PROBLEM DEPARTMENT

ASHLEY AHLIN AND HAROLD REITER*

DRAFT - Fall 2011

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to `hbreiter@uncc.edu`. *Electronic submissions using LaTeX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by March 1, 2012. Solutions identified as by students are given preference.*

Problems for Solution.

1243. *Proposed by M. B. Kulkarni, B. Y. K. College, Nasik India and M. N. Deshpande, Nagpur, India*

We toss an unbiased coin n times. The results of the random experiment are written in a linear array. Suppose R_n and X_n respectively denote the number of runs of like symbols and number of double HH's (overlapping allowed). Show that $Cov(R_n, X_n) = \frac{-n+1}{8}$.

1244. *Proposed by Cecil Rousseau, University of Memphis, Memphis, TN.*

The function $\cos(\log x)$ has an interesting property: its nth derivative is given simply by

$$\frac{d^n}{dx^n}\cos(\log x) = \frac{a_n\cos(\log x) - b_n\sin(\log x)}{x^n},$$

where a_n and b_n are integers.

1. Prove the last statement.
2. Find recurrences for (a_n) and (b_n), and use them to construct a table (n, a_n, b_n) for $n = 1, 2, \ldots, 10$.
3. The unsigned Stirling number of the first kind $\left[\begin{matrix} n \\ k \end{matrix}\right]$ is the number of permutations of $[n]$ that have k cycles. Use the change of basis identity

$$x^{\underline{n}} = \sum_k \left[\begin{matrix} n \\ k \end{matrix}\right] (-1)^{n-k} x^k,$$

to prove

$$a_n = (-1)^n \sum_k \left[\begin{matrix} n \\ 2k \end{matrix}\right] (-1)^k \qquad \text{and} \qquad b_n = (-1)^{n+1} \sum_k \left[\begin{matrix} n \\ 2k+1 \end{matrix}\right] (-1)^k.$$

Provide a combinatorial interpretation of a_n and b_n.

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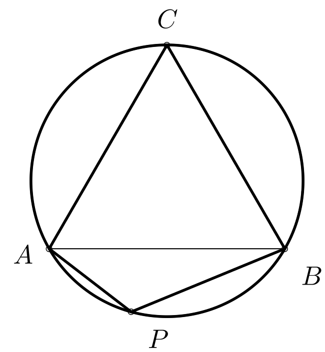
Note. The author gratefully acknowledges the assistance of Floridia Jackson in the early phase of the exploration that led to this problem proposal.

1245. *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA)*

Let ABC be an equilateral triangle with edge length c inscribed in a circle. Let P be a point on minor arc $\widehat{AB}$. Let $PA = a$ and $PB = b$. Is it possible for a, b, and c to all be distinct positive integers?



1246. *Proposed by Bessem Samet, Tunis College of Sciences, Tunisia.*

Characterize the set of functions $f : (0, \infty) \to (0, \infty)$ satisfying the following properties:

(a) f is a bounded function,
(b) f is of class C^2 on $(0, +\infty)$, and
(c) $xg''(x) + (1 + xg'(x))g'(x) \geq 0,\ \forall\, x > 0$, where $g(x) = \ln(f(x))$.

1247. *Proposed by Proposed by H. A. ShahAli, Tehran, IRAN.*

Given integers k and m with $1 < k < m$; also given m vectors of a finite dimensional vector space such that the sum of every k vectors is equal to k times one of the vectors. Prove that all of the vectors are equal.

1248. *Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain.*

Let T_n be the nth triangular number defined by $T_n = \binom{n+1}{2}$ for all $n \geq 1$. Show that for all integers $k \geq 2$, the sequence $\{T_n^k\}_{n\geq 1}$ does not contain any infinite subsequence with all terms in arithmetic progression.

1249. *Proposed by Peter Linnell, Vinginia Polytechnic University, Blacksburg, VA.* This was problem 5 on the annual VPI Regional College Math Contest, 1994.

Let $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ be a function which satisfies $f(0,0) = 1$ and

$$f(m,n) + f(m+1,n) + f(m,n+1) + f(m+1,n+1) = 0$$

for all $m, n \in \mathbb{Z}$ (where $\mathbb{Z}$ and $\mathbb{R}$ denote the set of all integers and all real numbers, respectively). Prove that $|f(m,n)| \geq 1/3$, for infinitely many pairs of integers (m,n).

1250. *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abington, PA.*

If on the sides of an arbitrary triangle ABC three similar triangles AKB, BLC, and CNA are drawn outward (or inward), then the triangles KLN and ABC have the same centroid G.

1251. *Proposed by Arthur Holshouser, Charlotte, NC.*

This problem appeared on the 2011 Lower Michigan Mathematics Competition.

The operator $(R, \odot)$ on the real numbers R is defined by

$$x \odot y = \frac{xy}{xy + (1-x)(1-y)}$$

when $xy + (1-x)(1-y) \neq 0$. $x \odot y$ is not defined when $xy + (1-x)(1-y) = 0$.

1. Show that $(R, \odot)$ has an identity $I \in R$ such that $x \odot I = I \odot x = x$ for all $x \in R$.
2. Show that each $x \in R \backslash \{0, 1\}$ has an inverse x^{-1} such that $x \odot x^{-1} = x^{-1} \odot x = I$.
3. Show that $(R, \odot)$ is both commutative and associative when all operations involved are defined.
4. Show that for all $x \in R$, $x \odot x \odot \cdots \odot x$ (n-times) is always defined and $x \odot x \odot \cdots \odot x$ (n-times) $= \frac{x^n}{x^n + (1-x)^n}$.
5. Discuss the possibility of "patching up" $(R, \odot)$ so that $(R, \odot)$ is a true group.

1252. *Proposed by Neculai Stanciu, Emil Palade Secondary School, Buzau, Romania.*

Let p be a prime not 2 or 5, let a be a digit and let m and n be positive integers. Show that there exist infinitely many numbers of type $A = a \cdot p^n$ whose last m digits are $0 \ldots 0a$.

1253. *Proposed by Arthur Holshouser, Charlotte, NC and Patrick Vennebush, NCTM, Reston, VA.*

Let P be a $2n$-sided regular polygon. Suppose $k \geq 3$ points are randomly and uniformly selected from the boundary of P. Find the probability that the convex hull of the k points includes the center of P.

1254. *Proposed by Perfetti Paolo, Dipartimento di matematica, Università degli Studi di Roma "Tor Vergata", Rome, Italy.*

Let $[a]$ the integer part of a and $\{a\} = a - [a]$. Evaluate

$$\int_0^1 \int_0^1 \frac{\left\{\frac{x}{y}\right\}}{\left[\frac{x}{y}\right] + 1} dx dy - \int_{x=0}^1 \int_{y=0}^x \ln \left[\frac{x}{y}\right] dx dy$$

Solutions. The editors regret that the spring issue of the Journal, we failed to acknowledge solutions of problem 1232A by **Joseph Dence**.

1235. *Proposed by Parviz Khalili, Christopher Newport University, Newport News, VA.*

Let x, y, and z be positive real numbers such that $x + y + z = 1$. Show that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 3 + 2\sqrt{\frac{x^x\, y^y\, z^z}{x\ y\ z}}.$$

Solution by **Dorottya Fekete**, *'Fejéntaláltuka Szeged' problem solving group, University of Szeged, Hungary.*

We know that $1 = x+y+z$, so the left-hand side of the inequality can be written in the following way:

$$\begin{aligned}\frac{x+y+z}{x} + \frac{x+y+z}{y} + \frac{x+y+z}{z} &= \frac{x}{x} + \frac{y}{x} + \frac{z}{x} + \frac{x}{y} + \frac{y}{y} + \frac{z}{y} + \frac{x}{z} + \frac{y}{z} + \frac{z}{z} \\ &= 3 + \frac{x}{y} + \frac{y}{x} + \frac{z}{y} + \frac{y}{z} + \frac{x}{z} + \frac{z}{x}\end{aligned}$$

Subtracting 3 from both sides, then dividing them by 2:

$$\frac{\frac{y}{x} + \frac{z}{x}}{2} + \frac{\frac{x}{y} + \frac{z}{y}}{2} + \frac{\frac{x}{z} + \frac{y}{z}}{2} \geq \sqrt{\frac{x^x y^y z^z}{xyz}}. \tag{0.1}$$

The right-hand side of the inequality can be simplified as follows:

$$\sqrt{\frac{x^x y^y z^z}{xyz}} = \frac{1}{x^{\frac{1-x}{2}} y^{\frac{1-y}{2}} z^{\frac{1-z}{2}}}.$$

But $1 = x+y+z$, thus:

$$\frac{1}{x^{\frac{1-x}{2}} y^{\frac{1-y}{2}} z^{\frac{1-z}{2}}} = \frac{1}{x^{\frac{y+z}{2}} y^{\frac{x+z}{2}} z^{\frac{x+y}{2}}} = \left(\frac{1}{x}\right)^{\frac{y+z}{2}} \left(\frac{1}{y}\right)^{\frac{x+z}{2}} \left(\frac{1}{z}\right)^{\frac{x+y}{2}}.$$

Since $\frac{y+z}{2} + \frac{x+z}{2} + \frac{x+y}{2} = 1$, we can apply the inequality of weighted arithmetic and geometric means, hence we get that

$$\begin{aligned}\left(\frac{1}{x}\right)^{\frac{y+z}{2}} \left(\frac{1}{y}\right)^{\frac{x+z}{2}} \left(\frac{1}{z}\right)^{\frac{x+y}{2}} &\leq \frac{y+z}{2}\frac{1}{x} + \frac{x+z}{2}\frac{1}{y} + \frac{x+y}{2}\frac{1}{z} \\ &= \frac{\frac{y}{x} + \frac{z}{x}}{2} + \frac{\frac{x}{y} + \frac{z}{y}}{2} + \frac{\frac{x}{z} + \frac{y}{z}}{2},\end{aligned}$$

where the last expression is exactly the left-hand side of (0.1).

Equality holds if and only if $x = y = z = \frac{1}{3}$. This completes the proof.

Also solved by **Paul S. Bruckman**, Nanaimo, BC; **Hongwei Chen**, Christopher Newport University, Newport News, VA; **Moti Levy**, Rehovot, Israel; **David E. Manes**, Oneonta, NY; **Yoshinobu Murayoshi**, Naha City, Okinawa, Japan; **Paolo Perfetti**, Dipartimento di matematica, Università degli Studi di Roma "Tor Vergata", Rome, Italy; **Henry Ricardo**, Tappan, NY; and the **Proposer**.

1236. *Proposed by Mohsen Soltanifar, University of Saskatchewan, Saskatoon, Canada.*

Prove or give a counterexample: Let $U_i (1 \leq i \leq n)$, be finite dimensional subspaces of a vector space V. Then, the dimension of $\sum_{i=1}^{n} U_i$ is given by:

$$\dim(\sum_{i=1}^{n} U_i) = \sum_{r=1}^{n}(-1)^{r+1}\sum_{i_1<i_2<...<i_r:n}\dim(U_{i_1}\cap U_{i_2}\cap ...\cap U_{i_r}),$$

where the summation $\sum_{i_1<i_2<...<i_r:n}\dim(U_{i_1}\cap U_{i_2}\cap ...\cap U_{i_r})$ is taken over all of the $\binom{n}{r}$ possible subsets of the set $1, 2, ..., n$.

Solution by **Kathleen E. Lewis**, *University of the Gambia, Brikama, Republic of the Gambia.*

The statement is not true. Consider the following subspaces of $\mathbb{R}^3$:

$$U_1 = \{(a, 0, 0) : a \in \mathbb{R}\},\ U_2 = \{(0, a, 0) : a \in \mathbb{R}\},\ U_3 = \{(a, a, 0) : a \in \mathbb{R}\}.$$

Then $U_1 + U_2 + U_3 = \{(a, b, 0) : a, b \in \mathbb{R}\}$, which has dimension 2, but

$$[\dim(U_1)+\dim(U_2)+\dim(U_3)]-[\dim(U_1\cap U_2)+\dim(U_1\cap U_3)+\dim(U_2\cap U_3)]+\dim(U_1\cap U_2\cap U_3) =$$

$$(1 + 1 + 1) - (0 + 0 + 0) + 0 = 3.$$

Also solved by the **Proposer**.

1237. *Proposed by Thomas Dence, Ashland University, Ashland, OH and Joseph Dence, St. Louis, MO.*

For each integer $n \geq 2$, determine the values of the integrals

$$I_{n,3} = \int_0^{\pi} \sin^3 x \sin(nx)\ dx \text{ and } I_{n,5} = \int_0^{\pi} \sin^5 x \sin(nx)\ dx.$$

Solution by **Elsie M. Campbell, Dionne T. Bailey, Charles Diminnie, and Andrew Siefker**, *Angelo State University, San Angelo, TX.*

Using basic trigonometric identities, it can be shown that

$$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x) \tag{0.1}$$

and

$$\sin^5 x = \frac{1}{16}(20\sin^3 x - 5\sin x + \sin 5x). \tag{0.2}$$

It can also be shown that

$$\int_0^{\pi} \sin(mx)\sin(nx)\ dx = \begin{cases} 0, & \text{if } m \neq n, \\ \frac{\pi}{2}, & \text{if } m = n. \end{cases} \tag{0.3}$$

Then, using (0.1) and (0.3),

$$\begin{aligned} I_{n,3} &= \int_0^{\pi} \sin^3 x \sin(nx)\ dx \\ &= \frac{1}{4}\int_0^{\pi} (3\sin x - \sin 3x)\sin(nx)\ dx \\ &= \frac{3}{4}\int_0^{\pi} \sin x \sin(nx)\ dx - \frac{1}{4}\int_0^{\pi} \sin 3x \sin(nx)\ dx. \end{aligned}$$

Therefore, for each integer $n \geq 2$,

$$I_{n,3} = \begin{cases} 0, & \text{if } n \neq 3, \\ -\frac{\pi}{8}, & \text{if } n = 3. \end{cases} \tag{0.4}$$

Similarly, using (0.2), (0.3), and (0.4),

$$\begin{aligned} I_{n,5} &= \int_0^\pi \sin^5 x \sin(nx)\ dx \\ &= \frac{1}{16}\int_0^\pi (20\sin^3 x + \sin 5x - 5\sin x)\sin(nx)\ dx \\ &= \frac{5}{4}\int_0^\pi \sin^3 x \sin(nx)\ dx + \frac{1}{16}\int_0^\pi \sin 5x \sin(nx)\ dx - \frac{5}{16}\int_0^\pi \sin x \sin(nx)\ dx. \end{aligned}$$

Thus, for each integer $n \geq 2$,

$$I_{n,5} = \begin{cases} 0, & \text{if } n \neq 3, 5, \\ -\frac{5\pi}{32}, & \text{if } n = 3, \\ \frac{\pi}{32}, & \text{if } n = 5. \end{cases}$$

Remark: In a similar way, it can be shown that, for each integer $n \geq 2$,

$$I_{n,7} = \begin{cases} 0, & \text{if } n \neq 3, 5, 7, \\ -\frac{21\pi}{128}, & \text{if } n = 3, \\ \frac{7\pi}{128}, & \text{if } n = 5, \\ -\frac{\pi}{128}, & \text{if } n = 7. \end{cases}$$

Also solved by **Natham Aguirre, student**, Binghamton University, NY; **Paul S. Bruckman**, Nanaimo, BC; **Cal Poly Pomona Problem Solving Group**, Cal Poly Pomona, Pomona, CA; **Elsie M. Campbell, Dionne T. Bailey, Charles Diminnie, and Andrew Siefker**, Angelo State University, San Angelo, TX; **Hongwei Chen**, Christopher Newport University, Newport News, VA; **Pat Costello**, Eastern Kentucky University, Richmond, KY; **Kenny Davenport**, Dallas, PA; **Alan Dyson, student**, Elizabethtown College, Elizabethtown, PA; **Miguel Lerma**, Northwestern University Problem Solving Group, Evanston, IL; **Carl Libis**, Middle Tennessee State University, Murfreesboro, TN; **Peter Lindstrom**, Batavia, NY; **Aaron Milauskas, Daniel Perrine, and Kari Webster, students**, Taylor University, Upland, IN; **Yoshinobu Murayoshi**, Naha City, Okinawa, Japan; **John R. Piccolo, J, P & R Group**, Weber State University, Ogden, UT; **Skidmore College Problem Group**, Saratoga Springs, NY; and the **Proposer**.

Solvers Pat Costello and Carl Libis pointed out that Mathematica can be used to solve the problem.

1238. *Proposed by Tuan Le, student, Fairmont High School, Fairmont, CA.*

Given $a, b, c, d \in [0, 1]$ such that no two of them are equal to 0. Prove that:

$$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+d^2} + \frac{1}{d^2+a^2} \geq \frac{8}{3+abcd}$$

Solution by **Valmir Bucaj, student**, *Texas Lutheran University, Seguin, TX.*

By the $AM - HM$ inequality we get:

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+d^2}+\frac{1}{d^2+a^2}\geq\frac{8}{a^2+b^2+c^2+d^2}.$$

Now, to conclude the proof, it suffices to show that

$$a^2+b^2+c^2+d^2\leq 3+abcd.$$

First, since $0\leq a,b,c,d\leq 1$, then $a^2+b^2+c^2+d^2\leq a+b+c+d$.

So,

$$\begin{aligned} a^2+b^2+c^2+d^2-abcd &\leq a+b+c+d-abcd\\ &= a(1-bcd)+b+c+d\\ &\leq 1-bcd+b+c+d\\ &= 1+b(1-cd)+c+d\\ &\leq 1+1+c(1-d)+d\\ &\leq 1+1+1-d+d\\ &= 3, \end{aligned}$$

which concludes the proof.

Solution by **Paolo Perfetti**, *Dipartimento di matematica, Università degli Studi di Roma "Tor Vergata", Rome, Italy.*

By Cauchy–Schwarz we have

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+d^2}+\frac{1}{d^2+a^2}\geq\frac{16}{2(a^2+b^2+c^2+d^2)}\geq\frac{8}{3+abcd}$$

or

$$a^2+b^2+c^2+d^2-3-abcd\leq 0$$

The function $f(a,b,c,d)=a^2+b^2+c^2+d^2-3-abcd$ is convex in each variable ($f_{aa}=f_{bb}=f_{cc}=f_{dd}=2$) thus the maximum is attained at one of the sixteen vertices of the four–dimensional cube $[0,1]^4$. Since $f(a,b,c,d)\leq 0$ if each coordinate equals 0 or 1, the result is achieved.

Also solved by **Dionne T. Bailey, Elsie M. Campbell, Charles Diminnie**, Angelo State University, San Angelo, TX; **Paul S. Bruckman**, Nanaimo, BC; **Tamás Dékány, "Fejéntaláltuka Szeged" problem solving group, University of Szeged, Hungary; Moti Levy**, Rehovot, Israel; **David E. Manes**, Oneonta, NY; **Yoshinobu Murayoshi**, Naha City, Okinawa, Japan; **Henry Ricardo**, Tappan, NY; and the **Proposer**.

1239. *Proposed by Matthew McMullen, Otterbein College, Westerville, OH.*

For $i\in\{1,2,\ldots,9\}$, define D_i to be the set of all positive integers that begin with i. For all positive integers n, define

$$a_{n,i}=\frac{1}{n}\cdot\left|D_i\cap\{1,2,\ldots,n\}\right|.$$

Find $\limsup_{n\to\infty} a_{n,i}$ and $\liminf_{n\to\infty} a_{n,i}$.

Solution by **R. Keith Roop-Eckart**, *Columbus State University, Columbus, GA.*

For a given i the Supremums occur at numbers of the form $k = (i+1)10^n - 1$. This is because these numbers are of the form $i99...9$. Increasing k does not increase D_i while decreasing k will decrease D_i, so any change in k will reduce $a_{k,i}$. Thus supremums have the form $(i+1)10^n - 1$. Therefore, for $k = (i+1)10^n - 1$, the number of numbers which begin with digit i and are not exceeding k is $\sum_{j=0}^{n} 10^j = \frac{1-10^{n+1}}{-9}$.

We get:

$$\lim_{n\to\infty} sup\ a_{n,i} = \lim_{n\to\infty} a_{(i+1)10^n-1,i} = \frac{10}{9(i+1)}.$$

Similarly, for a given i, the Infimums occur at numbers of the form $k = i \cdot 10^n - 1$. This is because these numbers are of the form $(i-1)99...9$. Increasing k will cause D_i to increase while decreasing k will leave D_i the same, so any change in k will increase $a_{k,i}$. Thus Infimums have the form $k = i \cdot 10^n - 1$.

Therefore, for $k = i \cdot 10^n - 1$, the number of numbers which begin with i and are not exceeding k is $\sum_{j=0}^{n-1} 10^j = \frac{1-10^n}{-9}$, so $a_{i\cdot 10^n-1,i} = \frac{10^n - 1}{-9i \cdot (10^n - 9)}$.

We get:

$$\lim_{n\to\infty} inf\ a_{n,i} = \lim_{n\to\infty} a_{i\cdot 10^n-1,i} = \frac{1}{9i}.$$

Solution by **Alan Dyson, student**, *Elizabethtown College, Elizabethtown, PA.*

Define $a_{n,i}$ as follows $a_{n,i} = |\{1, 2, ..., 10^k, 10^k+1, ..., 10^{k+1}-1\} \cap D_i| = \sum_{n=1}^{k+1} 10^{n-1}$, i $\in \{1, 2, ..., 9\}$ and $k \geq 0$.

However, note that $a_{n,i}$ is maximized at the lowest possible n such that

$$|\{1, 2, ..., 10^k, 10^k + 1, ..., n\} \cap D_i| = \sum_{n=1}^{k+1} 10^{n-1}.$$

Likewise, $a_{n,i}$ is minimized at the highest possible n such that

$$|\{1, 2, ..., 10^k, 10^k + 1, ..., n\} \cap D_i| = \sum_{n=1}^{k} 10^{n-1}$$

As a result, $a_{n,9} \leq a_{n,i} \leq a_{n,1}$. Let $i = 1$. Choose $n = 2 \cdot 10^k - 1$ such that

$$a_{n,1} = \frac{|\{1, 2, ..., 10^k, ..., 2 \cdot 10^k - 1\} \cap D_i|}{n} = \frac{\sum_{n=1}^{k+1} 10^{n-1}}{2 \cdot 10^k - 1}$$

At this point, $a_{j,i} < a_{n,1}\ \forall j > n$, where $i \in \{1, 2, ..., 9\}$, $j \in \mathbb{N}$.

Define E_p and b_p as follows:

$$E_p = \left\{ \frac{\sum_{n=1}^{k+1} 10^{n-1}}{2 \cdot 10^k - 1} : k \geq p \right\}, p \in \mathbb{N}$$

$$b_p = sup\{E_p = a_{2 \cdot 10^p - 1, 1}\}$$

$$\limsup_{n \to \infty} a_{n,i} = \lim_{p \to \infty} b_p = \frac{10 \sum\limits_{p=1}^{\infty} \frac{1}{10^p}}{1 + 90 \sum\limits_{p=1}^{\infty} \frac{1}{10^{p+1}}} = \frac{5}{9}$$

Let $i = 9$. Choose $n = 9 \cdot 10^k - 1$ such that

$$a_{n,9} = \frac{|\{1, 2, ..., 10^k, ..., 9 \cdot 10^k - 1\} \cap D_i|}{n} = \frac{\sum\limits_{n=1}^{k} 10^{n-1}}{9 \cdot 10^k - 1}$$

At this point, $\mathrm{a}_{j,i} > \mathrm{a}_{9,n}$ $\forall j > n$, where i $\in \{1, 2, ..., 9\}$, $j \in \mathbb{N}$.
Define G_p and c_p as follows:

$$G_p = \left\{ \frac{\sum_{n=1}^{k} 10^{n-1}}{9 \cdot 10^k - 1} : k \geq p \right\}, p \in \mathbb{N}$$

$$c_p = inf\{G_p = a_{9 \cdot 10^p - 1, 9}\}$$

$$\liminf_{n \to \infty} a_{n,i} = \lim_{p \to \infty} c_p = \frac{\sum\limits_{p=1}^{\infty} \frac{1}{10^p}}{8 + 9 \sum\limits_{p=1}^{\infty} \frac{1}{10^p}} = \frac{1}{81}$$

Also solved by **Paul S. Bruckman**, Nanaimo, BC; and the **Proposer**.

1240. *Proposed by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Rome, Italy.*

Let x, y be positive real numbers. Prove that

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} \leq \sqrt{xy} + \frac{x+y}{2} + \frac{(L(x,y) - \sqrt{xy})^2}{\frac{2xy}{x+y}}$$

where $L(x, y) = (x - y)/(\ln(x) - \ln(y))$ if $x \neq y$ and $L(x, x) = x$.

Solution by **Moti Levy**, *Rehovot, Israel.*

Using the notation,

$$H=\frac{2xy}{x+y};\ G=\sqrt{xy};\ L=L\left(x,y\right)=\frac{x-y}{\ln x-\ln y};\ A=\frac{x+y}{2};\ M=\sqrt{\frac{x^2+y^2}{2}}$$

the original inequality may be re-written as

$$\frac{\left(L-G\right)^2}{H}\geq M-A-G+H. \tag{0.1}$$

Using $\frac{G^2}{H}=A$, the inequality becomes

$$\left(\frac{L}{G}-1\right)^2\geq\frac{M-A-G+H}{A}$$

We will later show that

$$M-A-G+H\geq 0. \tag{0.2}$$

Thus our goal is to prove that

$$\frac{L}{G}\geq 1+\sqrt{\frac{M-A-G+H}{A}}. \tag{0.3}$$

Remark: It is well known that $L\geq G$, so inequality (0.3) is sharper version of $\frac{L}{G}\geq 1$.

For $x=y$, both sides of the inequality (0.1) are equal to zero, so w.l.o.g we may assume that $y>x$.

Substitute (after Tung-Po Lin, "The Power Mean and the Logarithmic Mean", in Mathematical Notes, October 1974)

$$\frac{y}{x}=\left(\frac{1+w}{1-w}\right)^2,\ 0\leq w<1$$

then

$$\frac{M-A-G+H}{A}=\frac{\left(1+w^2\right)\sqrt{w^4+6w^2+1}+w^4-4w^2-1}{\left(1+w^2\right)^2}$$

and

$$\frac{L}{G}=\frac{\left(\frac{1+w}{1-w}\right)^2-1}{2\left(\frac{1+w}{1-w}\right)\ln\left(\frac{1+w}{1-w}\right)}=\frac{w/\left(1-w^2\right)}{\frac{1}{2}\ln\left(\frac{1+w}{1-w}\right)}.$$

The inequality (0.3) in terms of the variable w is

$$\frac{w/\left(1-w^2\right)}{\frac{1}{2}\ln\left(\frac{1+w}{1-w}\right)}\geq 1+\frac{\sqrt{\left(1+w^2\right)\sqrt{w^4+6w^2+1}+w^4-4w^2-1}}{1+w^2}$$

Letting

$$f(w) := 1 + \frac{\sqrt{(1+w^2)\sqrt{w^4+6w^2+1}+w^4-4w^2-1}}{1+w^2},$$

our inequality becomes

$$\frac{\frac{w}{1-w^2}}{f(w)} \geq \frac{1}{2}\ln\left(\frac{1+w}{1-w}\right) \tag{0.4}$$

We will show later that

$$1.643\,6 \cong 1+\sqrt{\sqrt{2}-1} > f(w) \geq 1 \quad for \quad 0 \leq w < 1. \tag{0.5}$$

The Maclaurin series of $\frac{w}{1-w^2}$ and $\frac{1}{2}\ln\left(\frac{1+w}{1-w}\right)$ are:

$$\frac{w}{1-w^2} = \sum_{n=0}^{\infty} w^{2n+1} \tag{0.6}$$

and

$$\frac{1}{2}\ln\left(\frac{1+w}{1-w}\right) = \sum_{n=0}^{\infty}\frac{1}{2n+1}w^{2n+1}. \tag{0.7}$$

Now we use (0.6) and (0.7) to manipulate (0.4):

$$\frac{\frac{w}{1-w^2}}{f(w)} = \frac{\sum_{n=0}^{\infty} w^{2n+1}}{f(w)} = \frac{w+w^3+w^5}{f(w)} + \frac{\sum_{n=3}^{\infty} w^{2n+1}}{f(w)};$$

Since $f(w) \leq 1+\sqrt{\sqrt{2}-1} \leq 2n+1$ for $n \geq 3$ then

$$\begin{aligned}\frac{\frac{w}{1-w^2}}{f(w)} &\geq \frac{w+w^3+w^5}{f(w)} + \sum_{n=3}^{\infty}\frac{1}{2n+1}w^{2n+1} \\ &= \frac{w+w^3+w^5}{f(w)} - \left(w+\frac{1}{3}w^3+\frac{1}{5}w^5\right) + \frac{1}{2}\ln\left(\frac{1+w}{1-w}\right)\end{aligned}$$

It follows that if we prove

$$\frac{w+w^3+w^5}{f(w)} - \left(w+\frac{1}{3}w^3+\frac{1}{5}w^5\right) \geq 0 \quad for \quad 0 \leq w < 1$$

or

$$\frac{w+w^3+w^5}{w+\frac{1}{3}w^3+\frac{1}{5}w^5} \geq f(w) \quad for \quad 0 \leq w < 1 \tag{0.8}$$

then we are done.

Inequality (0.8) involves only radicals, so its proof is straightforward but requires some tedious calculations (a nice computer algebra system will do). The following sequence of equivalent inequalities proves inequality (0.8):

$$\frac{1+w^2+w^4}{1+\frac{1}{3}w^2+\frac{1}{5}w^4} \geq 1+\frac{\sqrt{(1+w^2)\sqrt{w^4+6w^2+1}+w^4-4w^2-1}}{1+w^2}$$

$$\left(1+w^2\right)\left(\frac{1+w^2+w^4}{1+\frac{1}{3}w^2+\frac{1}{5}w^4}-1\right) \geq \sqrt{(1+w^2)\sqrt{w^4+6w^2+1}+w^4-4w^2-1}$$

$$\frac{12w^6+22w^4+10w^2}{3w^4+5w^2+15} \geq \sqrt{(1+w^2)\sqrt{w^4+6w^2+1}+w^4-4w^2-1}$$

$$\left(\frac{12w^6+22w^4+10w^2}{3w^4+5w^2+15}\right)^2 \geq \left(1+w^2\right)\sqrt{w^4+6w^2+1}+w^4-4w^2-1$$

$$\left(\frac{12w^6+22w^4+10w^2}{3w^4+5w^2+15}\right)^2-\left(w^4-4w^2-1\right) \geq \left(1+w^2\right)\sqrt{w^4+6w^2+1}$$

$$\frac{135w^{12}+534w^{10}+738w^8+780w^6+590w^4+1050w^2+2251}{\left(3w^4+5w^2+15\right)^2\left(1+w^2\right)} \geq \sqrt{w^4+6w^2+1}$$

$$\left(\frac{135w^{12}+534w^{10}+738w^8+780w^6+590w^4+1050w^2+2251}{\left(3w^4+5w^2+15\right)^2\left(1+w^2\right)}\right)^2 \geq w^4+6w^2+1$$

$$\begin{aligned}&\left(135w^{12}+534w^{10}+738w^8+780w^6+590w^4+1050w^2+2251\right)^2\\&-\left(3w^4+5w^2+15\right)^4\left(1+w^2\right)^2\left(w^4+6w^2+1\right)\\&\geq 0\end{aligned}$$

The left hand side is the polynomial

$$\begin{aligned}&18144w^{24}+142992w^{22}+475992w^{20}+957\,216w^{18}+1387\,928w^{16}\\&+1648\,000w^{14}+2302\,540w^{12}+3320968w^{10}+3268176w^8+2758560w^6\\&+2435\,680w^4+4254\,600w^2+5016\,376\end{aligned}$$

which is indeed non-negative for $0 \leq w < 1$. This completes the proof.

Proof of $M-A-G+H \geq 0$:

It is enough to show that $\left(1+w^2\right)\sqrt{w^4+6w^2+1}+\left(w^4-4w^2-1\right) \geq 0$ for $0 \leq w < 1$ or that

$$\left(1+w^2\right)\sqrt{w^4+6w^2+1} \geq 1+4w^2-w^4$$

Since $1+4w^2-w^4 \geq 0$ for $0 \leq w \leq 1$, then we may square both sides of the inequality

$$\left(1+w^2\right)^2\left(w^4+6w^2+1\right) \geq \left(1+4w^2-w^4\right)^2.$$

But

$$\left(1+w^2\right)^2\left(w^4+6w^2+1\right)-\left(1+4w^2-w^4\right)^2=16w^6,$$

hence $\left(1+w^2\right)\sqrt{w^4+6w^2+1}+w^4-4w^2-1 \geq 0$, which implies that

$$\frac{M-A-G+H}{A} \geq 0.$$

Proof that $f(w)$ is monotone increasing in the interval $[0,1]$:
We differentiate $f(w)$,

$$\frac{df}{dw}=\left(\sqrt{w^4+6w^2+1}\left(w^2-1\right)-w^4+1\right)P(w)$$

for a certain positive function $P(w) \geq 0$.

$$\sqrt{w^4+6w^2+1}\left(w^2-1\right)-w^4+1 \geq \left(w^2-1\right)-w^4+1=w^2\left(1-w^2\right) \geq 0.$$

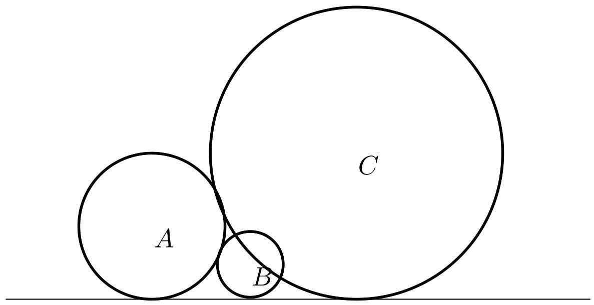
We have shown that $\frac{df}{dw} \geq 0$, hence $f(w)$ is monotone increasing in $[0,1]$. It follows that $1+\sqrt{\sqrt{2}-1}=f(1)>f(w) \geq f(0)=1$ for $0 \leq w<1$.

Also solved by **Paul S. Bruckman**, Nanaimo, BC; **Hongwei Chen**, Christopher Newport University, Newport News, VA; and the **Proposer**.

1241. *Proposed by Arthur Holshouser, Charlotte, NC. and Johannas Winterink, Albuquerque, NM.*

For each $i=0,1,2,3,\ldots,8$, does there exist a set $\{A,B,C\}$ of three circles in the plane such that there are exactly i circles in the plane that are tangent to each of A,B, and C?

Solution by **Richard Hess**, *Rancho Palos Verdes, CA.* The solution is a sequence of nine pictures, one for each value if i from 0 to 8. We give here just the picture for $i=3$. Circles A and B are externally tangent. Circles A and C are also externally tangent. All three circles are tangent to the x-axis. Circle B hits circle C in two places. The three circles tangent to each of A,B and C are as follows. One is in the little 'triangular' region exterior to all three circles, one is internal to C and the third is internal to B.



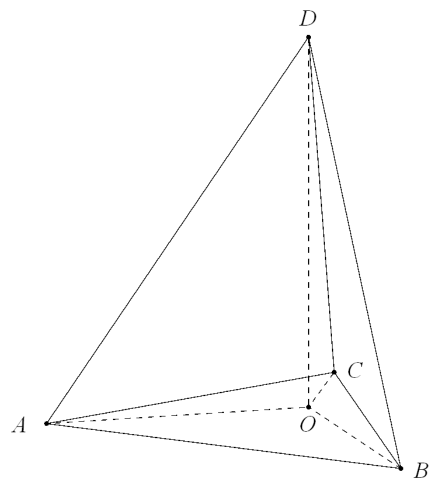
Also solved by the **Proposers**.

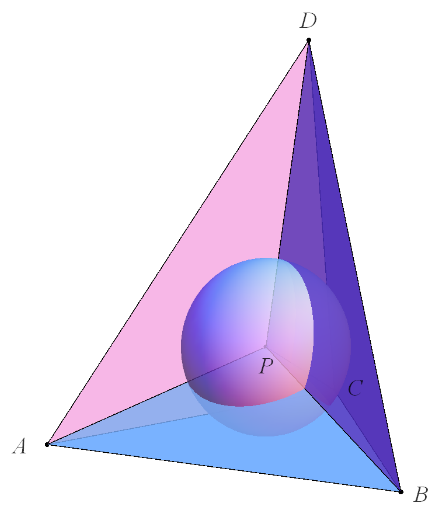
1242. *Proposed by Stas Molchanov, University of North Carolina Charlotte.*

Given pyramid $ABCD$ with bottom face triangle ABC with $BC = a, AC = b, AB = c$. Let the lateral faces BCD, ACD, ABD form with the bottom angles α, β, γ, in radians, all acute angles. Find the radius r of the sphere inscribed in the pyramid.

Solution by **Michael Cheung, student**, *Elizabethtown College, Elizabethtown, PA.*

We find the radius of the inscribed sphere by relating it to the volume of the tetrahedron. Let A_B be the area of the base, which we define to be the triangle ABC. Let h be the corresponding height from the base to the point D. Let O be the perpendicular projection of point D onto the base. First we compute A_B by splitting it into three smaller triangles BOC, COA, and AOB, the heights of which are $h \cot \alpha$, $h \cot \beta$, and $h \cot \gamma$, respectively. The area of the base of the tetrahedron is then $A_B = h\left(a \cot \alpha + b \cot \beta + c \cot \gamma\right)/2$. From Heron's formula, we also have $A_B = \sqrt{a+b-c}\sqrt{a-b+c}\sqrt{-a+b+c}\sqrt{a+b+c}$.





Let P be the center of the inscribed sphere. We compute the volume of the tetrahedron, $V = A_B h/3$, by splitting it into four smaller tetrahedra, $ABCP$, $BCDP$, $CDAP$, and $DABP$, whose bases are the four sides of the original tetrahedron and whose common height is the radius r of the inscribed sphere. Then

$$V = \frac{1}{3} r \left[\frac{1}{2} h \left(a \csc \alpha + b \csc \beta + c \csc \gamma \right) + A_B \right].$$

Equating the two expressions for V and solving for r gives

$$r = \frac{A_B}{\frac{1}{2} \left(a \csc \alpha + b \csc \beta + c \csc \gamma \right) + A_B/h},$$

and substituting the two expressions for A_B to remove the dependency on h gives

$$r = \frac{\sqrt{a+b-c}\sqrt{a-b+c}\sqrt{-a+b+c}\sqrt{a+b+c}}{2 \left[a \left(\csc \alpha + \cot \alpha \right) + b \left(\csc \beta + \cot \beta \right) + c \left(\csc \gamma + \cot \beta \right) \right]}.$$

Also solved by **Eugen J. Ionascu**, Columbus State University, Columbus, GA; **Nikolay Rangelov, student**, Skidmore College Problem Group, Saratoga Springs, NY; and the **Proposer**.