PROBLEM DEPARTMENT

ASHLEY AHLIN AND HAROLD REITER*

DRAFT - SPRING 2011

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to `hbreiter@uncc.edu`. *Electronic submissions using LaTeX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by October 1, 2011. Solutions identified as by students are given preference.*

Problems for Solution.

1235. *Proposed by Parviz Khalili, Christopher Newport University, Newport News, VA.*

Let x, y, and z are positive real numbers and $x + y + z = 1$. Show that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 3 + 2\sqrt{\frac{x^x\, y^y\, z^z}{x\ y\ z}}.$$

1236. *Proposed by Mohsen Soltanifar, University of Saskatchewan, Saskatoon, Canada.*

Prove or give a counterexample:Let $U_i(1 \leq i \leq n)$, be finite dimensional subspaces of a vector space V. Then, the dimension of $\sum_{i=1}^{n} U_i$ is given by :
$\dim(\sum_{i=1}^{n} U_i) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1<i_2<...<i_r:n} \dim(U_{i_1} \cap U_{i_2} \cap ... \cap U_{i_r})$,
where the summation $\sum_{i_1<i_2<...<i_r:n} \dim(U_{i_1} \cap U_{i_2} \cap ... \cap U_{i_r})$ is taken over all of the $\binom{n}{r}$ possible subsets of the set $1, 2, ..., n$.

1237. *Proposed by Thomas Dence, Ashland University, Ashland, OH and Joseph Dence, St. Louis, MO.*

For each integer $n \geq 2$, determine the values of the integrals

$$I_{n,3} = \int_0^{\pi} \sin^3 x \sin(nx)\ dx \text{ and } I_{n,5} = \int_0^{\pi} \sin^5 x \sin(nx)\ dx.$$

1238. *Proposed by Tuan Le, student, Fairmont High School, Fairmont, CA.*

Given $a, b, c, d \in [0, 1]$ such that no two of them are simultaneously equal to 0. Prove that:

$$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+d^2} + \frac{1}{d^2+a^2} \geq \frac{8}{3+abcd}$$

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*University of North Carolina Charlotte

1239. *Proposed by Matthew McMullen, Otterbein College, Westerville, OH.*

For $i \in \{1, 2, \ldots, 9\}$, define D_i to be the set of all positive integers that begin with i. For all positive integers n, define

$$a_{n,i} = \frac{1}{n} \cdot \Big| D_i \cap \{1, 2, \ldots, n\} \Big|.$$

Find $\limsup_{n\to\infty} a_{n,i}$ and $\liminf_{n\to\infty} a_{n,i}$.

1240. *Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Rome, Italy.*

Let x, y be positive real numbers. Prove that

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} \leq \sqrt{xy} + \frac{x+y}{2} + \frac{(L(x,y) - \sqrt{xy})^2}{\frac{2xy}{x+y}}$$

where $L(x, y) = (x - y)/(\ln(x) - \ln(y))$ if $x \neq y$ and $L(x, x) = x$.

1241. *Proposed by Arthur Holshouser, Charlotte, NC. and Johannas Winterink, Albuquerque, NM.*

For each $i = 0, 1, 2, 3, \ldots, 8$, does a set $\{A, B, C\}$ of three circles in the plane exist such that there are exactly i circles in the plane that are tangent to each of A, B, and C?

1242. *Proposed by Stas Molchanov, University of North Carolina Charlotte.*

This problem has an interesting history. It is mentioned in the book **Mathematicians Also Like Jokes**, by A. Fonin, Moskow, 'Nauka' 2010. The problem appeared on the admissions test for Moskow State University. The exams were very hard, given that there was one place for every 700 applicants. The applicant took the problem to physicist E. Lefshitz, who himself was unable to solve. Lefshitz asked his friend Lev Landau, nobel prize winning physicist about the problem. Landau rightly considered himself an expert in elementary mathematics and the same evening called Lefshitz back to say he'd solved it in an hour, and that nobody, 'except possibly Yakov Zelodovich could solve it faster'. Landau sent the problem to Zeldovich, and indeed he solved the problem in 45 minutes.

Given pyramid $ABCD$ with bottom face triangle ABC with $BC = a, AC = b, AB = c$. Let the lateral faces BCD, ACD, ABD form with the bottom angles α, β, γ, in radians, all acute angles. Find the radius r of the sphere inscribed in the pyramid.

Solutions. The editors regret that the fall 2010 issue of the Journal, we failed to acknowledge solutions of problems 1218, 1221, and 1223 by **M. Parvez Shaikh**, Albany Medical Center, Albany, NY.

1226. *Proposed by Jonathan Hulgan and Cecil Rousseau, University of Memphis, Memphis, TN.*

In part two of his justly famous *Textbook of Algebra*, G. Chrystal poses two intriguing problems of a similar nature. Suitably generalized, problem 2 on page 33 reads

> Segments of length $1, 2, \ldots, n$ are given. The number of triangles that can be built using three of these segments as edges is _________.

Problem 9 on page 34 reads

> Out of n straight lines $1, 2, \ldots, n$ inches long respectively, four can be chosen to form a pericyclic quadrilateral in _________ ways.

The answers have been excised. Without solving either of these two problems, prove that the answers are the same. A pericyclic quadrilateral is one that has an inscribed circle.

Solution by the **Proposers**.

Consider the set of all triples (a, b, c) of positive integers satisfying $1 < c < b < a \leq n$. There is a triangle with sides a, b, c if and only if the triangle inequality $b+c > a$ holds. Consider the same triple (a, b, c) as three sides of a would-be pericyclic quadrilateral. Then a, b, c, d are the sides of such a quadrilateral if and only if $a+d = b+c$. In other words, set $d := b+c-a$; then $\{a, b, c, d\}$ is a good choice in problem 9 if and only if $d \in \mathbb{Z}^+$. To see that the two answers are the same, just observe that

$$b + c > a \Leftrightarrow d := b + c - a \in \mathbb{Z}^+$$

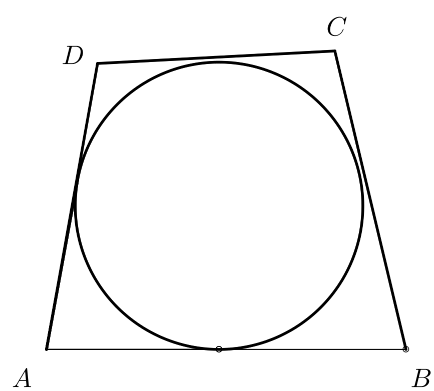
■

For the record,

$$f(n) = \begin{cases} \frac{1}{24}\, n(n-2)(2n-5), & n \text{ even}, \\ \\ \frac{1}{24}\, (n-1)(n-3)(2n-1), & n \text{ odd} \end{cases}$$

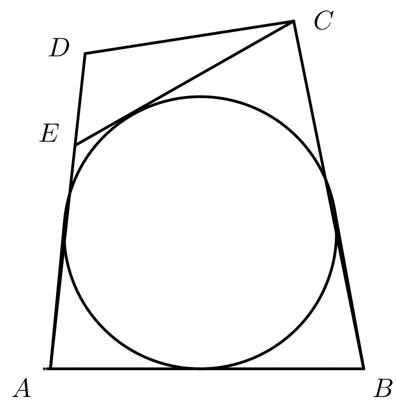
is the answer to both problem 2 and problem 9. .2in] The following proofs are intended as an aid for the editors, and not as part of the published solution.
1. Proposition. Convex quadrilateral $ABCD$ has an inscribed circle if and only if

$$AB + CD = AD + BC.$$



Proof. Suppose $ABCD$ has an inscribed circle. Then $AB + CD = AD + BC$ follows easily using equal tangents. Conversely, suppose $AB + CD = AD + BC$.

There exists a circle that is tangent to $\overline{DA}, \overline{AB}$, and $\overline{BC}$. The center of this circle is the common point of the angle bisectors of $\angle DAB$ and $\angle ABC$. If this circle is tangent to $\overline{CD}$, we are done. Otherwise, draw from C a line tangent to the circle and intersecting $\overline{AD}$ at $E \neq D$.



Assume that E is between A and D; the other case is similar. Because $ABCE$ has an inscribed circle,

$$AB + CE = AE + BC.$$

But

$$AB + CD = AD + BC$$

as well, so $CD - CE = AD - AE = ED$ and thus $CD = CE + ED$, which violates the triangle inequality. Hence $ABCD$ has an inscribed circle. ■

2. Calculation of the Number of Triples that Yield Pericyclic Quadrilaterals.

Let $f(n)$ count the triples (a, b, c) of positive integers that satisfy $1 < c < b < a \leq n$ where $d = b + c - a \in \mathbb{Z}^+$ and let $g(k)$ count those for which $a = k$. Then

$$f(n) = f(n-1) + g(n) \qquad \text{and} \qquad f(n) = \sum_{k=1}^{n} g(k).$$

To determine $g(k)$, count solutions of $k = a > b > c > d \geq 1$ and $a + d = b + c$. Fix $d \geq 1$ and let $|C|$ be the number of possible values of c. For each choice of b, the corresponding value of c is determined by $c = k + d - b$. Now $2c + 1 \leq b + c = k + d$ implies $d + 1 \leq c \leq (k + d - 1)/2$. Hence $|C| = \lfloor (k + d - 1)/2 \rfloor - d$.

Case (i). *k* is even. Set $k = 2m$. Then

$$|C| = m + \left\lfloor \frac{d-1}{2} \right\rfloor - d.$$

This gives $|C| = m - 1, m - 2, m - 2, \ldots, 1, 1$ for $d = 1, 2, \ldots, 2m - 1$, respectively.

Case (ii). *k* is odd. Set $k = 2m + 1$. Now

$$|C| = m + \left\lfloor \frac{d}{2} \right\rfloor - d.$$

This gives $|C| = m-1, m-1, m-2, m-2, \dots, 1, 1$ for $d = 1, 2, \dots, 2m-2$, respectively.

Putting these two results together gives

$$g(k) = \begin{cases} (m-1) + 2(m-2) + \cdots + 2 = (m-1)^2 & \text{if } k = 2m, \\ 2(m-1) + 2(m-2) + \cdots + 2 = m(m-1) & \text{if } k = 2m+1. \end{cases}$$

It is evident that "pericyclic quadrilaterals" can be replaced by "non-degenerate triangles" with no additional change. In the quadrilateral problem, d is the side opposite a; in the triangle problem, d is a "slack variable" used to convert the triangle inequality $b + c > a$ into the equation $a + d = b + c$. It follows that exactly the same function g applies in both problems 2 and 9, and therefore so does the same f.

3. Calculation of *f*.

If $n = 2m$ then

$$\begin{aligned} f(n) = f(2m) &= \sum_{k=1}^{m} g(2k) + \sum_{k=1}^{m} g(2k-1) \\ &= \sum_{k=1}^{m} (k-1)^2 + \sum_{k=1}^{m} (k-1)(k-2) \\ &= \sum_{k=1}^{m} (2k^2 - 5k + 3) = \frac{m(m+1)(2m+1)}{3} - 5\,\frac{m(m+1)}{2} + 3m \\ &= m\,\frac{2(m+1)(2m+1) - 15(m+1) + 18}{6} = m\,\frac{4m^2 - 9m + 5}{6} \\ &= m\,\frac{(m-1)(4m-5)}{6} = \frac{1}{24}\,n(n-2)(2n-5). \end{aligned}$$

If $n = 2m+1$ then

$$\begin{aligned} f(n) &= f(2m+1) = f(2m) + g(2m+1) = \frac{1}{6}\,m(m-1)(4m-5) + m(m-1) \\ &= \frac{1}{6}\,m(m-1)(4m+1) = \frac{1}{6}\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)(2n-1) \\ &= \frac{1}{24}\,(n-1)(n-3)(2n-1). \end{aligned}$$

■

Solution by **Michael Cheung, student**, *Elizabethtown College, Elizabethtown, PA.*

Also solved by **Paul S. Bruckman**, Nanaimo, BC.

1227. *Proposed by Stanislav Molchanov, Department of Mathematics and Statistics, University of North Carolina Charlotte*

1. Let A be a four-digit positive integer. For what k is it always possible to append to A a k-digit integer N such that $A \cdot 10^k + N$ is a perfect square?

2. Again let A be a four-digit positive integer. For what k is it always possible to append to A a k-digit integer N such that $A \cdot 10^k + N$ is a perfect cube?
3. Prove that for any positive integers A and n, it is possible to find a t-digit integer N such that $A \cdot 10^t + N$ is an n^{th} power.

Solution by **Michael Cheung, student**, *Elizabethtown College, Elizabethtown, PA.*

We are given integers A and n, from which we must find a t-digit integer N such that $S(N) = A \cdot 10^t + N$ is a perfect nth power.

We will prove the stronger statement that given any two integers a and n, we can find an integer t such that for all a-digit integers A, it is always possible to find a t-digit integer N such that $S(N) = A \cdot 10^t + N$ is a perfect nth power. Note that for any given a, $10^{a+t} < S \leq 10^{a+t-1}$ (t has yet to be chosen). The greatest difference between two consecutive nth powers p^n and $(p+1)^n$ occurs when $(p+1)^n$ is maximized. This maximum value cannot exceed 10^{a+t}, so the difference between p^n and $(p+1)^n$ cannot exceed $E = \left((10^{(a+t)/n}\right)^n - \left(10^{(a+t)/n} - 1\right)^n$. To ensure that one can always append a t-digit integer N to any given a-digit integer A such that $S(N)$ is an nth power, we must have $E \leq 9 \cdot 10^{t-1}$ for some integer t. Using basic algebra solving for t,

$$10^{t+a} - \left[10^{(t+a)/n} - 1\right]^n \leq 9 \cdot 10^{t-1} \iff t \geq 1 - n \log_{10}\left[10^{(a+1)/n} - (10^{a+1} - 9)^{1/n}\right].$$

Since $1 - n \log_{10}\left[10^{(a+1)/n} - (10^{a+1} - 9)^{1/n}\right]$ converges given any finite a and n, we can always find a finite lower bound for t such that a t-digit integer N can always be appended to any a-digit integer A to make a perfect nth power.

Specifically, for $a = 4$ and $n = 2$, we must have:

$$t \geq 1 - 2\log_{10}\left[10^{5/2} - (10^5 - 9)^{1/2}\right] \iff t \geq 4.6936,$$

or $t \geq 5$ since t must be an integer. Similarly, for $a = 4$ and $n = 3$, we must have:

$$t \geq 1 - 3\log_{10}\left[10^{5/3} - (10^5 - 9)^{1/3}\right] \iff t \geq 9.5686,$$

or $t \geq 10$ since t must be an integer.

Also solved by **Mark Evans**, Louisville, KY; and the **Proposer**.

1228. *Proposed by Hongwei Chen, Christopher Newport University, Newport News, VA.*

Let $a_0 = 0$ and $a_k > 0$ for all $1 \leq k \leq n$ with $\sum_{k=1}^n a_k = 1$. Prove that

$$1 \leq \sum_{k=1}^{n} \frac{a_k}{\sqrt{1 + a_0 + \cdots + a_{k-1}}\sqrt{a_k + \cdots + a_n}} \leq \frac{\pi}{2}.$$

Solution by **Henry Ricardo**, *Tappan, NY.*

We have, for $1 \leq k \leq n$,

$$\begin{aligned} A_k &= \frac{a_k}{\sqrt{1 + a_0 + \cdots + a_{k-1}}\sqrt{a_k + \cdots + a_n}} \\ &= \frac{a_k}{\sqrt{1 + a_0 + \cdots + a_{k-1}}\sqrt{1 - (a_0 + \cdots + a_{k-1})}} \\ &= \frac{(a_0 + \cdots + a_k) - (a_0 + \cdots + a_{k-1})}{\sqrt{1 - (a_0 + a_1 + \cdots + a_{k-1})^2}} \end{aligned}$$

If we partition the interval $[0,1]$ into the subintervals $\left[\sum_{k=0}^{m} a_k, \sum_{k=0}^{m+1} a_k\right]$ for $m = 0, 1, \ldots, n-1$, then

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{(a_0 + \cdots + a_k) - (a_0 + \cdots + a_{k-1})}{\sqrt{1 - (a_0 + a_1 + \cdots + a_{k-1})^2}}$$

is a left Riemann sum of the increasing convex function $\frac{1}{\sqrt{1-x^2}}$ over $[0,1]$. Thus $1 \cdot (1-0) \le \sum_{k=1}^{n} A_k \le \int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx = \frac{\pi}{2}$.

Let

$$b_k := \frac{a_k}{\sqrt{1 + a_0 + \cdots + a_{k-1}}\sqrt{a_k + \cdots + a_n}}.$$

Observe that

$$b_k = \frac{a_k}{\sqrt{1 - (a_0 + a_1 + \cdots + a_{k-1})^2}} = \frac{(a_0 + \cdots + a_k) - (a_0 + \cdots + a_{k-1})}{\sqrt{1 - (a_0 + a_1 + \cdots + a_{k-1})^2}}.$$

The proposed sum can be viewed as a Riemann sum for the function $1/\sqrt{1-x^2}$ on the interval $[0,1]$ when the partition is

$$\{0 = a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots, a_0 + a_1 + \cdots + a_n = 1\}$$

It follows at once that

$$1 \cdot (1-0) \le \sum_{k=1}^{n} b_k \le \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}.$$

Also solved by **Pedro Henrique O. Pantoja, student**, Natal-RN, Brazil; **Paolo Perfetti**, Dipartimento di matematica, Università degli Studi di Roma "Tor Vergata", Rome, Italy; and the **Proposer**. *Editor's Note.* One solver pointed out that this problem appeared in a Chinese mathematics contest of 1996.

1229. *F. Hoffman and S.C. Locke, Florida Atlantic University, Boca Raton, FL.*

For a non-zero integer s, let $h(s) = t$, where $2^t \mid s$ and $2^{t+1} \nmid s$. Let $(a_1, \ldots, a_n)$ be a sequence of non-zero integers, let $m = \max\{h(a_j) : 1 \leqslant j \leqslant n\}$, and let $Y = \{j : 1 \leqslant j \leqslant n,\ h(a_j) = m\}$.

Part (a). Suppose that $|Y|$ is odd and that $S = \sum_{j=1}^{n} \frac{1}{a_j} = \frac{p}{q}$, with $\gcd(p,q) = 1$. Prove that p is odd.

Part (b). Let $T = \sum_{j=k}^{n} \frac{1}{j} = \frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{n} = \frac{p}{q}$, with $\gcd(p,q) = 1$. Prove that p is odd.

Solution by **Kathleen E. Lewis, University of the Gambia, Brikama, Republic of the Gambia.**

a) Let k be the least common multiple of the denominators. Since the highest power of 2 that appears in any denominator is m, it follows that $k = 2^m * r$, with r odd. When each fraction in the sum is written with denominator k, the fractions that originally had smaller powers of 2 in the denominator now have even numerators.

Only those fractions that originally had a multiple of 2^m in the denominator now have odd numerators. Therefore, if there are an odd number of such terms, the sum of all the numerators must be an odd number even before the fraction is reduced to lowest terms.

b) In the sequence of consecutive positive integers $k, k+1, \ldots, n$, consider the highest power of 2, say 2^m, that occurs as a factor. If it occurs more than once in the sequence, then successive occurrences must be of the form $j * 2^m$ and $(j+1) * 2^m$ for some natural number j. But either j or $j+1$ must be even, which means that either $j * 2^m$ or $(j+1) * 2^m$ is actually a multiple of 2^{m+1}. But this contradicts the choice of 2^m as the highest power of 2 appearing. Since one is an odd number, this means that the sequence meets the conditions of part (a), so the sum of the fractions, reduced to lowest terms, has an odd numerator.

Also solved by **Paul S. Bruckman**, Nanaimo, BC; **Michael Cheung, student**, Elizabethtown College, Elizabethtown, PA; **Proposer**.

1230. *Proposed by Mohammad K. Azarian, University of Evansville, Indiana.*

Let

$$P(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n,$$

be a polynomial of degree n $(n \geq 1)$ where $a_n \neq 0$. If $r_1, r_2, \ldots, r_n$ are the roots of $P(x)$, then show that

$$(\sum_{i=1}^{n} r_i).(\sum_{i=1}^{n} \frac{1}{r_i}).(\prod_{i=1}^{n} r_i) = (-1)^n a_1 a_{n-1}.$$

Dionne T. Bailey, Elsie M. Campbell, Charles Diminnie, Angelo State University, San Angelo, TX;

Let

$$P(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n,$$

be a polynomial of degree n $(n \geq 1)$ where $a_n \neq 0$. If $r_1, r_2, \ldots, r_n$ are the roots of $P(x)$, then show that

$$\left(\sum_{i=1}^{n} r_i\right) \cdot \left(\sum_{i=1}^{n} \frac{1}{r_i}\right) \cdot \left(\prod_{i=1}^{n} r_i\right) = (-1)^n a_1 a_{n-1}.$$

Given the roots $r_1, r_2, \ldots, r_n$ of $P(x)$, then

$$P(x) = (x - r_1)(x - r_2) \cdots (x - r_n). \qquad (1)$$

From (1),

$$a_1 = \sum_{i=1}^{n} (-r_i) = -\sum_{i=1}^{n} r_i, \qquad (2)$$

and

$$\begin{aligned} a_{n-1} &= \sum_{i=1}^{n} \left(\prod_{j \neq i} (-r_j) \right) \\ &= (-1)^{n-1} \sum_{i=1}^{n} \left(\prod_{j \neq i} r_j \right) \\ &= (-1)^{n-1} \sum_{i=1}^{n} \left(\frac{1}{r_i} \left(\prod_{j=1}^{n} r_j \right) \right) \\ &= (-1)^{n-1} \left(\prod_{j=1}^{n} r_j \right) \left(\sum_{i=1}^{n} \frac{1}{r_i} \right). \end{aligned} \tag{3}$$

Thus, by (2) and (3),

$$\left(\sum_{i=1}^{n} r_i \right) \cdot \left(\sum_{i=1}^{n} \frac{1}{r_i} \right) \cdot \left(\prod_{i=1}^{n} r_i \right) = (-1)^n a_1 a_{n-1}.$$

Also solved by **Gezim Basha, student**, Kosovo, Serbia; **Paul S. Bruckman**, Nanaimo, BC; **Hongwei Chen**, Christopher Newport University, Newport News, VA; **Thomas Dence**, Ashland University, Ashland, OH; **Stephen Gendler**, Clarion University, Clarion PA; **Gerhardt Hinkle, student**, Central High School, Springfield, MO; **Brian Klatt, student**, Saint Joseph's University, Philadelphia, PA; **Kathleen E. Lewis**, University of the Gambia, Brikama, Gambia; **Peter A. Lindstrom**, Batavia, NY; **Yoshinobu Murayoshi**, Naha City, Okinawa, Japan; **Paolo Perfetti**, Dipartimento di matematica, Università degli Studi di Roma "Tor Vergata", Rome, Italy; **Juan L. Vargas, student**, University of North Carolina Charlotte; and the **Proposer**.

1231. *Proposed by Arthur L. Holshouser, Charlotte, NC.*

A real number is called a Cantor number if it can be written in base 3 notation without the use of the digit 1. For example $1/3 = 0.1 = 0.0\overline{2}$ is a Cantor number. Prove that every real number is the sum of two Cantor numbers.

Solution by **Robert Graham, student, and Melissa Cangialosi, student**, *Elizabethtown College, Elizabethtown, PA.*

Let c be a real number with ternary expansion $c_n c_{n-1} \ldots c_2 c_1 c_0 . c_{-1} c_{-2} \ldots$ and let $S = \{i \leq n | c_i = 1\}$. If S is empty, then c is a Cantor number, and if c is a Cantor number, then it can be written trivially as a sum of two Cantor numbers, $c = c + 0$, since zero is a Cantor number.

Note that if $j > k$, then $3^j - 3^k = \displaystyle\sum_{i=k}^{j-1} 2 \cdot 3^i$ is a Cantor number. Suppose that c is not a Cantor number and $S = \{s_1, s_2, s_3, s_4, \ldots\}$ where $s_1 > s_2 > s_3 > s_4 > \ldots$. If $|S| = 2m$ for some integer m, then $a = \displaystyle\sum_{i=1}^{m} (3^{s_{2i-1}} - 3^{s_{2i}})$ is a Cantor number since S is strictly ordered. The difference $b = c - a$ is also a Cantor number because $b_{s_{2i-1}} = 0$, $b_{s_{2i}} = 2$, $i = 1, \ldots, m$, and $b_j = c_j$, otherwise. Therefore, $c = a + b$ is written as a sum of two Cantor numbers. If $|S| = 2m + 1$ for some integer m, then

$a = \sum_{i=1}^{m} (3^{s_{2i-1}} - 3^{s_{2i}}) + \sum_{i=-\infty}^{s_{2m+1}-1} 2 \cdot 3^i$ is a Cantor number and $b = c - a$ is also a Cantor number. Finally, if $|S|$ is infinite, then $a = \sum_{i=1}^{\infty} (3^{s_{2i-1}} - 3^{s_{2i}})$ and $b = c - a$ are both Cantor numbers, so $c = a + b$ is the sum of two Cantor numbers.

Solution by **Miguel Lerma**, *Northwestern University Problem Solving Group, Evanston, IL.*

Without loss of generality we may assume that the real number is in the interval $[0, 1)$, since any real number can be mapped to that interval by possibly a change of sign and right shift of its base-3 digits.

So, assume $r \in [0, 1)$, with digits r_i $(i = 0, 1, 2, \dots)$ in base 3, i.e.:

$$r = \sum_{i=0}^{\infty} \frac{r_i}{3^i}, \quad r_0 = 0\,, \text{ and } r_i \in \{0, 1, 2\} \text{ for all } i > 0\,.$$

We will describe how to find two Cantor numbers $a = \sum_{i=0}^{\infty} \frac{a_i}{3^i}$, $b = \sum_{i=0}^{\infty} \frac{b_i}{3^i}$, with $a_i, b_i \in \{0, 2\}$ for all $i \geq 0$, such that $r = a + b$.

We define the sequences a_i and b_i, together with a third sequence c_i recursively in the following way: for $i = 0$, $a_0 = b_0 = c_0 = 0$, and for $i > 0$:

$$\begin{aligned} a_i &= \text{ the greatest of 0 or 2 such that } a_i \leq r_i + 3c_{i-1}\,, \\ b_i &= \text{ the greatest of 0 or 2 such that } a_i + b_i \leq r_i + 3c_{i-1}\,, \\ c_i &= r_i + 3c_{i-1} - a_i - b_i\,, \end{aligned}$$

By induction we see that $c_i \in \{0, 1\}$ for all $i \geq 0$. On the other hand we will prove also by induction that for $n \geq 0$:

$$\sum_{i=0}^{n} \frac{r_i}{3^i} - \sum_{i=0}^{n} \frac{a_i}{3^i} - \sum_{i=0}^{n} \frac{b_i}{3^i} = \frac{c_n}{3^n}\,, \tag{1}$$

In fact, for $n = 0$ the equality is trivially true. Next, assuming that the equality holds for some value of $n \geq 0$, we get for $n + 1$:

$$\sum_{i=0}^{n+1} \frac{r_i}{3^i} - \sum_{i=0}^{n+1} \frac{a_i}{3^i} - \sum_{i=0}^{n+1} \frac{b_i}{3^i} = \frac{c_n}{3^n} + \frac{r_{n+1} - a_{n+1} - b_{n+1}}{3^{n+1}} = \frac{3c_n + r_{n+1} - a_{n+1} - b_{n+1}}{3^{n+1}} = \frac{c_{n+1}}{3^{n+1}}\,.$$

This completes the induction.

Finally, the desired result follows from (1) by taking into account that the left hand side tends to $r - a - b$, and $c_n/3^n \to 0$.

Also solved by **Paul S. Bruckman**, Nanaimo, BC; **Stephen Gendler**, Clarion University, Clarion PA; **Kathleen E. Lewis** University of the Gambia, Brikama, Republic of the Gambia; and the **Proposer**.

1232A. *Proposed by Paul Bruckman, Nanaimo, BC.*

Given $0 < x < 1$, define $F(x)$ by $F(x) = \frac{1}{x} + \frac{1}{\ln(1-x)}$. Show that $\frac{1}{2} < F(x) < 1$. Show that F may be extended continuously to $[0, 1]$.

Solution by **Hongwei Chen**, *Christopher Newport University, Newport News, VA.*

Let $t = -\ln(1-x)$ and define $G(t)$ by

$$G(t) = F(x) = \frac{e^t}{e^t - 1} - \frac{1}{t} = \frac{te^t - e^t + 1}{t(e^t - 1)}.$$

We now show that $G'(t) > 0$ for all $0 < t < \infty$. To this end, the quotient rule yields that

$$G'(t) = \frac{(e^t-1)^2 - t^2e^t}{t^2(e^t-1)^2} = \frac{(e^t - 1 + te^{t/2})(e^t - 1 - te^{t/2})}{t^2(e^t-1)^2}.$$

Notice that $G'(t) > 0$ is equivalent to $e^t - 1 - te^{t/2} > 0$. For any $n \geq 3$, we have $n < 2^{n-1}$ and so

$$e^t - 1 - te^{t/2} = \sum_{n=1}^{\infty} \left(1 - \frac{n}{2^{n-1}}\right) \frac{t^n}{n!} > 0 \text{ for all } t > 0.$$

This confirms that $G'(t) > 0$. Moreover, by the L'Hôpital's rule, we have

$$\lim_{t\to 0} G(t) = \frac{1}{2}, \ \lim_{t\to\infty} G(t) = 1,$$

which implies that $\frac{1}{2} < F(x) < 1$. Defining $F(0) = 1/2$ and $F(1) = 1$ yields that $F(x)$ is continuous on $[0, 1]$

Also solved by **Frank Battles**, Massachusetts Maritime Academy, Buzzards Bay, MA; **Thomas Dence**, Ashland University, Ashland, OH; **Richard Hess**, Rancho Palos Verdes, CA; **Miguel Lerma**, Northwestern University Problem Solving Group, Evanston, IL; **Peter A. Lindstrom**, Batavia, NY; **Yoshinobu Murayoshi**, Naha City, Okinawa, Japan; **Paolo Perfetti**, Dipartimento di matematica, Università degli Studi di Roma "Tor Vergata", Rome, Italy; **Angel Plaza**, University of Las Palmas de Gran Canaria, Spain; **Raul A. Simon**, Chile. and the **Proposer**.

1232B. *Proposed by David Wells, Penn State New Kensington, Upper Burrell, PA.*

Numbers r, s, and t are chosen independently and uniformly at random from the interval $(0, 1]$. Circles with radii r, s, and t are then constructed so that they are pairwise externally tangent. What is the probability that the circles can be enclosed in a triangle, each of whose sides is tangent to two of the circles?

Solution by **the Proposer**.

Let the circles with radii r, s, and t have centers R, S, and T, respectively, and assume with no loss of generality that $r \geq s \geq t$. Of the two lines that are tangent to both circles R and S, let L_1 be the one whose points of tangency with R and S are on the opposite side of line RS from T. Let L_2 (respectively, L_3), tangent to both circles R and T (respectively, S and T) be chosen in a similar manner. The required triangle, if it exists, must have sides on L_1, L_2, and L_3. The circles may be placed in a coordinate plane so that $R = (0, 0)$, S is in the first quadrant, and L_1 is the line $x = r$.

Lemma: The required triangle exists if and only if $4st > r^2$. **Proof:** Consider first the case in which L_2 is the line $x = -r$. In that case the required triangle does not exist. Also $S = (r - s, 2\sqrt{rs})$, $T = (t - r, 2\sqrt{rt})$, and

$$(s + t)^2 = ST^2 = (2r - s - t)^2 + (2\sqrt{rs} - 2\sqrt{rt})^2.$$

Simplifying gives $4st = r^2$.

If circle T extends to the left of the line $x = -r$, then $4st > r^2$, which implies that $s > r/2$. Together with the condition $t \le s$, this implies that the line L_3 has a positive slope and intersects L_1 above all three circles. Because $t \le s \le r$, the line L_2 has a negative slope and intersects L_1 below all three circles. Therefore in this case the required triangle exists. If circle T lies entirely to the right of the line $x = -r$, then $4st < r^2$. In this case the lines L_2 and L_3 both intersect L_1 in the first quadrant, so the required triangle cannot exist. ■

Because the required triangle exists if and only if $s > r/2$ and $r^2/4s < t \le s$, it exists with probability

$$\frac{\int_0^1 \int_{r/2}^r \int_{r^2/4s}^s dtdsdr}{\int_0^1 \int_0^r \int_0^s dtdsdr} = 6\int_0^1 \int_{r/2}^r \left(s - \frac{r^2}{4s}\right) dsdr$$

$$= 6\int_0^1 \left(\frac{s^2}{2} - \frac{r^2}{4}\ln|s|\right)\Big|_{r/2}^{r} dr = 6\int_0^1 r^2\left(\frac{3}{8} - \frac{\ln 2}{4}\right) dr = \frac{3-\ln 4}{4}.$$

1233. *Proposed by Perfetti Paolo, Dipartimento di matematica, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy.*

Evaluate

$$\sum_{k=1}^{\infty} \frac{k(2k+1)H_{k-1}^2 + k(1+k)^2 H_{k-1} + (1+k)^2}{k^3(k+1)^2},$$

where $H_k = 1 + 1/2 + \ldots + 1/k$, $H_0 = 0$.

Solution by **the Proposer**.

The answer is 3. First note that

$$\frac{k(2k+1)H_{k-1}^2 + k(1+k)^2 H_{k-1} + (1+k)^2}{k^3(k+1)^2} =$$
$$\frac{2k+1}{k^2(k+1)^2}H_{k-1}^2 + \frac{1}{k^2}\left(H_{k-1} + \frac{1}{k}\right) =$$
$$\frac{2k+1}{k^2(k+1)^2}H_{k-1}^2 + \frac{H_k}{k^2} \doteq S.$$

Moreover

$$\frac{2k+1}{k^2(k+1)^2}H_{k-1}^2 = H_{k-1}^2\left(\frac{1}{k^2} - \frac{1}{(1+k)^2}\right)$$
$$= \left(\frac{H_k^2}{k^2} - 2\frac{H_{k-1}}{k^3} - \frac{1}{k^4}\right) + \left(-\frac{H_k^2}{(k+1)^2} + 2\frac{H_{k-1}}{k(k+1)^2} + \frac{1}{k^2(1+k^2)}\right)$$

We make use of the following equalities 1) and 2) (see for example the article D.Borwein and J.M.Borwein *On an intriguing Integral and some Series Related to* $\zeta(4)$ Proceedings of the Amer. Math. Soc., Vol 123, No.4 (Apr.,1995) pp.1191–1198)

$$1)\ \sum_{k=1}^{\infty}\frac{H_k^2}{k^2}=\frac{17}{4}\zeta(4), 2)\ \sum_{k=1}^{\infty}\frac{H_k^2}{(k+1)^2}=\frac{11}{4}\zeta(4),$$
$$3)\ \sum_{k=1}^{\infty}\frac{H_k}{k^3}=\frac{5}{4}\zeta(4), 4)\ \sum_{k=1}^{\infty}\frac{H_k}{k^2}=2\zeta(3)$$

3) can be obtained by 1) and 2) because $H_{k-1}=H_k-\frac{1}{k}$ and then $H_{k-1}^2=H_k^2-2\frac{H_k}{k}+\frac{1}{k^2}$.

Hence

$$2\sum_{k=1}^{\infty}\frac{H_k}{k^3}=\sum_{k=1}^{\infty}\left(\frac{1}{k^4}+\frac{H_k^2}{k^2}-\frac{H_{k-1}^2}{k^2}\right)=\zeta(4)\left(1+\frac{17}{4}-\frac{11}{4}\right)=\frac{5}{2}\zeta(4).$$

4) is an old result going back to Euler yielding

$$\sum_{k=1}^{\infty}\frac{H_{k+1}}{(k+1)^2}=\sum_{k=1}^{\infty}\frac{H_k}{k^2}-1=2\zeta(3)-1.$$

Moreover

$$\sum_{k=1}^{\infty}\frac{H_{k-1}}{k^3}=\sum_{k=1}^{\infty}\frac{H_k}{k^3}-\sum_{k=1}^{\infty}\frac{1}{k^4}=\frac{\zeta(4)}{4}.$$

Now

$$\sum_{k=1}^{\infty}\frac{1}{k^2(1+k^2)}=\sum_{k=1}^{\infty}\left(\frac{1}{k^2}+\frac{1}{(1+k)^2}+\frac{2}{k+1}-\frac{2}{k}\right)=\zeta(2)+(\zeta(2)-1)-2=2\zeta(2)-3.$$

Next

$$\begin{aligned}\frac{H_{k-1}}{k(1+k)^2}&=\frac{H_{k-1}}{k}-\frac{H_{k-1}}{k+1}-\frac{H_{k-1}}{(1+k)^2}\\&=\left[\frac{H_k}{k}-\frac{1}{k^2}\right]-\left[\frac{H_{k+1}}{k+1}-\frac{1}{(k+1)^2}-\frac{1}{k(k+1)}\right]-\left[\frac{H_{k+1}}{(1+k)^2}-\frac{1}{(1+k)^3}-\frac{1}{k(1+k)^2}\right]\\&=\left[\frac{H_k}{k}-\frac{H_{k+1}}{k+1}\right]+\left[\frac{1}{(k+1)^2}-\frac{1}{k^2}\right]+\frac{1}{k(k+1)}-\frac{H_{k+1}}{(1+k)^2}+\frac{1}{(1+k)^3}+\frac{1}{k(1+k)^2}.\end{aligned}$$

Telescoping

$$\sum_{k=1}^{\infty}\left(\frac{H_k}{k}-\frac{H_{k+1}}{k+1}+\frac{1}{(k+1)^2}-\frac{1}{k^2}\right)=0,$$

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)}=\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{(k+1)}\right)=1,\ \text{and}$$

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)^2} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} - \frac{1}{(k+1)^2} \right) = 1 - (\zeta(2) - 1) = 2 - \zeta(2).$$

Hence

$$\sum_{k=1}^{\infty} \frac{H_{k-1}}{k(1+k)^2} = 1 - (2\zeta(3) - 1) + \zeta(3) - 1 + 2 - \zeta(2) = 3 - \zeta(3) - \zeta(2).$$

Summing all the contributions we get

$$S = \zeta(4) \left(\frac{17}{4} - 2\frac{1}{4} - 1 - \frac{11}{4} \right) + 2(3 - \zeta(3) - \zeta(2)) + 2\zeta(2) - 3 + 2\zeta(3) = 3.$$

Also solved by **Paul S. Bruckman**, Nanaimo, BC; **Kenny Davenport**, Dallas, PA; **Juan L. Vargas, student**, University of North Carolina Charlotte.

1234. *Proposed by Sharon Sternadel Harada, Corvallis, Oregon and Leo Schneider, John Carroll University, Cleveland, Ohio.*

The streets in Mathville look like a grid formed by an x-y coordinate system. All the even-numbered east-west streets, $y = \ldots, -2, 0, 2, 4, \ldots,$ are one-way east (the direction of increasing x), and all the odd-numbered east-west streets $y = \ldots, -1, 1, 3, \ldots$ are one-way west. Similarly, all the even-numbered north-south streets $x = \ldots, -2, 0, 2, 4, \ldots,$ are one-way north (the direction of increasing y), and all the odd numbered north- south streets are one-way south. Compute the (Euclidean) area of the smallest convex quadrilateral that contains all the points (x, y) that can be reached by driving no more than n blocks, starting at the origin and traversing these one-way streets.

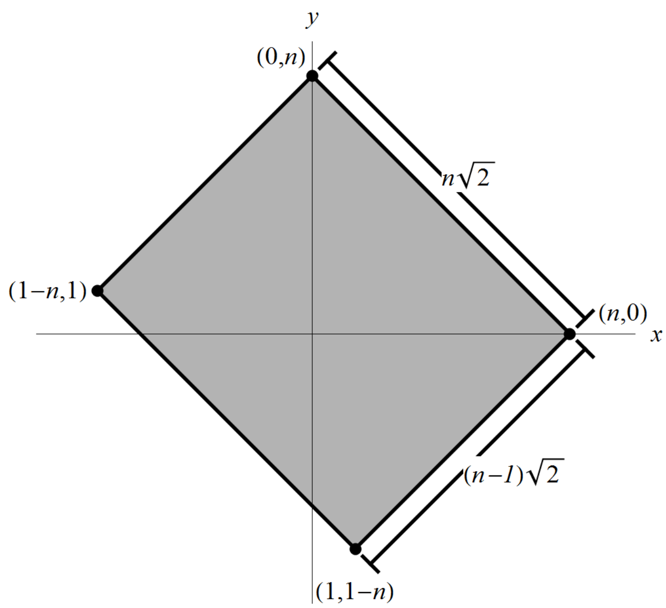
Solution by **Nathan Caudill, student**, *Elizabethtown College, Elizabethtown, PA.*

The area is $2n(n-1)$. If N, S, E, and W are the total number of blocks traveled in the north, south, east, and west directions, respectively, then the final position is

$$\begin{bmatrix} x \\ y \end{bmatrix} = (E - W) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (N - S) \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since you can drive no more than n blocks, $N + S + E + W \leq n$. Also, since travel in the first direction must be either towards the north or east, then $N + E \geq 1$. If $S = W = 0$, then $x = E$, $y = N$, and $x + y \leq n$. Similarly, if $N = W = 0$, then $x = E$, $y = -S$, and $x - y \leq n$ and if $S = E = 0$, then $x = -W$, $y = N$, and $-x + y \leq n$. Finally, since N and E cannot both be zero, if $E = 0$ and $N = 1$, then $x = -W$, $y = 1 - S$, and $-x - y \leq n - 1$. Similarly, if $E = 1$ and $N = 0$, then $-x - y \leq n - 1$. All four inequalities together give the region shown in the figure.

The corners can be reached by taking $N = S = W = 0$ and $E = n$ to arrive at $(n, 0)$, $S = E = W = 0$ and $N = n$ to arrive at $(0, n)$, $N = W = 0$, $S = n - 1$, and $E = 1$ to arrive at $(1, 1 - n)$, and $S = E = 0$, $N = 1$, and $W = n - 1$ to arrive at $(1 - n, 1)$. Therefore, the rectangle in the figure is the smallest convex quadrilateral that contains all of the points that can be reached by driving no more than n blocks. The lengths of the sides of the rectangle are $n\sqrt{2}$ and $(n-1)\sqrt{2}$, so the area is $2n(n-1)$.



If x is the number of blocks traveled in the x direction and y is the number of blocks traveled in the y direction, then $x+y \le n$. Turning south or west after driving the first block will make it possible to reach the points $(1, 1-n)$ and $(1-n, 1)$. Just as before, it is not possible to reach a further point northwest because of the nature of driving in only the coordinate directions.

Connecting these furthest points yields a rectangle with edges of length $n\sqrt{2}$ and $(n-1)\sqrt{2}$. This results in an area of $2n(n-1)$, which is the smallest possible area that fits the given criteria.

Also solved by **Richard Hess**, Rancho Palos Verdes, CA; and the **Proposer**.