\left(2^{2n}+\binom{2n}{n}\right)}\,.$$

*A composite of similar solutions by George Apostolopoulos, Messolonghi, Greece; and Albert Stadler, Herrliberg, Switzerland.*

By using the elementary facts that $\binom{2n}{k}=\binom{2n}{2n-k}$ for $0\leq k\leq 2n$ and $k\binom{2n}{k}=2n\binom{2n-1}{k-1}$ for $1\leq k\leq 2n$, and also the Cauchy–Schwarz Inequality, we have that

$$
\begin{aligned}
\left[\sum_{k=0}^{n}\sqrt{k}\binom{2n}{k}\right]^2 &= \left[\sum_{k=0}^{n}\sqrt{\binom{2n}{k}}\sqrt{k}\sqrt{\binom{2n}{k}}\right]^2 \\
&\leq \left[\sum_{k=0}^{n}\binom{2n}{k}\right]\left[\sum_{k=0}^{n}k\binom{2n}{k}\right] \\
&= \frac{1}{2}\left[\binom{2n}{n}+\sum_{k=0}^{2n}\binom{2n}{k}\right]\left[2n\sum_{k=1}^{n}\binom{2n-1}{k-1}\right] \\
&= \frac{1}{2}\left[\binom{2n}{n}+2^{2n}\right]\left[2n\sum_{k=0}^{n-1}\binom{2n-1}{k}\right] \\
&= \frac{1}{2}\left[\binom{2n}{n}+2^{2n}\right]\left[2n\cdot\frac{1}{2}\sum_{k=0}^{2n-1}\binom{2n-1}{k}\right] \\
&= \frac{1}{2}\left[2^{2n}+\binom{2n}{n}\right]\left(n\cdot 2^{2n-1}\right) \\
&= n\left(2^{2n}+\binom{2n}{n}\right)\cdot 2^{2n-2}\,,
\end{aligned}
$$

from which the claimed inequality follows immediately.

*Also solved by ARKADY ALT, San Jose, CA, USA; DIONNE CAMPBELL, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer.*

**3490**. [2009 : 515, 518] *Proposed by Michael Rozenberg, Tel-Aviv, Israel.*

Let $a$, $b$, and $c$ be nonnegative real numbers such that $a+b+c=1$. Prove that

(a) $\sqrt{9-32ab}+\sqrt{9-32ac}+\sqrt{9-32bc}\geq 7$;

(b) $\sqrt{1-3ab}+\sqrt{1-3ac}+\sqrt{1-3bc}\geq\sqrt{6}$.

*Solution to part (a) by Oliver Geupel, Brühl, NRW, Germany; solution to part (b) by George Apostolopoulos, Messolonghi, Greece, modified by the editor.*

(a) For nonnegative integers $\ell$, $m$, and $n$, let $[\ell,m,n]=\sum\limits_{\text{symm.}} a^\ell b^m c^n$.

The following inequality is a consequence of Muirhead's Theorem,

$$\begin{aligned}
&27\prod_{\text{cyclic}}\Big(9(a+b+c)^2-32ab\Big)-\Big(11(a+b+c)^2+16(ab+bc+ca)\Big)^3\\
&=9176\,[6,0,0]+34320\,[5,1,0]-36336\,[4,2,0]+50184\,[4,1,1]\\
&\qquad-\;54352\,[3,3,0]+100320\,[3,2,1]-103312\,[2,2,2]\\
&\geq 0\,.
\end{aligned}$$

We put $a+b+c=1$ in the above, and we observe that by the AM-GM Inequality $\sum\limits_{\text{cyclic}}\sqrt{(9-32ab)(9-32bc)}\geq 3\Big(\prod\limits_{\text{cyclic}}(9-32ab)\Big)^{1/3}$. It follows that $\sum\limits_{\text{cyclic}}\sqrt{(9-32ab)(9-32bc)}\geq 11+16(ab+bc+ca)$, and we deduce

$$\left(\sum_{\text{cyclic}}\sqrt{9-32ab}\right)^2\;\geq\;49\,,$$

from which the inequality in (a) follows.

(b) Let $x=3a$, $y=3b$, and $z=3c$. Then $x$, $y$, and $z$ are nonnegative real numbers such that $x+y+z=3$, and we are to show that

$$\sum_{\text{cyclic}}\sqrt{3-xy}\;\geq\;3\sqrt{2}\,. \tag{1}$$

Note first that $\sum_{\text{cyclic}} \sqrt{\frac{(3+z)^2}{8}} = \frac{1}{\sqrt{8}} \sum_{\text{cyclic}} (3+z) = \frac{1}{\sqrt{8}}(9+3) = 3\sqrt{2}$.

Also,

$$\begin{aligned} \frac{(3+z)^2}{8} - \left(3 - \frac{(x+y)^2}{4}\right) &= \frac{(3+z)^2}{8} - 3 + \frac{(3-z)^2}{4} \\ = \frac{1}{8}\left(9 + 6z + z^2 - 24 + 18 - 12z + 2z^2\right) &= \frac{3}{8}(z-1)^2 \,. \qquad (2) \end{aligned}$$

Hence,

$$\frac{(3+z)^2}{8} \;\geq\; 3 - \frac{(x+y)^2}{4}\,, \qquad (3)$$

and (1) is equivalent to

$$\begin{aligned} &\sum_{\text{cyclic}} \left(\sqrt{3 - xy} - \sqrt{3 - \frac{(x+y)^2}{4}}\right) \\ &\qquad\qquad \geq \sum_{\text{cyclic}} \left(\sqrt{\frac{(3+z)^2}{8}} - \sqrt{3 - \frac{(x+y)^2}{4}}\right) . \qquad (4) \end{aligned}$$

Let $H$ and $K$ denote the left and right side of (4), respectively. Then

$$\begin{aligned} H &= \frac{1}{4} \sum_{\text{cyclic}} \frac{(x-y)^2}{\sqrt{3 - xy} + \sqrt{3 - \frac{(x+y)^2}{4}}} \\ &\geq \frac{1}{4} \sum_{\text{cyclic}} \frac{(x-y)^2}{\sqrt{3} + \sqrt{3}} = \frac{1}{8\sqrt{3}} \sum_{\text{cyclic}} (x-y)^2 \,. \qquad (5) \end{aligned}$$

On the other hand, using (2) and (3), we have

$$\begin{aligned} K &= \sum_{\text{cyclic}} \frac{\frac{(3+z)^2}{8} - 3 + \frac{(x+y)^2}{4}}{\sqrt{\frac{(3+z)^2}{8}} + \sqrt{3 - \frac{(x+y)^2}{4}}} = \frac{3}{8} \sum_{\text{cyclic}} \frac{(z-1)^2}{\sqrt{\frac{(3+z)^2}{8}} + \sqrt{3 - \frac{(x+y)^2}{4}}} \\ &\leq \frac{3}{8} \sum_{\text{cyclic}} \frac{(z-1)^2}{2\sqrt{3 - \frac{(x+y)^2}{4}}} \leq \frac{3}{8} \sum_{\text{cyclic}} \frac{(z-1)^2}{2\sqrt{3 - \frac{9}{4}}} = \frac{\sqrt{3}}{8} \sum_{\text{cyclic}} (z-1)^2 \,. \qquad (6) \end{aligned}$$

Finally,

$$\begin{aligned} \sum_{\text{cyclic}} (x-y)^2 &= 3\left(\sum_{\text{cyclic}} x^2\right) - \left(\sum_{\text{cyclic}} x\right)^2 = 3\left(\sum_{\text{cyclic}} x^2\right) - 9 \\ &= 3\left(\sum_{\text{cyclic}} x^2\right) - 6\left(\sum_{\text{cyclic}} x\right) + 9 = 3\left(\sum_{\text{cyclic}} (z-1)^2\right) . \qquad (7) \end{aligned}$$

From (5), (6), and (7) we get $H \geq K$, establishing (4), and hence (1).

*Part (b) was also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer. Two incomplete solutions were submitted.*

*The case of equality was not requested, though Geupel claimed equality precisely when* $a = b = c = 1/3$, *but the proposer noted that equality also occurs when* $a = b = 1/2$, $c = 0$.

**3491**. [2009 : 515, 518] *Proposed by Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Romania.*

Let $a_1, a_2, \ldots, a_{n+1}$ be positive real numbers where $a_{n+1} = a_1$. Prove that

$$\sum_{i=1}^{n} \frac{a_i^4}{(a_i + a_{i+1})(a_i^2 + a_{i+1}^2)} \ \geq\ \frac{1}{4} \sum_{i=1}^{n} a_i \,.$$

*Solution by George Apostolopoulos, Messolonghi, Greece.*

Let

$$A \ =\ \sum_{i=1}^{n} \frac{a_i^4}{(a_i + a_{i+1})(a_i^2 + a_{i+1}^2)}\,,$$

$$B \ =\ \sum_{i=1}^{n} \frac{a_{i+1}^4}{(a_i + a_{i+1})(a_i^2 + a_{i+1}^2)}\,.$$

Then

$$A - B = \sum_{i=1}^{n} \frac{a_i^4 - a_{i+1}^4}{(a_i + a_{i+1})(a_i^2 + a_{i+1}^2)} = \sum_{i=1}^{n} a_i - a_{i+1} = 0\,,$$

and hence $A = B$.

We now show that for all positive real numbers $a$ and $b$ we have

$$a^4 + b^4 \ \geq\ \frac{(a+b)^2(a^2+b^2)}{4}\,.$$

Indeed, using the inequality $(x+y)^2 \leq 2(x^2+y^2)$ twice we obtain

$$(a+b)^2(a^2+b^2) \ \leq\ 2(a^2+b^2)^2 \ \leq\ 4(a^4+b^4)\,.$$

Hence,

$$\begin{aligned} 2A \ =\ A + B \ &=\ \sum_{i=1}^{n} \frac{a_i^4 + a_{i+1}^4}{(a_i + a_{i+1})(a_i^2 + a_{i+1}^2)} \\ &\geq\ \frac{1}{4}\sum_{i=1}^{n}(a_i + a_{i+1}) \ =\ \frac{1}{2}\sum_{i=1}^{n} a_i\,. \end{aligned}$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

*Also solved by ARKADY ALT, San Jose, CA, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer.*

**3492★**. [2009 : 515, 518] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $P$ be a point in the interior of tetrahedron $ABCD$ such that each of $\angle PAB$, $\angle PBC$, $\angle PCD$, and $\angle PDA$ is equal to $\arccos\sqrt{\frac{2}{3}}$. Prove that $ABCD$ is a regular tetrahedron and that $P$ is its centroid.

The problem remains open. The only submission, from Peter Y. Woo, Biola University, La Mirada, CA, USA, gave a counterexample where $ABCD$ is a degenerate tetrahedron. In particular, he provided an elegant proof that if $P$ is the centre of a parallelogram $ABCD$ with sides $AD = BC = 3\sqrt{2}$ and $AB = CD = \sqrt{6}$, and diagonals $AC = 2\sqrt{3}$ and $AC = 6$, then

$$\angle PAB = \angle PBC = \angle PCD = \angle PDA = \arccos\sqrt{\frac{2}{3}}\,.$$

This certainly addresses the question that was asked, and it suggests that there are infinitely many tetrahedra with an interior point $P$ that satisfies the given angle requirement, but it fails to provide an explicit nondegenerate example.

**3494**. [2009 : 516, 518] *Proposed by Michel Bataille, Rouen, France.*

Let $n > 1$ be an integer and for each $k = 1, 2, \ldots, n$ let

$$\sigma(n,k) \;=\; \sum_{1\le i_1<\cdots<i_k\le n} i_1 i_2\cdots i_k\,.$$

Prove that

$$\sum_{k=1}^{n}\frac{\ln n}{n+1-k}\cdot\sigma(n,k) \;\sim\; (n+1)! \;\sim\; \sum_{k=1}^{n}\frac{n+1-k}{\ln n}\cdot\sigma(n,k)\,,$$

where $f(n) \sim g(n)$ means that $\dfrac{f(n)}{g(n)} \to 1$ as $n \to \infty$.

*Solution by the proposer.*

Let

$$\begin{aligned} P_n(x) &= (x+1)(x+2)\cdots(x+n) \\ &= x^n + \sigma(n,1)x^{n-1} + \cdots + \sigma(n,n-1)x + \sigma(n,n)\,. \end{aligned}$$

If $U_n$ denotes $\sum\limits_{k=1}^{n}\dfrac{\sigma(n,k)}{n+1-k}$, then

$$U_n \;=\; \left(\int_0^1 P_n(x)\,dx\right) - \frac{1}{n+1}\,.$$

Clearly, $\dfrac{P_n'(x)}{P_n(x)} = \dfrac{1}{x+1} + \dfrac{1}{x+2} + \cdots + \dfrac{1}{x+n}$, so that for all $x \in [0,1]$,

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1} \;\leq\; \frac{P_n'(x)}{P_n(x)} \;\leq\; 1 + \frac{1}{2} + \cdots + \frac{1}{n}\,. \tag{1}$$

Multiplying by $P_n(x)$ and integrating over $[0,1]$ leads to

$$(H_{n+1} - 1)\left(U_n + \frac{1}{n+1}\right) \;\leq\; P_n(1) - P_n(0) \;\leq\; H_n\left(U_n + \frac{1}{n+1}\right)\,,$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denotes the $n^{\text{th}}$ harmonic number. Since we also have $P_n(1) - P_n(0) = (n+1)! - n! = \frac{n}{n+1} \cdot (n+1)!$, we obtain

$$\frac{n}{n+1} \cdot \frac{(n+1)!}{H_n} - \frac{1}{n+1} \;\leq\; U_n \;\leq\; \frac{n}{n+1} \cdot \frac{(n+1)!}{H_{n+1} - 1} - \frac{1}{n+1}$$

for all positive integers $n$. Recalling that $H_n \sim \ln(n)$, the Squeeze Theorem for limits yields $\lim\limits_{n\to\infty} \dfrac{U_n \ln(n)}{(n+1)!} = 1$, that is,

$$\sum_{k=1}^{n} \frac{\ln(n)}{n+1-k} \cdot \sigma(n,k) \sim (n+1)!\,.$$

Let $V_n = \sum\limits_{k=1}^{n} (n+1-k)\sigma(n,k)$. From (1) and $P_n(1) = (n+1)!$, we deduce that

$$(H_{n+1} - 1)(n+1)! \;\leq\; P_n'(1) \;\leq\; H_n(n+1)!\,.$$

Also, for $n > 1$,

$$\begin{aligned} V_n &= \sum_{k=1}^{n} (n-k)\sigma(n,k) + \sum_{k=1}^{n} \sigma(n,k) \\ &= P_n'(1) - n + (n+1)! - 1 \\ &= P_n'(1) + (n+1)! - (n+1)\,, \end{aligned}$$

so that

$$\begin{aligned} \frac{H_{n+1} - 1}{\ln(n)} + \frac{1}{\ln(n)} - \frac{1}{n!\ln(n)} &\leq \frac{V_n}{(n+1)!\ln(n)} \\ &\leq \frac{H_n}{\ln(n)} + \frac{1}{\ln(n)} - \frac{1}{n!\ln(n)}\,. \end{aligned}$$

Again, the Squeeze Theorem yields $(n+1)! \sim \sum\limits_{k=1}^{n} \dfrac{n+1-k}{\ln(n)} \cdot \sigma(n,k)$, and the proof is complete.

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; and ALBERT STADLER, Herrliberg, Switzerland.*

**3495**. [2009 : 516, 518] *Proposed by Cosmin Pohoaţă, Tudor Vianu National College, Bucharest, Romania.*

Let $a$, $b$, $c$ be positive real numbers with $a+b+c=2$. Prove that

$$\frac{1}{2}+\sum_{\text{cyclic}}\frac{a}{b+c} \;\le\; \sum_{\text{cyclic}}\frac{\left(a^2+bc\right)}{b+c} \;\le\; \frac{1}{2}+\sum_{\text{cyclic}}\frac{a^2}{b^2+c^2}\,.$$

*A combination of solutions by George Apostolopoulos, Messolonghi, Greece and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy, modified by the editor.*

For vectors $a=(a_1,a_2,\ldots,a_n)$ and $(b_1,b_2,\ldots,b_n)$ with real entries, the notation $a\prec b$ means that $a_1+a_2+\cdots+a_n=b_1+b_2+\cdots+b_n$ and $a_1+a_2+\cdots+a_j\le b_1+b_2+\cdots+b_j$ holds for each $j=1,\,2,\,\ldots,\,n-1$.

Since $a+b+c=2$, the inequality on the left is equivalent to

$$\left(\frac{1}{2}+\sum_{\text{cyclic}}\frac{a}{b+c}\right)\frac{a+b+c}{2}\le\sum_{\text{cyclic}}\frac{a^2+bc}{b+c}$$

or

$$\sum_{\text{symmetric}}\left(a^4+a^2bc\right)\ge\sum_{\text{symmetric}}\left(a^4+a^2bc\right).$$

Schur's Inequality yields

$$\sum_{\text{symmetric}}\left(a^4+a^2bc\right)\;\ge\;2\sum_{\text{symmetric}}\left(a^3b\right).$$

Now using Muirhead's inequality for $(2,2,0)\prec(3,1,0)$ we obtain

$$\sum_{\text{symmetric}}\left(a^3b\right)\ge\sum_{\text{symmetric}}\left(a^2b^2\right),$$

which proves the inequality on the left.

Now the inequality on the right is equivalent to

$$\sum_{\text{cyclic}}\frac{a^2+bc}{b+c}\le\left(\frac{1}{2}+\sum_{\text{cyclic}}\frac{a^2}{b^2+c^2}\right)\frac{a+b+c}{2},$$

or

$$\sum_{\text{symmetric}}\left(2a^9b+4a^8bc+7a^7b^2c+a^7b^3+2a^4b^4c^2\right)$$
$$\ge\sum_{\text{symmetric}}\left(2a^6b^4+a^5b^5+5a^5b^3c^2+a^4b^3c^3+5a^5b^4c+2a^6b^3c\right).$$

Using Muirhead's inequality repeatedly we obtain:

$$(6,4,0) \prec (9,1,0) \implies \sum_{\text{symmetric}} 2a^9b \geq \sum_{\text{symmetric}} 2a^6b^4$$

$$(6,2,2) \prec (7,2,1) \implies \sum_{\text{symmetric}} 2a^7b^2c \geq \sum_{\text{symmetric}} 2a^6b^2c^2$$

$$(6,3,1) \prec (8,1,1) \implies \sum_{\text{symmetric}} 2a^8bc \geq \sum_{\text{symmetric}} 2a^6b^3c$$

$$(5,4,1) \prec (8,1,1) \implies \sum_{\text{symmetric}} 2a^8bc \geq \sum_{\text{symmetric}} 2a^5b^4c$$

$$(5,5,0) \prec (7,3,0) \implies \sum_{\text{symmetric}} a^7b^3 \geq \sum_{\text{symmetric}} a^5b^5$$

$$(5,3,2) \prec (7,2,1) \implies \sum_{\text{symmetric}} a^7b^2c \geq \sum_{\text{symmetric}} a^5b^3c^2$$

$$(4,3,3) \prec (7,2,1) \implies \sum_{\text{symmetric}} a^7b^2c \geq \sum_{\text{symmetric}} a^4b^3c^3$$

$$(5,4,1) \prec (7,2,1) \implies \sum_{\text{symmetric}} a^7b^2c \geq \sum_{\text{symmetric}} a^5b^4c$$

Also, by the AM-GM Inequality, we have

$$\sum_{\text{symmetric}} (2a^6b^2c^2 + 2a^4b^4c^2) \geq \sum_{\text{symmetric}} 4a^5b^3c^2\,.$$

We add all these inequalities, and we are done.

*Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TITU ZVONARU, Comănești, Romania; and the proposer. One incomplete solution was submitted.*

*Zvonaru observed that this problem appeared in the book* Old And New Inequalities, Vol. 2, *by Vo Quoc Ba Can and Cosmin Pohoata, Gil Publishing House, 2008.*

**3496**. [2009 : 516, 519] *Proposed by Elias C. Buissant des Amorie, Castricum, the Netherlands.*

Prove the following equations:

(a) $\tan 72^\circ = \tan 66^\circ + \tan 36^\circ + \tan 6^\circ$.

(b)★ $\tan 84^\circ = \tan 78^\circ + \tan 72^\circ + \tan 60^\circ$;

[*Ed.: The proposer gave six more relations of the form* $f(\theta) = \sum_{i=1}^{4} \tan k_i\theta = 0$ *for* $k_i \in \mathbb{Z}$ *and* $\theta = 2\pi/n$ *with* $n|360$, *not included here for lack of space.*]

*Composite of solutions by Kee-Wai Lau, Hong Kong, China and D.J. Smeenk, Zaltbommel, the Netherlands.*

With the help of appropriate trigonometric identities, both equations can be reduced to properties of the golden section $\tau = \frac{1+\sqrt{5}}{2}$, which is the positive root of the quadratic equation

$$\tau^2 \;=\; \tau + 1\,. \tag{1}$$

Because $\tau$ is the ratio of a diagonal to a side of a regular pentagon, it satisfies

$$\cos 36^\circ = \frac{\tau}{2} \quad \text{and} \quad \cos 72^\circ = \frac{1}{2\tau}\,. \tag{2}$$

(a) The following equations are equivalent.

$$\begin{aligned}
\tan 72^\circ &= \tan 66^\circ + \tan 36^\circ + \tan 6^\circ\,,\\
\tan 72^\circ - \tan 36^\circ &= \tan 66^\circ + \tan 6^\circ\,,\\
\frac{\sin(72^\circ - 36^\circ)}{\cos 72^\circ \cos 36^\circ} &= \frac{\sin(66^\circ + 6^\circ)}{\cos 66^\circ \cos 6^\circ}\,,\\
2\sin 36^\circ \cos 66^\circ \cos 6^\circ &= 2\sin 72^\circ \cos 72^\circ \cos 36^\circ \;=\; \sin 144^\circ \cos 36^\circ\,,\\
2\sin 36^\circ \cos 66^\circ \cos 6^\circ &= \sin 36^\circ \cos 36^\circ\,,\\
2\cos 66^\circ \cos 6^\circ &= \cos 36^\circ\,,\\
\cos 72^\circ + \cos 60^\circ &= \cos 36^\circ\,,\\
\cos 72^\circ - \cos 36^\circ + \frac{1}{2} &= 0\,,
\end{aligned}$$

and the last equality follows immediately from equations (1) and (2).

(b) The following equations are equivalent.

$$\begin{aligned}
\tan 84^\circ &= \tan 78^\circ + \tan 72^\circ + \tan 60^\circ\,,\\
\tan 84^\circ - \tan 60^\circ &= \tan 78^\circ + \tan 72^\circ\,,\\
\frac{\sin(84^\circ - 60^\circ)}{\cos 84^\circ \cos 60^\circ} &= \frac{\sin(78^\circ + 72^\circ)}{\cos 78^\circ \cos 72^\circ} \;=\; \frac{1}{2\cos 78^\circ \cos 72^\circ}\,,\\
\cos 84^\circ &= 4\sin 24^\circ \cos 72^\circ \cos 78^\circ\,,\\
\sin 6^\circ &= 2(\sin 96^\circ - \sin 48^\circ)\cos 78^\circ\,,\\
\sin 6^\circ &= (\sin 174^\circ + \sin 18^\circ) - (\sin 126^\circ - \sin 30^\circ)\,,\\
\sin 6^\circ &= \sin 6^\circ + \cos 72^\circ - \cos 36^\circ + \frac{1}{2}\,,
\end{aligned}$$

and the last equality follows immediately from the equations (1) and (2) just as in part (a).

*Both parts were also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; DIONNE*

*CAMPBELL, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; JAN VERSTER, Kwantlen University College, BC; and TITU ZVONARU, Comănești, Romania. STAN WAGON, Macalester College, St. Paul, MN, USA gave a computer verification.*

*Part (a) was also solved by ARKADY ALT, San Jose, CA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.*

*Wagon used* Mathematica *to confirm that there are seven* $4$*-tuples* $(a, b, c, d)$ *of distinct integers between* $0$ *and* $90$ *(other than the pair featured in our problem) that satisfy the relation* $\tan a^\circ = \tan b^\circ + \tan c^\circ + \tan d^\circ$*, namely*

$$(60; 42, 36, 6),\quad (72; 60, 42, 24),\quad (78; 66, 60, 36),\quad (78; 72, 42, 36)$$
$$(60; 50, 20, 10),\quad (70; 60, 40, 10),\quad (80; 70, 60, 50).$$

*The first four are clearly related to the golden section as in our featured pair, while the final three seem to be related to the regular enneagon (or nonagon, if you prefer) as discussed in "Trigonometry and the Nonagon" by Andrew Jobbings (see www.arbelos.co.uk/papers.html). It is amusing to note that the proposer thought that he had found one that fails to fit either of the two patterns, but it turns out that* $\tan 62^\circ$ *differs from* $\tan 48^\circ + \tan 24^\circ + \tan 18^\circ$ *by about* $10^{-5}$*. Wagon further produced a list of* $49$ *such equations allowing repeated angles, and determined that there were no such* $3$*-term equations and no such* $5$*-term equations.*

**3497**. [2009 : 516, 519] *Proposed by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.*

Let $P$ be a point in the interior of triangle $ABC$, and let $r$ be the inradius of $ABC$. Prove that $\max\{AP, BP, CP\} \geq 2r$.

**I.** *Solution by Roy Barbara, Lebanese University, Fanar, Lebanon.*

Recall that the convex hull of a triangle $T$ is the union of its interior and boundary. If $C$ is a circle with radius $r$ in the convex hull of a triangle $T_1$ with inradius $r_1$, then $r \leq r_1$. (Here is a proof of this simple fact: Consider the three tangents to $C$ that are parallel to the sides of $T_1$ and separate the centre of $C$ from the corresponding sides; they form a triangle that is similar to $T_1$ for which $C$ is the incircle. Since all points of $C$ are inside or on $T_1$, the ratio of the sides of the new triangle to the sides of $T_1$—which is also the ratio of the inradii—could be at most $1$; that is, $r \leq r_1$.) Let $T = \triangle ABC$ be an arbitrary triangle with incircle $C$ and inradius $r$, and let $P$ be a point in the convex hull of $T$. Without loss of generality, we may assume that max$\{AP$, $BP$, $CP\} = AP$ and show that $AP \geq 2r$. Extend (if necessary) the segments $PB$ to $PB_1$ and $PC$ to $PC_1$ such that $PB_1 = PC_1 = PA$. Then $P$ is the circumcentre of triangle $T_1 = \triangle AB_1C_1$, and $PA$ its circumradius; let $r_1$ denote its inradius. Note that because $P$ is assumed to lie in the convex hull of $T$, $T$ must lie in the convex hull of $T_1$; consequently the incircle of $T$ also lies in that convex hull, so that (from our simple fact)

$$r_1 \geq r\,.$$

By Euler's inequality, $AP \geq 2r_1$, whence $AP \geq 2r$, as desired.

**II.** *Solution by Michel Bataille, Rouen, France.*

Generalization: The following result holds for any point $P$ in the plane of $\triangle ABC$. Let $R$, $O$, $a$, $b$, and $c$ be the circumradius, circumcentre, and sides of $\triangle ABC$, and let $M = \max\{AP, BP, CP\}$; then

(a) *if $\triangle ABC$ is acute, $M \geq R \geq 2r$, with $M = 2r$ if and only if $P = O$ and the triangle is equilateral*;

(b) *if $\triangle ABC$ is not acute, $M \geq \dfrac{\max\{a,b,c\}}{2} \geq 2r$, with $M = 2r$ if and only if $P$ is the midpoint of the longest side.*

Let $A'$, $B'$, and $C'$ be the midpoints of the sides opposite vertices $A$, $B$, and $C$, respectively. For part (a) we fix points $D$, $E$, and $F$ on the perpendicular bisectors of the sides so that the rays $[OD)$, $[OE)$, and $[OF)$ are opposite the rays $[OA')$, $[OB')$, and $[OC')$, respectively. The whole plane is the union of the nonoverlapping angles $\angle EOF$, $\angle FOD$, and $\angle DOE$. Without loss of generality we can assume that $P$ is in or on the sides of angle $\angle EOF$ (bounded by the rays $[OE)$ and $[OF)$) so that $M = PA$. Let $E_0$ on $AB$ and $F_0$ on $AC$ be such that $OE_0 || AC$ and $OF_0 || AB$. Note that because $O$ is in the interior of $\triangle ABC$, $E_0$ and $F_0$ belong to the rays $[AB)$ and $[AC)$, while $OE_0 \perp OE$ and $OF_0 \perp OF$. Since $A$ is in the interior of $\angle E_0OF_0$, the angle $\angle POA$ is obtuse, hence $M = PA \geq OA = R$, with equality exactly when $P = O$. The inequality $R \geq 2r$ is Euler's inequality, with $R = 2r$ exactly when $\triangle ABC$ is equilateral, so the proof of part (a) is complete.

For part (b) we first suppose that $\angle BAC$, say, is obtuse. Then $O$ is exterior to $\triangle ABC$ with line $BC$ separating $O$ from $A$, and the plane is the union of the three angles $\angle EOF$, $\angle EOA'$, and $\angle FOA'$. If $P$ is in $\angle EOF$ then $M = PA \geq R > \frac{a}{2}$ (much as in part (a)). Otherwise, without loss of generality, we can suppose that $P$ is in $\angle EOA'$, in which case $M = PC \geq A'C = \frac{a}{2}$. To check that the minimum value of $M$, namely $\frac{a}{2}$, occurs when $P = A'$, note that $A$ and $A'$ are on the same side of the perpendicular bisector of the segment $AC$, so that $A'A < A'C$; that is, if $P = A'$, then $M = A'C = A'B = \frac{a}{2}$. If $\angle BAC = 90^\circ$, this argument can easily be adapted to show that $M \geq \frac{a}{2} = R$. To complete the proof we show that in the present case we have $\frac{a}{2} \geq 2r$. Let $h = AH$ be the altitude from $A$, and let $A_0$ be the point on the ray $[HA)$ such that $\angle BA_0C = 90^\circ$. We want to show that $ah \geq 4rh$; that is, that $a+b+c \geq 4h$ (since $\frac{ah}{2} = \frac{r(a+b+c)}{2} =$ area($\triangle ABC$)). But

$$h \leq HA_0 = \sqrt{HB \cdot HC} \leq \frac{HB + HC}{2} = \frac{a}{2}\,,$$

whence $a \geq 2h$; moreover, $b$, $c \geq h$, so that $a + b + c \geq 4h$, as desired.

*Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece (2 solutions); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo,* *Bosnia and Herzegovina; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; VICTOR PAMBUCCIAN, Arizona State University West, Phoenix, AZ, USA; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; GEORGE TSINTSIFAS, Thessaloniki, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There were two incomplete submissions.*

*Tsintsifas extended the result to $n$-dimensional Euclidean space: For a point $P$ in the interior of the simplex $A_1A_2\ldots A_{n+1}$, $\max\{A_1P,\, A_2P,\, \ldots,\, A_{n+1}P\} \geq nr$.*

*Janous pointed out that the inequality follows from the more general assertion that $AP + BP + CP \geq 6r$, which is item 12.14 of* O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff Publ., Groningen, 1969.

**3498**. [2009 : 517, 519] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $F_n$ be the $n^{\text{th}}$ Fibonacci number, that is, $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For each positive integer $n$, prove that

$$\sqrt{\frac{F_{n+3}}{F_n}} + \sqrt{\frac{F_n + F_{n+2}}{F_{n+1}}} > 1 + 2\left(\sqrt{\frac{F_n}{F_{n+3}}} + \sqrt{\frac{F_{n+1}}{F_n + F_{n+2}}}\right).$$

*Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.*

Let $x = \sqrt{\frac{F_{n+3}}{F_n}}$ and $y = \sqrt{\frac{F_n + F_{n+2}}{F_{n+1}}}$. The claimed inequality is successively equivalent to

$$\begin{aligned} x + y &> 1 + 2\left(\frac{1}{x} + \frac{1}{y}\right), \\ \left(1 - \frac{2}{xy}\right)(x+y) &> 1. \end{aligned}$$

It thus suffices to show that the following two inequalities hold:

$$1 - \frac{2}{xy} \geq \frac{1}{3}, \tag{1}$$

$$x + y > 3. \tag{2}$$

Set $\lambda = \frac{F_{n+1}}{F_n}$. Then

$$\begin{aligned} xy &= \sqrt{\frac{F_{n+3}}{F_n} \cdot \frac{F_{n+2} + F_n}{F_{n+1}}} \\ &= \sqrt{\frac{(2F_{n+1} + F_n)(F_{n+1} + 2F_n)}{F_nF_{n+1}}} \\ &= \sqrt{(2\lambda + 1)\left(1 + \frac{2}{\lambda}\right)}. \end{aligned}$$

Hence, (1) is equivalent to each of

$$\begin{aligned}\sqrt{(2\lambda+1)\left(1+\frac{2}{\lambda}\right)} &\geq 3\,,\\ \frac{2\,(\lambda-1)^2}{\lambda} &\geq 0\,,\end{aligned}$$

and the latter is clearly true.

By the AM–GM Inequality,

$$\begin{aligned}x+y &> 2\cdot\sqrt[4]{\frac{F_{n+3}\,(F_n+F_{n+2})}{F_nF_{n+1}}}\\ &= 2\cdot\sqrt[4]{(2\lambda+1)\left(1+\frac{2}{\lambda}\right)}\,.\end{aligned}$$

For (2), it thus suffices to show that

$$(2\lambda+1)\left(1+\frac{2}{\lambda}\right) > \frac{81}{16}\,,$$

which is equivalent to

$$\frac{32\lambda^2-\lambda+32}{16\lambda} > 0\,,$$

which is clearly true.

*Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. Two incomplete solutions were submitted.*

**3499★**. [2009 : 517, 519] *Proposed by Bernardo Recamán, Instituto Alberto Merani, Bogotá, Colombia.*

A building has $n$ floors numbered $1$ to $n$ and a number of elevators all of which stop at both floors $1$ and $n$, and possibly other floors. For each $n$, find the least number of elevators needed in this building if between any two floors there is at least one elevator that connects them non-stop.

For example, if $n=6$, nine elevators suffice: $(1,6)$, $(1,5,6)$, $(1,4,6)$, $(1,3,4,6)$, $(1,2,4,5,6)$, $(1,2,5,6)$, $(1,2,6)$, $(1,3,5,6)$, and $(1,2,3,6)$.

*Solution by George Apostolopoulos, Messolonghi, Greece.*

The answer is $\left\lfloor\dfrac{n^2}{4}\right\rfloor$.

To see that at least this many elevators are needed, consider the set

$$P = \left\{(x,y)\in\mathbb{Z}^2 : 1\le x,y\le n,\ x\le \frac{n}{2},\ y>\frac{n}{2}\right\}.$$

Any elevator can connect at most one pair of floors in the set $P$, and the cardinality of $P$ is $\left\lfloor \frac{n^2}{4}\right\rfloor$, so at least this many elevators are needed.

To show that $\left\lfloor \frac{n^2}{4}\right\rfloor$ elevators suffice, we give a construction in two cases.

**Case 1:** $n = 2k$. Here $k^2$ elevators are needed. Let integers $i$ and $j$ be restricted so that $1\le i\le k$ and $k+1\le j\le 2k$, and describe each elevator by the tuple of floors it stops at. The elevators are then

$$\begin{cases} (1,i,j,2k)\,, & \text{if } i+j = 2k+1\,,\\ (1,2k+1-j,i,j,2k)\,, & \text{if } i+j > 2k+1\,,\\ (1,i,j,2k+1-j,2k)\,, & \text{if } i+j < 2k+1\,. \end{cases}$$

**Case 2:** $n = 2k+1$. Here $k^2+k$ elevators are needed. Let integers $i$ and $j$ be restricted so that $1\le i\le k$ and $k+1\le j\le 2k+1$, and describe each elevator by the tuple of floors it stops at. The elevators are then

$$\begin{cases} (1,i,j,2k+1)\,, & \text{if } i+j = 2k+2\,,\\ (1,2k+2-j,i,j,2k+1)\,, & \text{if } i+j > 2k+2\,,\\ (1,i,j,2k+2-j,2k+1)\,, & \text{if } i+j < 2k+2\,. \end{cases}$$

This completes the proof.

*Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; D.P. MEHENDALE (Dept. of Electronics) and M.R. MODAK, (formerly of Dept. Mathematics), S. P. College, Pune, India; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; MORTEN H. NIELSEN, University of Winnipeg, Winnipeg, MB; and PETER Y. WOO, Biola University, La Mirada, CA, USA. Two incomplete solutions were submitted.*